Hypercharge Flux in Heterotic Compactifications

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Abstract

We study heterotic Calabi-Yau models with hypercharge flux breaking, where the visible E_8 gauge group is directly broken to the standard model group by a non-flat gauge bundle, rather than by a two-step process involving an intermediate GUT theory and a Wilson line. It is shown that the required alternative E_8 embeddings of hypercharge, normalized as required for gauge unification, can be found and we classify these possibilities. However, for all but one of these embeddings we prove a general no-go theorem which asserts that no suitable geometry and vector bundle leading to a standard model spectrum can be found. Intuitively, this happens due to the large number of index conditions which have to be imposed in order to obtain a correct physical spectrum in the absence of an underlying GUT theory.

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1 Introduction

Particle physics model building in the context of the $E_8 \times E_8$ heterotic string [1–4] on smooth Calabi-Yau manifolds (see [5–20] for example) has, traditionally, been based on intermediate GUT theories, typically with gauge group $SU(5)$ or $SO(10)$, and subsequent GUT breaking by a Wilson line. A crucial benefit of this approach is the relative ease with which a quasi-realistic particle spectrum can be obtained – a single index condition has to be imposed at the GUT level in order to guarantee three chiral families and this chiral asymmetry is preserved by the Wilson line.

In this letter, we would like to study a different model-building approach where the visible E_8 gauge group is broken to the standard model group directly by flux, without an intermediate GUT theory and a Wilson line. This approach, based on direct flux breaking, is popular in the context of F-theory models (see Refs. [21–23] and also the recent progress in [24–27]) but has also been considered in the heterotic context [5–7] (including in the context of heterotic orbifolds, e.g. [28]).

Here, we would like to study such models systematically. We assume that the non-Abelian part of the standard model group is embedded into E_8 via the maximal subgroup $SU_W(2) \times SU_c(3) \times SU(6) \subset E_8$. Further, we assume that the structure group of the (visible) bundle resides in $SU(6)$ and is of the general split type $S(U(n_1) \times \cdots \times U(n_f)) \subset SU(6)$, where $n_1 + \cdots + n_f = 6$. In this case, the low-energy gauge group is $SU_W(2) \times SU_c(3) \times S(U(1)^f)$ and hypercharge has to be embedded into $S(U(1)^f)$.

After setting out the general structure of these models in the next section, we study their detailed properties in two steps. In section 3 we focus on group-theoretical aspects. Specifically, we classify the possible embeddings of hypercharge which can lead to a viable physical spectrum and which have the correct normalization to be consistent with the standard picture of gauge unification. In section 4, we analyse the underlying Calabi-Yau geometries and bundles which might lead to such models.

2 Basic structure of models

To begin, we describe the general structure of the models we would like to consider in this paper. We are studying compactifications of the $E_8 \times E_8$ heterotic string on Calabi-Yau manifolds with holomorphic, polystable bundles. We will require that the two E_8 factors remain hidden from each other, so the entire standard model group and all standard model multiplets should originate from one E_8 only. The non-Abelian part of the standard model group is embedded into this E_8 factor via the sub-group chain $SU_W(2) \times SU_c(3) \subset SU_W(2) \times SU_c(3) \times SU(6) \subset E_8$ while hypercharge, $U_Y(1)$, resides in the $SU(6)$ factor. The conventional model building approach would be to embed the whole standard model group into E_8 via an intermediate (GUT) $SU(5)$ group. Here we do not demand this - after having fixed the embedding of $SU_W(2) \times SU_c(3)$ as described above the embedding of hypercharge is left arbitrary.

The structure group, H, of the bundle in the observable sector can be at most $SU(6)$ since we require that $SU_W(2) \times SU_c(3)$ is contained in the commutant of H within E_8. Here we will study the different possible unitary splittings of this maximal structure group, that is, we will consider structure groups

$$H = S(U(n_1) \times \cdots \times U(n_f)) , \quad \sum_{a=1}^{f} n_a = 6 , \tag{2.1}$$

which are classified by the partitions of 6. There are 11 such partitions which, in terms of the vector $\mathbf{n} = (n_1, \ldots, n_f)$, are given by

$$\begin{aligned} \mathbf{n} = &(6), \ (5,1), \ (4,2), \ (3,3), \ (4,1,1), \ (3,2,1), \ (2,2,2), \\ &(3,1,1,1), \ (2,2,1,1), \ (2,1,1,1,1), \ (1,1,1,1,1,1) . \end{aligned} \tag{2.2}$$

The low-energy gauge group is then

$$G = SU_W(2) \times SU_c(3) \times S(U(1)^f) \,, \tag{2.3}$$

with hypercharge embedded into $S(U(1)^f)$, in a way to be specified. The additional $U(1)$ symmetries in eq. (2.3) will generically be Green-Schwarz anomalous with associated super-massive gauge bosons. However, for specific choices they can be non-anomalous and massless, a situation which we need to engineer for hypercharge. We will state the required conditions for this in the next section. To work out the possible low-energy multiplets we start with the branching

$$\mathbf{248}_{E_8} \rightarrow \left[(\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{35}) \oplus (\mathbf{1},\mathbf{3},\overline{\mathbf{15}}) \oplus (\mathbf{1},\overline{\mathbf{3}},\mathbf{15}) \oplus (\mathbf{2},\mathbf{3},\mathbf{6}) \oplus (\mathbf{2},\overline{\mathbf{3}},\overline{\mathbf{6}}) \oplus (\mathbf{2},\mathbf{1},\mathbf{20}) \right] \quad (2.4)$$

of the adjoint of E_8 into $SU_W(2) \times SU_c(3) \times SU(6)$ representations. To further decompose the $SU(6)$ representations into representations of H we introduce the notation $\mathcal{F}_a$ and Adj_a for the fundamental and adjoint of $SU(n_a)$, respectively, and also write $S(U(1)^f)$ charges as vectors $\mathbf{q} = (q_1, \ldots, q_f)$ subject to the identification $\mathbf{q} \sim \mathbf{q}' \Leftrightarrow \mathbf{q} - \mathbf{q}' \in \mathbb{Z}\mathbf{n}$. With this notation, the $SU(6)$ representations in the decomposition eq. (2.4) further decompose as follows

$$
\begin{aligned}
\mathbf{6} &\rightarrow \bigoplus_{a=1}^{f} (\mathcal{F}_a)_{\mathbf{e}_a} \\
\overline{\mathbf{6}} &\rightarrow \bigoplus_{a=1}^{f} (\bar{\mathcal{F}}_a)_{-\mathbf{e}_a} \\
\mathbf{15} &\rightarrow \bigoplus_{a=1}^{f} (\wedge^2 \mathcal{F}_a)_{2\mathbf{e}_a} \oplus \bigoplus_{a<b} (\mathcal{F}_a \otimes \mathcal{F}_b)_{\mathbf{e}_a+\mathbf{e}_b} \\
\overline{\mathbf{15}} &\rightarrow \bigoplus_{a=1}^{f} (\wedge^2 \bar{\mathcal{F}}_a)_{-2\mathbf{e}_a} \oplus \bigoplus_{a<b} (\bar{\mathcal{F}}_a \otimes \bar{\mathcal{F}}_b)_{-\mathbf{e}_a-\mathbf{e}_b} \\
\mathbf{20} &\rightarrow \bigoplus_{a=1}^{f} (\wedge^3 \mathcal{F}_a)_{3\mathbf{e}_a} \oplus \bigoplus_{a \neq b} (\wedge^2 \mathcal{F}_a \otimes \mathcal{F}_b)_{2\mathbf{e}_a+\mathbf{e}_b} \oplus \bigoplus_{a<b<c} (\mathcal{F}_a \otimes \mathcal{F}_b \otimes \mathcal{F}_c)_{\mathbf{e}_a+\mathbf{e}_b+\mathbf{e}_c} \\
\mathbf{35} &\rightarrow \bigoplus_{a=1}^{f} (Adj_a)_{\mathbf{0}} \oplus \bigoplus_{a \neq b} (\mathcal{F}_a \otimes \bar{\mathcal{F}}_b)_{\mathbf{e}_a-\mathbf{e}_b} \,,
\end{aligned}
\tag{2.5}
$$

where the subscript denotes the $S(U(1)^f)$ charge and $\mathbf{e}_a$ are the six-dimensional standard unit vectors. To parametrize the embedding of hypercharge into $S(U(1)^f)$ we introduce a vector

$$\mathbf{y} = (y_1, \ldots, y_f) \,, \qquad \mathbf{n} \cdot \mathbf{y} = 0 \,, \tag{2.6}$$

such that $Y(F) = \mathbf{y} \cdot \mathbf{q}(F)$ is the hypercharge of a multiplet F with $S(U(1)^f)$ charge $\mathbf{q}(F)$.

The vector bundle with the required structure group H has the general form

$$V = \bigoplus_{a=1}^{f} U_a \,, \tag{2.7}$$

where U_a is a rank n_a bundle with structure group $U(n_a)$ and we require that $c_1(V) = \sum_a c_1(U_a) \overset{!}{=} 0$. The number of multiplets in the low-energy theory can be determined from the first cohomology of certain associated bundles which are constructed from the vector bundle, eq. (2.7). This information, together with the various group-theoretical details, is summarised in Table 1. In the last two rows of this table we have also listed the hypercharge of the multiplets in terms of the embedding vector eq. (2.6) and the physically required hypercharge. Finding hypercharge embeddings $\mathbf{y}$ and associated patterns of multiplets which do indeed lead to the correct values of hypercharge for all standard model multiplets - and no additional multiplets with exotic charges - is a strong model-building requirement which will be analysed in detail below.

3

$(SU(2) \times SU(3))_{\mathbf{q}}$	$(\mathbf{1},\mathbf{1})_{\mathbf{e}_a-\mathbf{e}_b}$	$(\mathbf{1},\mathbf{3})_{-\mathbf{e}_a-\mathbf{e}_b}$	$(\mathbf{1},\overline{\mathbf{3}})_{\mathbf{e}_a+\mathbf{e}_b}$	$(\mathbf{2},\mathbf{3})_{\mathbf{e}_a}$	$(\mathbf{2},\overline{\mathbf{3}})_{-\mathbf{e}_a}$	$(\mathbf{2},\mathbf{1})_{\mathbf{e}_a+\mathbf{e}_b+\mathbf{e}_c}$
constraint		$a \leq b$	$a \leq b$			$a \leq b \leq c$
particle, F	$e_{a,b},\ S_{a,b}$	$\tilde{d}_{a,b},\ \tilde{u}_{a,b}$	$d_{a,b},\ u_{a,b}$	Q_a	$\widetilde{Q}_a$	$L_{a,b,c},\ H_{a,b,c},\ \bar{H}_{a,b,c}$
bundle	$U_a \otimes U_b^*$	$U_a^* \otimes U_b^*$ $\wedge^2 U_a^*$	$U_a \otimes U_b$ $\wedge^2 U_a$	U_a	U_a^*	$U_a \otimes U_b \otimes U_c$ $\wedge^2 U_a \otimes U_b,$ $U_a \otimes \wedge^2 U_b,\ \wedge^3 U_a$
contained in	$V \otimes V^*$	$\wedge^2 V^*$	$\wedge^2 V$	V	V^*	$\wedge^3 V$
$Y(F)$	$y_a - y_b$	$-y_a - y_b$	$y_a + y_b$	y_a	$-y_a$	$y_a + y_b + y_c$
$Y_{\mathrm{phys}}(F)$	2, 0	$-2/3,\ 4/3$	$2/3,\ -4/3$	$1/3$	$-1/3$	$-1,\ -1,\ 1$

Table 1: *Particle content of models with bundle structure group H a unitary split of $SU(6)$. The multiplicity of each type of multiplet is determined by the first cohomology of the associated bundle. The indices $a, b, \ldots$ label the summands of the bundle, eq. (2.7), and are in the range $a, b, \ldots = 1, \ldots, f$.*

3   Embedding of hypercharge

As explained above, the embedding of hypercharge $U_Y(1)$ into $S(U(1)^f)$ is described by a vector $\mathbf{y}$ as in eq. (2.6). Using the decomposition eq. (2.4) and eq. (2.5), the normalization of $U_Y(1)$ can be computed as

$$\frac{g^2}{g_Y^2} \;=\; \frac{1}{120}\operatorname{Tr}(Y^2) \;=\; \frac{1}{2}|\mathbf{y}^2| \;=\; \frac{1}{2}\sum_{a=1}^{f} n_a\, y_a^2 \tag{3.1}$$

The standard normalization of hypercharge which is appropriate for gauge unification in its conventional form and which is realized for the usual embedding of hypercharge into $SU(5)$ is given by $g^2/g_Y^2 = 5/3$. Hence, if we wish to implement the conventional picture of gauge unification we should require that

$$\sum_{a=1}^{f} n_a\, y_a^2 \;\simeq\; \frac{10}{3} \tag{3.2}$$

is satisfied for our hypercharge embedding, either exactly or to sufficient accuracy.

We can now ask the following, purely group-theoretical question. For which embedding vectors $\mathbf{y}$ can we assign $S(U(1)^f)$ charges to one standard model family such that we obtain the correct hypercharge for all multiplets and such that the unification condition eq. (3.2) is satisfied? If we require the unification condition eq. (3.2) exactly, it turns out there is a very limited range of possibilities which is summarised in Table 2.

A few remarks about this table are in order. First, note that certain splitting types in eq. (2.2) are excluded right away and, hence, do not appear in the table. Clearly, $\mathbf{n} = (6)$ which corresponds to a bundle structure group $SU(6)$ is excluded since no $U(1)$ symmetry which could account for hypercharge is left over in this case. Further constraints on the splitting type arise as follows. The first Chern classes of the constituent bundles U_a have to satisfy

$$\sum_{a=1}^{f} c_1(U_a) = 0 \,, \qquad \sum_{a=1}^{f} y_a c_1(U_a) = 0 \,, \tag{3.3}$$

splitting type $\mathbf{n}$	allowed $\mathbf{y}$ vectors
$(4,1,1)$	~~$(1/3, 1/3, \ \ -5/3)$~~
$(3,2,1)$	~~$(1/3, 1/3, -5/3)$~~, $(-2/3, 1/3, 4/3)$
$(2,2,2)$	no solution
$(3,1,1,1)$	~~$(1/3, 1/3, 1/3, \ \ -5/3)$~~, $(-2/3, 1/3, 1/3, 4/3)$
$(2,2,1,1)$	~~$(1/3, 1/3, 1/3, \ \ -5/3)$~~, $(1/3, -2/3, -2/3, 4/3)$
$(2,1,1,1,1)$	~~$(1/3, 1/3, 1/3, 1/3, \ \ -5/3)$~~, $(1/3, -2/3, -2/3, -2/3, 4/3)$, $(-2/3, -2/3, 1/3, 1/3, 4/3)$ $(5/6, -7/6, -2/3, -1/6, 1/3)$, $(-5/21, -17/21, -11/21, 1/3, 31/21)$
$(1,1,1,1,1,1)$	~~$(1/3, 1/3, \ \ -5/3, 1/3, 1/3, 1/3)$~~, $(1/3, 4/3, -2/3, -2/3, -2/3, 1/3)$ $(1/3, 5/6, -7/6, -1/6, -2/3, 5/6)$, $(1/3, 7/12, -17/12, 1/12, -5/12, 5/6)$, $\ldots$

Table 2: *All embedding vectors* $\mathbf{y}$ *(modulo re-ordering) for the various splitting patterns eq. (2.2) which can lead to the correct hypercharge for one family and satisfy the unification condition eq. (3.2) exactly. The crossed-out vectors correspond to the conventional embedding of hypercharge into* $SU(5)$*.*

where the first equation simply states that $c_1(V) = 0$ and the second equation is the condition for the hypercharge gauge boson to be massless.[1] Now consider the splitting types $\mathbf{n} = (n_1, n_2)$ into two summands. In this case, both first Chern classes, $c_1(U_1)$ and $c_1(U_2)$, must vanish since the conditions eq. (3.3) are independent. But with $c_1(U_1) = c_1(U_2) = 0$ the bundle structure group reduces and is no longer of the type eq. (2.1). Hence, these cases have been discarded in Table 2.

Embedding vectors of the form $\mathbf{y} = (1/3, \ldots, 1/3, -5/3)$, together with eq. (3.3), imply that $c_1(U_f) = 0$ and, hence, also lead to a reduced structure group outside the class specified by eq. (2.1). In fact, these cases correspond to the conventional embedding of hypercharge into an intermediate $SU(5)$ GUT which is then broken by a Wilson line. This is the standard heterotic model-building route which is, of course, perfectly viable. However, in this paper we are focusing on a direct flux breaking to the standard model without any intermediate GUT theory and, hence, these embedding vectors have been crossed out in Table 2.

This leaves us with a fairly limited number of possibilities, one for each for the splitting types $\mathbf{n} = (3,2,1)$, $(3,1,1,1)$, $(2,2,1,1)$ and four for the splitting type $\mathbf{n} = (2,1,1,1,1)$. Only the Abelian case, $\mathbf{n} = (1,1,1,1,1,1)$, where the vector bundle is a sum of line bundles comes with a large number of possibilities, indicated by the dots in the last row of Table 2, although with increasingly complicated fractions. In fact, the general solution for the Abelian case can be written in the form

$$\mathbf{y} = \left(\frac{1}{3}, \alpha, \alpha - 2, \frac{2}{3} - \alpha, \frac{1}{2}(1 - \alpha - s), \frac{1}{2}(1 - \alpha + s) \right), \quad s = \frac{1}{3}\sqrt{-63\alpha^2 + 114\alpha - 31} \qquad (3.4)$$

where α is a free parameter which should be chosen such that the resulting $\mathbf{y}$ vector is rational.

Evidently, the above classification of hypercharge vectors is fairly restrictive. We can slightly relax our requirements by asking the unification condition, eq. (3.2) to be satisfied approximately, within 5%, rather than exactly and then redo the classification. For simplicity we will only carry this out for the simplest splitting type, the Abelian splitting into a sum of line bundles with $\mathbf{n} = (1,1,1,1,1,1)$. To be precise, we are asking the following question. For the Abelian splitting type, $\mathbf{n} = (1,1,1,1,1,1)$, which hypercharge

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[1]Note that the second condition in eq. (3.3) guarantees that the one-loop contribution to the mass of the hypercharge gauge boson explored in Refs. [7, 29] vanishes.

embeddings **y** allow for a pattern of $S(U(1)^6)$ charge assignments such that we obtain the correct hypercharges for all multiplets in one standard model family and the unification condition, eq. (3.2), is satisfied approximately, to within 5%? The answer to this question is the following four families of embedding vectors,

$$\mathbf{y}_1(\alpha, \beta) = \left(-\frac{5}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} - \alpha, \frac{1}{3} - \beta, \frac{1}{3} + \alpha + \beta \right) ,$$

$$\mathbf{y}_2(\alpha) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{4}{3}, -\frac{2}{3} - \alpha, -\frac{2}{3} + \alpha \right) ,$$

$$\mathbf{y}_3(\alpha, \beta) = \left(-\frac{5}{3}, \frac{1}{3}, \frac{1}{3} - \alpha, \frac{1}{3} + \alpha, \frac{1}{3} - \beta, \frac{1}{3} + \beta \right) , \quad (3.5)$$

$$\mathbf{y}_4(\alpha, \beta) = \left(\frac{1}{3}, -\frac{5}{3} - \alpha, \frac{1}{3} - \alpha, \frac{1}{3} + \alpha, \frac{1}{3} - \beta + \alpha, \frac{1}{3} + \beta \right) ,$$

which depend on one or two parameters. Again, this is a fairly restrictive result.

We end this section with a remark related to gauge unification. Normally, for models with hypercharge flux, one would expect threshold corrections to the gauge kinetic function. These might spoil "natural" gauge unification just as an incorrect normalization of hypercharge would. In Ref. [29], Eqs. (5.14) and (5.19), the difference between the $U_Y(1)$ and the $SU_W(2) \times SU_c(3)$ gauge kinetic functions has been calculated as

$$\delta f_{ab} \sim T^i \, d_{ijk} \, c_1^j(U_a) \, c_1^k(U_b) , \quad (3.6)$$

where T^i are the Kähler moduli. In general, this expression is non-vanishing but we have to specialize the gauge field to the hypercharge direction y^a. However, from the second condition in eq. (3.3) which guarantees that hypercharge is massless, we have $y^a c_1^i(U_a) = 0$. This implies that the correction eq. (3.6) vanishes if at least one of the gauge fields corresponds to hypercharge. Hence, there is neither an additional threshold correction for hypercharge nor kinetic mixing of hypercharge with any of the other $U(1)$ symmetries.

To summarise, we conclude that hypercharge can be embedded into E_8 in a number of non-standard ways, such that the physically correct hypercharges for the standard model fields can be obtained and "natural" gauge unification is realised. We emphasise that our viewpoint so far has been purely group-theoretical. In other words, the viable hypercharge embeddings we have found *allow* for patterns of $S(U(1)^f)$ charges such that all values for hypercharge come out correctly. Whether such patterns can actually be realised by an underlying geometry and vector bundle is another question to which we now turn.

4 A no-go argument

Conventionally, heterotic standard models are built based on an "intermediate" GUT theory, typically with $SU(5)$ or $SO(10)$ gauge group, which is subsequently broken by a Wilson line. For such models hypercharge is of course embedded into the GUT group, in the usual way. This approach requires a Calabi-Yau manifold with a non-trivial first fundamental group or, equivalently, a Calabi-Yau manifold with a freely-acting discrete symmetry, so that a Wilson line can indeed be introduced. Calabi-Yau manifolds with freely-acting discrete symmetries are relatively rare (see for instance Refs. [30, 31]) and, in addition, such discrete symmetries are often not easy to find, so this may be considered a disadvantage of the conventional model-building route. A considerable advantage of this approach is the relative ease with which a physically promising particle spectrum can be obtained. Let us briefly discuss this for the case of an intermediate $SU(5)$ GUT. One standard model family can be grouped into the $SU(5)$ representations **10** and **5̄** and, at the GUT level, the chiral asymmetry of these representations is given by the indices $\mathrm{ind}(V)$ and $\mathrm{ind}(\wedge^2 V)$,

respectively, where V is the relevant vector bundle with $SU(5)$ structure group (or a rank four sub-group thereof). In fact, for $SU(5)$ bundles these indices are equal, $\mathrm{ind}(\wedge^2 V) = \mathrm{ind}(V)$. Hence, to obtain a promising model at the GUT level, the only index condition we need to require is $\mathrm{ind}(V) = -3|\Gamma|$, where $|\Gamma|$ is the order of the freely-acting symmetry group Γ. Taking the quotient by Γ will reduce this to precisely three GUT families and, since the Wilson line does not affect the chiral asymmetry, this will lead to three chiral standard model multiplets of each type. In other words, a promising spectrum with three chiral families is obtained by imposing a single index condition on the vector bundle.

Let us now compare this situation to the one for the models discussed in this paper, where E_8 is broken to the standard model directly through flux, without the need for an intermediate GUT and Wilson lines. A clear advantage of pure flux-breaking is that we do not require a non-trivial first fundamental group for the Calabi-Yau manifold - such models can in principle be built on any Calabi-Yau manifold. The disadvantage becomes apparent from Table 1. A physically promising three-family spectrum with the correct values of hypercharge requires satisfying a large number of index conditions on the many bundles in Table 1, so that all standard model multiplets appear in the right $S(U(1)^f)$ charge sector to produce the correct physical hypercharge for a given embedding $\mathbf{y}$ and also to avoid the appearance of multiplets with exotic hypercharge. This is to be compared with just one index condition which needs to be imposed on models with underlying GUT symmetry.

Guided by this observation, we would now like to analyse the index constraints which should be imposed to obtain physically promising spectra in more detail. For now, we will do this for the simplest, Abelian split pattern with $\mathbf{n} = (1, 1, 1, 1, 1, 1)$ and later extend our results to the general case.

We begin with a line bundle sum

$$V = \bigoplus_{a=1}^{6} L_a \tag{4.1}$$

and also introduce the notation $x_a = c_1(L_a)$ for the first Chern classes of the constituent line bundles. Further, we fix a specific hypercharge embedding $\mathbf{y} = (y_1, \ldots, y_6)$, for example one of the cases found in the previous section. Then, the requirements of $c_1(V) = 0$ and vanishing hypercharge mass translate into the conditions

$$\sum_{a=1}^{6} x_a = 0 \,, \qquad \sum_{a=1}^{6} y_a x_a = 0 \,. \tag{4.2}$$

A glance at Table 1 indicates the conditions we need to require for a physical spectrum. In order to obtain three chiral families of quarks with the correct hypercharges we need

$$\sum_{a:y_a=1/3} \mathrm{ind}(L_a) = -3 \,, \qquad \sum_{a<b:y_a+y_b=2/3} \mathrm{ind}(L_a \otimes L_b) = -3 \,, \qquad \sum_{a<b:y_a+y_b=-4/3} \mathrm{ind}(L_a \otimes L_b) = -3 \,. \tag{4.3}$$

To avoid quarks with exotic hypercharges we also must impose that

$$\mathrm{ind}(L_a) = 0 \text{ if } y_a \neq 1/3 \tag{4.4}$$

$$\mathrm{ind}(L_a \otimes L_b) = 0 \text{ if } a < b \text{ and } y_a + y_b \notin \{2/3, -4/3\} \,. \tag{4.5}$$

Finally, a chiral asymmetry with the wrong sign - which would lead to mirror quarks - should be avoided in each line bundle sector so that we require

$$-3 \leq \mathrm{ind}(L_a) \leq 0 \,, \qquad -3 \leq \mathrm{ind}(L_a \otimes L_b) \leq 0 \,, \tag{4.6}$$

for all $a, b = 1, \ldots 6$. For the leptons, we cannot impose an overall constraint on the index (the relevant bundles are real and, hence, their index vanishes) but we still need to ensure the absence of leptons with

exotic hypercharges. This amounts to the vanishing conditions

$$\text{ind}(L_a \otimes L_b \otimes L_c) = 0 \text{ if } a < b < c \text{ and } y_a + y_b + y_c \notin \{-1, 1\} \tag{4.7}$$

$$\text{ind}(L_a \otimes L_b^*) = 0 \text{ if } y_a - y_b \notin \{-2, 0, 2\} . \tag{4.8}$$

How many independent conditions the above equations amount to depends on the structure of the hypercharge embedding $\mathbf{y}$. As a rule of thumb, the more complicated $\mathbf{y}$, the more conditions have to be satisfied.

To see if the above physical conditions can be satisfied, we now express the various indices in terms of the underlying topological data, so that

$$\text{ind}(L_a) = \frac{1}{6}x_a^3 + \frac{1}{12}x_a c_2(TX) \tag{4.9}$$

$$\text{ind}(L_a \otimes L_b) = \frac{1}{6}(x_a + x_b)^3 + \frac{1}{12}(x_a + x_b)c_2(TX) \tag{4.10}$$

$$\text{ind}(L_a \otimes L_b \otimes L_c) = \frac{1}{6}(x_a + x_b + x_c)^3 + \frac{1}{12}(x_a + x_b + x_c)c_2(TX) \tag{4.11}$$

$$\text{ind}(L_a \otimes L_b^*) = \frac{1}{6}(x_a - x_b)^3 + \frac{1}{12}(x_a - x_b)c_2(TX) . \tag{4.12}$$

Hence, all relevant line bundle indices depend on the six first Chern classes x_a and the second Chern class of the Calabi-Yau tangent bundle $c_2(TX)$. However, two of the six quantities x_a can be eliminated from these index expressions in favour of the remaining four by using the linear relations, eq. (4.2). For definiteness we assume that x_5, x_6 have been eliminated and that all index expressions are written in terms of x_α, where $\alpha = 1, \ldots, 4$ and $c_2(TX)$. To further parametrise our ignorance of the underlying geometry we introduce the variables $X_{\alpha\beta\gamma} = x_\alpha x_\beta x_\gamma$ and $Z_\alpha = x_\alpha c_2(TX)$, where $\alpha \leq \beta \leq \gamma$ and $\alpha, \beta, \gamma = 1, \ldots, 4$. All index expressions, eq. (4.9)–eq. (4.12), can then be written as linear functions of the 24 variables $X_{\alpha\beta\gamma}$ and Z_α. Combined with the physical conditions on the indices listed above this leads to a system of linear equations (and inequalities) for $X_{\alpha\beta\gamma}$ and Z_α.

We can now ask if this linear system has a solution, for a given hypercharge embedding $\mathbf{y}$. It turns out that for all possible $\mathbf{y}$ vectors satisfying the unification condition, eq. (3.2), exactly (as given in the last row of Table 2 or more generally by using an arbitrary rational vector of the form eq. (3.4)) the answer to this question (by direct computation) is "no". Hence, remarkably, we have shown that heterotic models with pure flux breaking of a single E_8 and exact unification normalization of hypercharge can never lead to a physically acceptable particle spectrum. We emphasise that this conclusion does not rely on any particular Calabi-Yau manifold or class of Calabi-Yau manifolds (since we have absorbed the relevant topological data of the Calabi-Yau manifold into the variables $X_{\alpha\beta\gamma} = x_\alpha x_\beta x_\gamma$ and $Z_\alpha = x_\alpha c_2(TX)$) but is completely general.

This argument can be repeated for the hypercharge embeddings $\mathbf{y}$ from eq. (3.5) which lead to approximate unification. We find that the associated systems of linear equations have no solutions in all cases but one.[2] The single remaining case which cannot be excluded in this way is based on the hypercharge embedding

$$\mathbf{y} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{5}{3}, \frac{1}{3} - \alpha, \frac{1}{3} + \alpha \right) , \tag{4.13}$$

---

[2]To be precise, for $\mathbf{y}$ vectors not in the family, eq. (4.13), we have confirmed that the the the Diophantine system for X_{abc} and Z_a has no solutions if the parameters α and β in eq. (3.5) take the form k/l, with $-50 \leq k, l \leq 50$, such that the unification condition, eq. (3.2), is satisfied approximately to within 5%.

a common sub-case of the three two-parameter families in eq. (3.5) with $\alpha \to 0, \beta \to \alpha$ (and suitably re-ordered). Moreover, this case leads to an essentially unique solution for the index conditions which reads

$$
\begin{aligned}
\mathrm{ind}(L_1) = \mathrm{ind}(L_2) = \mathrm{ind}(L_3) = -1 &\quad \to \quad Q_1, \, Q_2, \, Q_3 \\
\mathrm{ind}(L_1 \otimes L_4) = \mathrm{ind}(L_2 \otimes L_4) = \mathrm{ind}(L_3 \otimes L_4) = -1 &\quad \to \quad u_{1,4}, \, u_{2,4}, \, u_{3,4} \\
\mathrm{ind}(L_5 \otimes L_6) = -3 &\quad \to \quad 3 \, d_{5,6} \\
\mathrm{ind}(L_4 \otimes L_5 \otimes L_6) = -\mathrm{ind}(L_1 \otimes L_2 \otimes L_3) = -3 &\quad \to \quad 3 \, L_{4,5,6} \\
\mathrm{ind}(L_1 \otimes L_4^*) = \mathrm{ind}(L_2 \otimes L_4^*) = \mathrm{ind}(L_3 \otimes L_4^*) = -1 &\quad \to \quad e_{1,4}, \, e_{2,4}, \, e_{3,4} \\
\mathrm{ind}(L_1 \otimes L_2^*) = -\mathrm{ind}(L_2 \otimes L_1^*) = 2 - X_{112} &\quad \to \quad S_{1,2} \text{ or } S_{2,1} \\
\mathrm{ind}(L_1 \otimes L_3^*) = -\mathrm{ind}(L_3 \otimes L_1^*) = 2 + X_{111} + X_{112} &\quad \to \quad S_{1,3} \text{ or } S_{3,1} \\
\mathrm{ind}(L_2 \otimes L_3^*) = -\mathrm{ind}(L_3 \otimes L_2^*) = 6 - X_{112} + X_{222} &\quad \to \quad S_{2,3} \text{ or } S_{3,2}
\end{aligned}
\tag{4.14}
$$

Finally, we should discuss the other splitting types in eq. (2.2). Those splitting types require non-Abelian constituent bundles, U_a, so that the analogues of the index relations, eq. (4.9)–eq. (4.12), also depend on higher Chern classes, $c_2(U_a)$, $c_3(U_a)$, in addition to $c_1(U_a)$. However, the splitting principle[3][32] asserts that the total Chern class of V can be expressed as

$$
\prod_{a=1}^{f} c(U_a) = \prod_{a=1}^{f} \prod_{i=1}^{n_a} (1 + x_{ai})
\tag{4.15}
$$

for suitable classes x_{ai} of the second cohomology. When expressed in terms of x_{ai} the index relations assume precisely the same form as in the Abelian case and the above no-go argument can be applied in the same form, provided the hypercharge embedding $\mathbf{y}$ in the non-Abelian case is split into an Abelian counterpart accordingly (so that, for example the embedding $\mathbf{y} = (-2/3, 1/3, 4/3)$ for $\mathbf{n} = (3, 2, 1)$ becomes $\mathbf{y} = (-2/3, -2/3, -2/3, 1/3, 1/3, 4/3)$ in the Abelian case). Based on this argument the hypercharge embeddings which satisfy the unification condition exactly, as listed in Table 2, cannot lead to a standard model spectrum and are, hence, ruled out. Further, all hypercharge embeddings with approximate unification which split into one of the vectors in eq. (3.5) - with the exception of eq. (4.13) which we have not excluded - are ruled out on the same grounds.

We have checked the above no-go argument by constructing explicit models for the Abelian case, based on the splitting type $\mathbf{n} = (1, 1, 1, 1, 1, 1)$ and rank six line bundle sums eq. (4.1), generalising the model building approach described in Refs. [33–35]. We have indeed not found a single model, both for hypercharge embeddings with exact and approximate unification, consistent with a standard model spectrum.

In addition, we have carried out a dedicated search for models based on the hypercharge embedding eq. (4.13), the one case we were not able to exclude from general arguments. The same scanning techniques and Calabi-Yau geometries used in Refs. [33–35] were employed for an extensive search. Unfortunately, no viable models were found for this case either. The problem seems to be one of integrality. It is difficult to satisfy the two conditions eq. (4.2) for the $\mathbf{y}$ vector eq. (4.13), together with the index conditions eq. (4.14) for all x_a being integral – as required if these quantities are to represent first Chern classes of line bundles. We do not currently know if this problem is general or related to the specific class of models we have studied.

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[3]It should be briefly noted here that the splitting principle must be applied with care. It is *not* the case that non-Abelian bundles can be split over the Calabi-Yau manifold. Rather, for any bundle $V \to X$ there exists a Flag space and a map $s : F(V) \to X$ such that $s^*(V)$ decomposes as a direct sum of (complex, not necessarily holomorphic) line bundles and $c_k(s^*(V)) = s^*(c_k(V))$). We are fortunate that the list of conditions in eq. (4.9) - eq. (4.12) do not depend explicitly on either holomorphy of the bundle or the Calabi-Yau condition on the base, thus the no-go results of the Abelian case continue to hold here as well.

5 Conclusion

In this letter, we have studied $E_8 \times E_8$ heterotic Calabi-Yau models based on flux breaking of the visible E_8 group down to the standard model group, without an intermediate GUT theory and Wilson lines. The non-Abelian part of the standard model group has been embedded into E_8 via the maximal sub-group $SU_W(2) \times SU_c(3) \times SU(6) \subset E_8$ which leads to the correct $SU_W(2) \times SU_c(3)$ representations required for a standard model spectrum. We have used bundle structure groups $S(U(n_1) \times \cdots \times U(n_f)) \subset SU(6)$, where $\sum_{a=1}^{f} n_a = 6$, so that the low-energy gauge group is $SU_W(2) \times SU_c(3) \times S(U(1)^f)$ and hypercharge embedding into $S(U(1)^f)$ is described by a vector $\mathbf{y} = (y_1, \ldots, y_f)$.

We have studied these models in two steps. First, we have considered the purely group-theoretical aspects of model building. In this context, we have classified all hypercharge embeddings $\mathbf{y}$ which can lead to the correct standard model hypercharges and have the standard normalization required for "natural" gauge unification. The results are given in Table 2 (for $\mathbf{y}$ vectors satisfying the unification condition eq. (3.2) exactly) and in eq. (3.5) (for $\mathbf{y}$ vectors satisfying the unification condition approximately).

In a second step, we have then attempted to build explicit models for these hypercharge embeddings. It turns out that obtaining a realistic spectrum in these cases leads to a highly constrained problem, whereby many index conditions have to be imposed on the internal bundle and its various tensor powers. We have shown for all hypercharge embeddings $\mathbf{y}$ which satisfy the unification condition eq. (3.2) exactly, that these conditions have no solution for any underlying Calabi-Yau manifold and bundle thereon. Further, for the case of approximate unification we have obtained a similar no-go result for all hypercharge embeddings except for a single case given by eq. (4.13). These results have been checked by explicit model building, based on the approach outlined in Refs. [33–35].

In summary, we have shown that heterotic $E_8 \times E_8$ models with flux breaking of the visible E_8 group to the standard model and "natural" gauge unification can never lead to a realistic particle physics spectrum - barring one marginal case with approximate unification. Intuitively, obtaining the standard model directly requires many topological index conditions, in fact, too many to be satisfied by any underlying geometry and bundle. This result highlights the benefits of heterotic model-building based on an intermediate GUT theory and Wilson line breaking. In this case, a promising spectrum with three chiral families can be obtained by imposing a single index condition at the GUT level while the subsequent Wilson line breaking preserves the chiral asymmetry for each standard model multiplet.

There are several generalisations and modifications of the models studied in this letter which we have not discussed explicitly. Firstly, it is possible to consider other embeddings of $SU_W(2) \times SU_c(3)$ into E_8, although these tend to lead to exotic representations in the branching of the adjoint of E_8. Further, it is possible to consider general hypercharge embeddings into $E_8 \times E_8$, as has been done in Ref. [5] (see also [36–38]), rather than into a single E_8. In this case, the second E_8 is not truly hidden and this gives rise to a range of additional model-building problems. For this reason, we have not considered such models but we expect that the approach presented in this letter - and quite possibly some of the no-go results - will extend to these cases.

Finally, we can speculate about the possible relevance of our results for other model-building approaches. Essentially, we have found that the standard model spectrum can be too complicated and fragmented to result from string theory directly. We require the organising principle of an intermediate GUT theory, broken by a Wilson line in order to preserve chiral asymmetries, for successful model building. Hypercharge flux remains the preferred breaking mechanism in F-theory models and it is conceivable that similar no-go results (or at least constraints) can be obtained in global F-theory models in this context. We hope to return to this problem in a future publication.

Acknowledgments

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