APS physics

CHORUS

This is the accepted manuscript made available via CHORUS. The article has been published as:

Baryon spectrum from superconformal quantum mechanics and its light-front holographic embedding

Guy F. de Téramond, Hans Günter Dosch, and Stanley J. Brodsky

Phys. Rev. D **91**, 045040 — Published 27 February 2015

DOI: 10.1103/PhysRevD.91.045040

SLAC–PUB–16130

Baryon Spectrum from Superconformal Quantum Mechanics and its Light-Front Holographic Embedding

Guy F. de Téramond

Universidad de Costa Rica, San José, Costa Rica[*]

Hans Günter Dosch

Institut für Theoretische Physik, Philosophenweg 16, D-6900 Heidelberg, Germany[†]

Stanley J. Brodsky

SLAC National Accelerator Laboratory,

Stanford University, Stanford, California 94309, USA[‡]

Abstract

We describe the observed light-baryon spectrum by extending superconformal quantum mechanics to the light front and its embedding in AdS space. This procedure uniquely determines the confinement potential for arbitrary half-integer spin. To this end, we show that fermionic wave equations in AdS space are dual to light-front supersymmetric quantum mechanical bound-state equations in physical space-time. The specific breaking of conformal invariance explains hadronic properties common to light mesons and baryons, such as the observed mass pattern in the radial and orbital excitations, from the spectrum generating algebra. The holographic embedding in AdS also explains distinctive and systematic features, such as the spin-J degeneracy for states with the same orbital angular momentum, observed in the light baryon spectrum.

[*] gdt@asterix.crnet.cr

[†] h.g.dosch@thphys.uni-heidelberg.de

[‡] sjbth@slac.stanford.edu

I. INTRODUCTION

The classical Lagrangian of QCD is invariant under scale and conformal transformations in the limit of massless quarks [1, 2]. However, meson and baryon bound-states have well-defined ground states and towers of excited states with well defined and measurable properties such as mass and spin. A simple but fundamental question in hadron physics is thus to understand the mechanism which endows a nominally conformal theory with a mass scale, as well as to explain the remarkably similar linear Regge spectroscopy of both mesons and baryons.

In the quest for semiclassical equations to describe bound states, in the large distance strongly coupled regime of QCD, one can start by reducing the strongly correlated multiparton light-front Hamiltonian dynamical problem to an effective one-dimensional quantum field theory [3]. This procedure is frame-independent and leads to a semiclassical, relativistic light-front (LF) wave equation for the valence state (the lowest Fock state), analogous to the Schrödinger and Dirac equations in atomic physics. The complexities arising from the strong interaction dynamics of QCD and an infinite class of Fock components are incorporated in an effective potential U, but its determination from first principles remains largely an open question.

Thus, a second central problem in the theoretical search for a semiclassical approximation to QCD is the construction of the effective LF confining potential U which captures the underlying dynamics responsible for confinement, the emergence of a mass scale as well as the universal Regge behavior of mesons and baryons. Since our light-front semiclassical approach is based on a one-dimensional quantum field theory, it is natural to extend the framework introduced by V. de Alfaro, S. Fubini and G. Furlan (dAFF) [4] to the frame-independent light-front Hamiltonian theory, since it gives important insight into the QCD confinement mechanism [5]. Remarkably, dAFF show that a mass scale can appear in the Hamiltonian without breaking the conformal invariance of the action.

The dAFF construction [4] begins with the study of the spectrum of a conformally invariant one-dimensional quantum field theory which does not have a normalizable ground state. A new Hamiltonian is defined as a superposition of the generators of the conformal group and consequently it leads to a redefinition of the corresponding evolution parameter τ, the range of which is finite. This choice determines the quantum mechanical evolution of

the system in terms of a compact operator with normalizable eigenstates and a well defined ground state. A scale appears in the Hamiltonian while retaining the conformal invariance of the action [4]. This remarkable result is based on the isomorphism of the algebra of the one-dimensional conformal group $Conf(R^1)$ to the algebra of generators of the group $SO(2,1)$. One of the generators of this group, the rotation in the 2-dimensional space, is compact. As a result, the form of the evolution operator is fixed and includes a confining harmonic oscillator potential, thus equally spaced eigenvalues [6, 7]. Since the generators of $Conf(R^1)$ have different dimensions, their relations with the generators of $SO(2,1)$ imply a scale, which according to dAFF may play a fundamental role [4, 5].

A third important feature in the construction of semiclassical equations in QCD, is the correspondence between the equations of motion for arbitrary spin in Anti–de Sitter (AdS) space and the light-front Hamiltonian equations of motion for relativistic light hadron bound-states in physical space-time [3, 8]. This approach is inspired by the AdS/CFT correspondence [9] where, in principle, one can compute physical observables in a strongly coupled gauge theory in terms of a weakly coupled classical gravity theory defined in a higher dimensional space [9–11]. In fact, an additional motivation for using AdS/CFT ideas to describe strongly coupled QCD follows from the vanishing of the β-function in the infrared, which leads to a conformal window in this regime [12–14].

The procedure, known as light-front holography [3, 15, 16], allows one to establish a precise relation between wavefunctions in AdS space and the LF wavefunctions describing the internal structure of hadrons. As a result, the effective LF potential U derived from the AdS embedding is conveniently expressed, for arbitrary integer spin representations, in terms of a dilaton profile which is determined by the dAFF procedure described above [5, 8]. The result is a light-front wave equation which reproduces prominent aspects of hadronic data, such as the mass pattern observed in the radial and orbital excitations of the light mesons [16], and in particular a massless pion in the chiral limit.

The light-front holographic embedding for baryons is not as simple as for mesons, since a dilaton term in the AdS fermionic action can be rotated away by a redefinition of the fermion fields in AdS [16, 17], and therefore it has no dynamical effects on the spectrum. In practice, one can introduce an effective interaction in the fermion action, a Yukawa term, which breaks the maximal symmetry in AdS and consequently the conformal symmetry in Minkowski space. This leads to a linear confining interaction in a LF Dirac equation for

baryons whose eigensolutions generate a baryonic Regge spectrum [18, 19]. The confining interaction term can be constrained by the condition that the square of the Dirac equation leads to a potential which matches the form of the dilaton-induced potential for integer spin, but this procedure appears to be *ad-hoc*.

There are some striking similarities between the spectra of the observed light mesons and baryons: they are of similar mass, the slope and spacing of the quantum orbital excitations in L and their daughter spacing in n, the radial quantum number, is the same. This behavior in the meson sector is related to the introduction of a scale within the framework of the conformal algebra. This procedure leaves the action invariant [4, 5]. Since supersymmetry is related with boson-fermion symmetry, it is compelling to examine the properties of the supersymmetric algebra and its superconformal extension to describe baryons in complete analogy to the bosonic case, where the confining potential was determined by the conformal algebra of one-dimensional quantum field theory [4, 5]. In fact, it is straightforward to translate a quantum mechanical model into its supersymmetric (SUSY) counterpart by following Witten's construction [20]. Superconformal quantum mechanics, the supersymmetric extension [21, 22] of conformal quantum mechanics [4], then follows from the properties of the superconformal algebra.

We shall show in this article that the structure of supersymmetric quantum mechanics is encoded holographically in the AdS equations for arbitrary half-integer spin for any superpotential. Most important for the present discussion, we will show that superconformal quantum mechanics [22] has an elegant representation on the light front and its holographic embedding in AdS space. Remarkably, this procedure uniquely determines the form of the confinement potential for arbitrary half-integer spin. If one extends with Fubini and Rabinovici [22], the method of de Alfaro, Fubini and Furlan [4] to the superconformal algebra, the form of the potential in the new evolution equations is completely fixed. We will also discuss in this article how the different embeddings of mesons and baryons in AdS space [8] lead to distinct systematic features of meson and baryon spectroscopy. In particular, we will show that the integrability methods used to construct baryonic light-front equations [19] are the light-front extension of the usual formulation of supersymmetric Hamiltonian quantum mechanics [20, 23]. In fact, a possible indication of a supersymmetric connection was already mentioned in Ref. [19], but a proof was not actually given there [24].

This article is organized as follows: In Sec. II we review for convenience light-front con-

formal quantum mechanics and its holographic embedding in AdS space. In Sec. III we extend supersymmetric quantum mechanics to the light-front and describe its embedding in AdS space. We show in particular that properly taking the square root of the light-front Hamiltonian operator leads to a linear relativistic invariant Dirac equation. In Sec. IV superconformal quantum mechanics is extended to light-front holographic QCD. The application of the method to the complex patterns observed in baryon spectroscopy is discussed in Sec. V. Some final comments and conclusions are given in Sec. VI. In Appendix we discuss briefly the specific action of the supercharges.

II. LIGHT-FRONT CONFORMAL QUANTUM MECHANICS AND ITS HOLOGRAPHIC EMBEDDING

Following Ref. [4] one starts with the one-dimensional action

$$S[x] = \frac{1}{2}\int dt \left(\dot{x}^2 - \frac{g}{x^2}\right), \tag{1}$$

where $x(t)$ is a field operator, the constant g is dimensionless, and t has dimensions of length squared. The action (1) is invariant under conformal transformations in the variable t, thus in addition to the Hamiltonian H there are two more invariants of motion, namely the dilatation operator D and the operator of special conformal transformations K, corresponding to the generators of the conformal group $Conf(R^1)$ with commutation relations

$$[H, D] = i\,H, \quad [H, K] = 2\,i\,D, \quad [K, D] = -i\,K. \tag{2}$$

Specifically, if one introduces the new variable τ defined through the relation

$$d\tau = \frac{dt}{u + v\,t + w\,t^2}, \tag{3}$$

it then follows that the operator

$$G = u\,H + v\,D + w\,K, \tag{4}$$

generates the quantum mechanical unitary evolution in τ [4]

$$G|\psi(\tau)\rangle = i\frac{d}{d\tau}|\psi(\tau)\rangle. \tag{5}$$

One can show that G is a compact operator for $4uw - v^2 > 0$ [4]. In terms of the fields x and $p = \dot{x}$ the new Hamiltonian G is given by

$$G(x,p) = \frac{1}{2}u\left(p^2 + \frac{g}{x^2}\right) - \frac{1}{4}v\,(xp + px) + \frac{1}{2}wx^2, \tag{6}$$

at $t = 0$. In the Schrödinger representation $x(0)$ is represented by the position operator and $p \to -i\frac{d}{dx}$. The Hamiltonian is [4]

$$\begin{aligned} G =& \frac{1}{2}u\Big(-\frac{d^2}{dx^2} + \frac{g}{x^2}\Big) + \frac{i}{4}v\Big(x\frac{d}{dx} + \frac{d}{dx}x\Big) + \frac{1}{2}wx^2, \\ =& uH + vD + wK, \end{aligned} \tag{7}$$

with

$$H = \frac{1}{2}\Big(-\frac{d^2}{dx^2} + \frac{g}{x^2}\Big), \tag{8}$$

$$D = \frac{i}{4}\Big(x\frac{d}{dx} + \frac{d}{dx}x\Big), \tag{9}$$

$$K = \frac{1}{2}wx^2, \tag{10}$$

the superposition of the 'free' Hamiltonian H, the generator of dilatations D and the generator of special conformal transformations K in one dimension.

We now compare the dAFF Hamiltonian with the light-front Hamiltonian in the semiclassical approximation described in [3]. A physical hadron in four-dimensional Minkowski space has four-momentum P_μ and invariant hadronic mass squared $H_{LF} = P_\mu P^\mu = M^2$ [25, 26]. In the limit of zero quark masses the longitudinal modes decouple and the LF eigenvalue equation $H_{LF}|\phi\rangle = M^2|\phi\rangle$ is a light-front wave equation for ϕ [3]

$$\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + U\left(\zeta, J\right)\right)\phi(\zeta) = M^2\phi(\zeta), \tag{11}$$

a relativistic single-variable LF Schrödinger equation. The boost-invariant transverse-impact variable ζ [15] measures the separation of quark and gluons at equal light-front time [27], and it also allows one to separate the bound-state dynamics of the constituents from the kinematics of their LF internal angular momentum L in the transverse light-front plane [3]. The effective interaction U is instantaneous in LF time and acts on the lowest state of the LF Hamiltonian. To actually compute U in the semiclasscal approximation one must systematically express higher Fock components as functionals of the lower ones. This method has the advantage that the Fock space is not truncated and the symmetries of the Lagrangian are preserved [28].

Comparing the Hamiltonian G (7) with the light-front wave equation (11) and identifying the variable x with the light-front invariant variable ζ, we have to choose $u = 2,\ v = 0$ and relate the dimensionless constant g to the LF orbital angular momentum,

$$g = L^2 - \frac{1}{4}, \tag{12}$$

in order to reproduce the light-front kinematics. Furthermore $w = 2\lambda_M^2$ fixes the form of the confining light-front potential to that of a harmonic oscillator in the LF transverse plane [5],

$$U \sim \lambda_M^2\, \zeta^2. \tag{13}$$

In contrast to the baryonic case, which is discussed below, one can perform a constant level shift by adding an arbitrary constant, with dimension mass squared, to the confining term in the light front potential.

A. Light-Front Holographic Embedding

The next step is to determine the arbitrary constant term in the LF effective potential for arbitrary integer spin representations. Following Ref. [8] this constant term in the potential is determined by the embedding of the LF Hamiltonian equations in AdS space. To this end it is convenient to consider an effective action for a spin-J field in AdS_{d+1} space represented by a totally symmetric rank-J tensor field $\Phi_{N_1 \ldots N_J}$, where M, N are the indices of the $d+1$ higher dimensional AdS space with coordinates $x^M = (x^\mu, z)$. The coordinate z is the holographic variable and the x^μ are Minkowski flat space-time coordinates. In the presence of a dilaton background φ the effective action in [8] is

$$S_{\mathit{eff}} = \int d^d x\, dz\, \sqrt{g}\, e^{\varphi(z)}\, g^{N_1 N_1'} \cdots g^{N_J N_J'} \Big(g^{MM'} D_M \Phi^*_{N_1 \ldots N_J}\, D_{M'} \Phi_{N_1' \ldots N_J'} - \mu^2_{\mathit{eff}}(z)\, \Phi^*_{N_1 \ldots N_J}\, \Phi_{N_1' \ldots N_J'} \Big), \tag{14}$$

where $\sqrt{g} = (R/z)^{d+1}$ and D_M is the covariant derivative which includes the affine connection (R is the AdS radius). The dilaton $\varphi(z)$ effectively breaks the maximal symmetry of AdS, and the z dependence of the effective AdS mass μ_{eff} allows a clear separation of kinematical and dynamical effects. It is determined by the precise mapping of AdS to light-front physics [8].

In order to map to the LF Hamiltonian, one considers hadronic states with momentum P and a z-independent spinor $\epsilon_{\nu_1\cdots\nu_J}(P)$ with polarization components along the physical Minkowski coordinates. In holographic QCD such a state is described by the product of a free state with moment P, propagating in physical space-time, and z-dependent wave function Φ_J

$$\Phi_{\nu_1\cdots\nu_J}(x,z) = e^{iP\cdot x}\,\epsilon_{\nu_1\cdots\nu_J}(P)\,\Phi_J(z), \tag{15}$$

with invariant hadron mass $P_\mu P^\mu \equiv \eta^{\mu\nu}P_\mu P_\nu = M^2$. Variation of the action leads to the wave equation

$$\left[-\frac{z^{d-1-2J}}{e^{\varphi(z)}}\partial_z\left(\frac{e^{\varphi(z)}}{z^{d-1-2J}}\partial_z\right) + \frac{(\mu\,R)^2}{z^2}\right]\Phi_J = M^2\Phi_J, \tag{16}$$

where $(\mu\,R)^2 = (\mu_{\mathit{eff}}(z)R)^2 - Jz\,\varphi'(z) + J(d-J+1)$ is a constant determined by kinematical conditions in the light front [8]. Variation of the AdS action also gives the kinematical constraints required to eliminate the lower spin states $J-1,\ J-2,\cdots$ from the fully symmetric AdS tensor field $\Phi_{\nu_1\ldots\nu_J}$ [8]:

$$\eta^{\mu\nu}P_\mu\,\epsilon_{\nu\nu_2\cdots\nu_J} = 0, \qquad \eta^{\mu\nu}\,\epsilon_{\mu\nu\nu_3\cdots\nu_J} = 0. \tag{17}$$

We now perform the AdS mapping to light-front physics in physical space-time. To this end we factor out the scale $(1/z)^{J-(d-1)/2}$ and dilaton factors from the AdS field writing

$$\Phi_J(z) = (R/z)^{J-(d-1)/2}\,e^{-\varphi(z)/2}\,\phi_J(z). \tag{18}$$

Upon the substitution of the holographic variable z by the light-front invariant variable ζ and replacing Φ_J into the AdS eigenvalue equation (16), we obtain for $d = 4$ the QCD light-front frame-independent wave equation (11) with the effective LF potential [8, 29]

$$U(\zeta, J) = \frac{1}{2}\varphi''(\zeta) + \frac{1}{4}\varphi'(\zeta)^2 + \frac{2J-3}{2\zeta}\varphi'(\zeta). \tag{19}$$

The fifth dimensional AdS mass μ in (16) is related to the light-front internal orbital angular momentum L and the total angular momentum J of the hadron according to

$$(\mu R)^2 = -(2-J)^2 + L^2, \tag{20}$$

where the critical value $L = 0$ corresponds to the lowest possible stable solution [30].

From the holographic relation (19) it follows that the harmonic potential is holographically related to a unique dilaton profile, $\varphi = \lambda_M z^2$ provided that $\varphi(z) \to 0$ as $z \to 0$. From (19) we find the effective LF potential (13)

$$U(\zeta, J) = \lambda_M^2\zeta^2 + 2\lambda_M(J-1). \tag{21}$$

The term $\lambda_M^2\zeta^2$ is determined uniquely by the underlying conformal invariance of the one-dimensional effective theory, and the constant term $2\lambda_M(J-1)$ is determined by the spin representations in the embedding space.

For the effective potential (21) equation (11) has eigenfunctions

$$\phi_{n,L}(\zeta) = |\lambda_M|^{(1+L)/2}\sqrt{\frac{2n!}{(n+L)!}}\,\zeta^{1/2+L}e^{-|\lambda_M|\zeta^2/2}L_n^L(|\lambda_M|\zeta^2), \tag{22}$$

and eigenvalues

$$M^2_{n,J,L} = 4\lambda_M\left(n+\frac{J+L}{2}\right), \tag{23}$$

for $\lambda_M > 0$. The spectral predictions explain the essential features of the observed light meson spectrum [16], including a zero pion mass in the chiral limit, and Regge trajectories with the same slope in the quantum numbers n and L. The solution for $\lambda_M < 0$ leads to a meson spectrum in clear disagreement with observations. Since the effective interaction is determined from the conformal symmetry of the effective one-dimensional quantum field theory, which is not severely broken for small quark masses, the method can be successfully extended to describe, for example, the K and K^* excitation spectrum [16, 31].

III. LIGHT-FRONT SUPERSYMMETRIC QUANTUM MECHANICS AND ITS HOLOGRAPHIC EMBEDDING

Supersymmetric quantum mechanics is a simple realization of a graded Lie algebra which contains two fermionic generators, the supercharges, Q and $Q^\dagger$, and a bosonic generator, the Hamiltonian H, which are operators in a state space [20]. It closes under the graded algebra $sl(1/1)$:

$$\frac{1}{2}\{Q,Q^\dagger\} = H, \tag{24}$$

$$\{Q,Q\} = \{Q^\dagger,Q^\dagger\} = 0, \tag{25}$$

$$[Q,H] = [Q^\dagger,H] = 0. \tag{26}$$

It is useful to write down the SUSY formulation of quantum mechanics in terms of anti-commuting spinor operators χ. A minimal realization of the group generators is given in terms of the one-dimensional representation

$$Q = \chi\left(\frac{d}{dx}+W(x)\right), \tag{27}$$

and

$$Q^\dagger = \chi^\dagger \left(-\frac{d}{dx} + W(x) \right), \tag{28}$$

where $W(x)$ is called the superpotential in the context of supersymmetric theories. The spinor operators χ and $\chi^\dagger$ satisfy the anti-commutation relation

$$\{\chi, \chi^\dagger\} = 1. \tag{29}$$

Using a representation in terms of 2×2 Pauli-spin matrices we have

$$\chi = \frac{\sigma_1 + i\sigma_2}{2}, \quad \chi^\dagger = \frac{\sigma_1 - i\sigma_2}{2}, \tag{30}$$

and

$$[\chi, \chi^\dagger] = \sigma_3. \tag{31}$$

Thus the Hamiltonian is

$$H = \frac{1}{2}\{Q, Q^\dagger\} = \frac{1}{2}\left(-\frac{d^2}{dx^2} + W^2(x) + \sigma_3 W'(x) \right). \tag{32}$$

It can be written in matrix form:

$$H = \frac{1}{2}\begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\frac{d^2}{dx^2} + V_+(x) & 0 \\ 0 & -\frac{d^2}{dx^2} + V_-(x) \end{pmatrix}, \tag{33}$$

with effective potential

$$V_\pm(x) = W^2(x) \pm W'(x). \tag{34}$$

Since H commutes with Q and $Q^\dagger$ (26), it follows that the eigenvalues of H_+ and H_- are identical.

A. Supersymmetric Quantum Mechanics in the Light-Front

To give a relativistic formulation of supersymmetric quantum mechanics it is convenient to write the anti-commuting spinor operators in terms of a 4×4 matrix representation of the Clifford algebra, which acts on four-dimensional physical space where the LF spinors are defined. We use the Weyl representation where the chirality operator γ_5 is diagonal, and define the matrices α and β by

$$i\alpha = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{35}$$

where I a two-dimensional unit matrix. The matrices α and β are hermitian and anti-commuting

$$\alpha^\dagger = \alpha, \quad \alpha^2 = 1, \tag{36}$$

$$\beta^\dagger = \beta, \quad \beta^2 = 1, \tag{37}$$

$$\{\alpha, \beta\} = 0. \tag{38}$$

From the product of α and β we construct a third matrix γ_5, which corresponds to the usual chirality operator: $\gamma_5 = i\alpha\beta$

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{39}$$

The matrix γ_5 is also hermitian and anti commutes with α and β

$$\gamma_5^\dagger = \gamma_5, \quad \gamma_5^2 = 1, \tag{40}$$

$$\{\gamma_5, \alpha\} = \{\gamma_5, \beta\} = 0. \tag{41}$$

The SUSY LF Hamiltonian H_{LF} is given by the $sl(1/1)$ algebra

$$\{Q, Q^\dagger\} = H_{LF}, \tag{42}$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \tag{43}$$

but the supercharges Q and $Q^\dagger$ are now represented by 4×4 matrices. Furthermore, since the Hamiltonian operator $H_{LF} = P_\mu P^\mu = M^2$ is invariant, it implies that H_{LF} should depend on a frame independent variable. In impact space the relevant invariant variable is ζ, and thus the representation:

$$Q = \eta \left(\frac{d}{d\zeta} + W(\zeta) \right), \tag{44}$$

and

$$Q^\dagger = \eta^\dagger \left(-\frac{d}{d\zeta} + W(\zeta) \right), \tag{45}$$

where the spinor operators η and $\eta^\dagger$ satisfy the anti-commutation relation

$$\{\eta, \eta^\dagger\} = 1, \tag{46}$$

and are given by

$$\eta = \frac{\beta + i\alpha}{2}, \quad \eta^\dagger = \frac{\beta - i\alpha}{2}, \tag{47}$$

in the 4×4 matrix representation defined above. We also have

$$[\eta, \eta^\dagger] = \gamma_5. \tag{48}$$

The LF Hamiltonian is thus expressed as

$$H_{LF} = \{Q, Q^\dagger\} = -\frac{d^2}{d\zeta^2} + W^2(\zeta) + \gamma_5 W'(\zeta), \tag{49}$$

which is frame independent.

B. A Linear Dirac Equation from Supersymmetric Quantum Mechanics in the Light-Front

Since $\gamma_5^2 = 1$, the LF Hamiltonian (49) can be conveniently expressed as $H_{LF} = BB^\dagger$ where

$$B = \left(\frac{d}{d\zeta} + \gamma_5 W(\zeta)\right), \tag{50}$$

and

$$B^\dagger = \left(-\frac{d}{d\zeta} + \gamma_5 W(\zeta)\right). \tag{51}$$

The next step is to take the 'square root' of the Hamiltonian H_{LF}. For this purpose we write H_{LF} as a product of Hermitian operators which we label D_{LF}; thus $H_{LF} = D_{LF}^2$. Using the relation $i\alpha B = -iB^\dagger\alpha$ and equations (50) and (51), we have

$$D_{LF} = -i\alpha\left(-\frac{d}{d\zeta} + \gamma_5 W(\zeta)\right), \tag{52}$$

and thus the invariant Dirac equation [19]

$$(D_{LF} - M)\,\psi(\zeta) = 0, \tag{53}$$

where $\psi(\zeta)$ is a LF Dirac spinor. Premultiplying the linear Dirac wave equation (53) by the operator $D_{LF} + M$ and using the properties of the Dirac matrices given above, we recover the LF eigenvalue equation

$$H_{LF}\,\psi = D_{LF}^2\psi = M^2\psi, \tag{54}$$

where H_{LF} is given by (49). We thus reproduce the results obtained in Ref. [19] using an operator construction of the light-front Hamiltonian and the Dirac equation, but starting from light-front supersymmetric quantum mechanics [32].

It is convenient to separate the kinematic and dynamic contributions to the superpotential. We write

$$W(\zeta) = \frac{\nu + 1/2}{\zeta} + u(\zeta), \tag{55}$$

where ν is a dimensionless parameter representing the LF orbital angular momentum, and the dynamical effects are encoded in the function $u(\zeta)$. From (52) we can express the LF-invariant Dirac equation (53) for the superpotential (55) as a system of coupled linear differential equations

$$\begin{aligned} -\frac{d}{d\zeta}\psi_- - \frac{\nu + \frac{1}{2}}{\zeta}\psi_- - u(\zeta)\psi_- &= M\psi_+, \\ \frac{d}{d\zeta}\psi_+ - \frac{\nu + \frac{1}{2}}{\zeta}\psi_+ - u(\zeta)\psi_+ &= M\psi_-, \end{aligned} \tag{56}$$

where the chiral spinors are defined by $\psi_\pm = \frac{1}{2}(1 \pm \gamma_5)\,\psi$.

C. Holographic Embedding

We can now determine the LF superpotential $u(\zeta)$ in (55) for arbitrary half-integer spin by embedding the LF results in AdS space. We start with an effective action for Rarita-Schwinger (RS) spinors in AdS space $[\Psi_{N_1 \cdots N_T}]_\alpha$, which transform as symmetric tensors of rank T with indices $N_1 \ldots N_T$, and as Dirac spinors with index α [33]. In presence of an effective interaction $V(z)$ the effective action is given by [8]

$$\begin{aligned} S_{\mathit{eff}} = \frac{1}{2}\int d^d x\, dz\, \sqrt{g}\, g^{N_1 N_1'} \cdots g^{N_T N_T'} \\ \left[\bar{\Psi}_{N_1 \cdots N_T}\Big(i\,\Gamma^A\, e_A^M\, D_M - \mu - V(z)\Big)\Psi_{N_1' \cdots N_T'} + h.c.\right], \end{aligned} \tag{57}$$

where $\sqrt{g} = \left(\frac{R}{z}\right)^{d+1}$ and e_A^M is the inverse vielbein, $e_A^M = \left(\frac{z}{R}\right)\delta_A^M$. The covariant derivative D_M includes the affine connection and the spin connection. The tangent-space Dirac matrices obey the usual anti-commutation relation $\left\{\Gamma^A, \Gamma^B\right\} = 2\eta^{AB}$. We have not included a dilaton factor $e^{\varphi(z)}$ in (57) since it can be absorbed by redefining the RS spinor according to $\Psi_T \to e^{\varphi(z)/2}\Psi_T$ [8, 17]. This is a consequence of the linear covariant derivatives in the fermion action, which also prevents a mixing between dynamical and kinematical effects, and thus, in contrast to the effective action for integer spin fields (14), the AdS mass μ in Eq. (57) is constant. Since a dilaton factor has no dynamical consequences, one must

introduce an effective confining interaction $V(z)$ in the fermion action to break conformal symmetry and generate a baryon spectrum [18, 19].

It is shown below that the potential $V(z)$, which has been introduced hitherto *ad hoc*, is precisely related to the superpotential u (55). Furthermore, in Sec. IV it is shown that, in analogy with the boson case [5], the form of u is determined in the framework of the superconformal algebra.

A physical baryon has plane-wave solutions with four-momentum P_μ, invariant mass $P_\mu P^\mu = M^2$, and polarization indices along the physical coordinates. Factoring out the four-dimensional plane-wave and spinor dependence, as well as the scale factor $(1/z)^{T-d/2}$, we have

$$\Psi^{\pm}_{\nu_1\cdots\nu_T}(z) = e^{iP\cdot x}\, u^{\pm}_{\nu_1\cdots\nu_T}(P) \left(\frac{R}{z}\right)^{T-d/2} \Psi^{\pm}_T(z), \tag{58}$$

where $T = J - \frac{1}{2}$ and the fully symmetric RS chiral spinor $u^{\pm}_{\nu_1\ldots\nu_T} = \frac{1}{2}(1 \pm \gamma_5) u_{\nu_1\ldots\nu_T}$ satisfies the four-dimensional chirality equations

$$\gamma\cdot P\, u^{\pm}_{\nu_1\ldots\nu_T}(P) = M u^{\mp}_{\nu_1\ldots\nu_T}(P), \qquad \gamma_5 u^{\pm}_{\nu_1\ldots\nu_T}(P) = \pm\, u^{\pm}_{\nu_1\ldots\nu_T}(P). \tag{59}$$

Variation of the AdS action (57) leads for $d = 4$ to the Dirac equation

$$\begin{aligned} -\frac{d}{dz}\Psi^-_T - \frac{\mu R}{z}\Psi^-_T - \frac{R}{z}\, V(z)\Psi^-_T &= M\Psi^+_T, \\ \frac{d}{dz}\Psi^+_T - \frac{\mu R}{z}\Psi^+_T - \frac{R}{z}\, V(z)\Psi^+_T &= M\Psi^-_T, \end{aligned} \tag{60}$$

and the Rarita-Schwinger condition [33] in physical space-time [8]

$$\gamma^\nu \Psi_{\nu\nu_2\ldots\nu_T} = 0. \tag{61}$$

By identifying the holographic variable z with the invariant LF variable ζ and the AdS LF spinors by the mapping $\Psi^{\pm}_T(z) \to \psi^{\pm}(\zeta)$, we can compare (60) with (56). Provided that the AdS mass μ is related to the parameter ν by

$$\mu R = \nu + \frac{1}{2}, \tag{62}$$

the specific LF mapping gives a relation between the effective interaction $V(z)$ in the AdS action (57) and the function $u(\zeta)$ in the LF superpotential (55)

$$u(\zeta) = \frac{R}{\zeta}\, V(\zeta). \tag{63}$$

In fact they are identical (modulo a kinematic factor), and this relation thus leads to a J-independent potential. This is a remarkable result, since independently of the specific form of the potential, the value of the baryon masses along a given Regge trajectory depends only on the LF orbital angular momentum L [34]. Thus, in contrast with the vector mesons (21), there is no spin-orbit coupling, in agreement with the observed near-degeneracy in the baryon spectrum [36, 37].

IV. LIGHT-FRONT SUPERCONFORMAL QUANTUM MECHANICS

In order to fix the superpotential u (55) we follow Fubini and Rabinovici in Ref. [22], and consider a one-dimensional quantum field theory invariant under conformal and supersymmetric transformations. Imposing conformal symmetry leads to a unique choice of the superpotential W (27), namely

$$W(x) = \frac{f}{x}, \tag{64}$$

in order for f to be a dimensionless constant. In this case the graded-Lie algebra has, in addition to the Hamiltonian H and the supercharges Q and $Q^\dagger$, an additional generator S which is the square root of the generator of conformal transformations K. The enlarged algebraic structure is the superconformal algebra of Haag, Lopuszanski and Sohnius [22, 38, 39]. Using the one-dimensional quantum-mechanical representation of the operators

$$Q = \chi \left(\frac{d}{dx} + \frac{f}{x} \right), \tag{65}$$

$$Q^\dagger = \chi^\dagger \left(-\frac{d}{dx} + \frac{f}{x} \right), \tag{66}$$

$$S = \chi\, x, \tag{67}$$

$$S^\dagger = \chi^\dagger x, \tag{68}$$

it is simple to verify that the algebraic structure of the enlarged algebra is fulfilled. We find

$$\frac{1}{2}\{Q, Q^\dagger\} = H, \qquad \frac{1}{2}\{S, S^\dagger\} = K, \tag{69}$$

$$\frac{1}{2}\{Q, S^\dagger\} = \frac{f}{2} + \frac{\sigma_3}{4} - D, \tag{70}$$

$$\frac{1}{2}\{Q^\dagger, S\} = \frac{f}{2} + \frac{\sigma_3}{4} + D, \tag{71}$$

where the operators

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + \frac{f^2 - \sigma_3 f}{x^2}\right), \tag{72}$$

$$K = \frac{1}{2}x^2, \tag{73}$$

$$D = \frac{i}{4}\left(\frac{d}{dx}x + x\frac{d}{dx}\right). \tag{74}$$

satisfy the conformal algebra (2). The anticommutation of all other generators vanish: $\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = \{Q, S\} = \cdots = 0$.

In analogy with the dAFF procedure [4], we now define, following Fubini and Rabinovici [22], a new supercharge R as a linear combination of the generators Q and S

$$R = \sqrt{u}\, Q + \sqrt{w}\, S, \tag{75}$$

and compute a new Hamiltonian G

$$G = \frac{1}{2}\{R, R^\dagger\}. \tag{76}$$

We find

$$G = uH + wK + \frac{1}{2}\sqrt{uw}\,(2f + \sigma_3), \tag{77}$$

which is a compact operator for $uw > 0$.

The quantum mechanical evolution operator G (77) obtained by this procedure is analogous to the Hamiltonian (6) obtained by the procedure of de Alfaro, Fubini and Furlan [4]. Remarkably, in the superconformal case there appears beside the confining term $w\,K$ also a constant term $\frac{1}{2}\sqrt{uw}(2f \pm 1)$ in G, which, as we will describe below, plays a key role in explaining the correct phenomenology.

A. Superconformal Quantum Mechanics in the Light-Front

The light-front extension of the superconformal results follows from the LF superpotential

$$W(\zeta) = \frac{\nu + 1/2}{\zeta}, \tag{78}$$

which corresponds to a kinematic term in the LF Hamiltonian. We now extend the new Hamiltonian G (77) to a relativistic LF Hamiltonian by the method described in Sec. III A.

This amounts to replace the Pauli matrix σ_3 in (77) with γ_5 in (48). We obtain:

$$\begin{aligned} H_{LF} &= \{R, R^\dagger\} \\ &= -\frac{d^2}{d\zeta^2} + \frac{\left(\nu+\frac{1}{2}\right)^2}{\zeta^2} - \frac{\nu+\frac{1}{2}}{\zeta^2}\gamma_5 + \lambda_B^2\zeta^2 + \lambda_B(2\nu+1) + \lambda_B\gamma_5, \end{aligned} \tag{79}$$

where the arbitrary coefficients u and w in (77) are fixed to $u = 1$ and $w = \lambda_B^2$. Thus the supercharge R is the superposition

$$R = Q + \lambda_B S. \tag{80}$$

In 2×2 block-matrix form the light-front Hamiltonian (79) can be expressed as

$$H_{LF} = \begin{pmatrix} -\frac{d^2}{d\zeta^2} - \frac{1-4\nu^2}{4\zeta^2} + \lambda_B^2\zeta^2 + 2\lambda_B(\nu+1) & 0 \\ 0 & -\frac{d^2}{d\zeta^2} - \frac{1-4(\nu+1)^2}{4\zeta^2} + \lambda_B^2\zeta^2 + 2\lambda_B\nu \end{pmatrix}. \tag{81}$$

The light-front eigenvalue equation $H_{LF}|\psi\rangle = M^2|\psi\rangle$ has eigenfunctions

$$\psi_+(\zeta) \sim \zeta^{\frac{1}{2}+\nu} e^{-\lambda_B\zeta^2/2} L_n^\nu(\lambda_B\zeta^2), \tag{82}$$

$$\psi_-(\zeta) \sim \zeta^{\frac{3}{2}+\nu} e^{-\lambda_B\zeta^2/2} L_n^{\nu+1}(\lambda_B\zeta^2), \tag{83}$$

and eigenvalues

$$M^2 = 4\lambda_B(n+\nu+1). \tag{84}$$

As a consequence of parity invariance, the eigenvalues for the chirality plus and minus eigenfunctions are identical. One can also show that the probabilities for both components ψ_+ and ψ_- are the same (See appendix)

$$\int d\zeta\, \psi_+^2(\zeta) = \int d\zeta\, \psi_-^2(\zeta). \tag{85}$$

For $\lambda_B < 0$ no solution is possible.

V. SYSTEMATICS OF THE BARYON SPECTRUM

To determine how well the superconformal light-front holographic model encompasses the systematics of the baryon spectrum, we list in Table I the confirmed (3-star and 4-star) baryon states from the Particle Data Group [40]. The internal spin, light-front internal

orbital angular momentum and radial quantum number assignment of the N and Δ excitation spectrum is found from the total angular momentum-parity PDG assignment using the conventional $SU(6) \supset SU(3)_{flavor} \times SU(2)_{spin}$ multiplet structure [41], but other model choices are also possible [42]. Further details can be found in [16].

The lowest possible stable state, the nucleon $N\frac{1}{2}^+(940)$, corresponds to $n = 0$ and $\nu = 0$. This fixes the scale $\sqrt{\lambda}_B = M_P/2$. The resulting predictions for the spectroscopy of the positive-parity spin-$\frac{1}{2}$ light nucleons are shown in Fig. 1 (a) for the parent Regge trajectory for $n = 0$ and $\nu = 0, 2, 4, \cdots, L$, where L is the relative LF angular momentum between the active quark and the spectator cluster. Thus the dimensionless constant f in the superpotential (64) is $f = L + \frac{1}{2}$ for the plus parity nucleon trayectory. The predictions for the daughter trajectories for $n = 1$, $n = 2, \cdots$ are also shown in this figure. Only confirmed PDG [40] states are shown. The Roper state $N\frac{1}{2}^+(1440)$ and the $N\frac{1}{2}^+(1710)$ are well described in this model as the first and second radial excited states of the nucleon. The newly identified state, the $N\frac{3}{2}^+(1900)$ [40] is depicted here as the first radial excitation of the $N\frac{3}{2}^+(1720)$. The model is successful in explaining the J-degeneracy for states with the same orbital angular momentum observed in the light baryon spectrum, such as the $L = 2$ plus parity doublet $N\frac{3}{2}^+(1720) - N\frac{5}{2}^+(1680)$, which corresponds to and $J = \frac{3}{2}$ and $\frac{5}{2}$ respectively (See Fig. 1 (a)).

In Fig. 1 (b) we compare the positive parity spin-$\frac{1}{2}$ parent nucleon trajectory with the negative parity spin-$\frac{3}{2}$ nucleon trajectory. As it is shown in this figure, the gap scale 4λ determines not only the slope of the nucleon trajectories, but also the spectrum gap between the plus-parity spin-$\frac{1}{2}$ and the minus-parity spin-$\frac{3}{2}$ nucleon families, as indicated by arrows in this figure. This means the respective assignment $\nu = L$ and $\nu = L + 1$ for the lower and upper trajectories in Fig. 1 (b), or $f = L + \frac{1}{2}$ and $f = L + \frac{3}{2}$ respectively. The degeneracy of states with the same orbital quantum number L is also well described, as for example the degeneracy of the $L = 1$ minus-parity triplet $N\frac{1}{2}^-(1650)$, $N\frac{3}{2}^-(1700)$, and $N\frac{5}{2}^-(1675)$, which corresponds respectively to $J = \frac{1}{2}$, $\frac{3}{2}$ and $\frac{5}{2}$ (See: Fig. 1 (b)).

Baryons with negative parity and internal spin $S = \frac{1}{2}$, such as the $N\frac{1}{2}^-(1535)$, as well as baryon states with positive parity and internal spin $S = \frac{3}{2}$, such as the $\Delta\frac{3}{2}^+(1232)$ are well described by the assignment $\nu = L + \frac{1}{2}$, or $f = L + 1$. This means, for example, that the positive and negative-parity Δ states are in the same trajectory consistent with experimental observations, as depicted in Fig. 1 (d). The newly found state, the $N\frac{3}{2}^-(1875)$ [40], depicted

TABLE I. Classification of confirmed baryons listed by the PDG [40]. The labels L, S and n refer to the internal orbital angular momentum, internal spin and radial quantum number respectively. The even-parity baryons correspond to the **56** multiplet of $SU(6)$ and the odd-parity to the **70**.

$SU(6)$	S	L	n	Baryon State
56	$\frac{1}{2}$	0	0	$N\frac{1}{2}^+(940)$
	$\frac{3}{2}$	0	0	$\Delta\frac{3}{2}^+(1232)$
56	$\frac{1}{2}$	0	1	$N\frac{1}{2}^+(1440)$
	$\frac{3}{2}$	0	1	$\Delta\frac{3}{2}^+(1600)$
70	$\frac{1}{2}$	1	0	$N\frac{1}{2}^-(1535)$ $N\frac{3}{2}^-(1520)$
	$\frac{3}{2}$	1	0	$N\frac{1}{2}^-(1650)$ $N\frac{3}{2}^-(1700)$ $N\frac{5}{2}^-(1675)$
	$\frac{1}{2}$	1	0	$\Delta\frac{1}{2}^-(1620)$ $\Delta\frac{3}{2}^-(1700)$
56	$\frac{1}{2}$	0	2	$N\frac{1}{2}^+(1710)$
	$\frac{1}{2}$	2	0	$N\frac{3}{2}^+(1720)$ $N\frac{5}{2}^+(1680)$
	$\frac{3}{2}$	2	0	$\Delta\frac{1}{2}^+(1910)$ $\Delta\frac{3}{2}^+(1920)$ $\Delta\frac{5}{2}^+(1905)$ $\Delta\frac{7}{2}^+(1950)$
70	$\frac{3}{2}$	1	1	$N\frac{1}{2}^-$ $N\frac{3}{2}^-(1875)$ $N\frac{5}{2}^-$
	$\frac{3}{2}$	1	1	$\Delta\frac{5}{2}^-(1930)$
56	$\frac{1}{2}$	2	1	$N\frac{3}{2}^+(1900)$ $N\frac{5}{2}^+$
70	$\frac{1}{2}$	3	0	$N\frac{5}{2}^-$ $N\frac{7}{2}^-$
	$\frac{3}{2}$	3	0	$N\frac{3}{2}^-$ $N\frac{5}{2}^-$ $N\frac{7}{2}^-(2190)$ $N\frac{9}{2}^-(2250)$
	$\frac{1}{2}$	3	0	$\Delta\frac{5}{2}^-$ $\Delta\frac{7}{2}^-$
56	$\frac{1}{2}$	4	0	$N\frac{7}{2}^+$ $N\frac{9}{2}^+(2220)$
	$\frac{3}{2}$	4	0	$\Delta\frac{5}{2}^+$ $\Delta\frac{7}{2}^+$ $\Delta\frac{9}{2}^+$ $\Delta\frac{11}{2}^+(2420)$
70	$\frac{1}{2}$	5	0	$N\frac{9}{2}^-$ $N\frac{11}{2}^-$
	$\frac{3}{2}$	5	0	$N\frac{7}{2}^-$ $N\frac{9}{2}^-$ $N\frac{11}{2}^-(2600)$ $N\frac{13}{2}^-$

in Fig. 1 (c) is well accounted as the first radial excitation of the $N\frac{3}{2}^-(1520)$. The degeneracy of the $L = 1$ minus-parity doublet $N\frac{1}{2}^-(1535) - N\frac{3}{2}^-(1520)$ for $J = \frac{1}{2}$ and $\frac{3}{2}$ is also well described. Likewise, the $\Delta(1600)$ corresponds to the first radial excitation of the $\Delta(1232)$

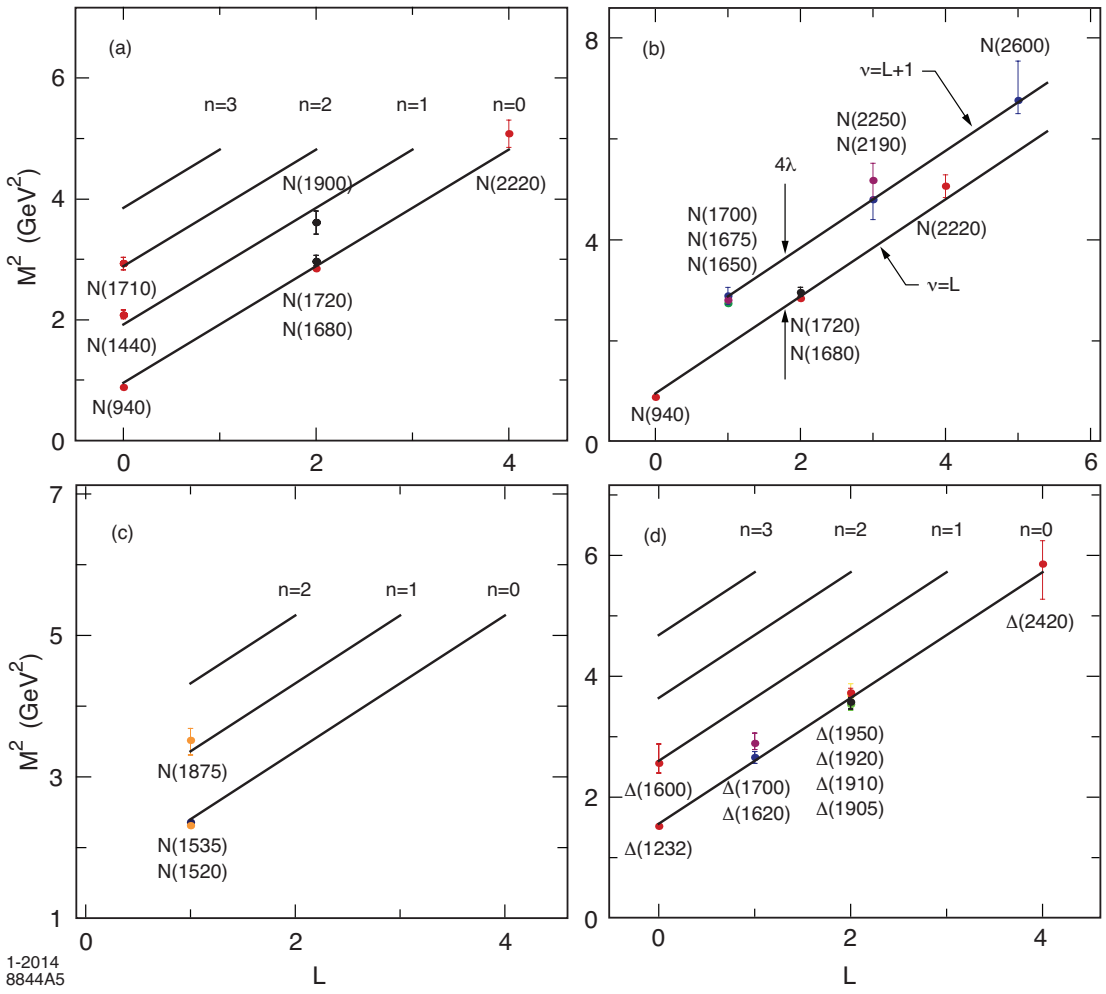

FIG. 1. Orbital and radial baryon excitation spectrum. Positive-parity spin-$\frac{1}{2}$ nucleons (a) and spectrum gap between the negative-parity spin-$\frac{3}{2}$ and the positive-parity spin-$\frac{1}{2}$ nucleons families (b). Minus parity spin-$\frac{1}{2}$ N (c) and plus and minus parity spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ Δ families (d). We have used in this figure the value $\sqrt{\lambda_B} = 0.49$ GeV for nucleons and 0.51 GeV for the Deltas.

as shown in Fig. 1 (d). The model explains the degeneracy of the $L = 2$ plus-parity quartet $\Delta\frac{1}{2}^+(1910)$, $\Delta\frac{3}{2}^+(1920)$, $\Delta\frac{5}{2}^+(1905)$, and $\Delta\frac{7}{2}^+(1950)$ which corresponds to $J = \frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ and $\frac{7}{2}$ respectively (See: Fig. 1 (d)). Our results for the Δ states agree with those of Ref. [43]. "Chiral partners" such as the $N\frac{1}{2}^+(940)$ and $N\frac{1}{2}^-(1535)$ nucleons with the same total angular momentum $J = \frac{1}{2}$, but with different orbital angular momentum and parity are non-degenerate from the onset. To recapitulate, the parameter f, has the internal spin S and parity P assignment given in Table II, which is equivalent to the assignment given in [44].

TABLE II. Orbital quantum number assignment for the superpotential parameter f for baryon trajectories according to parity P and internal spin S.

	$S=\frac{1}{2}$	$S=\frac{3}{2}$
P = +	$f=L+\frac{1}{2}$	$f=L+1$
P = −	$f=L+1$	$f=L+\frac{3}{2}$

This particular assignment successfully describes the full light baryon orbital and radial excitation spectrum, and in particular the gap between trajectories with different parity and internal spin [44]. The assignment $\nu = L$ for the lowest trajectory, the proton trajectory, is straightforward and follows from the stability of the ground state, the proton, and the mapping of AdS to light-front physics. The assignment for other spin and parity baryons states, given in Table II, is motivated by the observed spectrum. It is hoped that further analysis of the different quark configurations and symmetries of the baryon wave function [36, 45, 46] will indeed explain the assignment of the dimensionless parameter f.

If we follow the non-$SU(6)$ quantum number assignment for the $\Delta\frac{5}{2}^{-}(1930)$ given in Ref. [36], namely $S = 3/2$, $L = 1$, $n = 1$, we find with the present model the value $M_{\Delta(1930)} = 4\sqrt{\lambda}_B = 2M_P$, also consistent with the experimental result 1.96 GeV [40]. An important feature of light-front holography and supersymmetric LF quantum mechanics is the fact that it predicts a similar multiplicity of states for mesons and baryons, consistent with experimental observations [36]. This property is consistent with the LF cluster decomposition of the holographic variable ζ, which describes a system of partons as an active quark plus a cluster of $n-1$ spectators [37]. From this perspective, a baryon with 3 quarks looks in light-front holography as a quark–diquark system.

Another interesting consequence of the supersymmetric relation between the plus and minus chirality states, is the equal equal probability expressed by (85). This remarkable equality means that in the light-front holographic approach described here the proton's spin $J^z = L^z + S^z$ is carried by the quark orbital angular momentum: $\langle J^z\rangle = \langle L^z_q\rangle = \pm 1/2$ since $\langle S^z_q\rangle = 0$.

VI. CONCLUSIONS AND OUTLOOK

In this article we have shown how superconformal quantum mechanics [21, 22] can be extended to the light-front and how it can be precisely mapped to holographic QCD. We have also examined the higher half-integer spin representations of the model by embedding the resulting Dirac invariant light-front wave equation in AdS space. This procedure introduces a scale in the Hamiltonian equations and completely fixes the light-front potential in the Dirac equation introduced in Refs. [18, 19]. In this approach the main features of the observed light-baryon spectrum are described.

The construction procedure is similar to that of bosons [4, 5]. Both are based on the breaking of conformal invariance within the algebraic structure, by a redefinition of the quantum mechanical evolution in terms of a superposition of the operators of the conformal or superconformal algebras. Since the generators have different dimensions this amounts to the introduction of a scale in the Hamiltonian while maintaining a conformal action. Compared with the holographic construction for baryons, this unified approach is more satisfactory. In contrast to the meson case, the dilaton in the fermion action has no effect on the baryon spectrum. Consequently, a Yukawa potential must be introduced by hand to break conformal invariance. Here, the same underlying principle is used to introduce a mass scale and generate the masses for mesons and baryons from a spectrum generating algebra. For baryons the quantum mechanical evolution is determined from a supercharge which is a superposition of elements of the superconformal algebra [22]. In fact, the introduction of the generator S (the square root of the generator of conformal transformations K) is the key step for extending the dAFF [4, 5] procedure for obtaining a confining potential in the LF Dirac equation for baryons.

Mapping the results to light-front bound-state equations leads to a linear potential in the light-front Dirac equation and to a harmonic potential with additional constants in the quadratic Hamiltonian for fermions. In contrast to the case of mesons, there is no possibility to shift the energy levels by adding a constant to the linear potential in the light-front Dirac equation. Therefore superconformal quantum mechanics, together with the introduction of the scale according to Fubini and Rabinovici [22], fixes completely the fermionic Hamiltonian. The equations of motion obtained by following this procedure are equivalent to the holographic light-front equations obtained from the fermion Lagrangian in AdS_5, with a

Yukawa coupling providing the effective potential. In the bosonic case light-front holographic QCD yields a J-dependent constant from the holographic embedding – in addition to the confining harmonic potential obtained from conformal quantum mechanics [4] – which leads to a J-dependent level shift [5]. Such a level shift is not possible for fermions, and therefore there is a spin-J degeneracy for states at fixed L and n, an important characteristic which is actually observed in experiment. The model is also consistent with similar Regge meson and baryon spectra and similar multiplicity of states for mesons and baryons. In effect, the light-front Dirac equation for baryons described here is effectively a quark-diquark model. However, a quark-diquark construction is not imposed, but it is a natural consequence of the light-front cluster decomposition which follows from the LF embedding in AdS space [37]. In this approach the quark and diquark are both massless.

In this paper we have described a mechanism for the emergence of a confining light-front Hamiltonian for hadrons. A mass scale $\sqrt{\lambda}$ and confining potentials appear in the light-front Schrödinger and Dirac equations, consistent with the conformal invariance of the action, by applying the group-theoretical methods of Refs. [4, 22]. We have given a relation between the dimensionless quantities L, f or g, and μR occurring in the light-front Hamiltonian, the quantum mechanical evolution operator in the algebraic approach, and the wave equations in AdS_5, respectively (See Eqs. (12), (20), (62) and Table II). We expect that further analysis of the different quark configurations and symmetries of the hadron wavefunctions will shed further light on the detailed relations between these dimensionless parameters.

Even if a supersymmetric connection inspired by the universality of the Regge trajectories for bosons and baryons was our starting point, in the context of this article the supersymmetric construction of baryonic states refers to the "supersymmetry" between positive and negative chirality of light-front spinors. In this case supersymmetry is not broken since there is a perfect pairing for each baryonic state including the ground state, consistent with parity invariance. This does not exclude the possible supersymmetric connections between mesons and baryons which would be manifest as a consequence of confinement dynamics. In fact, although the form of the potential is fixed in both cases by the dAFF procedure and its extension to the superconformal algebra, the numerical values of the confining scales are *a priori* not related. Nevertheless the values of λ for the coefficient of the confining potentials come out to be similar in both cases with similar spacing between the orbital and radial hadronic excitations. This suggests a supersymmetric relation between the underlying

dynamics of the observed bosonic and fermionic hadrons. In this case, supersymmetry is broken since the ground state, the pion, is massless in the chiral limit and is not paired. We shall treat this subject elsewhere.

ACKNOWLEDGMENTS

We thank Vittorio de Alfaro, Cedric Lorcé and Michael Peskin for helpful comments. This work is supported by the Department of Energy contract DE–AC02–76SF00515.

Appendix: Supercharges and Ladder Operators

The supercharge operator R (80) in the light-front quantum mechanical representation discussed in Sect. IV can be expressed as

$$R = Q + \lambda S = \eta\, b, \tag{A.1}$$

$$R^\dagger = Q^\dagger + \lambda S^\dagger = \eta^\dagger b^\dagger, \tag{A.2}$$

where the spinor operators η and $\eta^\dagger$ in a 4×4 matrix representation are

$$\eta = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \qquad \eta^\dagger = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \tag{A.3}$$

with I a two-dimensional unit matrix, and the operators b and $b^\dagger$ are given by

$$b_\nu = \frac{d}{d\zeta} + \frac{\nu + \frac{1}{2}}{\zeta} + \lambda\zeta, \tag{A.4}$$

$$b_\nu^\dagger = -\frac{d}{d\zeta} + \frac{\nu + \frac{1}{2}}{\zeta} + \lambda\zeta. \tag{A.5}$$

The LF Hamiltonian H_{LF} (81) is conveniently factorized in terms of the linear operators b

$$H_{LF}^\nu = \{R, R^\dagger\} = \begin{pmatrix} b_\nu b_\nu^\dagger & 0 \\ 0 & b_\nu^\dagger b_\nu \end{pmatrix}, \tag{A.6}$$

and is thus integrable [47, 48].

Consider the eigenvalue equation for $b_\nu b_\nu^\dagger$

$$\left(-\frac{d^2}{dx^2} - \frac{1 - 4\nu^2}{4x^2} + \kappa^2\zeta^2 + 2\kappa(\nu + 1)\right)\phi_\nu(x) = \phi_\nu(x), \tag{A.7}$$

where $x = \zeta M$ and $\kappa = \lambda/M$. Equation (A.7) is equivalent to $b_\nu b_\nu^\dagger|\nu\rangle = |\nu\rangle$. It is also simple to verify that $b_\nu^\dagger|\nu\rangle \sim |\nu+1\rangle$ or

$$\left(-\frac{d}{d\zeta} + \frac{\nu+\frac{1}{2}}{\zeta} + \lambda\,\zeta\right)\phi_\nu(\zeta) \sim \phi_{\nu+1}(\zeta). \tag{A.8}$$

Likewise, one can show that $b_\nu|\nu\rangle \sim |\nu-1\rangle$.

We now construct a new supercharge T and its adjoint $T^\dagger$ as the linear superposition [22]

$$T = Q^\dagger - \lambda S^\dagger = \eta^\dagger a, \tag{A.9}$$

$$T^\dagger = Q - \lambda S = \eta\, a^\dagger, \tag{A.10}$$

where

$$a_\nu = -\frac{d}{d\zeta} + \frac{\nu+\frac{1}{2}}{\zeta} - \lambda\zeta, \tag{A.11}$$

$$a_\nu^\dagger = \frac{d}{d\zeta} + \frac{\nu+\frac{1}{2}}{\zeta} - \lambda\zeta. \tag{A.12}$$

One can show that the operator (A.11) lowers the radial quantum number n by one unit and raises ν by one unit

$$a|n,\nu\rangle \sim |n-1,\nu+1\rangle. \tag{A.13}$$

For a given ν the lowest possible state corresponds to $n = 0$. Consequently the state $|n=0,\nu\rangle$ is annihilated by the action of the operator a, $a|n=0,\nu\rangle = 0$, or equivalently

$$\left(\frac{d}{d\zeta} - \frac{\nu+\frac{1}{2}}{\zeta} + \lambda\zeta\right)\phi_\nu^{n=0}(\zeta) = 0, \tag{A.14}$$

with solution

$$\phi_\nu^{n=0}(\zeta) = C_\nu \zeta^{1/2+\nu} e^{-\lambda\zeta^2/2}, \tag{A.15}$$

where C_ν is a constant. Writing

$$\phi_\nu(\zeta) = C_\nu \zeta^{1/2+\nu} e^{-\lambda\zeta^2/2} G_\nu(\zeta), \tag{A.16}$$

and substituting in (A.8) we get

$$2x\, G_\nu(x) - G'_\nu(x) \sim x\, G_{\nu+1}(x), \tag{A.17}$$

with $x = \sqrt{\lambda}\,\zeta$, a relation which defines the confluent hypergeometric function $U(n,\nu+1,x)$ in terms of $U(n,\nu,x)$ [49]

$$U(n,\nu+1,x) = U(n,\nu,x) - U'(n,\nu,x), \tag{A.18}$$

or equivalently

$$2x\,U(n,\nu+1,x^2) = 2x\,U(n,\nu,x^2) - \frac{dU'(n,\nu,x^2)}{dx}. \tag{A.19}$$

Thus the normalizable solution to the eigenvalue equation $bb^\dagger\phi(\zeta) = M^2\phi(\zeta)$:

$$\phi_{n,\nu}(\zeta) = C_\nu \zeta^{1/2+\nu} e^{-\lambda\zeta^2/2} L_n^\nu(\lambda\zeta^2). \tag{A.20}$$

The solution also follows from the iterative application of the ladder operators following the procedure described in [50]. We find

$$\phi(\zeta)_{n,\nu} \sim \zeta^{1/2-\nu} e^{\lambda\zeta^2/2} \left(\frac{1}{\zeta}\frac{d}{d\zeta}\right)^n \zeta^{2(n+\nu)} e^{-\lambda\zeta^2}, \tag{A.21}$$

with eigenvalues

$$M^2 = 4\lambda(n+\nu+1). \tag{A.22}$$

Since we know the general solution for the upper component of the spinor wavefunction ϕ_ν, it is straightforward to compute the lowest component $b^\dagger\phi_\nu$, with identical mass, by applying the supercharge operators. We find

$$T\begin{pmatrix}\phi_{n,\nu}\\ \\ 0\end{pmatrix} = 0, \tag{A.23}$$

$$R^\dagger\begin{pmatrix}\phi_{n,\nu}\\ \\ 0\end{pmatrix} = C_{n,\nu}\begin{pmatrix}0\\ \\ \phi_{n,\nu+1}\end{pmatrix}, \tag{A.24}$$

with

$$C_{n,\nu} = \sqrt{\frac{\lambda}{n+\nu+1}}. \tag{A.25}$$

Thus the solution

$$\psi(\zeta) = \psi_+ u_+ + \psi_- u_- \tag{A.26}$$

$$= C z^{\frac{1}{2}+\nu} e^{-\lambda\zeta^2/2}\left[L_n^\nu\left(\lambda\zeta^2\right)u_+ + \frac{\sqrt{\lambda}\zeta}{\sqrt{n+\nu+1}} L_n^{\nu+1}\left(\lambda\zeta^2\right)u_-\right], \tag{A.27}$$

with normalization

$$\int d\zeta\,\psi_+^2(\zeta) = \int d\zeta\,\psi_-^2(\zeta). \tag{A.28}$$

Identical results follow by solving directly the Dirac equation (56) for the conformal superpotential (55) with $u = \lambda\zeta$.

The light-front quantum mechanical evolution operator H^{ν}_{LF} (A.6) was constructed in terms of the supercharges R and $R^{\dagger}$. We can also construct a light-front Hamiltonian $\overline{H}^{\nu}_{LF}$ in terms of the supercharges T and $T^{\dagger}$ given by (A.9) and (A.10):

$$\overline{H}^{\nu}_{LF} = \{R, R^{\dagger}\} = \begin{pmatrix} a^{\dagger}_{\nu} a_{\nu} & 0 \\ 0 & a_{\nu} a^{\dagger}_{\nu} \end{pmatrix}. \tag{A.29}$$

The light-front Hamiltonians $\overline{H}_{LF}$ (A.29) and H_{LF} (A.6) are related by $\overline{H}_{LF}(\lambda) = H_{LF}(-\lambda)$.

---

[1] G. Parisi, "Conformal invariance in perturbation theory," Phys. Lett. B **39**, 643 (1972).

[2] V. M. Braun, G. P. Korchemsky and D. Mueller, "The uses of conformal symmetry in QCD," Prog. Part. Nucl. Phys. **51** (2003) 311 [arXiv:hep-ph/0306057].

[3] G. F. de Teramond and S. J. Brodsky, "Light-front holography: a first approximation to QCD," Phys. Rev. Lett. **102**, 081601 (2009) [arXiv:0809.4899 [hep-ph]].

[4] V. de Alfaro, S. Fubini and G. Furlan, "Conformal invariance in quantum mechanics," Nuovo Cim. A **34**, 569 (1976).

[5] S. J. Brodsky, G. F. de Teramond and H. G. Dosch, "Threefold complementary approach to holographic QCD," Phys. Lett. B **729**, 3 (2014) [arXiv:1302.4105 [hep-th]].

[6] Harmonic confinement also follows from the covariant Hamiltonian description of mesons given in H. Leutwyler and J. Stern, "Harmonic confinement: A fully relativistic approximation to the meson spectrum," Phys. Lett. B **73**, 75 (1978).

[7] A linear potential in the instant-form of dynamics corresponds to a harmonic potential in front-form dynamics at large distances. See: A. P. Trawinski, S. D. Glazek, S. J. Brodsky, G. F. de Teramond and H. G. Dosch, "Effective confining potentials for QCD," Phys. Rev. D **90**, 074017 (2014) [arXiv:1403.5651 [hep-ph]].

[8] G. F. de Teramond, H. G. Dosch and S. J. Brodsky, "Kinematical and dynamical aspects of higher-spin bound-state equations in holographic QCD," Phys. Rev. D **87**, 075005 (2013) [arXiv:1301.1651 [hep-ph]].

[9] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38**, 1113 (1999) [arXiv:hep-th/9711200].

[10] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[11] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[12] From a physical perspective an infrared fixed point appears to be a natural consequence of confinement. See: S. J. Brodsky and R. Shrock, "Maximum wavelength of confined quarks and gluons and properties of quantum chromodynamics," Phys. Lett. B **666**, 95 (2008) [arXiv:0806.1535 [hep-th]].

[13] S. J. Brodsky, G. F. de Teramond and A. Deur, "Nonperturbative QCD coupling and its β-function from light-front holography," Phys. Rev. D **81**, 096010 (2010) [arXiv:1002.3948 [hep-ph]].

[14] A. Deur, S. J. Brodsky and G. F. de Teramond, "Scheme-independent determination of the perturbative QCD scale Λ_s from confinement dynamics in holographic QCD," arXiv:1409.5488 [hep-ph].

[15] S. J. Brodsky and G. F. de Teramond, "Hadronic spectra and light-front wavefunctions in holographic QCD," Phys. Rev. Lett. **96**, 201601 (2006) [arXiv:hep-ph/0602252].

[16] For a review see: S. J. Brodsky, G. F. de Teramond, H. G. Dosch and J. Erlich, "Light-front holographic QCD and emerging confinement," arXiv:1407.8131 [hep-ph].

[17] I. Kirsch, "Spectroscopy of fermionic operators in AdS/CFT," JHEP **0609**, 052 (2006) [arXiv:hep-th/0607205].

[18] Z. Abidin, C. E. Carlson, "Nucleon electromagnetic and gravitational form factors from holography," Phys. Rev. **D79**, 115003 (2009) [arXiv:0903.4818 [hep-ph]].

[19] S. J. Brodsky and G. F. de Teramond, "AdS/CFT and light-front QCD," In Search for the Totally Unexpected in the LHC Era, Proceedings of the International School of Subnuclear Physics, Vol. **45**, World Scientific Publishing Co. Singapore, 2009, arXiv:0802.0514 [hep-ph].

[20] E. Witten, "Dynamical breaking of supersymmetry," Nucl. Phys. B **188**, 513 (1981).

[21] V. P. Akulov and A. I. Pashnev, "Quantum superconformal model in (1,2) space," Theor. Math. Phys. **56**, 862 (1983) [Teor. Mat. Fiz. **56**, 344 (1983)].

[22] S. Fubini and E. Rabinovici, "Superconformal quantum mechanics," Nucl. Phys. B **245**, 17 (1984).

[23] For a review see: F. Cooper, A. Khare and U. Sukhatme, "Supersymmetry and quantum mechanics," Phys. Rept. **251**, 267 (1995) [arXiv:hep-th/9405029].

[24] Recently it was shown that the structure of the light-front wave equations for nucleons with harmonic confinement is indeed supersymmetric. See: A. Pallares-Rivera and M. Kirchbach, "The Higgs oscillator on the hyperbolic plane and light-front holography," arXiv:1408.3078 [quant-ph].

[25] For a hadron with four-momentum $P^\mu = (P^+, P^-, \mathbf{P}_\perp)$, $P^\pm = P^0 \pm P^3$, the invariant light-front Hamiltonian is $H_{LF} = P_\mu P^\mu = P^+P^- - \mathbf{P}_\perp^2$, where P^+ and $\mathbf{P}_\perp$ are kinematical operators and the Hamiltonian $P^- = i\frac{\partial}{\partial x^+}$ generates the quantum evolution of the system with respect to the light-front time $x^+ = x^0 + x^+$.

[26] For a review see: S. J. Brodsky, H. C. Pauli and S. S. Pinsky, "Quantum chromodynamics and other field theories on the light cone," Phys. Rept. **301**, 299 (1998) [arXiv:hep-ph/9705477].

[27] For a two-parton bound state $\zeta = \sqrt{x(1-x)}|\mathbf{b}_\perp|$, where x is the longitudinal momentum fraction and $\mathbf{b}_\perp$ is the transverse-impact distance between the two quarks and $L = \max|L^3|$.

[28] H. C. Pauli, "On confinement in a light cone Hamiltonian for QCD," Eur. Phys. J. C **7**, 289 (1999) [arXiv:hep-th/9809005].

[29] G. F. de Teramond and S. J. Brodsky, "Gauge/Gravity duality and hadron physics at the light-front," AIP Conf. Proc. **1296**, 128 (2010) [arXiv:1006.2431 [hep-ph]].

[30] For $J = 0$ the AdS mass μ is related to the orbital momentum L by $(\mu R)^2 = -4 + L^2$ and thus $(\mu R)^2 \geq -4$, the stability bound in AdS. See: P. Breitenlohner and D. Z. Freedman, "Stability in gauged extended supergravity," Annals Phys. **144**, 249 (1982).

[31] G. F. de Teramond, S. J. Brodsky and H. G. Dosch, "Light-front holography in QCD and hadronic physics," arXiv:1405.2451 [hep-ph].

[32] For applications of supersymmetric quantum mechanics to the Dirac equation in physical space-time see Ref. [23]. See also: R. L. Hall and O. Yesiltas, "Supersymmetric analysis for the Dirac equation with spin-symmetric and pseudo-spin-symmetric interactions," Int. J. Mod. Phys. E **19**, 1923 (2010) [arXiv:1006.4628 [math-ph]], and references therein.

[33] W. Rarita and J. Schwinger, "On a theory of particles with half integral spin," Phys. Rev. **60**, 61 (1941).

[34] This result was also found in Ref. [35].

[35] T. Gutsche, V. E. Lyubovitskij, I. Schmidt, A. Vega, "Dilaton in a soft-

wall holographic approach to mesons and baryons," Phys. Rev. D **85**, 076003 (2012) [arXiv:1108.0346 [hep-ph]].

[36] E. Klempt and J. M. Richard, "Baryon spectroscopy," Rev. Mod. Phys. **82**, 1095 (2010) [arXiv:0901.205 [hep-ph]].

[37] For n partons the invariant LF variable ζ is the x-weighted definition of the transverse impact variable of the $n-1$ spectator system [15]: $\zeta = \sqrt{\frac{x}{1-x}}\left|\sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j}\right|$ where $x = x_n$ is the longitudinal momentum fraction of the active quark, and L is the relative orbital angular momentum between the active quark and the spectator system.

[38] R. Haag, J. T. Lopuszanski and M. Sohnius, "All possible generators of supersymmetries of the S-matrix," Nucl. Phys. B **88**, 257 (1975).

[39] Superconformal quantum mechanics can emerge in the infrared limit of $d = 11$ supergravity or M theories with extended supersymmetry. See: T. Okazaki, "Membrane quantum mechanics," arXiv:1410.8180 [hep-th], and references therein.

[40] K. A. Olive *et al.* [Particle Data Group Collaboration], "Review of Particle Physics," Chin. Phys. C **38**, 090001 (2014).

[41] See for example: D. B. Lichtenberg, "Unitary Symmetry and Elementary Particles," Academic Press, New York (1978).

[42] In particular the $\Delta\frac{5}{2}^{-}(1930)$ state (also shown in Table I) has been given the non-$SU(6)$ assignment $S = 3/2$, $L = 1$, $n = 1$ in Ref. [36].

[43] H. Forkel, M. Beyer, T. Frederico, "Linear square-mass trajectories of radially and orbitally excited hadrons in holographic QCD," JHEP **0707**, 077 (2007) [arXiv:0705.1857 [hep-ph]].

[44] G. F. de Teramond, S. J. Brodsky and H. G. Dosch, "Hadron Spectroscopy and Dynamics from Light-Front Holography and Conformal Symmetry," arXiv:1401.5531 [hep-ph].

[45] H. Forkel and E. Klempt, "Diquark correlations in baryon spectroscopy and holographic QCD," Phys. Lett. B **679**, 77 (2009) [arXiv:0810.2959 [hep-ph]].

[46] A. Selem and F. Wilczek, "Hadron systematics and emergent diquarks," arXiv:hep-ph/0602128.

[47] L. Infeld, "On a new treatment of some eigenvalue problems", Phys. Rev. **59**, 737 (1941).

[48] L. Infeld and T. E. Hull, "The factorization method," Rev. Mod. Phys. **23**, 21 (1951).

[49] M. Abramowitz and I.A. Stegun, editors, "Handbook of Mathematical Functions," Washington (1964).

[50] M. Arik, M. Baykal and A. Baykal, "Factorization method for d-dimensional isotropic harmonic oscillator and the Generalized Laguerre Polynomials," arXiv:0808.2289 [math-ph].