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Phys. Rev. D **91**, 064026 — Published 9 March 2015

DOI: 10.1103/PhysRevD.91.064026

Conserved quantities and dual turbulent cascades in Anti–de Sitter spacetime

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We consider the dynamics of a spherically symmetric massless scalar field coupled to general relativity in Anti–de Sitter spacetime in the small-amplitude limit. Within the context of our previously developed two time framework (TTF) to study the leading self-gravitating effects, we demonstrate the existence of two new conserved quantities in addition to the known total energy E of the modes: The particle number N and Hamiltonian H of our TTF system. Simultaneous conservation of E and N implies that weakly turbulent processes undergo dual cascades (direct cascade of E and inverse cascade of N or vice versa). This partially explains the observed dynamics of 2-mode initial data. In addition, conservation of E and N limits the region of phase space that can be explored within the TTF approximation and in particular rules out equipartition of energy among the modes for general initial data. Finally, we discuss possible effects of conservation of N and E on late time dynamics.

I. INTRODUCTION

We are interested in the question of stability of Anti-de Sitter (AdS) spacetime in general relativity. Do small perturbations generically collapse to a black hole or do they propagate forever? In contrast to Minkowski spacetime, field perturbations of AdS are effectively confined by the asymptotically AdS boundary condition and cannot dissipate by dispersing to null infinity. Thus, even for arbitrarily small perturbations, self-interactions eventually play a dominant role in the dynamics, and the question of collapse is much more difficult.

In order to make the problem more tractable, we restrict to spherically symmetric perturbations. A real, minimally coupled, free, massless scalar field (coupled to gravity) is studied to provide a dynamical degree of freedom. This system was first treated numerically in AdS_4 in [1] (for the non-spherically symmetric gravitational case see Ref. [2], and for the complex scalar field see Ref. [3]). The authors of Ref. [1] considered initial data with a Gaussian profile, and they varied the amplitude of the initial pulse. For sufficiently large pulse amplitude, they found that black hole collapse occurs promptly, consistent with expectations in asymptotically flat spacetimes. Moreover, collapse is no longer prompt once the amplitude is decreased below a critical threshold [4]. Instead, the pulse propagates to infinity, and then is reflected back by the AdS boundary. Once it reaches the origin, it has another opportunity to collapse to a black hole. This time, however, gravitational focusing has caused the pulse to become more peaked, and the chance of collapse is increased. As the amplitude is decreased further, multiple bounces may be required before collapse. Reference [1] found that collapse always occurred even for very small-amplitude initial data, albeit after a very large number of bounces. This behavior motivated the conjecture that collapse to a black hole was unavoidable in AdS regardless of the initial perturbation and its strength (except for single-mode initial data [1, 5]).

The observed behavior is a manifestation of a weakly turbulent cascade, where energy is transferred to short distance scales through gravitational focusing. A free, non-gravitating, spherically symmetric, real scalar field in AdS_4 is characterized by[1] a set of normal modes $e_j(x)$ ($j = 0, 1, 2, \ldots$), with frequencies $\omega_j = 2j + 3$. In mode-space, the cascade corresponds to a transfer of energy to high-frequency modes. Since the frequency spectrum is commensurate, the nonlinear gravitational interactions are resonant, which means energy is transferred between modes quite readily.

This picture was complicated, however, by subsequent numerical simulations [6] that showed that if the Gaussian pulse profile was broadened (within a certain range), there was a critical amplitude, below which collapse was no longer seen. It was argued that there was a nonlinear field dispersion process that competes with the gravita-

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[1] Adding a mass or changing the spacetime dimension does not significantly alter this statement. The mode functions and frequencies change, but the frequencies remain linear in j.

tional focusing. Thus, if dispersion dominates, then collapse can potentially be averted. The problem was thus more complicated than originally thought. Because of the intrinsic limitations of finite, numerical simulations, it demands a more comprehensive perturbative analysis.

In previous work [7], we analyzed the leading resonant interactions in a two-timescale analysis. We introduced a slow time $\tau \equiv \epsilon^2 t$, and expanded the scalar field as $\phi(t,x) = \epsilon\phi_{(1)}(t,\tau,x) + O(\epsilon^3)$ (similar expansions apply for the metric). The general solution to the leading-order scalar field equation is

$$\phi_{(1)}(t,\tau,x) = \sum_{j=0}^{\infty} \left(A_j(\tau)e^{-i\omega_j t} + \bar{A}_j(\tau)e^{i\omega_j t}\right) e_j(x), \quad (1)$$

where the functions $A_j(\tau)$ are—at this point—undetermined. Were we to not introduce the slow time, A_j would be constant, and the resonant self-gravity interactions would lead to secular growths and a breakdown of perturbation theory at $O(\epsilon^3)$ [1]. However, as we showed in [7], the secular growths can be eliminated if we choose the mode amplitudes to satisfy the "two time framework" (TTF) equations

$$-2i\omega_j \frac{dA_j}{d\tau} = \sum_{klm} \mathcal{S}^{(j)}_{klm} \bar{A}_k A_l A_m, \quad (2)$$

where the $\mathcal{S}^{(j)}_{klm}$ are (real) numerical coefficients arising from overlap integrals of mode functions. (We computed the coefficients up to $j = j_{\max} = 47$.) The TTF equations capture (a) the effect of ϕ on the metric g_{ab}, and (b) the backreaction of g_{ab} on ϕ. In other words, ϕ can only self-interact via g_{ab} as an intermediary, and for this reason the TTF equations have cubic interactions[2].

The specific form of the equations (2) follows from the fact that the only resonances that are actually present in our system are those such that

$$\omega_j + \omega_k = \omega_l + \omega_m. \quad (3)$$

Indeed from the basic structure of the Einstein-scalar field equations it is *a priori* possible to have resonances satisfying $\omega_j \pm \omega_k = \pm\omega_l \pm \omega_m$ for arbitrary sign combinations, and this would introduce additional terms on the right hand side of (2) (e.g., $\bar{A}_k\bar{A}_l A_m$, etc.). However we found [7] that in practice the only non-zero terms are those of (2) (see also [1]). This statement was rigorously proven in [9].

We argued in [7] that solutions to the TTF equations provide a good approximation to solutions of the full system in the limit $\epsilon \to 0$ and for time scales $t \sim 1/\epsilon^2$. By solving the system of coupled ODEs (2), one can study, e.g., the transfer of energy $E_j(\tau) = 4\omega_j^2|A_j(\tau)|^2$ between the modes—without having to consider the rapid normal-mode harmonic oscillations. We checked previously that, while $E_j(\tau)$ can have very nontrivial time-dependence, the sum $E \equiv \sum_j E_j$ is conserved.

As we will describe in section II, there are two additional quantities that are conserved by the system of TTF equations[3]. The first,

$$N \equiv \sum_j 4\omega_j |A_j|^2, \quad (4)$$

can be interpreted as the "particle number" of the field. In terms of spacetime fields, this is the charge current of the positive frequency part of ϕ. (Since ϕ is real, the charge current of the field itself vanishes.) The second additional conserved quantity is the Hamiltonian function H of our TTF system (2). This quantity is quartic in A_j and appears to represent the interaction energy of the TTF fields. All three quantities (N, E and H) are only well-defined and conserved at the perturbative level captured by TTF, and it is not clear to what degree they extend to the fully nonlinear theory.

In section III, we describe how the presence of multiple conserved quantities (as opposed to solely the energy) implies that the turbulent dynamics must be more complex than originally thought [1]. This is well-known, for example, in the context of incompressible, inviscid two-dimensional hydrodynamics, where the existence of a second conserved quantity induces an inverse cascade of energy, in contrast to the standard direct energy cascade in higher dimensions [10, 11]. Likewise, in the TTF system (2), conservation of E and N implies that the energy cannot all be transferred to high-j modes, as this would violate conservation of N. As higher frequency modes have higher energy per particle, a given amount of energy gives rise to a lower population of particles. To conserve particle number as energy transfers to high modes, lower modes must also become more populated—an effect we have previously observed for 2-mode initial data [7]. Thus, if there is a direct cascade of E, there must simultaneously be an inverse cascade of N and vice versa. We therefore have an alternative explanation for the process of competing dispersion and focusing described in [6].

Finally, in section IV we explore the implications of conservation laws on possible end-states of evolution. Simultaneous conservation of N and E excludes certain regions of phase space and, in particular, generically, does not allow for energy equipartition. Instead, one can use N and E conservation to associate our initial data to recently uncovered quasi-periodic solutions [7]. In particular, if a given solution lies sufficiently close to its associated quasi-periodic solution then it is really a perturbation about that solution. It follows that if the

[2] TTF is universal when Einstein gravity is replaced with Gauss-Bonnet gravity [8]; however it does require intrinsic self-interactions of the scalar field(s) to be subleading compared to gravitationally induced self-interactions.

[3] These properties apply also for massive fields and in different dimensions, since the structure of the TTF equations is the same.

solution lies within the radius of stability of the quasi-periodic solution, then it cannot collapse. This is a generalization of the claim that broad Gaussian initial data is "close" to stable time-periodic solutions [12]. An alternative possibility is that a non-collapsing behavior can be described by stochasticity among the stable basins of these quasi-periodic solutions [13] as argued by Ref. [14] for the Fermi-Pasta-Ulam problem [15].

In this paper we follow the notation and definitions of [7]. We note that as we were preparing this manuscript a separate work also identified the new conserved quantities [16] within the perturbative resummation technique presented in [9] (see also [17] for the case of a non-gravitating probe scalar). As discussed in [9], the equations of motion obtained to third order are identical to those from the TTF approach.

II. CONSERVED QUANTITIES

A. Total mode energy and total particle number

By using the TTF equations (2) and the specific coefficients $\mathcal{S}^{(j)}_{klm}$ we derived up to $j = j_{\max} = 47$, it is easy to check directly that $E = \sum_j 4\omega_j^2 |A_j|^2$ and $N = \sum_j 4\omega_j |A_j|^2$ are conserved in time (up to $j = j_{\max}$). (We did this for E in [7], and this was also how we first identified N.) However, the conservation laws are more general in that they follow simply from symmetry properties of $\mathcal{S}^{(j)}_{klm}$ (and not the specific values these coefficients), as we will show below.

Because of its similarity to expressions in QFT, we interpret the quantity N as the total "particle number." However, we note that this is a continuous system, and there are no discrete particles.

1. *Derivation*

To proceed, we assume only certain basic symmetry properties, which hold[4] for the coefficients $\mathcal{S}^{(j)}_{klm}$,

(i) $\mathcal{S}^{(j)}_{klm} = \mathcal{S}^{(k)}_{jlm}$,

(ii) $\mathcal{S}^{(j)}_{klm} = \mathcal{S}^{(j)}_{kml}$,

(iii) $\mathcal{S}^{(j)}_{klm} = \mathcal{S}^{(l)}_{mjk}$,

(iv) $\mathcal{S}^{(j)}_{klm} = 0$ unless $j + k = l + m$.

Given symmetries (i)–(iii), we now show that conservation of E is precisely equivalent to (iv). The latter reflects the resonance condition (3). Conservation of N follows automatically from (i)–(iii) alone.

We first note that, as a result of (2),

$$\begin{aligned}\frac{d}{d\tau}|A_j|^2 &= \bar{A}_j \frac{dA_j}{d\tau} + A_j \frac{d\bar{A}_j}{d\tau} \\ &= \frac{i}{2\omega_j} \sum_{klm} \mathcal{S}^{(j)}_{klm} \left(\bar{A}_j \bar{A}_k A_l A_m - A_j A_k \bar{A}_l \bar{A}_m \right) \\ &= \frac{1}{\omega_j} \sum_{klm} \mathcal{S}^{(j)}_{klm} \Im \left(A_j A_k \bar{A}_l \bar{A}_m \right). \end{aligned} \tag{5}$$

Conservation of E and N are then

$$0 = \frac{dE}{d\tau} = \sum_{jklm} 4\omega_j \mathcal{S}^{(j)}_{klm} \Im \left(A_j A_k \bar{A}_l \bar{A}_m \right), \tag{6}$$

$$0 = \frac{dN}{d\tau} = \sum_{jklm} 4\mathcal{S}^{(j)}_{klm} \Im \left(A_j A_k \bar{A}_l \bar{A}_m \right), \tag{7}$$

for all possible values of the A_j.

Consider the energy E first. We look at the contribution of terms with particular choices of indices.

Case 1. j, k, l, m all identical

We have $\Im \left(A_j A_k \bar{A}_l \bar{A}_m \right) = \Im \left(|A_j|^4 \right) = 0$, so this type of term does not contribute, and the right hand side of (6) vanishes automatically, imposing no constraints on the coefficients $\mathcal{S}^{(j)}_{klm}$.

Case 2. $j = l = m$, k distinct

Imposing the symmetries (i)–(iii), terms with these particular indices contribute

$$\begin{aligned}\frac{dE}{d\tau} &\supset 8\omega_j S^{(j)}_{jjk} \Im (A_j^2 \bar{A}_j \bar{A}_k) \\ &\quad + 4(\omega_j + \omega_k) S^{(j)}_{kjj} \Im (A_j A_k \bar{A}_j^2) \\ &= 4\Im (A_j^2 \bar{A}_j \bar{A}_k) \left[2\omega_j S^{(j)}_{jjk} - (\omega_j + \omega_k) S^{(j)}_{kjj} \right]. \end{aligned} \tag{8}$$

This vanishes if (iv) holds.

Conversely, assume that E is conserved. The right hand side of (6) must vanish for all values of the A_j's. Now setting only A_j, A_k to be nonzero implies $S^{(j)}_{jjk} = S^{(j)}_{kjj} = 0$.

Case 3. $j = l$, $k = m$ distinct

This contributes

$$\frac{dE}{d\tau} \supset 4\Im (A_j^2 \bar{A}_k^2) \left[\omega_j \mathcal{S}^{(j)}_{jkk} - \omega_k \mathcal{S}^{(k)}_{kjj} \right]. \tag{9}$$

Since $\omega_j \neq \omega_k$, (iv) here is equivalent to the vanishing of this contribution, as above.

[4] For j, k, l, m pairwise distinct (e.g., $j, k, l, m = 0, 1, 0, 1$) symmetries (i) and (iii) only hold in the gauge where the metric function $\delta = 0$ at the AdS boundary. This gauge was used in [3, 16], whereas [1, 7, 9] set $\delta = 0$ at the origin. (For 1, 3 or 4 distinct indices $S^{(j)}_{klm}$ satisfies all the symmetries above in both gauges). Since the symmetries are needed in Sec. II B to write down a Hamiltonian formulation, we therefore assume we are working in the gauge where $\delta = 0$ at the AdS boundary. See [16] for further discussion of this point.

Case 4. $j = m$, k, l distinct

The symmetries imply that the contribution to $dE/d\tau$ is

$$\frac{dE}{d\tau} \supset 4\Im(A_j^2 \bar{A}_k \bar{A}_l)\mathcal{S}^{(j)}_{jkl}\left[2\omega_j - (\omega_k + \omega_l)\right]. \qquad (10)$$

Vanishing of the multiplier implies either $\mathcal{S}^{(j)}_{jkl} = 0$ or $2\omega_j = \omega_k + \omega_l$.

Case 5. j, k, l, m all distinct

In this last case, the contribution is

$$\begin{aligned}\frac{dE}{d\tau} \supset 8 \Big\{ & \Im(A_j A_k \bar{A}_l \bar{A}_m)\mathcal{S}^{(j)}_{klm}\left(\omega_j + \omega_k - \omega_m - \omega_l\right) \\ & + \Im(A_j A_l \bar{A}_k \bar{A}_m)\mathcal{S}^{(j)}_{lkm}\left(\omega_j + \omega_l - \omega_k - \omega_m\right) \\ & + \Im(A_j A_m \bar{A}_k \bar{A}_l)\mathcal{S}^{(j)}_{mkl}\left(\omega_j + \omega_m - \omega_k - \omega_l\right)\Big\} \end{aligned} \qquad (11)$$

Since the coefficients containing the A_j's can be made to be nonzero independently, we again see the equivalence of symmetry (iv) and the vanishing of the term.

Thus, it follows that conservation of E is precisely equivalent to condition (iv), given (i)–(iii). Conservation of N can be analyzed by replacing all ω_j's in the expressions above by 1. All terms then vanish by (i)–(iii).□

2. Discussion

From a spacetime perspective, conservation of N and E is somewhat surprising. Indeed, E is *not* the total energy of the system, as it neglects interaction energy. Meanwhile, our scalar field is real so we do not expect any sort of associated charge N. What this TTF analysis shows is that at a low (but nonlinear) perturbative order, and for times $t \sim 1/\epsilon^2$, these quantities are in fact conserved.

To see how N and E arise as spacetime quantities, consider a treatment of ϕ at the *linear* level. It should be kept in mind that at the linear level, there is no exchange of energy between modes, so conservation of N and E is trivial. Nevertheless, this treatment is useful as there is a precise correspondence between our definitions of E and N and spatial integrals of field quantities. Indeed, at this order, E is just the spatial integral of the time-time component of the stress-energy of $\phi^{(1)}$, living in an exact AdS background,

$$E = \int_0^{\pi/2} \left(|\partial_t \phi^{(1)}|^2 + |\partial_x \phi^{(1)}|^2\right) \tan^2 x \, \mathrm{d}x. \qquad (12)$$

Plugging in the expression (1) with each $A_j(\tau) =$ constant, we recover the standard mode sum expression for E.

For the particle number, split the field into positive and negative frequency parts, $\phi^{(1)} = \phi^{(1)}_+ + \phi^{(1)}_-$, with

$$\phi^{(1)}_+ = \sum_j A_j e^{-i\omega_j t} e_j(x). \qquad (13)$$

The positive frequency part of $\phi^{(1)}$ *is* a complex field, so one can define its charge current. One may verify that the integral of its time component is the particle number,

$$N = 2i \int_0^{\pi/2} \left(\overline{\phi^{(1)}_+}\partial_t \phi^{(1)}_+ - \phi^{(1)}\partial_t \overline{\phi^{(1)}_+}\right) \tan^2 x \, \mathrm{d}x. \qquad (14)$$

Beyond linear order it is not obvious how to define the positive frequency part of the field, nor is it obvious that N and E are conserved from a spacetime perspective. Fortunately, TTF provides a way to perturbatively define quantities that are conserved.

B. Hamiltonian

In addition, the TTF system is Hamiltonian. To see this, we first rescale our fields by defining $\hat{A}_j = \sqrt{\omega_j} A_j$. Then the Hamiltonian takes the form

$$H = -\frac{1}{4} \sum_{jklm} \frac{\mathcal{S}^{(j)}_{klm}}{\sqrt{\omega_j \omega_k \omega_l \omega_m}} \bar{\hat{A}}_j \bar{\hat{A}}_k \hat{A}_l \hat{A}_m. \qquad (15)$$

It may be verified that Hamilton's equations

$$i \frac{d\hat{A}_i}{d\tau} = \frac{\partial H}{\partial \bar{\hat{A}}_i} \qquad (16)$$

reproduce the equations of motion (2). Indeed, Hamilton's equations give

$$\begin{aligned} & -2i\omega_i \frac{dA_i}{d\tau} \\ &= -2i\sqrt{\omega_i}\frac{d\hat{A}_i}{d\tau} \\ &= -2i\sqrt{\omega_i}\frac{\partial H}{\partial \bar{\hat{A}}_i} \\ &= \frac{\sqrt{\omega_i}}{2} \sum_{jklm} \frac{\mathcal{S}^{(j)}_{klm}}{\sqrt{\omega_j \omega_k \omega_l \omega_m}} \left(\delta^i_j \bar{\hat{A}}_k \hat{A}_l \hat{A}_m + \bar{\hat{A}}_j \delta^i_k \hat{A}_l \hat{A}_m\right) \\ &= \sum_{jkl} \frac{\mathcal{S}^{(i)}_{jkl}}{\sqrt{\omega_j \omega_k \omega_l}} \bar{\hat{A}}_j \hat{A}_k \hat{A}_l \\ &= \sum_{jkl} \mathcal{S}^{(i)}_{jkl} \bar{A}_j A_k A_l. \end{aligned} \qquad (17)$$

[On the second last line we used the symmetry property (i) of $\mathcal{S}^{(j)}_{klm}$.] H itself is thus a conserved quantity.

It may be tempting to try to associate the Hamiltonian with the next-order contribution to the ADM mass

M of the spacetime. (Note that H itself is not the total energy because we have performed a two-time expansion to factor out the fast time.) Indeed, the ADM mass is a conserved quantity in general relativity, and it includes all of the kinetic and potential energy in the field ϕ, as well as the metric. The energy E, by contrast, only contains the energy of ϕ to quadratic order, while the expression (15) for H certainly looks like a potential energy that could be the fourth order contribution to M. However, one would also expect additional contributions to M at quartic order, such as contributions from higher order TTF corrections (involving additional slower time variables). Further analysis would therefore be required to elucidate any relationship between H and the ADM mass.

It is somewhat surprising that E and H are conserved independently. This could simply be an unphysical artifact of our expansion, or it could be indicative of a whole family of higher order conserved quantities, which would indicate integrability of the system. It may be illuminating to numerically monitor the behavior of E and H within the fully nonlinear system, at least over time scales where we expect validity of TTF ($t \sim 1/\epsilon^2$).

III. DUAL CASCADE

For the remainder of this paper, we restrict our analysis to examining the implications of simultaneous conservation of E and N. Within the context of turbulence, the presence of a second conserved quantity (in addition to the energy) indicates the occurrence of dual cascades. That is, if one quantity is cascading to higher modes, the other must be simultaneously cascading to low modes. As noted in the introduction, this has been observed, for instance, in inviscid incompressible two-dimensional fluid dynamics[5], where the *enstrophy* Ω (the integral of the vorticity squared) is conserved in addition to the energy $\mathcal{E}$. It can be shown that enstrophy undergoes a direct cascade, which forces energy to cascade in the opposite direction, leading to the formation of increasingly large vortices [10, 11].

In momentum space,

$$\mathcal{E} = \int E(k)\,\mathrm{d}k, \qquad \Omega = \int k^2 E(k)\,\mathrm{d}k, \tag{18}$$

with $E(k)$ the fluid energy at wavenumber k. On the other hand, for the TTF system we have

$$E = \sum_j E_j, \qquad N = \sum_j (2j+3)^{-1} E_j. \tag{19}$$

[5] This cascading behavior is also present in the viscous/compressible case, but its analysis is sightly more involved.

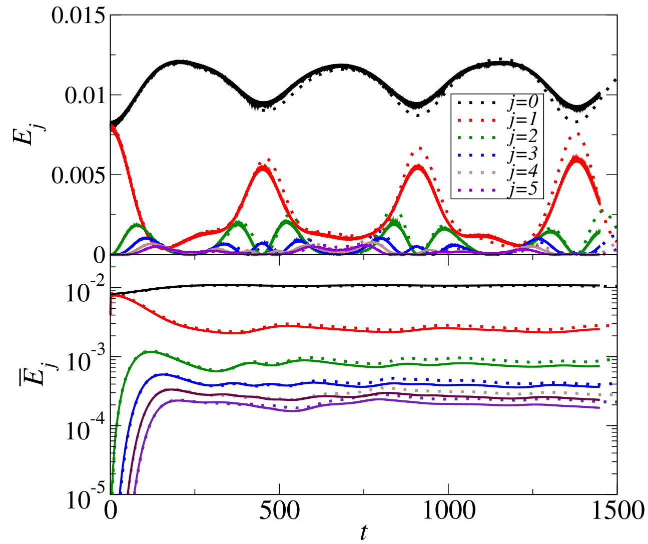


FIG. 1. Full numerical (solid) and TTF (dotted) energy (top panel) and running time-average energy (bottom panel) per mode, for 2-mode equal-energy initial data. The transfer of energy from mode $j = 1$ into mode $j = 0$ is a consequence of N conservation. Note that there is some uncertainty as to whether collapse occurs around $t \approx 1080$ [18], but this is not relevant to our discussion of dual turbulent cascades.

There is thus a natural parallel in both cases. However, the difference in the exponent of the wave number (k or j) has an important consequence. Namely, in the hydrodynamic case, energy must flow to longer wavelengths and enstrophy to higher ones irrespective of the wavenumber (assuming there is no upper limit to the wavelengths allowed). On the other hand, in the scalar case energy flows both to low and high wavenumbers by similar amounts in the high wavenumber regime (where the 3 can be neglected compared to $2j$).

For now, let us concentrate on understanding the dual cascade process in the scalar case. Suppose that, under time evolution, all of the energy E could be transferred to higher-j modes. In that case, since higher-j modes have more energy per particle ($E_j/N_j = \omega_j$), this process would violate particle number conservation. Instead, in order to conserve N some energy must transfer to lower-j modes, which are less energetic.

This phenomenon has in fact already been observed in our previous work [7], when we studied the evolution of initial data with the energy distributed evenly between the two lowest modes. We reproduce in figure 1 our prior results. Notice that the energy initially flows *into* mode $j = 0$ from mode $j = 1$. This behavior is now understood in light of the second conserved quantity, for if all the energy were to flow to higher modes, then N would not be conserved. The correspondence between full numerics and TTF shown in figure 1 also indicates that conservation of N is not merely an effect of our perturbation scheme.

IV. NON-EQUIPARTITION OF ENERGY AND STABILITY

That the Fermi-Pasta-Ulam (FPU) system of nonlinearly coupled oscillators did not reach equipartition for small initial energy was quite surprising and was seminal in the development of nonlinear dynamics [15, 19, 20]. In the current context of a scalar field in AdS, if a black hole does not form, then the question of thermalization of the holographically dual CFT is complicated. Recall that evaporation of small black holes gives rise to thermal states and so naturally bridges the initial state on the dual CFT to its final thermal one, represented by a partially evaporated black hole in equilibrium with surrounding Hawking radiation [21]. Consequently, failure to yield a black hole raises the question of whether and how the corresponding CFT will achieve a thermal state. One should take note here that our calculations within Einstein gravity are limited to the classical regime (thus ignoring $1/N^2$ corrections) and also ignore possible higher curvature modifications[6].

With such caveats in mind, one can show that within TTF, the fact that both N and E are conserved implies that energy equipartition cannot occur. To see this, assume some initial state has energy E and particle number N. Then, if equipartition were to occur, each mode would have energy $E_j = E/(j_{\max}+1)$ (where we truncate our system at $j = j_{\max}$). But that would imply that the total particle number is

$$N_{\text{final}} = \sum_{j=0}^{j_{\max}} \frac{E_j}{\omega_j} = \sum_{j=0}^{j_{\max}} \frac{E}{\omega_j (j_{\max}+1)} = \frac{H_{j_{\max}+\frac{3}{2}} - 2 + \log 4}{2(j_{\max}+1)} E, \tag{20}$$

where H_n is the nth harmonic number. Unless finely tuned initially, this value will not equal N and therefore will be excluded dynamically.

In fact, one can go further and place bounds on regions of phase space that are excluded by conservation of N and E. We illustrate this in figure 2. The blue curve in figure 2a represents the path of the 2-mode solution of figure 1 in (E_0, E_1)-space. It is confined to lie within the shaded blue region between the lines drawn on the plot. The two upper boundary lines arise because of E (dashed line) and N (dotted line) conservation,

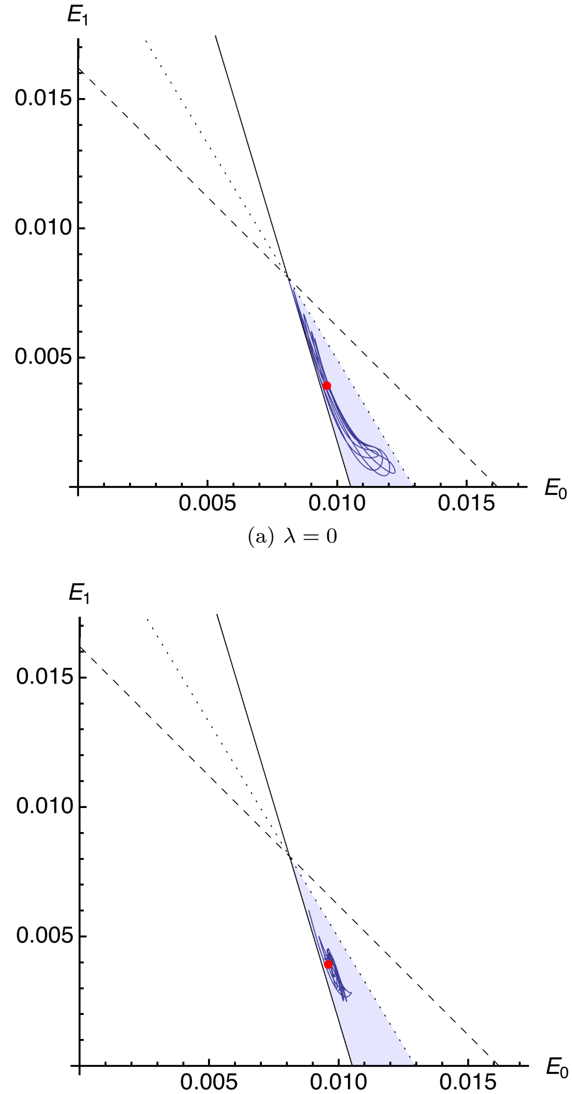
$$E_0 + E_1 \leq E, \tag{21}$$

$$\frac{E_0}{3} + \frac{E_1}{5} \leq N = \frac{4}{15} E. \tag{22}$$

The lower bound (solid line) arises because only a certain amount of energy can flow to the higher modes. The energy that can flow to higher modes is maximized if it flows into mode $j = 2$ because its particles are least energetic. So we write E and N conservation as

$$E_0 + E_1 + E_2 + \sum_{j>2} E_j = E, \tag{23}$$

$$\frac{E_0}{3} + \frac{E_1}{5} + \frac{E_2}{7} + \sum_{j>2} \frac{E_j}{2j+3} = N = \frac{4}{15} E. \tag{24}$$



(a) $\lambda = 0$

(b) $\lambda = 0.5$

FIG. 2. Paths of E_0 and E_1 for (a) 2-mode initial data of figure 1 (blue curve; TTF solution with $j_{\max} = 47$) and (b) interpolation between 2-mode and quasi-periodic initial data. Red dot represents the quasi-periodic solution with the same values of total E and N as the 2-mode solution. Solution is constrained by E and N conservation to lie within the blue shaded region.

[6] Within the class of Gauss-Bonnet corrections, it has been recently pointed out that many configurations avoid black hole collapse [8, 22].

Eliminating E_2, we obtain our bound,

$$\frac{4}{3}E_0 + \frac{2}{5}E_1 = \frac{13}{15}E + \sum_{j>2}\left(1 - \frac{7}{2j+3}\right)E_j$$
$$\geq \frac{13}{15}E, \qquad (25)$$

since the last term on the first line is non-negative.

Given the bounds, it is clear that energy equipartition is not possible, since equipartition occurs at $(E_0, E_1) = (1/(j_{\max}+1), 1/(j_{\max}+1)) \to (0,0)$ as $j_{\max} \to \infty$. More generally, the bounds arising from the conservation laws limit the amount of energy that can cascade to high modes.

As equipartition is ruled out, one can consider other possible late time configurations where some energy has transferred to high-j modes. For example, power laws of the form $E_j \sim (j+1)^{-\alpha}$ were observed in numerical evolutions just prior to collapse for Gaussian initial data [23, 24]. It is straightforward to determine, for a given α, the corresponding value of E/N for the configuration. Larger α (steeper power laws) correspond to smaller values of E/N, with $E/N \searrow 3$ as $\alpha \to \infty$ in AdS$_4$. It is intriguing to note that the $\sigma = 1/16$ Gaussian initial data of [1] is characterized by the power law with $\alpha = 1.15$; not that far from the value $\alpha = 1.2$ observed just prior to collapse [23]. The $\sigma = 0.4$ Gaussian initial data of [6]—that appears to avoid collapse—has $E/N = 3.43$, which corresponds to a much steeper power law of $\alpha = 2.60$. The 2-mode equal-energy configuration has $E/N = 3.75$, corresponding to $\alpha = 2.15$. The quantity E/N may therefore indicate whether given initial data is likely to collapse for small initial amplitude. However this cannot be the full story, since single-mode initial data has $E/N = \omega_j$ (which can be arbitrarily large for large j), and is expected not to collapse [5].

In light of the fact that we know that both N and E are conserved, we turn now to a re-examination of a class of quasi-periodic solutions we previously identified within the context of TTF. In [7], we looked for solutions with harmonic τ-dependence in each mode, $A_j(\tau) = \alpha_j e^{-i\beta_j\tau}$, with $\beta_j \in \mathbb{R}$ (so that the energy in each mode is constant). We found that, given a particular discrete choice of a dominant mode, there exists a two-parameter family of quasi-periodic solutions that generalize[7] the one-parameter periodic solutions of [1]. These solutions have approximately exponential energy spectrum to both sides of the dominant mode (see figure 1 of [7]), with the exponential decay rate one of the two parameters (the other being the overall amplitude). We could only find such solutions for sufficiently fast exponential fall-off. Under fully nonlinear numerical evolution, the solutions appeared to be stable.

[7] The periodic solutions of [1] were extended to higher nonlinear order in [5]. We have not attempted this analysis for our quasi-periodic solutions, however numerical evidence indicates stability in the full theory.

Instead of taking the two parameters characterizing a quasi-periodic solution to be the amplitude and the exponential decay rate of the energy spectrum, we can equivalently take them to be N and E. Thus, given a choice of a dominant mode, and values for N and E within a suitable range, there is a quasi-periodic TTF solution that appears to be stable under the full numerical evolution.

For our 2-mode initial data, we can extract N and E, and construct a quasi-periodic solution with these values, based around the dominant mode $j = 0$. This is presented as the red dot in figure 2a. We see that the 2-mode solution seems to oscillate around the quasi-periodic solution rather than filling the entire shaded blue region available to it. Following the q-breather approach to understanding the FPU problem [25, 26], we can ask whether the 2-mode solution should be thought of as a perturbation about the associated quasi-periodic solution (which itself is a generalization of a q-breather to the case with both N and E conserved). To study this possibility, we considered initial data that interpolates (with parameter λ) between the 2-mode initial data ($\lambda = 0$) and the quasi-periodic initial data ($\lambda = 1$). In figure 2b we present the results for $\lambda = 0.5$. Notice that the solution is confined to a smaller region in (E_0, E_1)-space around the quasi-periodic solution. This is to be expected if it can be treated as a perturbation about the quasi-periodic solution.

As λ is varied from 0 to 1, the dynamics of $E_j(\tau)$ smoothly deform to smaller oscillations, but maintain the overall structure of recurrences seen in figure 1. In particular, the time of the first recurrence changes by less than a factor of two. This supports the claim that the 2-mode solution can be regarded as a perturbation (albeit a large one) of the quasi-periodic solution. In that case, the recurrences are merely oscillations about the quasi-periodic solution. (This doesn't preclude collapse, as the 2-mode solution can in principle be an unstable perturbation of the quasi-periodic solution[8].) If in the full theory, the quasi-periodic solutions persist and are stable, and if a given solution lies within the radius of stability, then it must avoid collapse.

Alternatively, as argued in [13, 14], this behavior could be illustrating stochasticity between the stability regions of each individual mode. Both of these options are plausible explanations for the possible stability of 2-mode data.

V. FINAL COMMENTS

The question of stability of AdS has recently received a great deal of attention, as it is of interest within general relativity, and it has holographic applications. While the

[8] There is some disagreement in the literature as to whether our particular 2-mode data collapses or not in numerical studies [7, 18].

full answer to this question is still outstanding, a combination of numerical simulations and perturbative studies has provided many important insights (see e.g. [1–3, 5–9, 16, 18, 27, 28]). Using the perturbative approach introduced in [7], we studied conserved quantities in the spherically symmetric Einstein-scalar system. We identified the total mode-energy E, the Hamiltonian H, and the particle number N as conserved quantities, and we explored their dynamical effects. Based on our understanding of TTF, we expect these quantities to be conserved for timescales $t \sim 1/\epsilon^2$ in the full theory, and so they are particularly useful for studying small-amplitude perturbations.

We have shown that conservation of E and N together imply that an inverse cascade of particles must accompany any energy flow to higher wavenumbers and vice versa. Conservation of E and N also causes parts of phase space (including energy equipartition) to be dynamically excluded. Additionally, we have discussed the possibility of quasi-periodic q-breathers as providing yet further examples of stability islands.

Conservation laws therefore play a very important role in constraining the dynamics of the system. With three constants of motion already identified for the problem of spherical collapse in AdS, it is tantalizing to imagine that others are still waiting to be discovered. Numerical evidence suggests there are several families of initial data that do not lead to black hole formation. Nonthermalization in 1-dimensional systems such as FPU chains is evidence that the system may be integrable; i.e., that it contains an infinite number of conserved quantities.

Integrable systems in more than 1 spatial dimension, however, are quite rare, and so in that sense instability is more likely for the full gravitational problem in the absence of spherical symmetry. This is related to the fact that the density of states grows more rapidly with energy in higher dimensions. On the other hand, black hole formation is generally thought to be easiest *with* spherical symmetry. This tension remains to be resolved.

ACKNOWLEDGMENTS

We would like to thank D. Abanin, J. Santos, P. Vieira, J. R. Westernacher-Schneider, and H. Yang for interesting discussions. During the course of our work, Ref. [16] was posted, which independently identified the two new conserved quantities, and presented a thorough analysis. This work was supported by the NSF under grant PHY-1308621 (LIU), by NASA under grant NNX13AH01G, by NSERC through a Discovery Grant (to A.B. and L.L.) and by CIFAR (to L.L.). Research at Perimeter Institute is supported through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation.

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