Positive energy theorem for $P(X, \phi)$ theories

Benjamin Elder, Austin Joyce, Justin Khoury, and Andrew J. Tolley

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A Positive Energy Theorem for $P(X, \phi)$ Theories

Benjamin Elder,[1] Austin Joyce,[2] Justin Khoury,[1] and Andrew J. Tolley[3]

[1] *Center for Particle Cosmology, University of Pennsylvania, Philadelphia, PA 19104, USA*

[2] *Enrico Fermi Institute and Kavli Institute for Cosmological Physics, University of Chicago, Chicago, IL 60637*

[3] *CERCA/Department of Physics, Case Western Reserve University, 10900 Euclid Ave, Cleveland, OH 44106, USA*

We describe a positive energy theorem for Einstein gravity coupled to scalar fields with first-derivative interactions, so-called $P(X, \phi)$ theories. We offer two independent derivations of this result. The first method introduces an auxiliary field to map the theory to a Lagrangian describing two canonical scalar fields, where one can apply a positive energy result of Boucher and Townsend. The second method works directly at the $P(X, \phi)$ level and uses spinorial arguments introduced by Witten. The latter approach follows that of recent work by Nozawa and Shiromizu [41], but the end result is considerably less restrictive. We point to the technical step where our derivation deviates from theirs, which substantially expands the class of Lagrangians encompassed by the theorem. One of the more interesting implications of our analysis is to show it is possible to have positive energy in cases where dispersion relations following from locality and S-Matrix analyticity are violated. This indicates that these two properties are logically distinct, *i.e.*, it is possible to have positive energy even when the S-matrix is non-analytic, and vice-versa.

In recent years there has been much interest in derivatively coupled scalar theories, particularly in cosmology, but also in other areas of high-energy physics and condensed matter. The novelty of these theories is that, in certain cases, they can have large classical non-linearities while remaining radiatively stable, allowing for a range of interesting phenomena. For example, ghost condensation [1] and galileons [2] possess time-dependent solutions that can violate the Null Energy Condition (NEC) [3–7] and yield novel cosmologies [8–13]. These examples are free of ghost or gradient instabilities, but may have other unwelcome features, such as superluminality or conflict with black hole thermodynamics [14], casting doubt on whether they admit a local ultraviolet (UV) completion [15].

It is natural to wonder if there are any statements one can make about the viability of these theories in the presence of gravity. One desirable property is that the vacuum be classically stable. This will be the case if the theory admits a positive energy theorem for asymptotically flat solutions, *i.e.*, that the ADM mass is always non-negative and is zero for Minkowski space only.[1] It was originally shown [16] that Einstein gravity plus matter has positive energy if the matter obeys the dominant energy condition (DEC).[2] This proof was later simplified using a spinor technique due to Witten [17–19]. (These arguments were originally developed for asymptotically flat geometries, but can be extended to include asymptotically anti-de Sitter[3] spacetimes [20–23].) The result was extended by Boucher and Townsend, who showed that demanding that the matter satisfy the DEC is not necessary to ensure positive energy [26, 27]. See also [28]. For a nonlinear σ-model with N scalars coupled to Einstein gravity,

$$\mathcal{L} = -\frac{1}{2} f_{IJ}(\phi)\partial_\mu \phi^I \partial^\mu \phi^J - V(\phi^I) \, , \tag{1}$$

where f_{IJ} is positive-definite, positivity is guaranteed for asymptotically Minkowski/AdS solutions where $\phi^I \to \bar{\phi}^I$ at infinity so long as $V(\phi^I)$ is derivable from a "superpotential" $W(\phi^I)$ obeying the equation:[4]

$$V(\phi^I) = 8 f^{IJ} W_{,\phi^I} W_{,\phi^J} - 12 W^2 \, , \tag{2}$$

assuming that $V(\phi^I)$ admits a minimum with $V(\bar{\phi}^I) \leq 0$.

In this note, we further extend this result and derive a positive energy theorem for scalar theories of the form

$$\mathcal{L} = P(X, \phi) \, , \tag{3}$$

by similarly constraining the functional form that $P(X, \phi)$ can take. Here X is the canonical kinetic term: $X = -\frac{1}{2}(\partial\phi)^2$. (We use the mostly-plus sign convention.) This class of theories has a long history, especially in cosmology. They can be used for inflation [29–31], dark energy [32, 33], bouncing cosmologies [8, 9, 11], and display screening around heavy sources [34–39]. We are therefore motivated to ask: in what situations can it be

[1] For the ADM mass to be well-defined, we focus on flat (or AdS) asymptotics, where $\phi \to$ const. at spatial infinity. This immediately rules out time-dependent asymptotics, $\phi \to \phi(t)$, which may be more realistic for cosmology.

[2] The DEC states that: *i)* for any time-like u, $T_{\mu\nu}u^\mu u^\nu \geq 0$; and *ii)* for any future-pointing and causal u, $-T^\mu_\nu u^\nu$ is also future-pointing and causal. Roughly, these correspond respectively to the statements that the energy density is positive, and the energy-momentum flow is subluminal.

[3] These results do not generalize straightforwardly to asymptotically de Sitter spaces because of the lack of a time-like Killing vector. For a construction and analysis of a somewhat similar quantity on dS space, based upon a conformal Killing vector, see [24, 25].

[4] In fact, $V(\phi^I)$ must only satisfy the weaker inequality $V(\phi^I) \geq 8 f^{IJ} W_{,\phi^I} W_{,\phi^J} - 12W^2$.

shown that the energy of isolated gravitating systems is positive in theories of this type? Specific forms of P for which the energy can be shown to be positive in this setting are then somewhat better motivated to consider in other arenas, for example, cosmology.

Another motivation for this work is to disentangle the various criteria typically imposed upon theories. It is clearly desirable for a theory to have positive energy—an unbounded Hamiltonian leads to ghost instabilities upon quantization of the theory. Other oft-imposed criteria are that a theory not admit superluminal propagation or that the S-Matrix of the theory defined on Minkowski space be analytic, which leads to various dispersion relations obeyed by scattering amplitudes. Although violation of these last two criteria is less obviously bad, it is still somewhat undesirable. It is known that some $P(X,\phi)$ theories can violate one or both of these; therefore, we are compelled to see whether violation of any of these criteria necessarily implies negative energy, for example.

The result we establish is the following: consider Einstein gravity coupled minimally to a derivatively-interacting scalar field theory with a Lagrangian of the form (3) and zero cosmological constant.[5] For solutions where the metric is asymptotically flat (resp. AdS) and the scalar field goes to a constant[6] $\phi \to \bar{\phi}$ at infinity such that $P(\bar{X},\bar{\phi}) = 0$ (resp. $P(\bar{X},\bar{\phi}) = \text{const.} < 0$), the system has positive ADM energy so long as we can find two functions $\mathcal{W}(\phi)$ and $\mathcal{G}(X,\phi)$ such that P satisfies the following equation

$$P - XP_{,X} + 8\frac{\mathcal{W}_{,\phi}^2}{P_{,X}} + \mathcal{G}_{,X}^2 - 12\mathcal{W}^2 = 0 \ . \tag{4}$$

We establish this positive energy result in two different ways. First, at the classical level we map (3) to an equivalent two-derivative theory via an auxiliary field, following [40]. Turning on a small kinetic term for this second field, the action takes the form (1). We can then apply the result (2), which is translated to a statement about $P(X,\phi)$ upon solving for the auxiliary field.

Second, we will reproduce this result directly at the $P(X,\phi)$ level using Witten's spinor arguments. This approach was taken in [41], although we will see that their result was too restrictive; ultimately the models they consider can only violate the DEC by utilizing a pure potential for the scalar, making them morally equivalent to the σ-model example. Violation of the DEC through the kinetic part of the action is one of the main reasons for considering $P(X,\phi)$ models, so it is worth asking whether the class of models for which we can prove positive energy can be expanded. We will show that relaxing a small technical assumption in their argument allows for greater flexibility in choosing the functional form of $P(X,\phi)$.

This broader assortment of $P(X,\phi)$ theories consistent with positive energy allows for interesting phenomena. In particular, consider $P(X) = X + \alpha X^2$, arguably the simplest $P(X,\phi)$ example. With $\alpha > 0$, this theory obeys the DEC and hence has positive energy. Even with $\alpha < 0$, however, we will show that the theory allows positive energy, as long as we restrict to the region $P_{,X} > 0$. This is remarkable since this theory with $\alpha < 0$ both exhibits a screening mechanism and violates some of the dispersion relations following from S-matrix analyticity requirements of a local theory [15].

Two-Field Description: It is possible to map a $P(X,\phi)$ theory to a 2-derivative action by introducing an auxiliary field χ [40] so that the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2}P_{,\chi}(\partial\phi)^2 - \chi P_{,\chi} + P\,, \tag{5}$$

where $P = P(\chi,\phi)$. Indeed, the equation of motion for χ is $P_{,\chi\chi}(X-\chi) = 0$, which sets $\chi = X$, as long as $P_{,\chi\chi} \neq 0$. Substituting $\chi = X$ in (5) gives $\mathcal{L} = P(X,\phi)$, establishing the classical equivalence of the two descriptions. To put it in the form (1), we simply turn on a small kinetic term for χ:

$$\mathcal{L} = -\frac{1}{2}P_{,\chi}(\partial\phi)^2 - \frac{1}{2}Z^2(\partial\chi)^2 - \chi P_{,\chi} + P\ . \tag{6}$$

At this level, this is just a technical trick — at the end we will take $Z \to 0$. Upon making the identifications

$$f_{\chi\chi} = Z^2\ ; \qquad f_{\phi\phi} = P_{,\chi}\ ; \qquad V(\chi,\phi) = \chi P_{,\chi} - P\,, \tag{7}$$

this is of form (1). Note f_{IJ} must be positive-definite, imposing $P_{,\chi} > 0$. After solving for χ, this translates to $P_{,X} > 0$, which is equivalent to the Null Energy Condition,[7] and represents a minimal restriction on the theory. Violating this constraint necessarily implies that the perturbations in the ϕ field are either ghostlike or possess a gradient instability [7, 39]. In some cases this constraint will restrict the range of X, but this is acceptable because it is a *Lorentz-invariant* restriction on the space of allowed solutions. The condition $P_{,X} > 0$ is required for the validity of the single-field EFT which is partially UV completed by the two-field system (6) [40].

Substituting (7), the condition (2) yields

$$\chi P_{,\chi} - P = 8\frac{W_{,\phi}^2}{P_{,\chi}} + 8\frac{W_{,\chi}^2}{Z^2} - 12W^2\ . \tag{8}$$

To have a smooth $Z \to 0$ limit, the superpotential must take the form $W(\chi,\phi) = \mathcal{W}(\phi) + \frac{Z}{2\sqrt{2}}\mathcal{G}(\chi,\phi) + \mathcal{O}(Z^2)$,

[5] The generalization to negative cosmological constant is straightforward.

[6] In order for this asymptotic boundary condition on the field to be well-defined, it is necessary for the function P to satisfy $P_{,\phi}(\bar{X},\bar{\phi}) = 0$.

[7] The stress tensor for $P(X,\phi)$ is $T_{\mu\nu} = P_{,X}\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}P$. Contracting with a null vector n^μ, the NEC boils down to $0 \leq T_{\mu\nu}n^\mu n^\nu = P_{,X}(n^\mu\partial_\mu\phi)^2$, which requires $P_{,X} > 0$.

where the factor of $2\sqrt{2}$ is introduced to simplify later expressions. Substituting this into (8) and taking $\chi \to X$, the positive energy condition becomes

$$P - XP_{,X} + 8\frac{\mathcal{W}_{,\phi}^2}{P_{,X}} + \mathcal{G}_{,X}^2 - 12\mathcal{W}^2 = 0 \ . \tag{9}$$

This is our main result. It is the analogue of (2) for theories of the $P(\phi, X)$ type. Positivity of the energy requires the existence of two functions, $\mathcal{W}(\phi)$ and $\mathcal{G}(\phi, X)$, related to $P(\phi, X)$ through (9). Asymptotically, we assume $X \to 0$ and $\phi \to \bar{\phi}$ such that $P_{,\phi}(\bar{\phi}) = 0$. The asymptotic geometry can be either flat (if $P(\bar{X}, \bar{\phi}) = 0$) or AdS (if $P(\bar{X}, \bar{\phi}) < 0$).

The proof generalizes to N scalar fields with $P(X^{IJ}, \phi^K)$, where following [41] we have defined the tensor $X^{IJ} = -\frac{1}{2}\partial_\mu\phi^I\partial^\mu\phi^J$. This generalization is particularly interesting because the EFT of fluids [42] is a theory of this type. We introduce a matrix of scalar fields χ^{IJ}, and the generalization of (6) becomes

$$\mathcal{L} = -\frac{1}{2}P_{MN}\partial^\mu\phi^M\partial_\nu\phi^N - \frac{1}{2}Z^2 P_{KM}P_{LN}\partial_\mu\chi^{KL}\partial_\nu\chi^{MN} + P - \chi^{MN}P_{MN} \ , \tag{10}$$

where $P_{IJ} \equiv \partial P/\partial\chi^{IJ}$ is positive definite and invertible. Again, solving for χ and setting $Z \to 0$ gives $X^{IJ} = \chi^{IJ}$. Following the same steps as before, we find that the superpotential must take the form $W = \mathcal{W}(\phi^I) + \frac{Z}{2\sqrt{2}}\mathcal{G}(\phi^I, \chi^{MN}) + \mathcal{O}(Z^2)$. Writing the inverse of P_{IJ} as P^{IJ}, we arrive at the positivity condition

$$P - X^{MN}P_{MN} + 8P^{MN}\mathcal{W}_{,\phi^M}\mathcal{W}_{,\phi^N} + P^{KM}P^{LN}\mathcal{G}_{KL}\mathcal{G}_{MN} - 12\mathcal{W}^2 = 0 \ . \tag{11}$$

Direct derivation: We now re-derive the positive energy condition (9) directly at the level of $P(X, \phi)$. This method generally follows the presentation of Witten's proof of the positive energy theorem in [41], but with a crucial difference, which we will point out below.

The starting point is the Nester 2-form [17, 19]:

$$N^{\mu\nu} = -i\left(\bar{\epsilon}\gamma^{\mu\nu\rho}\hat{\nabla}_\rho\epsilon - \overline{\hat{\nabla}_\rho\epsilon}\gamma^{\mu\nu\rho}\epsilon\right) \ . \tag{12}$$

where we have defined the super-covariant derivative

$$\hat{\nabla}_\mu\epsilon = (\nabla_\mu + \mathcal{A}_\mu)\,\epsilon \ . \tag{13}$$

Some words on notation: ϵ is a *commuting* Dirac spinor [27], with conjugate $\bar{\epsilon} = i\epsilon^\dagger\gamma^0$; the Dirac matrices obey the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, and we have defined the anti-symmetric product $\gamma^{\mu\nu\rho} \equiv \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$.

The virtue of $N^{\mu\nu}$ is that its integral is simply related to the energy of a gravitating system [17, 19, 27]

$$E = \int_{\partial\Sigma} \mathrm{d}\Sigma_{\mu\nu}N^{\mu\nu} = \int_\Sigma \mathrm{d}\Sigma_\nu\nabla_\mu N^{\mu\nu} \ , \tag{14}$$

where Σ is an arbitrary space-like surface, with $\mathrm{d}\Sigma_\nu$ denoting the normal-pointing volume form. In order for (14) to be the energy, it is necessary that the spacetime admit an asymptotically time-like Killing vector, which along with our boundary conditions on the scalar field restricts us to asymptotically Minkowski or Anti-de Sitter spacetimes. The divergence of $N^{\mu\nu}$ is given by [41]

$$\nabla_\nu N^{\mu\nu} = 2i\overline{\hat{\nabla}_\nu\epsilon}\gamma^{\mu\nu\rho}\hat{\nabla}_\rho\epsilon - \frac{T^\mu_{\ \nu}}{M_{\rm Pl}^2}\,i\bar{\epsilon}\gamma^\nu\epsilon - i\bar{\epsilon}\gamma^{\mu\nu\rho}\mathcal{F}_{\nu\rho}\epsilon \ , \tag{15}$$

where $\mathcal{F}_{\nu\rho} = \nabla_\nu\mathcal{A}_\rho - \nabla_\rho\mathcal{A}_\nu + [\mathcal{A}_\nu, \mathcal{A}_\rho]$ is the curvature of the connection $\mathcal{A}_\mu$. The stress tensor for (3) is

$$T_{\mu\nu} = P_{,X}\partial_\mu\phi\partial_\nu\phi + Pg_{\mu\nu} \ . \tag{16}$$

The term $2i\overline{\hat{\nabla}_\nu\epsilon}\gamma^{\mu\nu\rho}\hat{\nabla}_\rho\epsilon$, gives a positive contribution to the energy, after imposing the Witten condition $\gamma^i\hat{\nabla}_i\epsilon = 0$ [17]. The other two terms are not manifestly positive. To proceed, we follow [41] and make the ansatz

$$\mathcal{A}_\mu = \mathcal{W}(\phi)\gamma_\mu \ , \tag{17}$$

for some $\mathcal{W}(\phi)$. The last term in (15) becomes

$$-i\bar{\epsilon}\gamma^{\mu\nu\rho}\mathcal{F}_{\nu\rho}\epsilon = -4i\bar{\epsilon}\gamma^{\mu\nu}\epsilon\mathcal{W}_{,\phi}\partial_\nu\phi + 12i\bar{\epsilon}\gamma^\mu\epsilon\mathcal{W}^2 \ . \tag{18}$$

Our goal is to write this as a sum of squares of spinors, plus a remainder piece. To do this, we define

$$\delta\lambda_1 = \frac{1}{\sqrt{2}}\left(\sqrt{P_{,X}}\gamma^\mu\partial_\mu\phi - 4\frac{\mathcal{W}_{,\phi}}{\sqrt{P_{,X}}}\right)\epsilon \ ; \qquad \delta\lambda_2 = \mathcal{G}_{,X}\epsilon \ , \tag{19}$$

so that

$$-i\bar{\epsilon}\gamma^{\mu\nu\rho}\mathcal{F}_{\nu\rho}\epsilon = i\sum_{i=1}^2\overline{\delta\lambda}_i\gamma^\mu\delta\lambda_i + i\bar{\epsilon}\gamma^\nu\epsilon P_{,X}\partial^\mu\phi\partial_\nu\phi + i\bar{\epsilon}\gamma^\mu\epsilon\left(XP_{,X} - 8\frac{\mathcal{W}_{,\phi}^2}{P_{,X}} - \mathcal{G}_{,X}^2 + 12\mathcal{W}^2\right) \ . \tag{20}$$

This is the key difference from the derivation in [41]. In that calculation, the authors only used one $\delta\lambda$ spinor field, which led to a strongly restricted class of solutions where X could only appear in the Lagrangian underneath a square root. Instead we expressed $-i\bar{\epsilon}\gamma^{\mu\nu\rho}\mathcal{F}_{\nu\rho}\epsilon$ as the sum of *two* squares of spinors. The second spinor introduces a new function $\mathcal{G} = \mathcal{G}(X, \phi)$, which allows us to derive a more general positivity constraint than [41], leading to Lagrangians that may contain arbitrary powers of X as long as the positivity constraint is satisfied.

Combining (15), (16) and (20), we obtain

$$\nabla_\nu N^{\mu\nu} = 2i\overline{\hat{\nabla}_\nu\epsilon}\gamma^{\mu\nu\rho}\hat{\nabla}_\rho\epsilon + i\sum_{i=1}^2\overline{\delta\lambda}_i\gamma^\mu\delta\lambda_i + i\bar{\epsilon}\gamma^\mu\epsilon\left(XP_{,X} - P - 8\frac{\mathcal{W}_{,\phi}^2}{P_{,X}} - \mathcal{G}_{,X}^2 + 12\mathcal{W}^2\right) \ . \tag{21}$$

The first line is positive-definite, whereas the second line is not. To ensure positivity of E, it is sufficient to set the second line to zero. This yields (9), which is precisely the energy condition obtained from the 2-field approach.

An alternative route to (9) is to not introduce $\mathcal{G}_{,X}$ through (19), but demand that the second line of (21) (with $\mathcal{G}_{,X} = 0$) be positive definite. This yields the inequality $XP_{,X} - P - 8\mathcal{W}_{,\phi}^2/P_{,X} + 12\mathcal{W}^2 \geq 0$. Since the left-hand side is positive-definite, we can write it as the square of some function. Calling this function $\mathcal{G}_{,X}$ yields (9).

The mass vanishes for $\hat{\nabla}_\mu \epsilon = \delta\lambda_a = 0$, which implies Minkowski or AdS space-time [27]. Having derived this constraint on the functional form of P, we now turn to solving this equation in a few situations of interest.

Pure $P(X)$: One simple but nontrivial case to consider is $P = P(X)$, *i.e.*, a field with purely derivative couplings and no potential. We simply assume that $\mathcal{W} \equiv \mathcal{W}_0$ is constant, and take $\mathcal{G} = \mathcal{G}(X)$. In this case, the positive energy condition (9) reduces to an ordinary differential equation for $\mathcal{G}$, which can be integrated:

$$\mathcal{G}(X) = \int \mathrm{d}X \left(XP_{,X} - P + 12\mathcal{W}_0^2\right)^{1/2} . \qquad (22)$$

In order for this integral to be real-valued, we must have $XP_{,X} - P \geq -12\mathcal{W}_0^2$. Note that this condition is weaker than the dominant energy condition: $XP_{,X} - P \geq 0$, indicating that we have actually gained some freedom with respect to the standard positive energy theorem.

As a simple example, consider the function

$$P(X) = X - \beta X^2 \; ; \qquad \beta \geq 0 . \qquad (23)$$

This theory violates the DEC for all X: $XP_{,X} - P = -\beta X^2 < 0$. Recall that our derivation requires $P_{,X} \geq 0$, so we must restrict ourselves to the range $|X| \leq 1/\sqrt{2}\beta$. In this case, (22) can be integrated, ensuring the existence of a suitable superpotential, and guaranteeing that the theory has positive energy in the allowed X range.

This theory with "wrong-sign" X^2 term is well-known to violate the standard dispersion relations following from local S-matrix theory [15], at least at tree level. Nevertheless, we have shown that the theory does allow positive energy, at least over the range of X where the NEC is satisfied, indicating that positive energy and S-matrix analyticity are independent criteria to place on a given EFT. This may seem paradoxical from the perspective of the 2-field action discussed earlier; after all, (6) describes two healthy scalars with some potential, and therefore should have an analytic S-matrix. The resolution is that the vacuum state $X = 0$ or, equivalently, $\chi = 0$, is tachyonic in the two-field language, hence its S-matrix is ill-defined.

Separable $P(X,\phi)$: A slightly more complicated case is where P is a separable function:

$$P(X,\phi) = K(\phi)\tilde{P}(X) - V(\phi) \; , \qquad (24)$$

with $K(\phi) \geq 0$ without loss of generality. This form has been widely-studied in the context of k-essence [29, 32].

It will prove convenient to redefine the arbitary function $\mathcal{G}(X,\phi)$ via

$$\mathcal{G}_{,X}^2 = \mathcal{H}(X,\phi) + 8\frac{\mathcal{W}_{,\phi}^2}{K(\phi)}\left(1 - \frac{1}{\tilde{P}_{,X}}\right) . \qquad (25)$$

Inserting this into (9), we find that P must satisfy

$$\tilde{P} - X\tilde{P}_{,X} + \frac{\mathcal{H}(X,\phi)}{K(\phi)} = \frac{1}{K(\phi)}\left(12\mathcal{W}^2 - 8\frac{\mathcal{W}_{,\phi}^2}{K} + V(\phi)\right) .$$

For this to be separable, $\mathcal{H}$ must factorize as $\mathcal{H}(X,\phi) = K(\phi)H(X)$. The above then implies two equations

$$\begin{aligned} H(X) &= X\tilde{P}_{,X} - \tilde{P}(X) - E \, ; \\ V(\phi) &= 8\frac{\mathcal{W}_{,\phi}^2}{K(\phi)} - 12\mathcal{W}^2 + EK(\phi) \, . \end{aligned} \qquad (26)$$

We must ensure that through all these redefinitions we maintain $\mathcal{G}_{,X}^2 \geq 0$. Combining (25)–(26), we find

$$X\tilde{P}_{,X} - \tilde{P}(X) \geq E - 8\frac{\mathcal{W}_{,\phi}^2}{K^2(\phi)}\left(1 - \frac{1}{\tilde{P}_{,X}}\right) . \qquad (27)$$

This allows for DEC-violation through the kinetic part of the action whenever the right-hand side is negative.

A few limiting cases of these results:

- If $\tilde{P} = X$, corresponding to the two-derivative Lagrangian $\mathcal{L} = K(\phi)X - V(\phi)$, we can set $E = 0$ and $\mathcal{G} = 0$. The second of (26) reduces to the standard result (2) for a single scalar field

$$V(\phi) = 8\frac{\mathcal{W}_{,\phi}^2}{K(\phi)} - 12\mathcal{W}^2 . \qquad (28)$$

- For the pure $P(X)$ case, corresponding to $K(\phi) = 1$ and $V(\phi) = 0$, the second of (26) allows us to choose $\mathcal{W} = \mathcal{W}_0 =$ constant, with $E = 12\mathcal{W}_0^2$. The first of (26), combined with (25), then implies

$$\mathcal{G}_{,X}^2 = \mathcal{H}(X) = XP_{,X} - P + 12\mathcal{W}_0^2 , \qquad (29)$$

whose integral reproduces (22).

Conclusions: We have derived, following two different methods, an extension of the positive energy theorem of General Relativity to the class of $P(X,\phi)$ scalar field theories coupled to gravity. The first method we used is a new technique we have introduced, inspired by [40], and may greatly simplify positivity calculations for more complicated models in the future. We found that as long as it is possible to write P in terms of two arbitrary superpotential-like functions, positive energy is guaranteed. This derivation generalizes the result of [26, 27] for two-derivative scalar theories with arbitrary potential, and reduces to the known condition as a particular case. This result allows for more general P than the recent result of [41], and we highlighted the technical step where our derivation deviates from theirs.

By examining a few special classes of P we showed that in the $P(X)$ context it is possible to have positive energy while violating the DEC. The derivation does however require that the NEC to be satisfied. More interestingly, it is possible to have positive energy in cases where the S-matrix fails to satisfy the usual analyticity requirements for a local theory and where tree-level superluminality appears. At the very least, this logically disentangles these constraints which are often placed upon low-energy EFTs.

For future work, it will be interesting to investigate BPS solutions admitted by the class of $P(X)$ Lagrangians with positive energy. It will also be interesting to extend our results to more general derivative interactions, such as galileons or massive gravity [43].

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