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In this paper, we study the effects of high-order operators on the non-relativistic Lifshitz holography in the framework of the Hořava-Lifshitz (HL) theory of gravity, which naturally contains high-order operators in order for the theory to be power-counting renormalizble, and provides an ideal place for such studies. In particular, we show that the Lifshitz space-time is still a solution of the full theory of the HL gravity. The effects of the high-oder operators on the space-time itself is simply to shift the Lifshitz dynamical exponent. However, while in the infrared the asymptotic behavior of a (probe) scalar field near the boundary is similar to that studied in the literature, it gets dramatically modified in the UV limit, because of the presence of the high-order operators in this regime. Then, according to the gauge/gravity duality, this in turn affects the two-point correlation functions.

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I. INTRODUCTION

Non-relativistic gauge/gravity duality has attracted lot of attention recently, as it may provide valuable tools to study strongly coupling systems encountered in condensed matter physics [1], which otherwise are not tractable with our current understanding. If such a duality indeed exists, instead of directly studying those strongly coupling systems, one can study the corresponding weakly coupling systems of gravity, which are much easier to handle, and often well within our abilities.

The non-relativistic quantum field theories (NQFT) are usually assumed to possess either the Schrödinger [2] or the Lifshitz [3] symmetry. In the latter, the symmetry algebra consists of the rotations M_{ij}, spatial translations P_i, time translations H, and dilatations D. These generators satisfy the standard commutation relations for M_{ij}, P_k and H [5], while with D the relations read,

$$[D, M_{ij}] = 0, \quad [D, P_i] = iP_i, \quad [D, H] = izH, \tag{1.1}$$

where z denotes the Lifshitz dynamical exponent, and determines the relative scaling between the time and spatial coordinates [4],

$$x^i \to \ell x^i, \quad t \to \ell^z t. \tag{1.2}$$

This algebra is often called the Lifshitz algebra, as it generalizes the symmetry of Lifshitz fixed points [1].

The gauge/gravity duality requires that the space-time in the gravitational side must possess the same symmetry. However, the symmetry of a space-time is usually defined by the existence of Killing vectors ζ_μ [6], satisfying the Killing equations,

$$\zeta_{\mu;\nu} + \zeta_{\nu;\mu} = 0, \tag{1.3}$$

where a semicolon ";" denotes the covariant derivative with respect to the spapcetime metric $g_{\mu\nu}$. It was found that this can be realized in the Lifshitz space-time [3],

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2, \tag{1.4}$$

where $d\vec{x}^2 \equiv \sum_{i=1}^d dx^i dx^i$. Then, the Killing vectors $\zeta^\mu \partial_\mu \equiv (M, P, H, D)$ of the above space-time, given by,

$$\begin{aligned} &M_{ij} = -i\left(x_i\partial_j - x_j\partial_i\right), \quad P_i = -i\partial_i, \\ &H = -i\partial_t, \quad D = -i\left(zt\partial_t + x^i\partial_i - r\partial_r\right), \end{aligned} \tag{1.5}$$

produce precisely the required Lifshitz algebra, where $x_i \equiv \delta_{ij}x^j$. The corresponding NQFT lives on the boundary $r = \infty$.

Note that the metric is invariant under the rescaling (1.2), provided that r is scaling as $r \to \ell^{-1}r$. Clearly, this is non-relativistic for $z \neq 1$, and to produce such a space-time in Einstein's theory of general relativity (GR), matter fields must be present, in order to create such a preferred direction. In [3], this was realized by two p-form gauge fields with $p = 1, 2$, and was soon generalized to other cases [7].

On the other hand, to construct a viable theory of quantum gravity, Hořava [8] recently proposed a theory based on the anisotropic scaling (1.2), the so-called Hořava-Lifshitz (HL) theory of quantum gravity, and has attracted a great deal of attention, due to its several remarkable features [9]. The HL theory is based on the perspective that Lorentz symmetry should appear as an

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emergent symmetry at long distances, but can be fundamentally absent at short ones [10]. In the UV regime, the system exhibits a strong anisotropic scaling between space and time, given by Eq.(1.2). To have the theory be power-counting renormalizable, the Lifshitz dynamical exponent z must be no less than D in the $(D+1)$-dimensional spacetime [8, 11]. At long distances, high-order curvature corrections become negligible, and the lowest order terms take over, whereby the Lorentz invariance is expected to be "accidentally restored."

Since in the HL gravity the anisotropic scaling (1.2) is built in [1], it is natural to expect that the HL gravity provides a minimal holographic dual for non-relativistic Lifshitz-type field theories. Indeed, recently it was showed that the Lifshitz spacetime (1.4) is a vacuum solution of the HL gravity in (2+1) dimensions, and that the full structure of the $z=2$ anisotropic Weyl anomaly can be reproduced in dual field theories [12], while its minimal relativistic gravity counterpart yields only one of two independent central charges in the anomaly. This speculation has been further confirmed by the existence of other types of the Lifshitz spacetimes, including Lifshitz solitons [13, 14].

In this paper, we study another important issue: the effects of high-order operators in non-relativistic Lifshitz holography. Since high-order operators are necessarily appear in the HL gravity in order to be power-counting renormalizable, it provides an ideal place to study such effects. In the framework of GR, this was studied in [15], and found that these effects only shift the values of z. In this paper, we shall first show that this is true also in the HL gravity. Then, we study the effects on a scalar field and the corresponding two-point correlation functions. We find that, while in the infrared the asymptotic behavior of a (probe) scalar field near the boundary is similar to that studied in [3], it gets dramatically modified in the UV limit, because of the presence of the high-order operators in this regime. Then, according to the gauge/gravity duality, this in turn affects the two-point correlation functions. This is expected, as in the UV the high-order operators will dominate, and the asymptotic behavior of the scalar field will be determined by these high-order operators.

Specifically, the paper is organized as follows: In Section II, we shall give a brief introduction to the non-projectable HL gravity in (2+1)-dimensional spacetimes, and find out the stability and ghost-free conditions in terms of the independently coupling constants of the theory. In Section III, we show that the Lifshitz space-time (1.4) is not only a solution of the HL gravity in the IR limit, but also a solution of the full theory. The only difference is that the Lifshitz dynamical exponent z is shifted. In Section IV, we study a scalar field propagating on the Lifshitz background (1.4). To compare our results with the ones obtained in [3], in this section (and also the next) we set $z=2$. In Section V, we calculate the two-point correlation functions, and find their main properties in the IR as well as in the UV limit. In Section V, we present our main conclusions.

II. NON-PROJECTABLE HL THEORY IN (2+1) DIMENSIONS

Because of the anisotropic scaling (1.2) [see also Footnote 1], the gauge symmetry of the theory is broken down to the foliation-preserving diffeomorphism, Diff(M, $\mathcal{F}$),

$$\delta t = -f(t), \quad \delta x^i = -\zeta^i(t,\mathbf{x}), \tag{2.1}$$

for which the lapse function N, shift vector N^i, and 3-spatial metric g_{ij}, first introduced in the Arnowitt-Deser-Misner (ADM) decompositions [16], transform as

$$\begin{aligned}\delta N &= \zeta^k\nabla_k N + \dot{N}f + N\dot{f},\\ \delta N_i &= N_k\nabla_i\zeta^k + \zeta^k\nabla_k N_i + g_{ik}\dot{\zeta}^k + \dot{N}_i f + N_i\dot{f},\\ \delta g_{ij} &= \nabla_i\zeta_j + \nabla_j\zeta_i + f\dot{g}_{ij},\end{aligned} \tag{2.2}$$

where $\dot{f} \equiv df/dt$, ∇_i denotes the covariant derivative with respect to g_{ij}, $N_i = g_{ik}N^k$, and $\delta g_{ij} \equiv \tilde{g}_{ij}\left(t,x^k\right) - g_{ij}\left(t,x^k\right)$, etc.

Due to the Diff(M, $\mathcal{F}$) diffeomorphisms (2.1), one more degree of freedom appears in the gravitational sector - a spin-0 graviton. Using the gauge freedom (2.1), without loss of the generality, one can always set

$$N^i = 0, \tag{2.3}$$

for which the remaining gauge freedom is

$$t = \hat{f}(t'), \quad x^i = \hat{\zeta}^i(x'). \tag{2.4}$$

In the rest of this section, we shall leave the gauge choice open, and in particular not restrict ourselves to the gauge (2.3).

The Riemann and Ricci tensors R_{ijkl} and R_{ij} of the 2D leaves $t=$ constant are uniquely determined by the 2D Ricci scalar R via the relations [17],

$$\begin{aligned}R_{ijkl} &= \frac{1}{2}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)R,\\ R_{ij} &= \frac{1}{2}g_{ij}R, \ (i,j=1,2).\end{aligned} \tag{2.5}$$

The general action of the HL theory without the projectability condition in (2+1)-dimensional spacetimes is given by [13]

$$S = \zeta^2\int dtd^2xN\sqrt{g}\Big(\mathcal{L}_K - \mathcal{L}_V + \zeta^{-2}\mathcal{L}_M\Big), \tag{2.6}$$

[1] It should be noted that in the HL gravity, all the spatial coordinates $\left(r,x^i\right)$ are scaling as $x^n \to \ell x^n$, where $n = r,i,(i = 1,2,3,...,d)$. This is different from that of the metric (1.4), in which r must be scaling as $r \to \ell^{-1}r$, in order to keep the metric invariant. Therefore, in principle the Lifshitz dynamical exponent z appearing in (1.4) is different from that considered in the HL theory: $x^n \to \ell x^n, \quad t \to \ell^z t$.

where $g = \det(g_{ij})$, $\zeta^2 = 1/(16\pi G)$, and

$$\begin{aligned}
\mathcal{L}_K &= K_{ij}K^{ij} - \lambda K^2, \\
\mathcal{L}_V &= \gamma_0\zeta^2 + \beta a_i a^i + \gamma_1 R \\
&+ \frac{1}{\zeta^2}\Big[\gamma_2 R^2 + \beta_1\left(a_i a^i\right)^2 + \beta_2\left(a^i{}_i\right)^2 \\
&+ \beta_3 a_i a^i a^j{}_j + \beta_4 a^{ij} a_{ij} \\
&+ \beta_5 a^i a_i R + \beta_6 a^i{}_i R\Big],
\end{aligned} \tag{2.7}$$

with $\Delta \equiv g^{ij}\nabla_i\nabla_j$, and

$$\begin{aligned}
K_{ij} &= \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right), \\
a_i &= \frac{N_{,i}}{N}, \quad a_{ij} = \nabla_i a_j.
\end{aligned} \tag{2.8}$$

$\mathcal{L}_M$ is the Lagrangian of matter fields. Then, the corresponding field equations and conservation laws are given explicitly in [13].

A. Stability and Ghost-free Conditions

It is easy to show that the Minkowski space-time

$$\left(\bar{N}, \bar{N}^i, \bar{g}_{ij}\right) = \left(1, 0, \delta_{ij}\right), \tag{2.9}$$

is a solution of the above HL gravity with $\gamma_0 = 0$. Then, its linear perturbations are given by

$$\begin{aligned}
\delta N &= n, \quad \delta N_i = \partial_i B - S_i, \\
\delta g_{ij} &= -2\psi\delta_{ij} + \left(\partial_i\partial_j - \delta_{ij}\partial^2\right)E + 2F_{(i,j)},
\end{aligned} \tag{2.10}$$

where $F_{(i,j)} \equiv (F_{i,j} + F_{j,i})/2$, and

$$\partial^i S_i = \partial^i F_i = 0. \tag{2.11}$$

It is interesting to note that in the decompositions (2.10) no tensor mode appears in δg_{ij}. This is closely related to the fact that in (2+1)-dimensional spacetimes, spin-2 massless gravitons do not exist.

Then, the infinitesimal gauge transformations (1.4) can be written as

$$f = \epsilon(t), \quad \zeta^i = \partial^i\zeta + \eta^i, \quad (\partial_i\eta^i = 0), \tag{2.12}$$

under which the quantities defined in Eq.(2.10) transfer as,

$$\begin{aligned}
\tilde{n} &= n + \dot{\epsilon}, \quad \tilde{B} = B + \dot{\zeta}, \\
\tilde{E} &= E + \zeta, \quad \tilde{\psi} = \psi - \frac{1}{2}\partial^2\zeta, \\
\tilde{S}_i &= S_i + \dot{\eta}_i, \quad \tilde{F}_i = F_i + \eta_i.
\end{aligned} \tag{2.13}$$

Thus, from the above we can construct three scalar and one vector gauge-invariants,

$$\begin{aligned}
\Psi &\equiv \psi + \frac{1}{2}\partial^2 E, \quad \Phi \equiv B - \dot{E}, \\
\Upsilon &\equiv \partial^2 n, \quad \Phi_i \equiv S_i - \dot{F}_i.
\end{aligned} \tag{2.14}$$

Using the above gauge freedom, without loss of the generality, we can set

$$E = 0, \quad F_i = 0, \tag{2.15}$$

which will uniquely fix the gauge freedom represented by ζ and η_i, while leave $\epsilon(t)$ unspecified. To further study the above linear perturbations, let us consider the scalar and vector perturbations, separately.

1. Scalar Perturbations

Under the gauge (2.15), the remaining scalars are n, B and ψ, with which it can be shown that the gravitational sector of the action to the second-order takes the form,

$$\begin{aligned}
S_g^{(2)} &= \zeta^2\int dtd^2x\Big\{2(1-2\lambda)\dot{\psi}^2 + 2(1+2\lambda)\dot{\psi}\partial^2 B \\
&+ (1-\lambda)(\partial^2 B)^2 + \beta n\partial^2 n - 2\gamma_1 n\partial^2\psi \\
&- \frac{1}{\zeta^2}\big[4\gamma_2(\partial^2\psi)^2 + (\beta_2+\beta_4)(\partial^2 n)^2 \\
&+ 2\beta_6(\partial^2 n)(\partial^2\psi)\big]\Big\}.
\end{aligned} \tag{2.16}$$

Its variations with respect to ψ, B and n yield, respectively,

$$\ddot{\psi} + \frac{1}{2}\partial^2\dot{B} + \frac{\gamma_1}{2(1-2\lambda)}\partial^2 n + \frac{4\gamma_2\partial^4\psi + \beta_6\partial^4 n}{2\zeta^2(1-2\lambda)} = 0, \tag{2.17}$$

$$(1-2\lambda)\dot{\psi} + (1-\lambda)\partial^2 B = 0, \tag{2.18}$$

$$\beta n - \gamma_1\psi - \frac{\beta_2+\beta_4}{\zeta^2}\partial^2 n - \frac{\beta_6}{\zeta^2}\partial^2\psi = 0. \tag{2.19}$$

From Eq.(2.18) we can find B in terms of ψ, and then substituting it into (2.16) we obtain,

$$\begin{aligned}
S_g^{(2)} &= \zeta^2\int dtd^2x\Big\{\frac{1-2\lambda}{1-\lambda}\dot{\psi}^2 + \beta n\partial^2 n - 2\gamma_1 n\partial^2\psi \\
&- \frac{1}{\zeta^2}\Big[4\gamma_2(\partial^2\psi)^2 + (\beta_2+\beta_4)(\partial^2 n)^2 \\
&+ 2\beta_6(\partial^2 n)(\partial^2\psi)\Big]\Big\}.
\end{aligned} \tag{2.20}$$

Then, the ghost-free condition require

$$\frac{1-2\lambda}{1-\lambda} \geq 0, \tag{2.21}$$

that is,

$$(i)\ \lambda > 1 \quad \text{or} \quad (ii)\ \lambda \leq \frac{1}{2}. \tag{2.22}$$

From Eqs.(2.17)-(2.19), on the other hand, we can get a master equation for ψ, which in momentum space can be written in the form

$$\ddot{\psi}_k + \omega_k^2 \psi_k = 0, \tag{2.23}$$

where

$$\begin{aligned} \omega_k^2 &= \frac{1-\lambda}{1-2\lambda}\left(\frac{4\gamma_2 k^4}{\zeta^2} + \left(\frac{\beta_6 k^4}{\zeta^2} - \gamma_1 k^2\right)\frac{\gamma_1 - \frac{\beta_6 k^2}{\zeta^2}}{\beta + \frac{(\beta_2+\beta_4)k^2}{\zeta^2}}\right) \\ &= \begin{cases} -\frac{1-\lambda}{1-2\lambda}\frac{\gamma_1^2 k^2}{\beta}, & k^2/\zeta \ll 1, \\ \frac{1-\lambda}{1-2\lambda}\left(4\gamma_2 - \frac{\beta_6^2}{\beta_2+\beta_4}\right)\frac{k^4}{\zeta^2}, & k^2/\zeta \gg 1. \end{cases} \end{aligned} \tag{2.24}$$

Thus, to have the mode be stable in the infrared (IR), we must require

$$\beta < 0, \tag{2.25}$$

while its stability condition in the ultraviolet (UV) requires

$$\gamma_2 \geq \frac{\beta_6^2}{4(\beta_2+\beta_4)}. \tag{2.26}$$

In the intermediate range, by properly choosing other free parameters the mode can be made always stable, and such requirement does not impose any severe constraints. So, in the following we do not consider it any further, and simply assume that it is always satisfied.

It should be noted that the conditions (2.22), (2.25) and (2.26) are valid only for the cases $\lambda \neq 1$, for which Eq.(2.25) tells that β must be strictly negative, and in particular cannot be zero.

When $\lambda = 1$, from Eq.(2.18) we find that

$$\dot{\psi} = 0, \tag{2.27}$$

that is, ψ does not represent a propagative mode, and we can always set it to zero by properly choosing the boundary conditions. Then, Eqs.(2.17) and (2.19) reduce to,

$$\dot{B} - \gamma_1 n - \frac{\beta_6}{\zeta^2}\partial^2 n = 0, \tag{2.28}$$

$$\frac{\beta_2+\beta_4}{\zeta^2}\partial^2 n - \beta n = 0. \tag{2.29}$$

From the last equation, we can see that n does not represent a propagative mode either, and can be set to zero by properly choosing the boundary conditions. Then, Eq.(2.28) yields $\dot{B} = 0$, that is, B is also not a propagative mode.

Therefore, in the case $\lambda = 1$ there is no gravitational propagative mode, similar to the relativistic case [17]. As a result, *all the free parameters in this case are free, as long as the stability and ghost-free conditions are concerned.*

As a corollary, we find that the HL theory with $\beta = 0$ is viable only when $\lambda = 1$. Otherwise, the corresponding scalar mode will become unstable, as one can see clearly from Eq.(2.24).

2. Vector Perturbations

Under the gauge (2.15), the remaining vector is S_i, with which it can be shown that the gravitational sector of the action to the second-order takes the form,

$$S_g^{(2)} = -\frac{\zeta^2}{2}\int dtd^2xN\sqrt{g}\left(S^i\partial^2 S_i\right), \tag{2.30}$$

from which we find that,

$$\partial^2 S^i = 0. \tag{2.31}$$

That is, there is no propagative vector mode in the HL gravity, even the Lorentz symmetry is violated.

In summary, the above analysis shows: *(i) In the case $\lambda \neq 1$, only spin-0 gravitons exist in the (2+1)-dimensional non-projectable HL gravity.* Their stability and ghost-free conditions require the independent coupling constants must satisfy the conditions of Eqs.(2.22), (2.25) and (2.26). *(ii) In the case $\lambda = 1$, the gravitational sector of the HL gravity has no free propagation mode, similar to its relativistic counterpart.* Then, all the free parameters in this case are free, as long as the stability and ghost-free conditions are concerned.

B. Detailed Balance Condition

To reduce the number of the coupling constants, Hořava imposed the detailed balance condition [8]. The main idea is to introduce a superpotential W on the leaves $t =$ Constant,

$$W = \int d^2x\sqrt{g}\mathcal{L}_W\left(R_{ij}, a_k, \nabla_l\right), \tag{2.32}$$

so that the potential part of the action is given by

$$\hat{\mathcal{L}}_V^{(DB)} = E_{ij}G^{ijkl}E_{kl}, \quad E_{ij} \equiv \frac{1}{\sqrt{g}}\frac{\delta W}{\delta g^{ij}}, \tag{2.33}$$

where G^{ijkl} denotes the generalized de Witt metric on the space of metrics, and is given by

$$G^{ijkl} \equiv \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda g^{ij}g^{kl}. \tag{2.34}$$

Power-counting renormalizibility requires that the dimension of $\mathcal{L}_W$ must be greater or equal to $2d$, that is, $[\mathcal{L}_W] \geq 2d$. Taking the lowest dimension, one can see that in (2+1)-dimensional space-times, $\mathcal{L}_W$ in general can be cast in the form,

$$\mathcal{L}_W = w\left(R + \mu a_i a^i - 2\Lambda_W\right), \tag{2.35}$$

where w, μ and Λ_W are three coupling constants. Plugging the above into Eq.(2.33) and taking Eq.(2.1) into account, we find that

$$\begin{aligned} E_{ij} &= w\left[\mu\left(a_i a_j - \frac{1}{2}g_{ij}a_k a^k\right) + \Lambda_W g_{ij}\right], \\ \hat{\mathcal{L}}_V^{(DB)} &= \frac{w^2}{2}\left[\mu^2\left(a_i a^i\right)^2 + 4\left(1-2\lambda\right)\Lambda_W^2\right]. \end{aligned} \tag{2.36}$$

To have a healthy IR limit, the detailed balance condition is frequently allowed to be broken softly [8, 18, 19] by adding all the low dimensional relevant terms, $R,\ a_ia^i,\ \Lambda$, into $\hat{\mathcal{L}}_V^{(DB)}$, so that the potential is finally given by

$$\mathcal{L}_V^{(DB)} = 2\Lambda + \beta a_i a^i + \gamma_1 R + \frac{\beta_1}{\zeta^2}\left(a_i a^i\right)^2, \tag{2.37}$$

where $\beta_1 \equiv w^2\mu^2/2$ and $\Lambda \equiv \gamma_0\zeta^2/2$. Comparing it with $\mathcal{L}_V$ given by Eq.(2.7), one can see that this is equivalent to set $\gamma_2 = 0 = \beta_n$ $(2 \le n \le 6)$.

III. LIFSHITZ SPACETIMES IN (2+1)-DIMENSIONS

In this section we are going to study static vacuum spacetimes with the ADM variables given by

$$\begin{aligned} N &= r^z f(r), \quad N^i = 0,\\ g_{ij} &= \text{diag.}\left(\frac{g^2(r)}{r^2}, r^2\right), \end{aligned} \tag{3.1}$$

in the coordinates (t, r, x), where z is the dynamical Lifshitz exponent. Then, we find that

$$\begin{aligned} R_{ij} &= \frac{rg'-g}{r^2 g}\delta^r_i\delta^r_j + \frac{r^2\left(rg'-g\right)}{g^3}\delta^\theta_i\delta^\theta_j,\\ a_i &= \frac{(zf+rf')}{rf}\delta^r_i, \quad K_{ij} = 0. \end{aligned} \tag{3.2}$$

Inserting the above into the general action (2.6), for the vacuum case $\mathcal{L}_M = 0$, we obtain

$$S_g = -V_x\zeta^2\int dtdr r^z fg\mathcal{L}_V\left(f^{(n)}, g^{(m)}, r\right), \tag{3.3}$$

where $V_x \equiv \int dx,\ I^{(n)} \equiv d^n I(r)/dr^n$, and $\mathcal{L}_V$ is given by Eq.(A.2). Then, it can be shown that in the present case there are only two independent equations, which can be cast in the forms,

$$\sum_{n=0}^{3}(-1)^n\,\frac{d^n}{dr^n}\left(\frac{\delta\mathcal{L}_g}{\delta f^{(n)}}\right) = 0, \tag{3.4}$$

$$\sum_{n=0}^{3}(-1)^n\,\frac{d^n}{dr^n}\left(\frac{\delta\mathcal{L}_g}{\delta g^{(n)}}\right) = 0, \tag{3.5}$$

where $\mathcal{L}_g \equiv r^z fg\mathcal{L}_V$. In terms of f, g and their derivatives, these two equations are given by Eqs.(A.3) and (A.4).

The Lifshitz spacetime corresponds to

$$f = f_0, \quad g = g_0, \tag{3.6}$$

where f_0 and g_0 are two constant. Then, the corresponding metric can be cast in the form,

$$ds^2 = L^2\left\{-\left(\frac{r}{\ell}\right)^{2z}dt^2 + \left(\frac{\ell}{r}\right)^2 dr^2 + \left(\frac{r}{\ell}\right)^2 dx^2\right\}, \tag{3.7}$$

where $L \equiv (f_0 g_0^z)^{1/(z+1)},\ \ell \equiv (g_0/f_0)^{1/(1+z)}$. Inserting Eq.(3.6) into Eqs.(3.4) and (3.5), we obtain

$$\begin{aligned} &2\zeta^2\Lambda g_0^4 - \zeta^2 g_0^2\left[z(2+z)\beta + 2\gamma_1\right] - z^3(4+3z)\beta_1\\ &\quad + 4\gamma_2 + z\Big[z(3+2z)\beta_2 + z\left(z^2-2\right)\beta_3\\ &\quad - (2+z)\left(\beta_4 - 2\beta_5 + 2\beta_6\right)\Big] = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &2\zeta^2\Lambda g_0^4 - z\zeta^2 g_0^2\left(z\beta + 2\gamma_1\right) - 4\gamma_2 + 2z\left(4\gamma_2 + \beta_6\right)\\ &\quad - z^2\Big\{\beta_2 + 3\beta_4 - 4\beta_5 + 4\beta_6 + z\Big[3z\beta_1 - 2\beta_2\\ &\quad - (z-2)\beta_3 + 2\beta_5\Big]\Big\} = 0. \end{aligned} \tag{3.9}$$

In the IR limit, all the fourth-order terms become negligible, and the above equations reduce to

$$2\Lambda g_0^2 - \left[z(2+z)\beta + 2\gamma_1\right] = 0, \tag{3.10}$$

$$2\Lambda g_0^2 - z\left(z\beta + 2\gamma_1\right) = 0, \tag{3.11}$$

which have the solutions,

$$z = \frac{\gamma_1}{\gamma_1 - \beta}, \quad \Lambda = \frac{\gamma_1^2(2\gamma_1 - \beta)}{2g_0^2(\gamma_1-\beta)^2}. \tag{3.12}$$

These are exactly what were obtained in [12].

When the higher-order operators are not negligible, the sum of Eqs.(3.8) and (3.9) yields,

$$\begin{aligned} \Lambda = \frac{\zeta^2\left[z\beta + (1-z)\gamma_1\right]}{\Delta}\Big\{&z^4\left[z\beta - (1+3z)\gamma_1\right]\beta_1\\ &+z^2\left[z\beta + \left(2z^2+z+1\right)\gamma_1\right]\beta_2\\ &+z^4\left[\beta + (z-1)\gamma_1\right]\beta_3\\ &+z^2\left[z(z+2)\beta + (1-z)\gamma_1\right]\beta_4\\ &+z^3\left[(z+2)(z-1)\beta + 4\gamma_1\right]\beta_5\\ &+z\left[z(z+2)(z+1)\beta - 2\gamma_1(z^2+1)\right]\beta_6\\ &-4\left[z\left(z^2+z-1\right)\beta + (z-1)\gamma_1\right]\gamma_2\Big\}, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \Delta &= 2\Big\{2z^3\beta_1 - 2z^2\beta_2 - z(z-3)\beta_6\\ &\quad + (1-z)\left[z^2\beta_3 + z\beta_4 - 4\gamma_2\right]\\ &\quad - z\left[2 + z(z-1)\right]\beta_5\Big\}^2. \end{aligned} \tag{3.14}$$

The difference of Eqs.(3.8) and (3.9), on the other hand, yields,

$$az^3 + bz^2 + cz + d = 0, \tag{3.15}$$

where

$$\begin{aligned} a &= -2\beta_1 + \beta_3 + \beta_5,\\ b &= 2\beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6,\\ c &= -\alpha^2(\beta-\gamma_1) - 4\gamma_2 - \beta_4 + 2\beta_5 - 3\beta_6,\\ d &= 4\gamma_2 - \alpha^2\gamma_1, \quad \alpha \equiv \zeta g_0, \end{aligned} \tag{3.16}$$

which can be used to determine the dynamical exponent z in terms of the coupling constants. In general, it has three different solutions for any given set of the coupling constants. On the other hand, Eq.(3.15) can be also used to determine the integration constant g_0 for any given z and a set of the coupling constants. In this case, we have

$$g_0^2 = \frac{az^3 + bz^2 + \hat{c}z + 4\gamma_2}{\zeta^2[\gamma_1 - (\gamma_1 - \beta)z]}, \tag{3.17}$$

where $\hat{c} \equiv -4\gamma_2 - \beta_4 + 2\beta_5 - 3\beta_6$. Clearly, for the metric to have a proper signature, z has to be chosen so that $g_0^2 > 0$ for any given set of the coupling constants (β_i, γ_j).

When the fourth-order corrections are small, we can expand z near its IR fixed point, z_0, given by Eq.(3.12). Writing the fourth-order coupling constants in the form $s = s_0 + \epsilon\hat{s}$, where $\epsilon \ll 1$, we find that

$$\begin{aligned} z &= z_0 + \epsilon\delta z, \\ a &= \epsilon(-2\hat{\beta}_1 + \hat{\beta}_3 + \hat{\beta}_5), \\ b &= \epsilon(2\hat{\beta}_2 - \hat{\beta}_3 + \hat{\beta}_4 - \hat{\beta}_5 + \hat{\beta}_6), \\ c &= c_0 + \epsilon(-4\hat{\gamma}_2 - \hat{\beta}_4 + 2\hat{\beta}_5 - 3\hat{\beta}_6), \\ d &= d_0 + 4\epsilon\hat{\gamma}_2, \end{aligned} \tag{3.18}$$

where

$$z_0 = \frac{\gamma_1}{\gamma_1 - \beta}, \;\; c_0 = -\alpha^2(\beta - \gamma_1), \;\; d_0 = -\alpha^2\gamma_1.$$

Thus, to the first-order of ϵ Eq.(3.15) yields,

$$\begin{aligned} &(-2\hat{\beta}_1 + \hat{\beta}_3 + \hat{\beta}_5)z_0^3 + (2\hat{\beta}_2 - \hat{\beta}_3 + \hat{\beta}_4 - \hat{\beta}_5 + \hat{\beta}_6)z_0^2 \\ &+(-4\hat{\gamma}_2 - \hat{\beta}_4 + 2\hat{\beta}_5 - 3\hat{\beta}_6)z_0 + 4\hat{\gamma}_2 + c_0\delta z = 0, \end{aligned} \tag{3.19}$$

from which we fin that,

$$\begin{aligned} \delta z \;=\; & \frac{1}{\alpha^2(\beta - \gamma_1)^4}\Big\{\gamma_1[\beta^2(\beta_4 - 2\beta_5 + 3\beta_6) \\ & -\beta\gamma_1(-2\beta_2 + \beta_3 + \beta_4 - 3\beta_5 + 5\beta_6) \\ & +2\gamma_1^2(\beta_1 - \beta_2 - \beta_5 + \beta_6)] \\ & +4\beta\gamma_2(\beta - \gamma_1)^2\Big\}. \end{aligned} \tag{3.20}$$

Note that in writing the above expression, without causing any confusions, we had dropped hats from all fourth-order parameters. To study the behavior of z in the UV, let us consider some particular cases.

A. Solutions with softly-breaking detailed balance condition

When the softly-breaking detailed balance condition is imposed, we have $\gamma_2 = \beta_i = 0, \; (i \geq 2)$. Then, Eqs.(3.15) and (3.13) reduce, respectively, to

$$z^3 + \frac{\alpha^2}{2\beta_1}(\beta - \gamma_1)z + \frac{\alpha^2}{2\beta_1}\gamma_1 = 0, \tag{3.21}$$

$$\Lambda = \frac{\zeta^2}{4z^2\beta_1}[z\beta + (1-z)\gamma_1][z\beta - (1+3z)\gamma_1]. \tag{3.22}$$

Eq.(3.21) in general has three roots, and depending on the signature of $\mathcal{D}$, the nature of these roots are different, where

$$\mathcal{D} \equiv \frac{\alpha^4}{16\beta_1^2}\left[\gamma_1^2 - \frac{2\alpha^2(\gamma_1 - \beta)^3}{27\beta_1}\right]. \tag{3.23}$$

Let us consider the cases $\mathcal{D} = 0$, $\mathcal{D} > 0$ and $\mathcal{D} < 0$, separately.

1. $\mathcal{D} = 0$

When $\mathcal{D} = 0$, we find that

$$\beta_1 = \frac{2\alpha^2(\gamma_1 - \beta)^3}{27\gamma_1^2}, \tag{3.24}$$

and Eq.(3.21) has three real roots, two of which are equal and given by

$$z_1 = \frac{3\gamma_1}{\beta - \gamma_1}, \quad z_2 = z_3 = -\frac{3\gamma_1}{2(\beta - \gamma_1)}. \tag{3.25}$$

Clearly, by properly choosing β and γ_1, they can take any real values, $z_i \in (-\infty, \infty)$.

2. $\mathcal{D} > 0$

In this case, Eq.(3.21) has only one real root, which can be written as

$$z = \sqrt[3]{\mathcal{D}^{1/2} - \frac{q}{2}} - \sqrt[3]{\mathcal{D}^{1/2} + \frac{q}{2}}, \tag{3.26}$$

where $q \equiv \alpha^2\gamma_1/(2\beta_1)$. In this case it is clear that z can also take any real values for different choices of $(\beta, \gamma_1, \beta_1)$. In particular, it has an extreme at $\beta = \gamma_1$, given by $z_m = -q^{1/3}$.

3. $\mathcal{D} < 0$

In this case, Eq.(3.21) has three real and different roots, given by

$$z_n = \sqrt{\frac{2\alpha^2(\gamma_1 - \beta)}{3\beta_1}}\cos\left(\theta + \frac{2n\pi}{3}\right), (n = 0, 1, 2), \tag{3.27}$$

where θ is defined as

$$\theta = \frac{1}{3}\arccos\left[\frac{\alpha^2\gamma_1}{4\beta_1}\left(\frac{6\beta_1}{\alpha^2(\gamma_1 - \beta)}\right)^{3/2}\right]. \tag{3.28}$$

Again, similar to the last two subcases, by choosing different values of the coupling constants, we can have different values of z_n. For example, taking $\alpha^2 = 4, \; \beta = -1, \; \beta_1 = 0.00001, \; \gamma_1 = 1$, we obtain $z_1 \simeq 632.205$.

B. Solutions with $\mathcal{L}_V = \mathcal{F}(R)$

Another interesting case is the $\mathcal{F}(R)$ models [20], for which we have

$$\mathcal{L}_V = \mathcal{F}(R), \tag{3.29}$$

where $\mathcal{F}(R)$ can be any function of R (possibly subjected to some stability and ghost-free conditions). In particular, one can take the form,

$$\mathcal{F}(R) = 2\Lambda + \gamma_1 R + \beta\mathcal{A}^2 + \frac{\gamma_2}{\zeta^2}R^2, \tag{3.30}$$

which corresponds to the potential given by Eq.(2.7) with $\beta_i = 0,\ (i = 1, ..., 6)$, where $\mathcal{A}^2 \equiv a_i a^i$. Note that in writing the above expression, we had kept the $a_i a^i$ term, in order to have a healthy IR limit for any given coupling constant λ [12, 13].

In this case, Eqs.(3.8) and (3.9) have the solutions,

$$\begin{aligned} z &= 1 - \frac{\alpha^2\beta}{4\gamma_2 - \alpha^2(\gamma_1 - \beta)}, \\ \Lambda &= \frac{\zeta^2}{2\alpha^4}\left\{\alpha^2\left[z(2+z)\beta + 2\gamma_1\right] - 4\gamma_2\right\}. \end{aligned} \tag{3.31}$$

C. Solutions with $\mathcal{L}_V = \mathcal{G}(\mathcal{A})$

Similar to the last case, the function $\mathcal{G}(\mathcal{A})$ can take any form in terms of $\mathcal{A}$. A particular case is the potential given by Eq.(2.7) with $\gamma_1 = \gamma_2 = \beta_5 = \beta_6 = 0$, for which we have

$$\begin{aligned} \mathcal{G}(\mathcal{A}) &= 2\Lambda + \beta a_i a^i \\ &\quad + \frac{1}{\zeta^2}\Big[\beta_1\left(a_i a^i\right)^2 + \beta_2\left(a^i{}_i\right)^2 \\ &\quad + \beta_3 a_i a^i a^j{}_j + \beta_4 a^{ij}a_{ij}\Big]. \end{aligned} \tag{3.32}$$

In this case, Eq.(3.15) reduces to

$$az^2 + bz + c = 0, \tag{3.33}$$

but now with

$$\begin{aligned} a &= -2\beta_1 + \beta_3, \\ b &= 2\beta_2 - \beta_3 + \beta_4, \\ c &= -\alpha^2\beta - \beta_4. \end{aligned} \tag{3.34}$$

Thus, in general there are two solutions,

$$z_\pm = \frac{1}{2(2\beta_1 - \beta_3)}\left[(2\beta_2 - \beta_3 + \beta_4) \pm \sqrt{D}\right], \tag{3.35}$$

where $D \equiv (2\beta_2 - \beta_3 + \beta_4)^2 + 4(\alpha^2\beta + \beta_4)(\beta_3 - 2\beta_1)$. Clearly, for $z_\pm$ to be real, we must assume that $D \geq 0$.

IV. SCALAR FIELD IN THE LIFSHITZ SPACETIME

The action of a scalar field in the HL theory takes the form,

$$\begin{aligned} S_M &= \int dt d^2x N\sqrt{g}\Big\{\frac{1}{2N^2}[\dot{\varphi} - N^i\nabla_i\varphi]^2 \\ &\quad - V(\varphi) - \mathcal{V}_\phi^{(2)} - \frac{1}{M_*^2}\mathcal{V}_\phi^{(4)}\Big\}, \end{aligned} \tag{4.1}$$

where $\mathcal{V}_\phi^{(2)}$ and $\mathcal{V}_\phi^{(4)}$ are, respectively, the second and forth order operators, made of $R_{ij},\ a_i,\ \nabla_i$ and ϕ, where

$$[R_{ij}] = 2, \quad [a_i] = 1 = [\nabla_i], \quad [\phi] = 0. \tag{4.2}$$

In general, they take the forms [21, 22],

$$\begin{aligned} \mathcal{V}_\phi^{(2)} &= \underline{\frac{1}{2}\left[1 + 2V_1(\varphi)\right](\nabla_i\varphi)^2} + \epsilon_1(\phi)a_i\nabla^i\phi + \epsilon_2(\phi)a_i a^i \\ &\quad + \epsilon_3(\phi)R + ..., \\ \mathcal{V}_\phi^{(4)} &= \underline{V_2(\varphi)\left(\nabla^2\varphi\right)^2} + \underline{V_4(\varphi)\nabla^4\varphi} + \delta_1(\phi)R_{ij}\nabla^i\phi\nabla^j\phi \\ &\quad + \delta_2(\phi)\left(a_i\nabla^i\phi\right)^2 + \delta_3(\phi)R^2 + ..., \end{aligned} \tag{4.3}$$

where V_i, ϵ_i and δ_i are arbitrary functions of ϕ only, and the elapsing terms are the mixed ones made of $R_{ij},\ a_i$ and $\nabla_i\phi$. When the background is fixed, these terms always give rise to low order operators in terms of the scalar field ϕ. For example, the term $\epsilon_1(\phi)a_i\nabla^i\phi$ appearing in $\mathcal{V}_\phi^{(2)}$ contributes to the equation of motion of the scalar field only with the first-order spatial derivative, $\nabla^i\left[\epsilon_1(\phi)a_i\right]$, while the term $\delta_1(\phi)R_{ij}\nabla^i\phi\nabla^j\phi$ appearing in $\mathcal{V}_\phi^{(4)}$ contributes only with the second-order spatial derivative, $\nabla^j\left[\delta_1(\phi)R_{ij}\nabla^j\phi\right]$. In addition, the term $\delta_3(\phi)R^2$ had contributions of the form, $\delta_3'(\phi)R^2$, which acts as a potential term once the background is fixed. Therefore, when the space-time background is fixed, the dominant terms in the UV are only the V_2 and V_4 terms appearing in Eq.(4.3). In the IR, on the other hand, their contributions must be so that the resulted action is of general covariance, in order to have a consistent theory with observations [23] [2]. Therefore, in this paper, without loss of the generality, we shall keep only the underlined $V_i(\phi)$ terms appearing in Eq.(4.3) and absorb the factor M_*^{-2} into $V_2(\phi)$ and $V_4(\phi)$. Then, the Variation of the action with respect to φ yields,

$$\frac{1}{\sqrt{g}}\partial_t\left[\frac{\sqrt{g}}{N}(\dot{\varphi} - N^i\nabla_i\varphi)\right] = \nabla_i\left[\frac{N^i}{N}(\dot{\varphi} - N^k\nabla_k\varphi)\right]$$

---

[2] The only possible contributions of these terms are in the intermediate energy scales. However, the study of them in these energy scales in general are very complicated, and are hardly carried out analytically. Thus, in this paper we shall not consider them.

$$+\nabla^i[N(\nabla_i\varphi)(1+2V_1)] - \nabla^2[2NV_2(\nabla^2\varphi)] - \nabla^4[NV_4] - N[V' + V_1'(\nabla\varphi)^2 + V_2'(\nabla^2\varphi)^2 + V_4'(\nabla^4\varphi)]. \quad (4.4)$$

To compare with the results obtained in [3], we first set $L = \ell = 1$, $z = 2$ and $u = 1/r$. Then, the metric (3.7) becomes,

$$ds^2 = -\frac{1}{u^4}dt^2 + \frac{1}{u^2}(dx^2 + du^2). \quad (4.5)$$

In the probe limit, the backreaction of the scalar field is neglected. Hence, taking the above space-time as the background, and choosing

$$\begin{aligned} V &= m^2\varphi^2,\ V_1 = a_1,\ V_2 = \frac{\hat{a_2}}{M_*^2} \equiv a_2, \\ V_4 &= \frac{\hat{a_4}}{M_*^2}\varphi \equiv a_4\varphi, \end{aligned} \quad (4.6)$$

where a_n are constants, we find that Eq.(4.4) reduces to,

$$\begin{aligned} u^2\partial_t^2\varphi &= (1+2a_1)\left(\partial_x^2\varphi + \partial_u^2\varphi - \frac{2}{u}\partial_u\varphi\right) - \frac{2}{u^2}m^2\varphi \\ &-a_4\Big[8\partial_x^2\varphi + 16\partial_u^2\varphi - \frac{32}{u}\partial_u\varphi + \frac{36\varphi}{u^2}\Big] \\ &-2u^2(a_2+a_4)(\partial_x^4\varphi + 2\partial_x^2\partial_u^2\varphi + \partial_u^4\varphi). \end{aligned} \quad (4.7)$$

At the boundary $u = 0$, the scalar field takes the asymptotical form,

$$\varphi \sim u^{\triangle}\varphi_1(t,x), \quad (4.8)$$

where $\triangle$ is one of the real roots of the equation,

$$\begin{aligned} &(1+2a_1)(\triangle^2 - 3\triangle) - 2m^2 - a_4(16\triangle^2 - 48\triangle + 36) \\ &-2(a_2+a_4)\triangle(\triangle-1)(\triangle-2)(\triangle-3) = 0. \end{aligned} \quad (4.9)$$

From the action (4.1), integrating it by parts and discarding boundary terms, we find that it takes the form,

$$\begin{aligned} S_M = \int dtd^2xN\sqrt{g}\Bigg\{ &-\frac{\varphi}{N\sqrt{g}}\partial_t(\frac{\sqrt{g}\dot{\varphi}}{2N}) \\ &-m^2\varphi^2 + \frac{(1+2a_1)\varphi}{2N}\nabla_i(N\nabla^i\varphi) \\ &-\frac{a_2\varphi}{N}\nabla^2\left(N\nabla^2\varphi\right) - a_4\varphi\nabla^4\varphi\Bigg\}. \end{aligned} \quad (4.10)$$

It can be shown that both actions (4.1) and (4.8) are finite for

$$\triangle > \frac{3}{2} \quad (4.11)$$

with the asymptotic condition (4.8).

In the IR, the V_2 and V_4 terms are very small, and can be set to zero safely. In addition, in this limit the scalar field should be relativistic, so $V_1 = 0$. Hence, the above equation reduces to

$$\triangle^2 - 3\triangle - 2m^2 = 0, \quad (4.12)$$

which has the solutions,

$$\triangle_{\pm} = \frac{1}{2}\left(3 \pm \sqrt{9+8m^2}\right). \quad (4.13)$$

For

$$m^2 > -\frac{9}{8}, \quad (4.14)$$

in contrast to the case considered in [3], now only the solution with $\Delta = \Delta_+$,

$$\varphi(u,t,x) \to u^{\triangle_+}\left(\varphi(t,x) + O(u^2)\right), \quad (4.15)$$

leads to a finite action either in the form of Eq.(4.1) or in the one of Eq.(4.10).

In the UV, on the other hand, the V_2 and V_4 terms dominate, and Eq.(4.9) becomes,

$$\begin{aligned} &(a_2+a_4)\triangle^4 - 6(a_2+a_4)\triangle^3 + (11a_2 + 27a_4)\triangle^2 \\ &-(6a_2 + 54a_4)\triangle + 36a_4 = 0. \end{aligned} \quad (4.16)$$

In the case $a_4 = 0$, the above equation reduces to

$$\triangle^3 - 6\triangle^2 + 11\triangle - 6 = 0, \ (a_4 = 0), \quad (4.17)$$

which has solutions

$$\triangle_1 = 1, \quad \triangle_2 = 2, \quad \triangle_3 = 3, \ (a_4 = 0). \quad (4.18)$$

If we choose $a_2 = -a_4$, Eq.(4.14) has the double root

$$\triangle = 6, \ (a_2 = -a_4). \quad (4.19)$$

From the above analysis, one can see that the scalar field has quite different behaviors at the boundary $u = 0$ in the two limits, IR and UV.

V. TWO-POINT CORRELATION FUNCTIONS

The bulk field $\varphi(u,x)$ can be written in the form

$$\varphi(u,t,x) = \int d^3x'\varphi(0,t',x')G(u,t,x;0,t',x'). \quad (5.1)$$

where $\varphi(0,t,x)$ is the scalar field on the boundary and $G(u,t,x;0,t',x')$ the boundary to bulk propagator. It is easy to work in the Fourier space due to the translational invariance in t and x. In the Fourier space, we have

$$\tilde{\varphi}(u,\omega,k) = \tilde{G}(u,\omega,k)\tilde{\varphi}(0,\omega,k). \quad (5.2)$$

A. In the IR

In the IR, we set $a_1 = a_2 = a_4 = 0$, Eq.(4.7) reduces to

$$-u^2\partial_\tau^2\varphi = \partial_x^2\varphi + \partial_u^2\varphi - \frac{2}{u}\partial_u\varphi - \frac{2}{u^2}m^2\varphi, \quad (5.3)$$

and $\tilde{G}(u,\omega,k)$ in Fourier space satisfies the equation,

$$\partial_u^2\tilde{G} - \frac{2}{u}\partial_u\tilde{G} - (\omega^2u^2 + |k|^2)\tilde{G} = 0, \qquad (5.4)$$

with the boundary conditions,

$$\begin{aligned} &(i)\ \ \tilde{G}(0,\omega,k) = 1,\\ &(ii)\ \ \tilde{G}(\infty,\omega,k)\ \text{ is finite.} \end{aligned} \qquad (5.5)$$

Note that in writing down Eq.(5.3), we had set $t = i\tau$. Then, the above conditions uniquely determine the propagator $\tilde{G}(u,\omega,k)$,

$$\begin{aligned} \tilde{G}(u,\omega,k) &= \frac{2}{\sqrt{\pi}}e^{-|\omega|u^2/2}\Gamma\left(\frac{k^2}{4|\omega|} + \frac{5}{4}\right)\\ &\times U\left(\frac{k^2}{4|\omega|} - \frac{1}{4}, -\frac{1}{2}, |\omega|u^2\right), \end{aligned} \quad (5.6)$$

where $U(a,b,u)$ is the confluent hypergeometric function of the second kind. Near $u = 0$, $\tilde{G}$ is given by

$$\tilde{G} = 1 - \frac{k^2}{2}u^2 + \frac{8\Gamma\left(\frac{k^2}{4|\omega|} + \frac{5}{4}\right)|\omega|^{3/2}}{3\Gamma\left(\frac{k^2}{4|\omega|} - \frac{1}{4}\right)}u^3 + O\left(u^4\right). \quad (5.7)$$

In the IR limit and $m = 0$, the action Eq.(4.1) yields

$$\begin{aligned} S_M^* &\equiv \frac{i}{2}S_M = \frac{1}{2}\int d\tau d^2xN\sqrt{g}\Big\{\frac{1}{N^2}{\varphi'}^2 + (\nabla\varphi)^2\Big\}\\ &= \frac{1}{2}\int d\tau d^2x\sqrt{^{(3)}g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi, \end{aligned} \quad (5.8)$$

where $\varphi' = \frac{\partial\varphi}{\partial\tau}$. Integrating by parts, one can show that the on-shell bulk action is determined by the values of the field on the boundary

$$\begin{aligned} S_M^* &= \int d\tau dx[\sqrt{^{(3)}g}g^{uu}\varphi\partial_u\varphi]_\epsilon^\infty\\ &= \int d\omega dk\tilde{\varphi}(0,k,\omega)\mathcal{F}(k,\omega)\tilde{\varphi}(0,-k,-\omega), \end{aligned} \quad (5.9)$$

where we had cut off the space at $u = \epsilon$ to regulate the bulk action, and the "flux factor" $\mathcal{F}$ is defined as

$$\mathcal{F}(k,\omega) = [\tilde{G}(u,k,\omega)\sqrt{^{(3)}g}g^{uu}\partial_u\tilde{G}(u,-k,-\omega)]_\epsilon^\infty. \quad (5.10)$$

Since the propagator $\tilde{G}$ vanishes at $u = \infty$, $\mathcal{F}$ only receives a contribution from the cutoff at $u = \epsilon$. The momentum space two-point function for the operator $\mathcal{O}_\varphi$ dual to φ is given by differentiating Eq.(5.9) twice with respect to $\varphi(0,k,\omega)$:

$$\langle\mathcal{O}_\varphi(k,\omega)\mathcal{O}_\varphi(-k,-\omega)\rangle = \mathcal{F}(k,\omega). \qquad (5.11)$$

Plugging Eq.(5.7) into Eq.(5.10), we pick out the leading non-polynomial piece in either k or ω. This gives the correlation function, after taking the limit $\epsilon \to 0$,

$$\langle\mathcal{O}_\varphi(k,\omega)\mathcal{O}_\varphi(-k,-\omega)\rangle = -\frac{8|\omega|^{3/2}\Gamma(a+\frac{3}{2})}{\Gamma(a)}, \quad (5.12)$$

where $a \equiv \frac{k^2}{4|\omega|} - \frac{1}{4}$. Since $\Gamma(a \simeq 0) \to \infty$, we find that $\langle\mathcal{O}_\varphi(k,\omega)\mathcal{O}_\varphi(-k,-\omega)\rangle \simeq 0$ as $a \to 0$. When $a \gg 1$, on the other hand, we find $\langle\mathcal{O}_\varphi(k,\omega)\mathcal{O}_\varphi(-k,-\omega)\rangle \simeq -8|\omega|^{1/2}(k^2 + |\omega|)$, which gives rise to correlations between points only with temporal separation.

In general, the divergence arising as $\epsilon \to 0$ from the term proportional to u^2 is removed via local boundary terms [3, 24], and the terms $\mathcal{O}(u^4)$ and higher vanish as the cutoff is removed when taking the limit $\epsilon \to 0$.

B. In the UV

In the UV limit, the last term in Eq.(4.7) dominates, and we find that

$$\partial_\tau^2\varphi = 2a_{24}(\partial_x^4\varphi + 2\partial_x^2\partial_u^2\varphi + \partial_u^4\varphi), \qquad (5.13)$$

where $a_{24} \equiv a_2 + a_4$. In the Fourier space, this becomes

$$\partial_u^4\tilde{G} - 2k^2\partial_u^2\tilde{G} + \left(k^4 + \frac{\omega^2}{2a_{24}}\right)\tilde{G} = 0, \qquad (5.14)$$

with the same boundary condition as in Eq.(5.5). Then, we find that

$$\begin{aligned} \tilde{G} &= c_1e^{-u\sqrt{\rho}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})}\\ &+(1-c_1)e^{-u\sqrt{\rho}(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2})}, \end{aligned} \qquad (5.15)$$

where c_1 is an integration constant, and

$$\rho\cos\theta = k^2, \quad \rho\sin\theta = \sqrt{\frac{w^2}{2a_{24}}}. \qquad (5.16)$$

Thus, with $m = 0$, the action (4.1) gives rise to,

$$\begin{aligned} iS_M &= \int d\tau d^2xN\sqrt{g}\Big\{\frac{1}{2N^2}{\varphi'}^2 + a_2(\nabla^2\varphi)^2\\ &\qquad + a_4\phi\nabla^4\phi\Big\}\\ &= \int d\omega dk\tilde{\varphi}(0,k,\omega)\int_\epsilon^\infty du\\ &\quad\times\Big\{\frac{\omega^2}{2}\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega)\\ &\quad+a_{24}k^4\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega) \end{aligned}$$

$$
\begin{aligned}
&-2a_{24}k^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega)\\
&+a_2\partial_u^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega)\\
&+a_4\tilde{G}(u,k,\omega)\partial_u^4\tilde{G}(u,-k,-\omega)\\
&+\frac{4a_4}{u}[\tilde{G}(u,k,\omega)\partial_u^3\tilde{G}(u,-k,-\omega)\\
&\quad-k^2\tilde{G}(u,k,\omega)\partial_u\tilde{G}(u,-k,-\omega)]\\
&+\frac{2a_4}{u^2}[\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega)\\
&\quad-k^2\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega)]\Big\}\tilde{\varphi}(0,-k,-\omega)\\
=&\int d\omega dk\tilde{\varphi}(0,k,\omega)\mathcal{F}(k,\omega)\tilde{\varphi}(0,-k,-\omega), \quad (5.17)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{F}(k,\omega) \;=\; &\int_\epsilon^\infty du\Big\{\frac{\omega^2}{2}\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega)\\
&+a_{24}k^4\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega)\\
&-2a_{24}k^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega)\\
&+a_2\partial_u^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega)\\
&+a_4\tilde{G}(u,k,\omega)\partial_u^4\tilde{G}(u,-k,-\omega)\\
&+\frac{4a_4}{u}[\tilde{G}(u,k,\omega)\partial_u^3\tilde{G}(u,-k,-\omega)\\
&\quad-k^2\tilde{G}(u,k,\omega)\partial_u\tilde{G}(u,-k,-\omega)]\\
&+\frac{2a_4}{u^2}[\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega)\\
&\quad-k^2\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega)]\Big\}. \quad (5.18)
\end{aligned}
$$

Plugging Eq.(5.15) into Eq.(5.18), and taking the limit $\epsilon \to 0$, we find that

$$
\mathcal{F}(k,\omega) \;=\; 4a_2c_1(1-c_1)\rho^{\frac{3}{2}}\sin\theta\sin\frac{\theta}{2}. \quad (5.19)
$$

VI. CONCLUSIONS

In this paper, we have investigated the effects of high-order operators on the non-relativistic Lifshitz holography in the framework of the Hořava-Lifshitz (HL) theory of gravity [8], which contains all the required high-order spatial operators in order to be power-counting renormalizble. The unitarity of the theory is also preserved, because of the absence of the high-order time operators. In this sense, the HL gravity is an ideal place to study the effects of high-order operators on the non-relativistic gauge/gravity duality.

In particular, we have first shown that the Lifshitz space-time (3.7) is not only a solution of the HL gravity in the IR, as first shown in [12] and later rederived in [13], but also a solution of the full theory. The effects of the high-oder operators on the Lifshitz dynamical exponent z is simply to shift it to different values, as these high-oder operators become more and more important, as shown explicitly in Section III. This is similar to the case studied in [15].

In Section IV, we have studied a scalar field that has the same symmetry in the UV as the HL gravity, the foliation-preserving diffeomorphism described by Eq.(1.4). While in the IR the asymptotic behavior of the scalar field near the boundary is similar to that given in the 4-dimensional spacetimes [3], its asymptotic behavior in the UV gets dramatically changed, so does the corresponding two-point correlation function, as shown in Section V. This is expected, because the high-order operators dominate the behavior of the scalar field in the UV. Then, according to the holographic correspondence, this in turn affects the two-point correlation functions.

It would be important to study the effects of high-order operators on other properties of the non-relativistic Lifshitz holography, including phase transitions and superconductivity of the corresponding non-relativistic quantum field theories defined on the boundary. In particular, it has been suggested that inflation may be described holographically by means of a dual field theory at the future boundary [25]. This might provide deep insights to the Planckian physics in the very early universe, where (non-perturbative) quantum gravitational effects are expected to play an important role. Recently, a powerful analytical approximation method, the so-called *uniform asymptotic approximation*, was developed [26, 27], which is specially designed to study such effects in the very early universe. With the arrival of the era of the precision cosmology [28, 29], such effects might be within the range of the detection of the forthcoming generation of experiments [30].

Another possible application of these high-order effects might be to Hawking radiation, where quantum gravitational effects also become important. Previous studies of such effects showed that the Hawking radiation is robust with respect to the UV corrections [31]. To study them in detail, one can equally apply the uniform asymptotic approximation method developed in [26] to the studies of Hawking radiation. In particular, in the spherical background, one can simply identify the radial coordinate r in the Hawking radiation with the time variable η used in the inflationary models. In the inflationary models, the initial conditions are normally the Bunch-Davies vacuum, but here in the studies of Hawking radiation they should be the Unruh vacuum.

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Appendix A: Field Equations for Satic Spacetimes

From Eq.(3.2) we find that

$$
\begin{aligned}
R &= \frac{2\left(rg'-g\right)}{g^3},\\
\Delta R &= \frac{2r}{g^7}\left[15r^2g'^3 - rgg'\left(21g'+10rg''\right)+g^2\left(6g'+r\left(6g''+rg^{(3)}\right)\right)\right],\\
a_{ij} &= \frac{f\left(g\left(f'+rf''\right)-rf'g'\right)-rgf'^2-zf^2g'}{rf^2g}\delta^r_i\delta^r_j + \left(\frac{r^2\left(zf+rf'\right)}{fg^2}\right)\delta^\theta_i\delta^\theta_j,\\
a_ia^i &= \frac{\left(zf+rf'\right)^2}{f^2g^2},\\
a^i{}_i &= \frac{zf^2\left(g-rg'\right)-r^2gf'^2+rf\left(f'\left(2g-rg'\right)+rgf''\right)}{f^2g^3},\\
a^{ij}a_{ij} &= \frac{1}{f^4g^6}\left\{f^2g^2\left(zf+rf'\right)^2+r^2\left[rgf'^2+zf^2g'-f\left(g\left(f'+rf''\right)-rf'g'\right)\right]^2\right\},
\end{aligned}
\tag{A.1}
$$

and

$$
\begin{aligned}
L_V &= \zeta^2 g_0 + \frac{1}{g^2}\left\{\beta z^2 - 2\gamma_1 + \beta r\frac{f'}{f}\left(2z + r\frac{f'}{f}\right)\right\} + 2\gamma_1 r\frac{g'}{g^3}\\
&+\frac{1}{\zeta^2g^4}\left\{2\left(2\gamma_2-\gamma_4z^2\right)+z^2\left(\beta_1z^2+\beta_2+\beta_3z+\beta_4\right)\right.\\
&+\frac{1}{f}\left(2rz\left(-2\gamma_4+2\beta_1z^2+2\beta_2+2\beta_3z+\beta_4\right)f'+r^2z\left(2\beta_2+\beta_3z\right)f''\right)\\
&+\frac{1}{f^2}\left(r^2\left(-2\gamma_4+6\beta_1z^2+2\left(2-z\right)\beta_2+z\left(5-z\right)\beta_3+2\beta_4\right)\left(f'\right)^2\right.\\
&\left.+2r^3\left(2\beta_2+\beta_3z+\beta_4\right)f''f'+r^4\left(\beta_2+\beta_4\right)\left(f''\right)^2\right)\\
&+\frac{1}{f}\left(2rz\left(-2\gamma_4+2\beta_1z^2+2\beta_2+2\beta_3z+\beta_4\right)f'+r^2z\left(2\beta_2+\beta_3z\right)f''\right)\\
&+\frac{1}{f^3}\left(2r^3\left(2\beta_1z-2\beta_2-\left(z-1\right)\beta_3-\beta_4\right)\left(f'\right)^3\right.\\
&\left.\left.-r^4\left(2\beta_2-\beta_3+2\beta_4\right)f''\left(f'\right)^2\right)+r^4\frac{\left(f'\right)^4}{f^4}\left(\beta_1+\beta_2-\beta_3+\beta_4\right)\right\}\\
&+\frac{1}{\zeta^2g^5}\left\{\left(2r\left(-4\gamma_2+6\beta_6+z^2\gamma_4\right)-rz^2\left(2\beta_2+z\beta_3\right)\right)g'+2r^2\beta_6\left(6g''+rg^{(3)}\right)\right.\\
&+\frac{g'}{f}\left(r^2z\left(4\gamma_4-6\beta_2-3\beta_3r-2\beta_4\right)f'-2r^3z\left(\beta_2+\beta_4\right)f''\right)\\
&+\frac{g'}{f^2}\left(r^3\left(2\gamma_4+2\left(z-2\right)\beta_2-3z\beta_3+2\left(z-1\right)\beta_4\right)\left(f'\right)^2\right.\\
&\left.\left.-2r^4\left(\beta_2+\beta_4\right)f''f'\right)+r^4\frac{\left(f'\right)^3}{f^3}\left(2\beta_2-\beta_3+2\beta_4\right)g'\right\}\\
&+\frac{1}{\zeta^2g^6}\left\{r^2\left(2\left(2\gamma_2-21\beta_6\right)+z^2\left(\beta_2+\beta_4\right)\right)\left(g'\right)^2\right.\\
&\left.-20\beta_6r^3g''g'+r^3\frac{f'}{f}\left(g'\right)^2\left(\beta_2+\beta_4\right)\left(2z+r\frac{f'}{f}\right)\right\}+30\beta_6r^3\frac{\left(g'\right)^3}{\zeta^2g^7}.
\end{aligned}
\tag{A.2}
$$

Then, the field equations (3.4) and (3.5) take the forms,

$$
0 = -r^z\zeta^2\gamma_0 g + \frac{r^z}{g}\left\{2\gamma_1 + r\beta\left(2(z+2)+\left[2z+4-r\frac{f'}{f}\right]\frac{f'}{f}+2rf''\right)\right\} - \frac{r^z}{g^2}\left\{2\gamma_1+2\beta r\left(z+r\frac{f'}{f}\right)\right\}g'
$$

$$
\begin{aligned}
&+\frac{r^z}{\zeta^2 g^3}\left\{-4\gamma_2-2z(z+2)\gamma_4+z^3(3z+4)\beta_1-z^2(2z+3)\beta_2-z^2(z^2-2)\beta_3+z(z+2)\beta_4\right.\\
&+\frac{2r}{f}\left(\left[-2(z+2)\gamma_4+6z^2(z+3)\beta_1-2(2z^2+5z+2)\beta_2+2z(z^2-2z+1)\beta_3-(z^2+3z+2)\beta_4\right]f'\right.\\
&+r\left[-2\gamma_4+6\beta_1 z^2-(z^2+11z+14)\beta_2-2z(z-1)\beta_3-(z^2+8z+13)\beta_4\right]f''\\
&\left.-2r^2(4+z)\left[\beta_2+\beta_4\right]f^{(3)}-r^3\left[\beta_2+\beta_4\right]f^{(4)}\right)\\
&+\frac{r^2}{f^2}\left(2\left[\gamma_4+3z(z+6)\beta_1+(z^2+6z+4)\beta_2-(z^2+6z-3)\beta_3+(z^2+6z+8)\beta_4\right](f')^2\right.\\
&+4r\left[12\beta_1 z+4z(z+5)\beta_2-2(2z-1)\beta_3+(2z+13)\beta_4\right]f''f'\\
&\left.+4r^2\left[\beta_2+\beta_4\right]f^{(3)}f'+3r^2\left[\beta_2+\beta_4\right](f'')^2\right)-\frac{4r^3}{f^3}\left(\left[(3z-4)\beta_1+(z+2)\beta_2-(z-2)\beta_3+(z+3)\beta_4\right]f'\right.\\
&\left.-r\left[3\beta_1-2\beta_2-\beta_3-2\beta_4\right]f''\right)(f')^2-\frac{3r^4}{f^4}\left(3\beta_1-\beta_2-\beta_3-\beta_4\right)(f')^4\\
&+\frac{r^z}{\zeta^2 g^4}\left\{\left(2r\left[4\gamma_2-6\beta_6-6\beta_1 z^3+z(z^2+9z+6)\beta_2-z^2(3-2z)\beta_3+z(z^2+4z+1)\beta_4\right]g'\right.\right.\\
&\left.+2r^2\left[-6\beta_6+2\gamma_4 z+2z(z+3)\beta_2+z(2z+5)\beta_4\right]g''+2r^3\left[-\beta_6+z(\beta_2+\beta_4)\right]g^{(3)}\right)\\
&+\frac{1}{f}\left(\left(2r^2\left[2(6+z)\gamma_4-18\beta_1 z^2+3(z^2+9z+8)\beta_2+6z(z-1)\beta_3+(3z^2+17z+18)\beta_4\right]f'\right.\right.\\
&\left.+2r^3\left[2\gamma_4+3(10+3z)\beta_2+(29+9z)\beta_4\right]f''+12r^4\left[\beta_2+\beta_4\right]f^{(3)}\right)g'\\
&+2r^3\left(\left[2\gamma_4+2(2z+5)\beta_2+(4z+9)\beta_4\right]g''+2r\left[\beta_2+\beta_4\right]g^{(3)}\right)f'\\
&\left.+8r^4\left[\beta_2+\beta_4\right]f''g''\right)+\frac{1}{f^2}\left(\left(-2r^3\left[\gamma_4+18\beta_1 z+6(2+z)\beta_2+3(1-2z)\beta_3+2(8+3z)\beta_4\right]f'\right.\right.\\
&\left.\left.\left.-18r^4\left[\beta_2+\beta_4\right]f''\right)f'g'-4r^4\left[\beta_2+\beta_4\right](f')^2g''\right)-\frac{4r^4}{f^4}\left(3\beta_1-2\beta_2-\beta_3-2\beta_4\right)(f')^3g'\right\}\\
&+\frac{r^z}{\zeta^2 g^5}\left\{\left(-r^2\left[4\gamma_2-42\beta_6+16\gamma_4 z+z(42+15z)\beta_2+z(34+15z)\beta_4\right]g'\right.\right.\\
&\left.+20r^3\left[\beta_6+z(\beta_2+\beta_4)\right]g''\right)g'+\frac{2}{f}\left(\left(-r^3\left[8\gamma_4+(36+15z)\beta_2+(32+15z)\beta_4\right]f'-30r^4\left[\beta_2+\beta_4\right]f''\right)(g')^2\right.\\
&\left.\left.-20r^4\left[\beta_2+\beta_4\right]f'g'g''\right)+\frac{15r^4}{f^2}\left(\beta_2+\beta_4\right)(f')^2(g')^2\right\}+\frac{30r^3}{g^6}\left\{-\beta_6+\left[\beta_2+\beta_4\right]\left(z+r\frac{f'}{f}\right)\right\}(g')^3\,, \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& -r^z\zeta^2 f\gamma_0+\frac{r^z}{g^2}f\left(2\left[z+\frac{f'}{f}\right]\gamma_1+\left[z^2+2zr\frac{f'}{f}+r^2\frac{(f')^2}{f^2}\right]\beta\right)\\
&+\frac{r^z}{\zeta^2 g^4}\left\{\left(4(1-2z)\gamma_2-2z(z^2-1)\beta_6+2z^2(z-2)\gamma_4+3\beta_1 z^4+z^2(2z-1)\beta_2-z^3(z-2)\beta_3+3\beta_4 z^2\right)f\right.\\
&+2r\left(\left[-4\gamma_2+3z(z+1)\beta_6+z(3z-2)\gamma_4+6\beta_1 z^3-4\beta_2 z^2-z^2(2z-3)\beta_3-z(z-1)\beta_4\right]f'\right.\\
&\left.+r\left[+3(z+1)\beta_6+2\gamma_4 z-z(z+3)\beta_2+z(z+4)\beta_4\right]f''+r^2\left[\beta_6-z(\beta_2+\beta_4)\right]f^{(3)}\right)\\
&+\frac{r^2}{f}\left(2\left[\gamma_4 z+9\beta_1 z^2+z(z-2)\beta_2-3z(z-1)\beta_3+z(z+2)\beta_4\right](f')^2\right.\\
&\left.+2r^3\left(\left[2\gamma_4+(z-2)\beta_2+(z-3)\beta_4\right]f''-r\left[\beta_2+\beta_4\right]f^{(3)}\right)f'+r^2\left[\beta_2+\beta_4\right](f'')^2\right)\\
&+\frac{2r^3}{f^2}\left(\left[-\gamma_4+6\beta_1 z-(2z-1)\beta_3+2\beta_4\right]f'+r\left[\beta_2+\beta_4\right]f''\right)(f')^2\\
&\left.+\frac{r^4}{f^3}\left[3\beta_1-\beta_2-\beta_3-\beta_4\right](f')^4\right\}+\frac{r^z}{\zeta^2 g^5}\left\{2r\left((z+2)\left[4\gamma_2-2z\beta_6+z^2(\beta_2+\beta_4)\right]g'\right.\right.\\
&\left.+r\left[4\gamma_2-2\beta_6 z+z^2(\beta_2+\beta_4)\right]g''\right)f+2r^2\left(\left[4\gamma_2-2(3+2z)\beta_6+3z(z+2)(\beta_2+\beta_4)\right]f'\right.\\
&\left.+2r\left[-\beta_6+z(\beta_2+\beta_4)\right]f''\right)g'+4r^3\left[-\beta_6+z(\beta_2+\beta_4)\right]g''f'
\end{aligned}
$$

$$+\frac{2r^3}{f}\left(\left((z+4)\left[\beta_2+\beta_4\right]f'+2r\left[\beta_2+\beta_4\right]f''\right)f'g'+r\left[\beta_2+\beta_4\right](f')^2g''\right)-\frac{2r^4}{f^2}\left[\beta_2+\beta_4\right](f')^3g'\Bigg\}$$
$$+\frac{5r^2r^z}{\zeta^2g^6}\left\{\left[-4\gamma_2+2\beta_6z-z^2(\beta_2+\beta_4)\right]-2r\left[\beta_6-z(\beta_2+\beta_4)\right]f'-\frac{r^2}{f}\left[\beta_2+\beta_4\right](f')^2\right\}(g')^2. \tag{A.4}$$

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