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Abstract

In this note we construct asymptotically Lifshitz spacetimes in the Chern-Simons formulation of three dimensional higher spin gravity and relate the resulting theories to integrable systems which are elements of the KdV hierarchy.

1 Introduction

Higher spin gravities in various dimensions as formulated by Vasiliev and collaborators (see [1, 2, 3, 4] for some reviews) provide an interesting new playground to explore the AdS/CFT correspondence. In the following we will consider only three dimensional higher spin theory, which can be formulated using Chern-Simons gauge theories [5, 6]. Interest in these theories is fueled by the proposal of an exact AdS/CFT duality linking such theories to W_N minimal model CFTs, due to Gaberdiel and Gopakumar [7, 8].

Three dimensional higher spin theories allow for the construction of non-AdS solutions [9, 10, 11, 12], such as asymptotically Lobachevsky, Schrödinger, warped AdS and Lifshitz spacetimes. In the following we will focus on the asymptotically Lifshitz solutions following the approach developed in [13]. On the field theory side systems with Lifshitz scaling, i.e. anisotropic scaling symmetries with respect to spatial and time directions, are ubiquitous and important in condensed matter systems near quantum critical points (see e.g. [14]).

The goal of the present paper is to construct an asymptotically Lifshitz spacetime using the Chern-Simons formulation with various integer values of the Lifshitz scaling exponent z and uncover the relation of these spacetimes to integrable systems, in particular members of the KdV hierarchy.

The structure of this paper is as follows: In section 2 we give a brief overview of the Chern-Simons formulation of three-dimensional higher spin gravity (more details can be found for example in [8, 15, 16, 17]). In section 3 we review some aspects of the integrable systems which are relevant for the present paper. In particular, we discuss the formulation of the KdV hierarchy in terms of pseudo differential operators and the Lax pair formulation. This formalism will be useful to make the connection to the asymptotically Lifshitz solutions of the Chern-Simons higher spin theory. In section 4 we briefly review the general properties of field theories with Lifshitz symmetry. In section 5 we construct the general algorithm to find asymptotic Lifshitz connections in the context of gauge algebra $hs(\lambda)$ with an integer scaling exponent z. We work out the detailed calculation for the special case of Lifshitz exponent $z = 3$ for the $hs(\lambda)$ algebra. Finally, the infinite dimensional set of equations is reduced to a finite dimensional set by setting $\lambda = 4$, for which $hs(\lambda)$ is truncated to $sl(4, \mathbb{R})$, and we obtain a $z = 3$ Lifshitz spacetime for $sl(4, \mathbb{R})$ Chern-Simons gravity. In section 6 the map between the $sl(4, \mathbb{R})$, $z = 3$ and $sl(3, \mathbb{R})$, $z = 2$ Lifshitz theories and members of KdV hierarchy is presented. A specific gauge choice, called KdV gauge, must be made to make the relation work. A conjecture on a general relation valid for all values of z and $sl(N,\mathbb{R})$ as well as for the infinite dimensional $hs(\lambda)$ case, is given. In section 7 the symmetry algebra for asymptotic Lifshitz connections in the $hs(\lambda)$ theory is constructed for arbitrary z. In addition the two specific cases $sl(4,\mathbb{R}), z = 3$ and $sl(3,\mathbb{R}), z = 2$ are worked out in KdV gauge. We close the paper in section 8 with a discussion of some open questions and future directions of research. Our conventions for the relevant gauge algebras and gauge choices are presented in appendix A and B respectively.

2 Review of Chern-Simon formulation of higher spin theories

The Chern-Simons formulation of three dimensional (higher-spin) gravity is based on two copies of Chern-Simons action at level k and $-k$ and gauge algebra $sl(N,\mathbb{R}) \times sl(N,\mathbb{R})$ or the higher spin algebra $hs(\lambda) \times hs(\lambda)$. For completeness we present our conventions for the algebras in appendix A. The action is given by,

$$S = S_{CS}[A] - S_{CS}[\bar{A}] \tag{2.1}$$

where the Chern-Simons action takes the familiar form,

$$S_{CS}[A] = \frac{k}{4\pi} \int \mathrm{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big) \tag{2.2}$$

The equations of motion impose the flatness of the gauge connection

$$F = dA + A \wedge A = 0, \qquad \bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} = 0 \tag{2.3}$$

It was shown in [18, 19] that Einstein gravity with negative cosmological constant is realized by choosing the gauge algebra to be $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$. The relation to the metric is obtained by expressing the vielbein and spin connection in terms of the Chern-Simons connection as follows

$$e_\mu = \frac{l}{2}(A_\mu - \bar{A}_\mu), \quad \omega_\mu = \frac{1}{2}(A_\mu + \bar{A}_\mu) \tag{2.4}$$

The metric can be calculated from the connections via

$$g_{\mu\nu} = \frac{1}{2} tr(e_\mu e_\nu) \tag{2.5}$$

The gauge transformations of the Chern-Simons theory

$$\delta A = d\Lambda + [A, \Lambda], \quad \delta\bar{A} = d\bar{\Lambda} + [A, \bar{\Lambda}] \tag{2.6}$$

correspond to diffeomorphisms and Lorentz frame rotations in the metric theory. For $N > 2$ the theory is a truncation of the three dimensional Vasiliev theory to fields of spin $s = 2, 3, \cdots, N$. The simplest case is $N = 3$ which corresponds to the gauge algebra $sl(3,\mathbb{R}) \times sl(3,\mathbb{R})$ was discussed in [20], where it was shown that the theory described gravity coupled to a massless spin three field which is given in terms of the gauge connection (2.4) by

$$\phi_{\mu\nu\rho} = \frac{1}{6}\,\mathrm{tr}(e_{(\mu}e_\nu e_{\rho)}) \tag{2.7}$$

analogous formulae can be obtained for larger N following [21], but will not be needed here. The analysis of the asymptotic symmetry is greatly simplified by special choice of gauge. We define a radial coordinate ρ, where the holographic boundary will be located at $\rho \to \infty$. In addition we define a time-like coordinate t and a space-like coordinate x, which can be either compact or non-compact and hence the boundary has either the topology of $\mathbb{R} \times S^1$ or $\mathbb{R} \times \mathbb{R}$. The "radial gauge" which we will use is constructed by defining $b = \exp(\rho L_0)$ and expressing the radial dependence via a ρ dependent gauge transformation

$$A_\mu(x,t,\rho) = b^{-1}a_\mu(x,t)\,b + b^{-1}\partial_\mu b, \quad \bar{A}_\mu(x,t,\rho) = b\,\bar{a}_\mu(x,t)b^{-1} + b\,\partial_\mu(b^{-1}) \tag{2.8}$$

Here L_0 is a Cartan generator of a $sl(2,\mathbb{R})$ sub-algebra of $sl(N,\mathbb{R})$, or its corresponding generator V^2_0 in $hs(\lambda)$. In this "radial gauge" the flatness condition reduces to a condition on the ρ independent connections a_t and a_x.

$$\partial_t a_x - \partial_x a_t + [a_t, a_x] = 0, \quad \partial_t \bar{a}_x - \partial_x \bar{a}_t + [\bar{a}_t, \bar{a}_x] = 0 \tag{2.9}$$

The Chern-Simons formulation of three dimensional higher spin theories has been used to define black holes in such theories via holonomy conditions [22, 23, 24, 25], as well as to calculate entanglement entropies using Wilson-loops [26, 27, 28].

3 Review of the KdV hierarchy

The KdV equation is a partial differential equation describing propagation of (shallow) water waves in channels, given by

$$4\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} \tag{3.1}$$

The KdV equation is an example of an integrable system, with infinitely many conserved and commuting charges, as well as soliton solutions with dispersion-free scattering. The KdV equation is a particular example of an infinite set of integrable systems, the so called KdV hierarchy. We review the treatment of the KdV hierarchy using pseudo-differential operators and Lax pairs. Pseudo differential operators allow for the introduction of negative powers of derivatives ∂ retaining the rules of differentiation, such as the Leibniz rule. More information about the formalism and its applications to integrable systems can be found in [29, 30].

The KdV hierarchy is characterized by two integers n and m and a differential operator L

$$L = \partial^n + u_2 \partial^{n-2} + \cdots + u_{n-1}\partial + u_n \tag{3.2}$$

Here $\partial = \frac{\partial}{\partial x}$ and $u_i = u_i(x,t)$. The formalism of pseudo differential operators allows to define fractional powers of L, in particular $L^{1/n}$.

$$L^{1/n} = \partial + \frac{1}{n} u \partial^{-1} + o(\partial^{-2}) \tag{3.3}$$

For another integer m one defines

$$P_m = \left(L^{m/n}\right)_+ \tag{3.4}$$

Where the subscript $()_+$ denotes the non-negative part of the pseudo differential operator, which has terms with $\partial^k, k \geq 0$. An integrable system is constructed due to the fact that P, L form a Lax pair, i.e. the evolution equation

$$\frac{\partial}{\partial t} L = [P_m, L] \tag{3.5}$$

gives a system of partial differential equations for $u_i(x,t)$. For the KdV hierarchy an infinite set of conserved quantities can be obtained by

$$q^{(k)} = \int \mathrm{res}\big(L^{\frac{k}{n}}\big) \tag{3.6}$$

Where "res" denotes the coefficient of the term multiplying ∂^{-1} in the pseudo differential operator. The charges are conserved if the equation of motion (3.5) is satisfied and the fields u_i fall off fast enough as $x \to \pm\infty$, so that total derivatives can be discarded. In the following we will present several members of the KdV hierarchy for low m and n, for which we will show that they are related to Lifshitz higher spin theories.

3.1 KdV equation: $n = 2, m = 3$

The original KdV equation, fits in the hierarchy by choosing $n = 2$ and $m = 3$. The Lax pair is given by

$$L = \partial^2 + u_2 \tag{3.7}$$

and

$$\begin{aligned} P_3 &= \left(L^{\frac{3}{2}}\right)_+ \\ &= \partial^3 + \frac{3}{2}u_2\partial + \frac{3}{4}u_2' \end{aligned} \tag{3.8}$$

The commutator gives

$$[P_3, L] = \frac{1}{4}u_2''' + \frac{3}{2}u_2u_2' \tag{3.9}$$

The Lax equation (3.5) takes the following form

$$4\dot{u}_2 = u_2''' + 6u_2u_2' \tag{3.10}$$

which reproduces the KdV equation (3.1).

3.2 Boussinesq equation: $n = 3, m = 2$

The next case is given by choosing $n = 3$ and $m = 2$. As we shall see later this case will be relevant for the $z = 2$ Lifshitz. The operator L is now of third order and contains two independent fields u_2 and u_3

$$L = \partial^3 + u_2\partial + u_3 \tag{3.11}$$

and

$$L^{1/3} = \partial + \frac{1}{3}u_2\partial^{-1} + \frac{1}{3}(u_3 - u_2')\partial^{-2} + o(\partial^{-3}) \tag{3.12}$$

Setting $m = 2$ the Lax operator becomes

$$P_2 = (L^{2/3})_+ = \partial^2 + \frac{2}{3}u_2 \tag{3.13}$$

The Lax equation (3.5) is equivalent to the following system of partial differential equations

$$\begin{aligned} \dot{u}_2 &= 2u_3' - u_2'' \\ \dot{u}_3 &= u_3'' - \frac{2}{3}u_2''' - \frac{2}{3}u_2u_2' \end{aligned} \tag{3.14}$$

Eliminating u_3 then gives an equation for u_2 alone

$$\ddot{u}_2 = -\frac{1}{3}u_2'''' - \frac{4}{3}(u_2 u_2')' \tag{3.15}$$

This equation is known as the Boussinesq equation [31], which has been studied in the context of propagation of waves. Using (3.6) we can easily calculate the first two conserved charges

$$\begin{aligned} q^{(1)} &= \int \mathrm{res}\big(L^{\frac{1}{3}}\big) = \frac{1}{3}\int u_2 \\ q^{(2)} &= \int \mathrm{res}\big(L^{\frac{2}{3}}\big) = \int (\frac{2}{3}u_3 - \frac{1}{3}u_2') = \frac{2}{3}\int u_3 \end{aligned} \tag{3.16}$$

3.3 $n = 4, m = 3$ member of KdV hierarchy

The case $n = 4$ $m = 3$ will be relevant for the $z = 3$ Lifshitz theory. The operator L is now of fourth order and contains three fields u_i, $i = 2, 3, 4$.

$$L = \partial^4 + u_2\partial^2 + u_3\partial + u_4 \tag{3.17}$$

from this we can evaluate

$$L^{1/4} = \partial + \frac{u_2}{4}\partial^{-1} + \frac{1}{4}(u_3 - \frac{3}{2}u_2^{'})\partial^{-2} + (\frac{1}{4}u_4 - \frac{3}{8}u_3^{'} + \frac{5}{16}u_2^{''} - \frac{3}{32}u_2^2)\partial^{-3} + o(\partial^{-4}) \tag{3.18}$$

The Lax operator is given by

$$P_3 = L_+^{3/4} = \partial^3 + \frac{3}{4}u_2\partial + \frac{3}{4}u_3 - \frac{3}{8}u_2^{'} \tag{3.19}$$

The Lax equation (3.5) is equivalent to

$$\begin{aligned} \dot{u_2} &= \frac{1}{4}u_2^{'''} - \frac{3}{2}u_3^{''} + 3u_4^{'} - \frac{3}{4}u_2u_2^{'} \\ \dot{u_3} &= -2u_3^{'''} + 3u_4^{''} + \frac{3}{4}u_2^{''''} - \frac{3}{4}u_2u_3^{'} - \frac{3}{4}u_3u_2^{'} \\ \dot{u_4} &= u_4^{'''} + \frac{3}{8}u_2^{'''''} - \frac{3}{4}u_3^{''''} + \frac{3}{4}u_2u_4^{'} \\ &\quad -\frac{3}{4}u_2u_3^{''} + \frac{3}{8}u_2u_2^{'''} - \frac{3}{4}u_3u_3^{'} + \frac{3}{8}u_3u_2^{''}. \end{aligned} \tag{3.20}$$

Using (3.6) we can calculate the conserved quantities and display the first three here

$$\begin{aligned} q^{(1)} &= \int \mathrm{res}(L^{\frac{1}{4}}) = \frac{1}{4}\int u_2 \\ q^{(2)} &= \int \mathrm{res}(L^{\frac{2}{4}}) = \int (\frac{1}{2}u_3 - \frac{1}{2}u_2') = \frac{1}{2}\int u_3 \\ q^{(3)} &= \int \mathrm{res}(L^{\frac{3}{4}}) = \int (\frac{3}{4}u_4 - \frac{3}{8}u_3' + \frac{1}{16}u_2'' - \frac{3}{32}u_2^2) = \int (\frac{3}{4}u_4 - \frac{3}{32}u_2^2) \end{aligned} \tag{3.21}$$

4 Field theories with Lifshitz scaling

Lifshitz theories are field theories which exhibit an anisotropic scaling symmetry with respect to space and time

$$t \to \lambda^z t, \qquad x \to \lambda x \tag{4.1}$$

where z is the Lifshitz scaling exponent and $z = 1$ corresponds to conformal scaling. The algebra of Lifshitz symmetries is given by the time translation H, the spatial translation P and the Lifshitz scaling D, which satisfy the following commutation relations

$$[P, H] = 0 \qquad [D, H] = zH \qquad [D, P] = P \tag{4.2}$$

For theories with Lifshitz scaling the stress tensor does not have to be symmetric, as there is no relativistic boost symmetry. The stress energy tensor contains four components: the energy density $\mathcal{E}$, the energy flux $\mathcal{E}^x$, the momentum density $\mathcal{P}_x$ and the stress density Π_x^x. These quantities satisfy the following conservation equations (see e.g. [33]).

$$\begin{aligned} \partial_t \mathcal{E} + \partial_x \mathcal{E}^x &= 0 \\ \partial_t \mathcal{P}_x + \partial_x \Pi_x^x &= 0 \end{aligned} \tag{4.3}$$

In addition, the Lifshitz scaling with exponent z implies a modified tracelessness condition

$$z\mathcal{E} + \Pi_x^x = 0. \tag{4.4}$$

Since the operator D generates scale transformation the commutation relations (4.2) imply that the momentum operator P has scaling dimension one, whereas the Hamiltonian H has scaling dimension z. We will give the precise definition of scaling dimension later.

It is an interesting question whether theories with Lifshitz scaling have a holographic description. For a two dimensional Lifshitz theory the three dimensional spacetime takes the following form:

$$ds^2 = d\rho^2 - e^{2z\rho}dt^2 + e^{2\rho}dx^2 \tag{4.5}$$

The shift $\rho \to \rho + \ln\lambda$ in the holographic radial coordinate induces a Lifshitz scaling transformation on t, x with scaling exponent z as in (4.1). Such metrics (and their higher dimensional generalizations) are not solutions of Einstein gravity and nontrivial matter interactions have to be added. The first solutions of this kind where found in [14] in four dimensional gravity coupled to anti-symmetric tensor fields. Subsequently, Many solutions which exhibit Lifshitz asymptotics have been constructed in supergravity theories.

5 Asymptotic Lifshitz connection

In this section we construct connections which are asymptotically Lifshitz in the Chern-Simons formulation of three dimensional higher spin gravity. A Lifshitz connection is a solution to flatness condition which produces a Lifshitz metric (4.5). The connections which reproduce a metric with integer scaling exponent z can be easily written down using the $hs(\lambda)$ algebra, the unbarred connection is given by

$$a \;=\; V_z^{z+1}dt + V_1^2 dx, \qquad A = V_z^{z+1}e^{z\rho}dt + V_1^2 e^{\rho}dx + V_0^2 d\rho \tag{5.1}$$

and the barred connection is given by

$$\bar{a} \;=\; V_{-z}^{z+1}dt + V_{-1}^2 dx, \qquad \bar{A} = V_{-z}^{z+1}e^{z\rho}dt + V_{-1}^2 e^{\rho}dx - V_0^2 d\rho \tag{5.2}$$

It follows from (2.5) that these connections realize a Lifshitz spacetime with an arbitrary integer z. For an integer $\lambda = N$ the algebra $hs(\lambda)$ is truncated to $sl(N,\mathbb{R})$. For example (5.1) in the $z = 2$ case, one reproduces the $sl(3,\mathbb{R})$ Lifshitz connections studied in [13] with the identification $V^3_{\pm 2} = W_{\pm 2}, V^2_{\pm 1} = L_{\pm 1}$ and $V_0^2 = L_0$.

In the following we will consider connections where the barred sector is determined in terms of the unbarred sector. This is possible due to an automorphism of $hs(\lambda)$ algebra, which is obtained from a conjugation $(V^s_m)^c = (-1)^{s+m+1}V^s_{-m}$. In particular the generator V_0^2 used in constructing the radial gauge transformations is self conjugate up to a sign, i.e. $(V_0^2)^c = -V_0^2$. Consequently, if A solves the flatness condition $F = 0$ in the radial gauge, the barred connection is chosen to be the conjugate $\bar{A} = A^c$. For this choice $\bar{A}$ automatically satisfies the flatness condition $\bar{F} = 0$ and the radial gauge. From now on we will leave out the barred sector as it is determined from the un-barred sector.

Though we have explicit expression for Lifshitz connections, they are static solutions without any dynamics. Here we want to consider asymptotic Lifshitz connection. Asymptotic Lifshitz connections are connections in which leading terms are Lifshitz connections given by (5.1) where additional terms are present with sub-leading powers e^{ρ}. Consequently such connections will lead to asymptotic Lifshitz spacetimes where the metric and tensor fields have additional terms which become negligible as $\rho \to \infty$ compared to the Lifshitz vacuum.

5.1 Constructing asymptotic Lifshitz connections

In this section we describe an algorithm to find solutions to the flatness conditions in the radial gauge

$$\partial_t a_x - \partial_x a_t + [a_t, a_x] = 0 \tag{5.3}$$

for asymptotically Lifshitz $hs(\lambda)$ connections. However the implementation of the algorithm becomes unwieldy for larger values of z and hence a detailed calculation is presented for the case $z = 3$ in section 5.2. Furthermore, if we set $\lambda = N$ the Lie algebra $hs(\lambda)$ is truncated to $sl(N, \mathbb{R})$ and a finite dimensional example is treated in section 5.4.

The algorithm proceeds in the following steps:

1. Adopt a "lowest weight gauge" for a_x such that it only contains lowest weight terms except for V_1^2. Lowest weight terms are of the form V^i_{-i+1}, whose weight is lowest for a given spin.
$$a_x = V_1^2 + \sum_{i=2}^{\infty} \alpha_i V^i_{-i+1} \tag{5.4}$$
Where $\alpha_i(x, t)$ are the dynamical fields and their evolution equations will be determined.

2. The ansatz for the time component of the connection is given by
$$a_t = (*a_x)^z|_{traceless} + \Delta a_t \tag{5.5}$$
Where the subscript $|_{traceless}$ denotes the removal of the V_0^1 component from the star product, see appendix A.3 for our conventions on the higher spin algebra. This ansatz

works for any integer value of z. The ansatz is motivated by the fact that for constant a_x, eq.(5.5) is a solution of the flatness condition with $\Delta a_t = 0$[1].

3. For x and t dependent α_i the flatness condition (5.3) now takes the form

$$\partial_t a_x - \partial_x (*a_x)^z|_{traceless} - \partial_x \Delta a_t + [\Delta a_t, a_x] = 0 \tag{5.6}$$

4. Calculate $(*a_x)^z|_{traceless}$ and note that the terms with the highest weight which appear in this expression are $V^i_{-i+2z-1}$ for sufficiently large spin i. Considering the only term that raises the weight is the commutator with V^2_1 in a_x, it is sufficient to include terms with weight up to $-i+2z-2$ for Δa_t in general, so the whole expression will be closed of terms with weight up to $-i+2z-1$. In addition, we can add a term $V^i_{-i+2z-1}$ to Δa_t if it happens to be a highest weight term, which has a vanishing commutator with V^2_1. Now we have the suitable ansatz for a_t.

5. Since $\partial_t a_x$ only contains lowest weight term, so all the non lowest weight terms in $-\partial_x (*a_x)^z|_{traceless} - \partial_x \Delta a_t + [\Delta a_t, a_x]$ must vanish. This will fix most coefficients in Δa_t in terms of coefficients in a_x.

6. Require lowest weight terms V^i_{-i+1} to cancel out in the flatness condition we get the time evolution of the α_i.

5.2 Detailed calculation for the $z = 3$ case

In this section we would like to work out the explicit example of $z = 3$ to illustrate the general algorithm. The ansatz for Δa_t is

$$\Delta a_t = \sum_{i=2}^{\infty} \beta_i V^i_{-i+1} + \sum_{i=2}^{\infty} \gamma_i V^i_{-i+2} + \sum_{i=2}^{\infty} \delta_i V^i_{-i+3} + \sum_{i=3}^{\infty} \sigma_i V^i_{-i+4} + \mu V^3_2. \tag{5.7}$$

[1] Note that this construction has been used to construct higher spin black holes [34].

In the following we will calculate each term in the flatness condition (5.6). The triple product

$$\begin{aligned}
a_x * a_x * a_x &= (V_1^2 + \sum_{i=2}^{\infty} \alpha_i V_{-i+1}^i)^3 \\
&= (V_1^2)^3 + \sum_{i=2}^{\infty} \alpha_i (V_1^2 * V_1^2 * V_{-i+1}^i + V_1^2 * V_{-i+1}^i * V_1^2 + V_{-i+1}^i * V_1^2 * V_1^2) \\
&+ \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} \alpha_i \alpha_j (V_1^2 * V_{-i+1}^i * V_{-j+1}^j + V_{-i+1}^i * V_{-j+1}^j * V_1^2 + V_{-j+1}^j * V_1^2 * V_{-i+1}^i) \\
&+ \sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} \alpha_i \alpha_j \alpha_k V_{-i+1}^i * V_{-j+1}^j * V_{-k+1}^k. \qquad (5.8)
\end{aligned}$$

The relevant products are evaluated as follows

$$V_1^2 * V_1^2 * V_{-i+1}^i + V_1^2 * V_{-i+1}^i * V_1^2 + V_{-i+1}^i * V_1^2 * V_1^2 = o(i,\lambda) V_{-i+3}^{i-2} + p(i,\lambda) V_{-i+3}^i + 3V_{-i+3}^{i+2} \quad (5.9)$$

and

$$\begin{aligned}
&V_1^2 * V_{-i+1}^i * V_{-j+1}^j + V_{-i+1}^i * V_{-j+1}^j * V_1^2 + V_{-j+1}^j * V_1^2 * V_{-i+1}^i \\
&\quad = r(i,j,\lambda) V_{-i-j+3}^{i+j-2} + s(i,j) V_{-i-j+3}^{i+j-1} + 3V_{-i-j+3}^{i+j} \qquad (5.10)
\end{aligned}$$

as well as

$$V_{-i+1}^i * V_{-j+1}^j * V_{-k+1}^k = V_{-i-j-k+3}^{i+j+k-2} \qquad (5.11)$$

where $o(i,\lambda)$, $p(i,\lambda)$, $r(i,j,\lambda)$ and $s(i,j)$ are coefficients which can be calculated from structure constants defined in appendix A.3. The commutator

$$[\Delta a_t, a_x] = [\sum_{i=2}^{\infty} \beta_i V_{-i+1}^i + \sum_{i=2}^{\infty} \gamma_i V_{-i+2}^i + \sum_{i=2}^{\infty} \delta_i V_{-i+3}^i + \sum_{i=3}^{\infty} \sigma_i V_{-i+4}^i + \mu V_2^3, V_1^2 + \sum_{i=2}^{\infty} \alpha_j V_{-j+1}^j] \qquad (5.12)$$

can be calculated using the commutation relation of the generators of the algebra

$$[V_1^2, V_n^t] = (t-n-1) V_{n+1}^t \qquad (5.13)$$

and

$$\begin{aligned}
&[V_{-i+1}^i, V_{-j+1}^j] = 0 \\
&[V_{-i+2}^i, V_{-j+1}^j] = (j-1) V_{-i-j+3}^{i+j-2} \\
&[V_{-i+3}^i, V_{-j+1}^j] = 2(j-1) V_{-i-j+4}^{i+j-2} \\
&[V_{-i+4}^i, V_{-j+1}^j] = w(i,j,\lambda) V_{-i-j+5}^{i+j-4} + 3(j-1) V_{-i-j+5}^{i+j-2} \\
&[V_{-i+5}^i, V_{-j+1}^j] = z(i,j,\lambda) V_{-i-j+6}^{i+j-4} + 4(j-1) V_{-i-j+6}^{i+j-2}. \qquad (5.14)
\end{aligned}$$

where $w(i,j,\lambda)$ and $z(i,j,\lambda)$ can be calculated from structure constants. After some simplification the flatness condition now reads

$$
\begin{aligned}
\sum_{i=2}^{\infty}\dot{\alpha}_i V^i_{-i+1} \;=\; & \sum_{i=2}^{\infty}\alpha^{'}_{i+2}o(i+2,\lambda)V^i_{-i+1}+\sum_{i=2}^{\infty}\alpha^{'}_{i}p(i,\lambda)V^i_{-i+3}+\sum_{i=4}^{\infty}\alpha^{'}_{i-2}3V^i_{-i+5} \\
& +\sum_{i=2}^{\infty}\sum_{j=2}^{i}(\alpha_{i-j+2}\alpha_j)^{'}r(i-j+2,j,\lambda)V^i_{-i+1}+\sum_{i=4}^{\infty}\sum_{j=2}^{i-2}(\alpha_{i-j}\alpha_j)^{'}3V^i_{-i+3} \\
& +\sum_{i=4}^{\infty}\sum_{j+k+l=i+2}(\alpha_j\alpha_k\alpha_l)^{'}V^i_{-i+1}+\sum_{i=2}^{\infty}\beta^{'}_iV^i_{-i+1}+\sum_{i=2}^{\infty}\gamma^{'}_iV^i_{-i+2}+\sum_{i=2}^{\infty}\delta^{'}_iV^i_{-i+3} \\
& +\sum_{i=3}^{\infty}\sigma^{'}_iV^i_{-i+4}+\mu^{'}V^3_2+\sum_{i=2}^{\infty}(2i-2)\beta_iV^i_{-i+2}+\sum_{i=2}^{\infty}(2i-3)\gamma_iV^i_{-i+3} \\
& +\sum_{i=2}^{\infty}(2i-4)\delta_iV^i_{-i+4}+\sum_{i=3}^{\infty}(2i-5)\sigma_iV^i_{-i+5}-\sum_{i=2}^{\infty}\sum_{j=2}^{i}\gamma_{i-j+2}\alpha_j(j-1)V^i_{-i+1} \\
& -\sum_{i=2}^{\infty}\sum_{j=2}^{i}\delta_{i-j+2}\alpha_j2(j-1)V^i_{-i+2}-\sum_{i=3}^{\infty}\sum_{j=2}^{i-1}\sigma_{i-j+2}\alpha_j3(j-1)V^i_{-i+3} \\
& -\sum_{i=2}^{\infty}\sum_{j=2}^{i+1}\sigma_{i-j+4}\alpha_jw(i-j+4,j,\lambda)V^i_{-i+1}-\sum_{i=2}^{\infty}\mu\alpha_{i+1}z(3,i+1,\lambda)V^i_{-i+2} \\
& -\sum_{i=3}^{\infty}4(i-2)\mu\alpha_{i-1}V^i_{-i+4}.
\end{aligned}
\tag{5.15}
$$

Vanishing of non lowest weight terms enables us to solve most coefficients in a_t in terms of the coefficients in a_x, i.e. α_i's. From equating the coefficients of the V^i_{-i+5} generators we find

$$
\begin{aligned}
\sigma_3 &= 0 \\
\sigma_i &= -\frac{3}{2i-5}\alpha^{'}_{i-2}, i\geq 4
\end{aligned}
\tag{5.16}
$$

Equating the coefficients of the V^i_{-i+4} generators gives

$$
\begin{aligned}
\delta_3 &= 0 \\
\delta_i &= \frac{3}{(2i-5)(2i-4)}\alpha^{''}_{i-2}, i\geq 4
\end{aligned}
\tag{5.17}
$$

Equating the coefficients of the V^i_{-i+3} generators gives

$$\begin{aligned}
\gamma_2 &= -p(2,\lambda)\alpha_2^{'} - \delta_2^{'} \\
\gamma_3 &= -\frac{1}{3}p(3,\lambda)\alpha_3^{'} \\
\gamma_i &= -\frac{1}{2i-3}p(i,\lambda)\alpha_i^{'} - \frac{1}{(2i-3)(2i-4)(2i-5)}\alpha_{i-2}^{'''} \\
&\quad -\frac{3}{2i-3}\sum_{j=2}^{i-2}(2+\frac{3(j-1)}{2(i-j)-1})\alpha_j\alpha_{i-j}^{'}, i \geq 4
\end{aligned} \tag{5.18}$$

Finally equating the coefficients of the V^i_{-i+2} generators gives

$$\begin{aligned}
\beta_2 &= \frac{1}{2}p(2,\lambda)\alpha_2^{''} + \frac{1}{2}\delta_2^{''} + \alpha_2\delta_2 \\
\beta_3 &= \frac{1}{12}p(3,\lambda)\alpha_3^{''} + \delta_2\alpha_3 \\
\beta_i &= \frac{1}{(2i-2)(2i-3)}p(i,\lambda)\alpha_i^{''} + \frac{1}{(2i-2)(2i-3)(2i-4)(2i-5)}\alpha_{i-2}^{''''} \\
&\quad +\frac{3}{(2i-2)(2i-3)}\sum_{j=2}^{i-2}(2+\frac{3(j-1)}{2(i-j)-1})(\alpha_j\alpha_{i-j}^{'})^{'} + \alpha_j\delta_2 \\
&\quad +\frac{3}{2i-2}\sum_{j=2}^{i-2}\frac{j-1}{(i-j)(2(i-j)-2)}\alpha_j\alpha_{i-j}^{''}, i \geq 4
\end{aligned} \tag{5.19}$$

There are two exceptions, δ_2 and μ cannot be determined by equations of motion (5.15) and can in principle be chosen arbitrarily. It can be shown that they are purely gauge. Vanishing of lowest weight terms gives us equations of motion

$$\begin{aligned}
&\dot{\alpha}_i - o(i+2,\lambda)\alpha_{i+2}^{'} - \sum_{j=2}^{i} r(i-j+2,j,\lambda)(\alpha_{i-j+2}\alpha_j)^{'} - \sum_{j+k+l=i+2}(\alpha_j\alpha_k\alpha_l)^{'} - \beta_i^{'} \\
&+\sum_{j=2}^{i}(j-1)\gamma_{i-j+2}\alpha_j + \sum_{j=2}^{i+1} w(i-j+4,j,\lambda)\sigma_{i-j+4}\alpha_j = 0
\end{aligned} \tag{5.20}$$

where the cubic term is understood to be there only for $i \geq 4$. We get the equations of motion in terms of α's with extra gauge freedom to choose δ_2 and μ arbitrarily.

5.3 Scaling dimension

An interesting feature of the equations of motion is their scaling structure. For concreteness we will work out the scaling structure for the $z = 3$ Lifshitz example. The results can easily be adapted for general integer values of z. A scaling transformation (4.1) acts on the space and time coordinates $x \to \lambda x$ $t \to \lambda^3 t$. We show in the following that the connections are invariant after an appropriate rescaling of the fields. A field variable has scaling dimension l if it is rescaled by a factor λ^{-l}.

In the triple product $a_x * a_x * a_x$ there are terms of the form $\alpha_i \alpha_j V_1^2 * V_{-i+1}^i * V_{-j+1}^j$ as well as terms of the form $\alpha_{i+j} V_1^2 * V_1^2 * V_{-i-j+1}^{i+j}$. Both kind of terms contain higher spin generators of of weight $3-i-j$ and hence should have the same scaling dimensions, hence the dimensions of $\alpha_i\alpha_j$ and α_{i+j} should be the same, or symbolically $[\alpha_i] + [\alpha_j] = [\alpha_{i+j}]$, where the square bracket means the scaling dimension of the quantity. Consequently, the scaling dimension of α_i is additive and determined by the weight of the generator V_{-i+1}^i.

Following the same argument, comparing the triple product of a_x to Δa_t gives the relation $[\alpha_i] = [\beta_{i-2}] = [\gamma_{i-1}] = [\delta_i] = [\sigma_{i+1}]$ and $[\mu] = [\sigma_2]$. Comparing $\partial_x a_t$ to $[a_t, a_x]$ terms we get $[\partial_x][\beta_{i-1}] = [\beta_i]$. Comparing $\partial_t a_x$ to $[a_t, a_x]$ terms gives $[\partial_t][\alpha_{i-3}] = [\alpha_i]$. It follows from (4.1) that $[\partial_x] = 1$ and $[\partial_t] = 3$. Summarizing the dimensions of all field variables

$$[\alpha_i] = i,\ [\beta_i] = i+2,\ [\gamma_i] = i+1,\ [\delta_i] = i,\ [\sigma_i] = i-1,\ [\mu] = 1, \tag{5.21}$$

The scaling dimensions (5.21) agree with the scaling of the fluctuating fields demanding the invariance of the connections $A_t dt$ and $A_x dx$ under a shift of the radial coordinate ρ accompanied by a Lifshitz scaling of x, t given by

$$\begin{aligned} \rho' &= \rho + \log\lambda \\ x' &= \lambda^{-1} x \\ t' &= \lambda^{-3} t \end{aligned} \tag{5.22}$$

To fix the gauge and to get equations of motion in terms of α's unambiguously we want to express δ_2 and μ in terms of α's or their derivatives. μ has to be zero since it's the only variable with dimension 1. δ_2 must be proportional to α_2, say, with proportionality constant c.

5.4 The $sl(4,\mathbb{R})$ and $z = 3$ example

Following the algorithm described above, we can in principle construct asymptotic Lifshitz connections with infinitely many terms in the context of $hs(\lambda)$. In this section we simplify further and consider a finite truncation of the infinite dimensional algebra. We set $\lambda = 4$ so the higher spin algebra is truncated to $sl(4,\mathbb{R})$. In the equations of motion (5.2) $\dot{\alpha}_i$ is coupled to α's with higher scaling dimension only via the second term. $o(5,\lambda)$ and $o(6,\lambda)$ are zero when $\lambda = 4$, because they must vanish due to their definition as coefficients of spin five and spin six $hs(\lambda)$ elements in (5.6). Even though there are still infinitely many α's, the dynamics of the first three $\alpha_2, \alpha_3, \alpha_4$ are decoupled from the others and it is consistent to set all the $\alpha_i, i > 4$ to zero. Consequently when the algebra is truncated, the equations of motion are also truncated to what we will get if we just start with ansatz in the finite algebra. As discussed in section 5.3 the scaling symmetry imposes the gauge choices $\mu = 0$ and $\delta_2 = c\alpha_2$. Now we have

$$
\begin{aligned}
&\sigma_3 = 0, \quad \sigma_4 = -\alpha_2^{'} \\
&\delta_2, \delta_3 = 0, \quad \delta_4 = \frac{1}{4}\alpha_2^{''} \\
&\gamma_2 = \frac{41}{5}\alpha_2^{'} - \delta_2^{'}, \quad \gamma_3 = 2\alpha_3^{'}, \quad \gamma_4 = -\frac{1}{20}\alpha_2^{'''} - \frac{3}{5}\alpha_4^{'} - \frac{9}{5}\alpha_2\alpha_2^{'} \\
&\beta_2 = \delta_2\alpha_2 + \frac{1}{2}\delta_2^{''} - \frac{41}{10}\alpha_2^{''}, \quad \beta_3 = \delta_2\alpha_3 - \frac{1}{2}\alpha_3^{''}, \\
&\beta_4 = \delta_2\alpha_4 + \frac{1}{120}\alpha_2^{''''} + \frac{1}{10}\alpha_4^{''} + \frac{3}{10}(\alpha_2^{'})^2 + \frac{23}{60}\alpha_2\alpha_2^{''}
\end{aligned}
\tag{5.23}
$$

and the equations of motion

$$
\begin{aligned}
\dot{\alpha_2} &= -(\frac{41}{10} - \frac{1}{2}c)\alpha_2^{'''} - (\frac{123}{5} - 3c)\alpha_2^{'}\alpha_2 + \frac{54}{5}\alpha_4^{'} \\
\dot{\alpha_3} &= -\frac{1}{2}\alpha_3^{'''} - (15 - c)\alpha_3^{'}\alpha_2 - (30 - 3c)\alpha_2^{'}\alpha_3 \\
\dot{\alpha_4} &= \frac{1}{10}\alpha_4^{'''} + \frac{1}{120}\alpha_2^{'''''} - (30 - 4c)\alpha_2^{'}\alpha_4 - (\frac{27}{5} - c)\alpha_2\alpha_4^{'} - 12\alpha_3^{'}\alpha_3 + \frac{13}{30}\alpha_2\alpha_2^{'''} \\
&\quad + \frac{59}{60}\alpha_2^{'}\alpha_2^{''} + \frac{24}{5}\alpha_2^2\alpha_2^{'}
\end{aligned}
\tag{5.24}
$$

6 Map to KdV

In this section we want to demonstrate that there is a map from the equations of motion of the Chern-Simons connection to the time evolution equations of the KdV hierarchy. We discuss two concrete examples $z = 2$, $sl(3,\mathbb{R})$ and the $z = 3$, $sl(4,\mathbb{R})$ and then propose a conjecture for the general case.

6.1 The $sl(3,\mathbb{R})$ and $z = 2$ example

Here we we want to consider a simpler case which has been worked out in the previous paper [13] and provide the map to the $n = 3, m = 2$ member of the KdV hierarchy[2] which was described in section 3.2. The asymptotic Lifshitz connection is[3]

$$\begin{aligned} a_t &= V_2^3 + 2\alpha_2 V_0^3 - \frac{2}{3}\alpha_2' V_{-1}^3 - 2\alpha_3 V_{-1}^2 + \left(\alpha_2^2 + \frac{1}{6}\alpha_2''\right) V_{-2}^3 && (6.1) \\ a_x &= V_1^2 + \alpha_2 V_{-1}^2 + \alpha_3 V_{-2}^3 && (6.2) \end{aligned}$$

The flatness condition implies the equations of motion

$$\begin{aligned} \dot{\alpha_2} &= -2\alpha_3' && (6.3) \\ \dot{\alpha_3} &= \frac{4}{3}(\alpha_2^2)' + \frac{1}{6}\alpha_2''' && (6.4) \end{aligned}$$

These equations are equivalent to the Boussinesq equations (3.14) via the following field identification

$$\begin{aligned} u_2 &= 4\alpha_2 \\ u_3 &= -4\alpha_3 + 2\alpha_2' && (6.5) \end{aligned}$$

or conversely

$$\begin{aligned} \alpha_2 &= \frac{1}{4}u_2 \\ \alpha_3 &= -\frac{1}{4}u_3 + \frac{1}{8}u_2'. && (6.6) \end{aligned}$$

[2] A different realization of the Boussinesq integrable system in Chern-Simons higher spin gravity was presented in [32].

[3] Here we have adapted the general notation, with $-\mathcal{L}$ in the previous paper [13] replaced by α_2 and $\mathcal{W}$ replaced by α_3.

6.2 The $sl(4,\mathbb{R})$ and $z=3$ example

Let's go back to the $sl(4,\mathbb{R}), z=3$ case, which has the novelty of gauge dependence described by the parameter c. We want to find a map from Chern-Simons connection variables to KdV variables such that the equations of motion of Chern-Simons connection (5.4) are equivalent to KdV with $n=4, m=3$ (3.19). The Chern-Simons variables have scaling dimensions according to the analysis in the section 5.3. The KdV variables also have scaling dimensions by the formulation of pseudo-differential operators. The fact that the scaling dimensions on both sides have to agree puts strong restrictions on the mapping of the variables. Hence we must use the ansatz $u_2 = k\alpha_2, u_3 = a\alpha_2^{'} + b\alpha_3$. For the second KdV equation

$$\dot{u_3} = -2u_3^{'''} + 3u_4^{''} + \frac{3}{4}u_2^{''''} - \frac{3}{4}u_2 u_3^{'} - \frac{3}{4}u_3 u_2^{'} \tag{6.7}$$

on the right hand side $\alpha_2\alpha_3^{'}$ and $\alpha_2^{'}\alpha_3$ have the same coefficients, on the left hand side the same kind of terms come from $\dot{\alpha_3} = -\frac{1}{2}\alpha_3^{'''} - (15-c)\alpha_3^{'}\alpha_2 - (30-3c)\alpha_2^{'}\alpha_3$, so we must have $(15-c) = (30-3c)$, obtaining $c = \frac{15}{2}$. Comparing terms and check integrability condition recursively one can obtain $k=10, a=10, b=24$ and the full map

$$\begin{aligned} u_2 &= 10\alpha_2 \\ u_3 &= 10\alpha_2^{'} + 24\alpha_3 \\ u_4 &= 3\alpha_2^{''} + 9\alpha_2^2 + 12\alpha_3^{'} + 36\alpha_4 \end{aligned} \tag{6.8}$$

establish the correspondence.

Here we see we must make the gauge choice $c = \frac{15}{2}$ to establish the relation between Chern-Simons Lifshitz theory and KdV hierarchy. We call this kind of gauge choice "KdV gauge". Explicitly, the equations of motion in the KdV gauge read

$$\begin{aligned} \dot{\alpha_2} &= -\frac{7}{20}\alpha_2^{'''} - \frac{21}{10}\alpha_2^{'}\alpha_2 + \frac{54}{5}\alpha_4^{'} \\ \dot{\alpha_3} &= -\frac{1}{2}\alpha_3^{'''} - \frac{15}{2}\alpha_3^{'}\alpha_2 - \frac{15}{2}\alpha_2^{'}\alpha_3 \\ \dot{\alpha_4} &= \frac{1}{10}\alpha_4^{'''} + \frac{1}{120}\alpha_2^{'''''} + \frac{21}{10}\alpha_2\alpha_4^{'} - 12\alpha_3^{'}\alpha_3 + \frac{13}{30}\alpha_2\alpha_2^{'''} + \frac{59}{60}\alpha_2^{'}\alpha_2^{''} + \frac{24}{5}\alpha_2^2\alpha_2^{'} \end{aligned} \tag{6.9}$$

6.3 General conjecture

In the two previous sections we have mapped the equations of motion for asymptotic Lifshitz connections to member of the KdV hierarchy in two particular cases, namely the $z=2$,

$sl(3,\mathbb{R})$ connection is mapped to the $n=3, m=2$ element of the KdV hierarchy and $z=3$, $sl(4,\mathbb{R})$ connection is mapped to the $n=4, m=3$ element of the KdV hierarchy. This result inspires us to propose general conjecture:

The asymptotic Lifshitz connection for $sl(N,\mathbb{R})$ and an arbitrary integer Lifshitz scaling exponent z can be mapped to the member of the KdV hierarchy with $n=N, m=z$.

Apart from the two cases worked out in this paper we have also checked the case $z=2$, $sl(4,\mathbb{R})$. The fact that $n=N$ can be deduced from the fact that lowest weight ansatz for a_x for $sl(N,\mathbb{R})$ contains $N-1$ fields a_i which has to be equated with the $n-1$ fields u_i on the KdV side. Furthermore, the dimensional analysis for the Lax equation (3.5) implies ∂_t has the same dimension as P_m, that is, the same dimension as ∂_x^m, so m is exactly the Lifshitz scaling exponent.

Now a natural question rises for the Lifshitz connection in the algebra $hs(\lambda)$ where λ is not an integer: Can the equations of motion, which involve infinite number of fields be mapped to some integrable hierarchy? Note that such an integrable system should reduce to the KdV hierarchy when $hs(\lambda)$ is truncated to $sl(N,\mathbb{R})$ upon setting $\lambda=N$. A candidate for such integrable systems is the KP hierarchy which we briefly review here.

The starting point is the following pseudo differential operator which contains infinitely many fields $v_i, i=2,3,\cdots$.

$$S=\partial+v_2\partial^{-1}+v_3\partial^{-2}+v_4\partial^{-3}+\dots \tag{6.10}$$

The Lax equation for the m-th element of the hierarchy[4] is defined by

$$\frac{\partial}{\partial t}S=[S_+^m,S] \tag{6.11}$$

The Lax equation gives equations of motion of the KP variables v's.

The connection of the KP hierarchy to the KdV hierarchy is obtained as follows: Note that the Lax equation above implies the following equation for the n-th power of the operator S

$$\dot{S^n}=[S_+^m,S^n] \tag{6.12}$$

[4]Note that the name "KP hierarchy" is usually reserved for the system of equations for all m where a different time variable t_m is associated with each element. We are interested in the a specific element of the hierarchy and denote the time simply by t.

With the definitions $L = S^n$ and $P_m = S_+^{\frac{m}{n}}$ we get the Lax equation of KdV defined in (3.5). At this point the pseudo differential operator L contains all possible powers of ∂, down to $\partial^{-\infty}$. It is possible to consistently restrict L to only non-negative powers of differentiation, which implies that the dynamics of the first $n-1$ variables is decoupled from the other, and they are just KdV hierarchy with the same values of m and n. Consequently, it is possible to perform a field redefinition to truncate KP to KdV. The map from $sl(N, \mathbb{R}), z$ Chern-Simons Lifshitz theory to KdV with $m = z, n = N$ can be regarded as a part of the whole map from $hs(N), z$ Chern-Simons Lifshitz theory to KP with $m = z$, with N being the parameter of the map.

In general it is possible to define powers of the pseudo differential operator S to for non integer exponents [36, 38]. We conjecture that by choosing N as a real number λ we will be able to construct a map between Chern-Simons Lifshitz theory with generic $hs(\lambda)$ and KP. We leave the explicit construction of this map for future work, but observe that there are several arguments that indicate that this correspondence indeed exists. First, finding the maps involves solving algebraic equations, as in the case of $\lambda = N$, but the recursive solution does in general not require N to be an integer. Second, the $hs(\lambda)$ Chern-Simons for a conformal theory provides a realization of the W_∞ nonlinear extension of the W_N algebras [8, 35, 21]. While the construction is slightly different many of the features of the relation such as the relation of the gauge transformations which preserve the highest weight gauge of a_c to the W-algebra transformation, carry over. When W algebras were first investigated in the early '90 a relation of the W_∞ algebra to the KP hierarchy was proposed in several papers [36, 37, 38, 39, 40, 41]

7 Lifshitz symmetry algebra for generic $hs(\lambda)$ and arbitrary z

In this section we show how Lifshitz symmetry algebra is realized in the generic case, that is, with gauge algebra $hs(\lambda)$ and an arbitrary integer Lifshitz scaling exponent z, with special gauge choice. We have the asymptotic Lifshitz connection

$$\begin{aligned} a_x &= V_1^2 + \sum_{i=2}^{\infty} \alpha_i V_{-i+1}^i \\ a_t &= (*a_x)^z + \Delta a_t. \end{aligned} \tag{7.1}$$

We choose a slightly different gauge which is called "non highest weight gauge" for a_t, that is, the only highest weight term in a_t is V_z^{z+1} (see appendix B for some details on this gauge choice and the nomenclature we are using).

The generic infinitesimal gauge transformation preserving the radial gauge is generated by the gauge parameter $\Lambda(\rho, x, t) = b(\rho)^{-1}\lambda(x,t)b(\rho)$ and the gauge transformation itself is

$$\delta a_\mu = [a_\mu, \lambda] + \partial_\mu \lambda \tag{7.2}$$

The three gauge parameters generating time translation, space translation and Lifshitz scaling are

$$\begin{aligned} \lambda_H &= -a_t \\ \lambda_P &= -a_x \\ \lambda_D &= x a_x + z t a_t - V_0^2, \end{aligned} \tag{7.3}$$

We can verify directly by the flatness condition that these gauge parameters generate the desired transformations and preserve lowest weight gauge for a_x. We will use the general formula

$$\delta Q(\Lambda) = -\frac{k}{2\pi}\int dx \mathrm{tr}(\Lambda \delta A_x) = -\frac{k}{2\pi}\int dx \mathrm{tr}(\lambda \delta a_x) \tag{7.4}$$

to obtain the boundary charges. The Hamiltonian can be expressed as follows:

$$\begin{aligned} \delta Q(\Lambda_H) &= \frac{k}{2\pi}\int dx \mathrm{tr}(a_t \delta a_x) = \frac{k}{2\pi}\mathrm{tr}(V_z^{z+1} * V_{-z}^{z+1})\int dx \delta\alpha_{z+1} \\ Q(\Lambda_H) &= \frac{k}{2\pi}\mathrm{tr}(V_z^{z+1} * V_{-z}^{z+1})\int dx \alpha_{z+1} \end{aligned} \tag{7.5}$$

The momentum is given by

$$\begin{aligned} \delta Q(\Lambda_P) &= \frac{k}{2\pi}\int dx \mathrm{tr}(a_x \delta a_x) = \frac{k}{2\pi}\mathrm{tr}(V_1^2 * V_{-1}^2)\int dx \delta\alpha_2 \\ Q(\Lambda_P) &= \frac{k}{2\pi}\mathrm{tr}(V_1^2 * V_{-1}^2)\int dx \alpha_2 \end{aligned} \tag{7.6}$$

The Lifshitz scaling charge takes the form

$$Q(\Lambda_D) = -\frac{k}{2\pi}\int dx x \alpha_2 \mathrm{tr}(V_1^2 * V_{-1}^2) + z t \alpha_{z+1}\mathrm{tr}(V_z^{z+1} * V_{-z}^{z+1}) \tag{7.7}$$

Now let's use the formula [21]

$$\{Q(\Lambda), Q(\Gamma)\} = \delta_\Lambda Q(\Gamma) = -\delta_\Gamma Q(\Lambda) \tag{7.8}$$

to verify the Lifshitz algebra.

$$\{Q(\Lambda_H), Q(\Lambda_P)\} = -\delta_{\Lambda_P} Q(\Lambda_H) = \frac{k}{2\pi}\mathrm{tr}(V_z^{z+1} * V_{-z}^{z+1}) \int dx \alpha'_{z+1} = 0 \tag{7.9}$$

with the by-product that $\dot{\alpha_2}$ must be a total derivative.

$$\{Q(\Lambda_D), Q(\Lambda_H)\} = \delta_{\Lambda_D} Q(\Lambda_H) = \frac{k}{2\pi}\mathrm{tr}(V_z^{z+1} * V_{-z}^{z+1}) \int dx \delta_{\Lambda_D} \alpha_{z+1} \tag{7.10}$$

using

$$\begin{aligned} \delta_{\Lambda_D} a_x &= \partial_x \lambda_D + [a_x, \lambda_D] = a_x + [V_0^2, a_x] + x a'_x + zt\dot{a_x} \\ \delta_{\Lambda_D} \alpha_{z+1} &= (z+1)\alpha_{z+1} + x\alpha'_{z+1} + zt\dot{\alpha}_{z+1} \end{aligned} \tag{7.11}$$

we get

$$\{Q(\Lambda_D), Q(\Lambda_H)\} = zQ(\Lambda_H). \tag{7.12}$$

At last

$$\{Q(\Lambda_D), Q(\Lambda_P)\} = \delta_{\Lambda_D} Q(\Lambda_P) = \frac{k}{2\pi}\mathrm{tr}(V_1^2 * V_{-1}^2) \int dx \delta_{\Lambda_D} \alpha_2 \tag{7.13}$$

using

$$\delta_{\Lambda_D} \alpha_2 = 2\alpha_2 + x\alpha'_2 + zt\dot{\alpha_2} \tag{7.14}$$

we get

$$\{Q(\Lambda_D), Q(\Lambda_P)\} = Q(\Lambda_P). \tag{7.15}$$

7.1 The $sl(3,\mathbb{R})$ and $z=2$ case

In this section with $sl(3,\mathbb{R})$, $z=2$, we will obtain boundary charges corresponding to time translation, space translation and Lifshitz scaling and that verify they satisfy Lifshitz algebra. Furthermore, we identify the components of the stress energy tensor and show it's consistent with conservation laws and Lifshitz scaling symmetry.

For $sl(3,\mathbb{R})$ The generic gauge parameter is

$$\lambda = \sum_{i=-1}^{1} \epsilon_i V_i^2 + \sum_{j=-2}^{2} \chi_j V_j^3. \tag{7.16}$$

By requiring a_x to be form-invariant, that is, it's still in the lowest weight gauge, we find only the coefficient of highest weight terms ϵ_1, χ_2 are free and all the other variables in the gauge parameter are expressed in terms of them. We can assign specific values to ϵ_1, χ_2 to get gauge parameters generating time translation, space translation and Lifshitz scaling

$$\begin{aligned}
&\epsilon_1 = 0, \chi_2 = -1, \lambda = \lambda_H \\
&\epsilon_1 = -1, \chi_2 = 0, \lambda = \lambda_P \\
&\epsilon_1 = x, \chi_2 = 2t, \lambda = \lambda_D
\end{aligned} \tag{7.17}$$

Using (7.4) we get the symmetry charges

$$\begin{aligned}
Q(\Lambda_H) &= \frac{2k}{\pi} \int dx \alpha_3 \\
Q(\Lambda_P) &= -\frac{2k}{\pi} \int dx \alpha_2 \\
Q(\Lambda_D) &= \frac{2k}{\pi} \int dx (x\alpha_2 - 2t\alpha_3)
\end{aligned} \tag{7.18}$$

Using (7.8) we can verify the Lifshitz algebra

$$\begin{aligned}
\{Q(\Lambda_H), Q(\Lambda_P)\} &= 0 \\
\{Q(\Lambda_D), Q(\Lambda_H)\} &= 2Q(\Lambda_H) \\
\{Q(\Lambda_D), Q(\Lambda_P)\} &= Q(\Lambda_P).
\end{aligned} \tag{7.19}$$

Identify the density of $Q(\Lambda_H)$ as the energy density, the density of $Q(\Lambda_P)$ as the momentum density

$$\begin{aligned}
\mathcal{E} &= \frac{2k}{\pi} \alpha_3 \\
\mathcal{P}_x &= -\frac{2k}{\pi} \alpha_2.
\end{aligned} \tag{7.20}$$

Use the Lifshitz symmetry condition $2\mathcal{E} + \Pi_x^x = 0$ to get $\Pi_x^x = -\frac{4k}{\pi}\alpha_3$, we can verify that the conservation of momentum

$$\partial_t \mathcal{P}_x + \partial_x \Pi_x^x = 0 \tag{7.21}$$

is guaranteed by the equations of motion. Plugging the expression for $\mathcal{E}$ into the equation of conservation of energy $\partial_t \mathcal{P} + \partial_x \mathcal{E}^x = 0$ one obtains the expression for energy flow

$$\mathcal{E}^x = -\frac{2k}{\pi}\left(\frac{2}{3}\alpha_2^2 + \frac{1}{6}\alpha_2''\right) \tag{7.22}$$

7.2 The $sl(4,\mathbb{R})$ and $z = 3$ case

Analogues to what we did for $sl(3,\mathbb{R})$, $z = 2$, we will obtain boundary charges, verify the symmetry algebra and study the stress energy tensor. Here we work with KdV gauge. In $sl(4,\mathbb{R})$, the generic gauge parameter for infinitesimal gauge transformation is

$$\lambda = \sum_{i=-1}^{1} \epsilon_i V_i^2 + \sum_{j=-2}^{2} \chi_j V_j^3 + \sum_{k=-3}^{3} \mu_k V_k^4. \tag{7.23}$$

By requiring a_x to be form-invariant, we find only the highest weight terms $\epsilon_1, \chi_2, \mu_3$ are free. Again by appropriately choosing values for these three variables, we get the desired gauge parameters

$$\begin{aligned}
&\epsilon_1 = \frac{7}{10}\alpha_2, \chi_2 = 0, \mu_3 = -1, \lambda = \lambda_H \\
&\epsilon_1 = -1, \chi_2 = 0, \mu_3 = 0, \lambda = \lambda_P \\
&\epsilon_1 = x - \frac{21}{10}\alpha_2 t, \chi_2 = 0, \mu_3 = 3t, \lambda = \lambda_D
\end{aligned} \tag{7.24}$$

For these three gauge parameters, we use (7.4) to calculate the boundary charges

$$\begin{aligned}
Q(\Lambda_H) &= \frac{k}{2\pi}\int dx(-36\alpha_4 + \frac{7}{2}\alpha_2^2) \\
Q(\Lambda_P) &= \frac{k}{2\pi}\int dx(-10\alpha_2) \\
Q(\Lambda_D) &= \frac{k}{2\pi}\int dx(10x\alpha_2 + 108t\alpha_4 + 21t\alpha_2^2)
\end{aligned} \tag{7.25}$$

Again use (7.8) we can verify the Lifshitz symmetry algebra. The density of $Q(\Lambda_H)$ is identified with the energy density up to a total derivative and the density of $Q(\Lambda_P)$ is identified with the momentum density

$$\begin{aligned}
\mathcal{E} &= \frac{k}{2\pi}(-36\alpha_4 + \frac{7}{2}\alpha_2^2 + \frac{7}{6}\alpha_2^{''}) \\
\mathcal{P}_x &= -\frac{k}{2\pi}10\alpha_2
\end{aligned} \tag{7.26}$$

Using the Lifshitz symmetry condition $3\mathcal{E} + \Pi_x^x = 0$ to get $\Pi_x^x = -3\mathcal{E}$, we can show that

$$\partial_t \mathcal{P}_x + \partial_x \Pi_x^x = 0. \tag{7.27}$$

The relation of the KdV conserved charges $q^{(i)}$ to the Chern-Simons variables is given by the following expressions

$$
\begin{aligned}
q^{(1)} &= \int dx \frac{1}{4} u_2 = \int dx \frac{5}{2} \alpha_2 \\
q^{(2)} &= \int dx \frac{1}{4} u_3 = \int dx\, 6 \alpha_3 \\
q^{(3)} &= \int dx \Big(\frac{3}{4} u_4 + \frac{1}{16} u_2^{''} - \frac{3}{32} u_2^2 - \frac{3}{8} u_3^{'}\Big) = \int dx \Big(27\, \alpha_4 - \frac{21}{8} \alpha_2^2\Big)
\end{aligned}
$$

Hence, the KdV charges $q^{(1)}$ and $q^{(3)}$ are proportional to $Q(\Lambda_P)$ and $Q(\Lambda_H)$ respectively.

8 Discussion

In the present paper we have explored the relation of asymptotic Lifshitz spacetimes in higher spin gravity theories to integrable systems in the KdV hierarchy. We were able to make this relation explicit in some specific examples. The evidence of the validity of the general conjecture is the match of the number of degrees of freedom, the agreement of the scaling symmetry of the fields on both sides and the presence of a residual gauge symmetry which can be used to construct the exact map of the equations. It's an interesting open question to find a general proof for our conjecture and we hope to come back to this question in future work.

The Chern-Simons formulation of higher spin gravity and W-algebras are strongly related. W-algebras are nonlinear extensions the 2-dimensional conformal algebra with higher spin fields. For example the standard Drinfeld-Sokolov reduction relates the symmetries of the asymptotically conformal $sl(N, \mathbb{R})$ connections to the symmetries and Ward identities of the W_N algebra. In the past there have been several generalizations of the result relating the Virasoro algebra to the KdV equation, relating the elements of the KdV hierarchy to W algebras in the context of conformal field theories and their deformations [42, 43, 44].

In the present paper we constructed Chern-Simons connections which do not have a conformal scaling symmetry but instead a time/space anisotropic Lifshitz scaling symmetry. It is an interesting question how our results are related to above mentioned work on the relation of W-algebras and KdV. One important difference in the construction of the connections is that we do not use light-cone coordinates $x^{\pm}$ (or complex coordinates $z, \bar{z}$ after Wick rotation). Light cone coordinates naturally lead to split into left and right movers (or holomorphic and anti-holomorphic) sectors characteristic of a CFT. On the CFT side the translations in both space and time are generated by modes of the stress tensor.

In the construction of the Lifshitz theories we use time t and a spatial coordinate x instead. The analysis of section 7 shows that spatial translations are generated by a charge associated with the stress tensor, whereas time translations are generated by a charge associated with the higher spin current in the W_N-algebra. This suggest an intriguing possibility to construct a Lifshitz theory with scaling exponent $z = N$ from a W_N CFT: define the theory on a spatial slice and replace the Hamiltonian which generates time evolution by the integrated spin N current. We leave the exploration of this suggestion for future work.

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A $sl(3,\mathbb{R})$, $sl(4,\mathbb{R})$ and $hs(\lambda)$ conventions

In this appendix we present a realization of the $sl(N,\mathbb{R})$ algebra which are used for calculations in the main body of the text.

A.1 $sl(3,\mathbb{R})$

The $sl(2,\mathbb{R})$ generators of the principal embedding are given by the following matrices

$$L_{-1} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, \qquad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{A.1}$$

and the spin 3 generators, on which we omit the superscript ${}^{(3)}$ for notational simplicity, are as follows:

$$W_{-2} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_{-1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_0 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{A.2}$$

$$W_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \qquad W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \tag{A.3}$$

If we define $(T_1, T_2, \ldots, T_8) = (L_1, L_0, L_{-1}, W_2, \ldots W_{-2})$, then traces of all pairs of generators are given by

$$\mathrm{tr}(T_i T_j) = \left(\begin{array}{ccc|ccccc} & & -4 & 0 & & \cdots & & 0 \\ & 2 & & \vdots & & \ddots & & \vdots \\ -4 & & & 0 & & \cdots & & 0 \\ \hline 0 & \cdots & 0 & & & & & 4 \\ & & & & & & -1 & \\ \vdots & \ddots & \vdots & & & \frac{2}{3} & & \\ & & & & -1 & & & \\ 0 & \cdots & 0 & 4 & & & & \end{array}\right) \tag{A.4}$$

A.2 $sl(4,\mathbb{R})$

The $sl(4,\mathbb{R})$ matrix representation we use is the following. The $sl(2,\mathbb{R})$ sub algebra given by

$$l_0 = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix} \quad l_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad l_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \tag{A.5}$$

$w_i, i = +2, +1, \cdots, -2$ form a spin 2 representation, whereas the $u_i, i = +3, +3, \cdots, -3$ form s spin 3 representation of the $sl(2,\mathbb{R})$ sub algebra.

$$\begin{gathered} w_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad w_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad w_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ w_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \quad w_{-2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ u_2 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad u_1 = \begin{pmatrix} 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix} u_0 = \begin{pmatrix} -\frac{3}{10} & 0 & 0 & 0 \\ 0 & \frac{9}{10} & 0 & 0 \\ 0 & 0 & -\frac{9}{10} & 0 \\ 0 & 0 & 0 & \frac{3}{10} \end{pmatrix} \\ u_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{6}{5} & 0 & 0 & 0 \\ 0 & \frac{12}{5} & 0 & 0 \\ 0 & 0 & -\frac{6}{5} & 0 \end{pmatrix} \quad u_{-2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{pmatrix} \quad u_{-3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -36 & 0 & 0 & 0 \end{pmatrix} \end{gathered} \tag{A.6}$$

The $w_i, i = +2, +1, \cdots, -2$ form a spin 2 representation, whereas the $u_i, i = +3, +3, \cdots, -3$ form s spin 3 representation of the $sl(2,\mathbb{R})$ sub algebra.

A.3 $hs(\lambda)$ conventions

Higher spin algebra elements V^s_m, $s = 1, 2, 3, \ldots$ and $m = -s+1, -s+2, \ldots, s-1$. We call s the spin and m the weight.

The lone star product is defined as

$$V_m^s * V_n^t = \frac{1}{2}\sum_{u=1}^{s+t-|s-t|-1} g_u^{st}(m,n,\lambda)V_{m+n}^{s+t-u} \tag{A.7}$$

The structure constants of the $hs(\lambda)$ algebra were defined in [45] and can be represented as follows

$$g_u^{st}(m,n;\lambda) = \frac{q^{u-2}}{2(u-1)!}\phi_u^{st}(\lambda)N_u^{st}(m,n) \tag{A.8}$$

q is a normalization constant which can be eliminated by a rescaling on the generators, we choose $q = 1/4$ to agree with the literature. The other terms in (A.8) are given by

$$\begin{aligned} N_u^{st}(m,n) &= \sum_{k=0}^{u-1}(-1)^k \begin{pmatrix} u-1 \\ k \end{pmatrix} [s-1+m]_{u-1-k}[s-1-m]_k[t-1+n]_k[t-1-n]_{u-1-k} \\ \phi_u^{st}(\lambda) &= {}_4F_3\left[\begin{array}{cccc|c} \frac{1}{2}+\lambda & \frac{1}{2}-\lambda & \frac{2-u}{2} & \frac{1-u}{2} & \\ \frac{3}{2}-s & \frac{3}{2}-t & \frac{1}{2}+s+t-u & & 1 \end{array}\right] \end{aligned} \tag{A.9}$$

The descending Pochhammer symbol $[a]_n$ is defined as,

$$[a]_n = a(a-1)...(a-n+1) \tag{A.10}$$

The commutator is defined as

$$[V_m^s, V_n^t] = V_m^s * V_n^t - V_n^t * V_m^s \tag{A.11}$$

V_0^1 is the unit element. The trace of a $hs(\lambda)$ element is defined as the coefficient of V_0^1 up to a multiplicative constant $\mathrm{tr}(V_0^1)$. When $\lambda = N$ where N is a positive integer, $hs(\lambda)$ is truncated to $sl(N,\mathbb{R})$. That means, we can consistently set V_m^s to be zero if $s > N$, and the remaining elements form $sl(N,\mathbb{R})$ with star product identified as matrix multiplication and trace identified as matrix trace.

B Gauge choices

B.1 Lowest weight gauge

The lowest weight gauge is that a_x only contains lowest weight terms except for V_1^2. Here we show how we can transform away all non lowest weight terms in a_x. Under an infinitesimal gauge transformation

$$\delta a_x = [a_x, \lambda] + \partial_x \lambda \tag{B.1}$$

We have V_1^2 in a_x, so we can put V_{s-2}^s in the gauge parameter λ to gain a highest weight term V_{s-1}^s in δa_x from the commutator. We can exponentiate this infinitesimal transformation to cancel the highest weight term in the original a_x. After eliminating all highest weight terms, we use V_{s-3}^s in λ to cancel V_{s-2}^s terms. Do this recursively we get to the lowest weight gauge.

B.2 Gauge freedom of a_t and non highest weight gauge

The construction of the asymptotic Lifshitz connection used the fact that $-\partial_x a_t + [a_t, a_x]$ should only contain lowest weight terms to determine coefficients of a_t in terms of coefficients of a_x up to some indeterminacy. The computation is the same as to find an infinitesimal gauge transformation preserving lowest weight gauge, with $-a_t$ playing the role of the gauge parameter. Therefore the indeterminacy in a_t is a gauge freedom. The indeterminacy encompasses actually all the coefficients of the highest weight terms in Δa_t. We can choose them to make highest weight terms in a_t to vanish except for the leading term. We call it non highest weight gauge for a_t.

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