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Spacetime Symmetries of the Quantum Hall Effect

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We study the symmetries of non-relativistic systems with an emphasis on applications to the fractional quantum Hall effect. A source for the energy current of a Galilean system is introduced and the non-relativistic diffeomorphism invariance studied in previous work is enhanced to a full spacetime symmetry, allowing us to derive a number of Ward identities. These symmetries are smooth in the massless limit of the lowest Landau level. We develop a formalism for Newton-Cartan geometry with torsion to write these Ward identities in a covariant form. Previous results on the connection between Hall viscosity and Hall conductivity are reproduced.

I. INTRODUCTION

The fractional quantum Hall (FQH) [1, 2] effect is one of the most difficult problems in condensed matter physics. In the integer quantum Hall effect, interactions do not play a large role and one can make much progress by studying the dynamics of free electrons moving in a uniform magnetic field in the presence of impurities. The FQH effect on the other hand, relies crucially on the interactions of particles within a single Landau level and cannot be analyzed using perturbative techniques. The lowest Landau level (LLL) constraint is especially difficult to deal with; the majority of proposed theoretical schemes break this constraint at some stage. For example, in the popular Chern-Simons field theories (in both the bosonic [3] of fermionic [4] varieties) the operation of flux attachment mixes states in different Landau levels. The consequence of this breaking is that many physical quantities which should depend only on the Coulomb (or interaction) energy scale, appear to be sensitive to the cyclotron energy. Schemes have been developed within the Chern-Simons field theory, to evade this unphysical sensitivity to the cyclotron energy, at the cost of introducing phenomenological elements into the theory. Some other theoretical approaches have been developed to deal with the LLL constraint explicitly (see, for example, Refs. [5, 6]), but most have only limited scope.

This paper proposes a new approach that emphasizes the symmetries of the LLL. In recent work we have demonstrated that non-relativistic particles moving in an external electromagnetic field possess a far larger degree of symmetry than was previously realized, namely invariance under arbitrary time-dependent diffeomorphisms of space [7, 8] which may further be enlarged to full spacetime diffeomorphism invariance by introducing a background source coupled to the energy current [9, 10].

In section II we recap this story, demonstrating how diffeomorphism invariance may be obtained by introducing a number of different sources. The source for the energy density was first introduced by Luttinger [11]; including a source coupled to the energy flux allows for local time reparameterizations. For specific interactions, most importantly for a delta-function contact interaction between bosons, this action is also Weyl invariant. We then demonstrate in section III how a regular massless limit may be taken after a special choice of parity breaking parameters. The resulting theory contains only particles confined to the LLL. Physical results of this limit will be considered in upcoming work.

In section IV we consider the complete set of one-point Ward identities that follow from non-relativistic diffeomorphism invariance. Spatial diffeomorphisms give rise to local momentum conservation in the presence of external electromagnetic and dilaton fields whereas temporal diffeomorphisms lead to the work-energy equation. In trivial backgrounds, these Ward identities were considered in Refs. [10, 12]. Here we present them in their full generality for nonzero spin and g-factor. In section V we rederive the viscosity-conductivity relations that were first found in Ref. [13].

Spacetime diffeomorphism invariance can be naturally treated using the formalism of Newton-Cartan geometry with torsion, which we present in section VI. In section VII we present a fully covariant treatment of the one-point Ward identities. The stress tensor and energy current as traditionally defined do not transform covariantly under general diffeomorphisms and need to be modified. We define these covariant currents and derive their Ward identities. The spacetime transformation properties of the new covariant stress, charge and energy densities both facilitate streamlined calculations and place strong constraints on the allowed response. Section VIII contains concluding remarks while various technical details are contained in the appendices.

II. SYMMETRIES

At it's most basic level, the FQH problem is that of particles moving in $2+1$ dimensions in the presence of a magnetic field

$$S=\int d^3x\left(i\psi^\dagger D_0\psi-\frac{1}{2m}|D_i\psi|^2-\lambda|\psi|^4\right). \tag{1}$$

Here $D_\mu=\partial_\mu-iA_\mu$ is the gauge covariant derivative and the theory is gauge invariant

$$\psi\to e^{i\alpha}\psi \qquad A_\mu\to A_\mu+\partial_\mu\alpha. \tag{2}$$

We have chosen a contact interaction for simplicity though more general interactions will be consistent with the symmetries we are about to discuss. Strictly speaking, the contact interaction requires a cutoff to be well-defined in $2+1$ dimensions due to the logarithmic running of the coupling constant λ. In the LLL limit $m\to 0$ that we will be especially interested in, the running of λ disappears. We thus will ignore the dependence on the cutoff altogether.

When the magnetic field $B=\epsilon^{ij}\partial_iA_j$ is large (here ϵ^{ij} is the antisymmetric symbol with $\epsilon^{12}=1$), the spectrum is stratified into Landau levels of energy $\frac{B}{m}(n+\frac{1}{2})$ that are well-separated compared to the intra-Landau level spacing (we choose units where $\hbar=c=1$). Since we are only concerned with the LLL, we would like to integrate out all states for which $n\geq 1$. One possible way of doing this is to take the $m\to 0$ limit in which the higher Landau levels tend to infinity and decouple from the theory. Unfortunately, this limit is not regular due to the infinite shift in the zero-point energy, but we shall see there is an easy way around this.

A. The g-factor

We will now systematically introduce a number of generalizations to the basic action (1) that will not affect the physics at the end of the day but are essential for our later analysis. In the process we introduce a number of external probes used to define response currents. Begin with an intrinsic angular momentum parameterized by a g-factor g

$$S=\int d^3x\left(i\psi^\dagger D_0\psi-\frac{1}{2m}|D_i\psi|^2+\frac{\mathrm{g}B}{4m}|\psi|^2-\lambda|\psi|^4\right). \tag{3}$$

In GaAs, this factor is close to zero (there g is the product of the Lande g-factor g^* [14]) and the ratio of the band mass m to the bare electron mass: $\mathrm{g}=g^*m/m_e\approx -0.03$), but it is easy to see, at least for constant B, that it's actual value is irrelevant. In this case the new term merely gives rise to a constant shift to the Hamiltonian, which has no physical significance.

When B is not uniform, the situation is somewhat more involved but not insurmountable. Notice that g enters the action in the combination $A_0+\frac{\mathrm{g}}{4m}B$. Defining a new electric potential

$$A_0'=A_0+\frac{\mathrm{g}-\mathrm{g}'}{4m}B \tag{4}$$

maps the action to itself, but with a new g-factor

$$S_{\mathrm{g}}[A_0]=S_{\mathrm{g}'}[A_0']. \tag{5}$$

We may just as well perform calculations with any g we like, so long as we use the shifted gauge field A_0'. In section III we shall see that when we select $\mathrm{g}=2$, the LLL is shifted to zero energy even in a non- uniform field and curved space. The massless limit is then regular and the projection onto the LLL proceeds without difficulty. This feature was exploited in Ref. [7] in the construction of an effective field theory for FQH states.

B. Curvature

Next introduce a nontrivial background metric g_{ij}

$$S=\int d^3x\sqrt{g}\left(\frac{i}{2}\psi^\dagger\overleftrightarrow{D}_0\psi-\frac{g^{ij}}{2m}D_i\psi^\dagger D_j\psi+\frac{\mathrm{g}B}{4m}|\psi|^2-\lambda|\psi|^4\right) \tag{6}$$

where $\psi^\dagger \overleftrightarrow{D}_0 \psi = \psi^\dagger D_0 \psi - D_0 \psi^\dagger \psi$. The magnetic field is now $B = \varepsilon^{ij}\partial_i A_j$ where $\varepsilon^{ij} = \frac{1}{\sqrt{g}}\epsilon^{ij}$ is the natural spatial volume element associated to the metric. There is some ambiguity in how we choose to couple the theory to geometry; we could for example have included higher curvature terms. These terms would change the equations of motion on curved backgrounds but leave the flat space dynamics unaltered. If at the end of the day one is only interested in flat space, we may choose the coupling however we like without fear of altering the physics. In the above we have chosen to couple the theory in the minimal way.

If the field ψ has spin s, even minimal substitution requires the introduction of a zweibein e^a_i that diagonalizes the metric

$$g_{ij} = \delta_{ab} e^a_i e^b_j \qquad e^a_i e^{bi} = \delta^{ab}. \tag{7}$$

The covariant derivative is then

$$D_\mu = \partial_\mu - iA_\mu + is\omega_\mu \tag{8}$$

where ω_μ is the spin connection

$$\begin{aligned} \omega_0 &= \frac{1}{2}\epsilon_{ab} e^{aj}\partial_0 e^b_j \\ \omega_i &= \frac{1}{2}\epsilon_{ab} e^{aj}\nabla_i e^b_j = \frac{1}{2}\epsilon_{ab} e^{aj}\partial_i e^b_j - \frac{1}{2}\varepsilon^{jk}\partial_j g_{ik}. \end{aligned} \tag{9}$$

Here ∇_i represents the spatial covariant derivative defined by g_{ij}. Under a local rotation of the zweibein by an angle $\theta(x)$, the spin connection transforms as a $U(1)$ gauge field

$$\omega_\mu \to \omega_\mu + \partial_\mu \theta \tag{10}$$

canceling the spin rotation of the field $\psi \to e^{-is\theta}\psi$. We notice that the same minimal coupling to gravity through spin connection was recently used in Ref. [15] to modify the conventional flux attachment procedure to derive the Hall viscosity and Wen-Zee term from the Chern-Simons gauge theories.

Even if one does not care about curved space dynamics and plans to set $g_{ij} = \delta_{ij}$, introducing a metric is a useful intermediate step for several reasons. First, it gives a natural definition of a symmetric stress tensor as the response of the action to geometric perturbations in the same way that the charge current is a response to electromagnetic perturbations

$$\delta S = \int d^3x \sqrt{g}\left(\frac{1}{2}T^{ij}_{\rm nc}\delta g_{ij} + j^\mu_{\rm nc}\delta A_\mu\right), \tag{11}$$

as is done in relativity theory. The subscript "nc" (as in "non-covariant") is to differentiate this notion of stress from the spacetime covariant one that we shall introduce later. In the usual case $\mathrm{g} = s = 0$ in flat space, these are the familiar expressions

$$\begin{aligned} j^0_{\rm nc} &= |\psi|^2 \qquad\qquad j^i_{\rm nc} = -\frac{i}{2m}\psi^\dagger \overleftrightarrow{D}^i \psi \\ T^{ij}_{\rm nc} &= \frac{1}{m}D^{(i}\psi^\dagger D^{j)}\psi + \left(\frac{i}{2}\psi^\dagger \overleftrightarrow{D}_0 \psi - \frac{1}{2m}D_k\psi^\dagger D^k\psi - \lambda|\psi|^4\right)g^{ij} \end{aligned} \tag{12}$$

The spin is often set to zero in the literature, although for spin polarized electrons in two spatial dimensions, the actual value would be 1/2. However, as with g, there is a simple mapping between theories of different spin. Like before, a redefinition

$$A'_\mu = A_\mu + (s' - s)\omega_\mu \tag{13}$$

sends the action to itself, but with s replaced by s'

$$S_s[A_\mu] = S_{s'}[A'_\mu]. \tag{14}$$

In what follows we will find the selection $s = 1$ to be particularly convenient. Note a pure zweibein rotation $A'_\mu \to A'_\mu$, $\omega_\mu \to \omega_\mu + \partial_\mu\theta$ of the new theory now corresponds to a zweibein rotation of angle θ plus a gauge transformation $A_\mu \to A_\mu - (s' - s)\partial_\mu\theta$ of the original theory. Thus in this new picture the field ψ has spin s' and local rotation invariance is still manifest.

The redefinitions (4) and (13) will affect the stress and charge current. In appendix B we derive the relationship between the primed and unprimed currents, which we present here in flat space for simplicity. If we imagine doing an experiment on a system with say $\mathrm{g}' = s' = 0$ and change to our preferred values $\mathrm{g} = 2$, $s = 1$, the new currents are

$$\begin{aligned} &j^0_{\rm nc} = j'^0_{\rm nc} \qquad\qquad j^i_{\rm nc} = j'^i_{\rm nc} + \frac{1}{2m}\epsilon^{ij}\partial_j j'^0_{\rm nc} \\ &T^{ij}_{\rm nc} = T'^{ij}_{\rm nc} - \epsilon^{k(i}\partial_k j'^{j)}_{\rm nc} - \frac{1}{2m}\left(B j'^0_{\rm nc}\delta^{ij} + \left(\partial^i\partial^j - g^{ij}\partial^2\right) j'^0_{\rm nc}\right). \end{aligned} \tag{15}$$

The primed currents are to be evaluated at the physical fields E'_i, B', whereas the unprimed currents are at

$$E_i = E'_i - \frac{1}{2m}\partial_i B' \qquad\qquad B = B'. \tag{16}$$

One of the main reasons for introducing g_{ij} is that it makes the symmetry of the action more apparent. The index structure makes clear that the theory is invariant under time-independent spatial diffeomorphisms $\xi^k = \xi^k(\mathbf{x})$

$$\begin{aligned} \delta\psi &= -\xi^k\partial_k\psi \\ \delta A_\mu &= -\xi^k\partial_k A_\mu - A_k\partial_\mu\xi^k \\ \delta g_{ij} &= -\xi^k\partial_k g_{ij} - g_{jk}\partial_i\xi^k - g_{ik}\partial_j\xi^k. \end{aligned} \tag{17}$$

In Refs. [7–9] it was found that this invariance may be extended to time-dependent diffeomorphisms $\xi^k(t,\mathbf{x})$ by adding a non-covariant part to the transformation of the vector potential

$$\begin{aligned} \delta\psi &= -\xi^k\partial_k\psi \\ \delta A_0 &= -\xi^k\partial_k A_0 - A_k\dot{\xi}^k + \frac{\mathrm{g}-2s}{4}\varepsilon^{ij}\partial_i(g_{jk}\dot{\xi}^k) \\ \delta A_i &= -\xi^k\partial_k A_i - A_k\partial_i\xi^k - m g_{ij}\dot{\xi}^j \\ \delta g_{ij} &= -\xi^k\partial_k g_{ij} - g_{jk}\partial_i\xi^k - g_{ik}\partial_j\xi^k \end{aligned} \tag{18}$$

which they called *non-relativistic general coordinate invariance*. The s part has not been considered in previous work. Note that for $\mathrm{g} = 2$, $s = 1$, these transformations take a particularly simple form. Taking $s = 1$ is mostly a choice made to make the formulas easier to work with, whereas using $\mathrm{g} = 2$ is crucial to ensure that the regularity of the $m \to 0$ limit.

C. A Source for the Energy Current

This symmetry may be enlarged further to show that the microscopic action is not only invariant under time-dependent spatial coordinate reparameterizations, but completely general changes of coordinates on spacetime. This allows for a fully spacetime covariant treatment of non-relativistic physics just as in relativity theory. To do so however, we begin with a seemingly unrelated question: how to define an energy current for our theory.

In general relativity, charged fields couple to both a vector potential and a Lorentzian metric and we can consider the system's response to infinitesimal variations in these quantities. As in (11) this defines a charge current j^μ and a stress-energy tensor $T^{\mu\nu}$ that collects both the energy current T^{0i} and the stress T^{ij} into a single object. It's well-known that in non-relativistic physics, we have an independent energy current which we will denote as $\varepsilon^\mu_{\rm nc}$ that is not tied to the stress $T^{ij}_{\rm nc}$ in any way [16].

A source for the energy current was considered in Refs. [9, 10]. This involves dilaton Φ and a spatial vector C_i in the following way

$$S = \int d^3x\sqrt{g}e^{-\Phi}\left(\frac{i}{2}e^{\Phi}\psi^\dagger\overleftrightarrow{D}_0\psi - \frac{1}{2m}\left(g^{ij} + \frac{i\mathrm{g}}{2}\varepsilon^{ij}\right)\tilde{D}_i\psi^\dagger\tilde{D}_j\psi - \lambda|\psi|^4\right) \tag{19}$$

where $\tilde{D}_i = D_i + C_i D_0$. The Hamiltonian now appears in the action with a factor $e^{-\Phi}$ and Φ is essentially the source introduced by Luttinger [11]. Note that we have also collected the magnetic momentum term into the kinetic term. Upon integration by parts, this merely becomes the $\frac{\mathrm{g}B}{4m}|\psi|^2$ coupling considered before, plus boundary terms that go as the derivatives of the new fields Φ and C_i.

Finally, for the symmetries we are about to consider to hold, we must also modify the spatial Christoffel symbol to be

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{kl} \left(\tilde{\partial}_i g_{jk} + \tilde{\partial}_j g_{ik} - \tilde{\partial}_k g_{ij} \right). \tag{20}$$

In our action this only affects the spin connection

$$\omega_i = \frac{1}{2} \epsilon_{ab} e^{aj} \nabla_i e^b_j = \frac{1}{2} \epsilon_{ab} e^{aj} \partial_i e^b_j - \frac{1}{2} \varepsilon^{jk} \tilde{\partial}_j g_{ik}. \tag{21}$$

For $\Phi = C_i = 0$, this is just the action (6) and so we have not altered the dynamics in these backgrounds, but we can now define $\varepsilon^\mu_{\rm nc}$ via

$$\delta S = \int d^3x \sqrt{g} e^{-\Phi} \left(\frac{1}{2} T^{ij}_{\rm nc} \delta g_{ij} + j^\mu_{\rm nc} \delta A_\mu + \varepsilon^0_{\rm nc} \delta \Phi + \varepsilon^i_{\rm nc} \delta C_i \right). \tag{22}$$

One might also wish to introduce a source for the momentum current, but in a Galilean invariant theory, the momentum is entirely determined by the charge current, so we do not include any further sources. We will see this in section IV and again in section VII B where we find a unique way to demonstrate this using Newton-Cartan geometry.

To motivate our placement of Φ and C_i, consider the energy current so defined for $\mathrm{g} = s = 0$ and a trivial background $\Phi = C_i = 0$, $g_{ij} = \delta_{ij}$

$$\begin{aligned} \varepsilon^0_{\rm nc} &= \frac{1}{2m} D_i \psi^\dagger D^i \psi + \lambda |\psi|^4 \\ \varepsilon^i_{\rm nc} &= -\frac{1}{2m} \left(D_0 \psi^\dagger D^i \psi + D^i \psi^\dagger D_0 \psi \right). \end{aligned} \tag{23}$$

We immediately recognize $\varepsilon^0_{\rm nc}$ as the total energy of the system. One may also check using the equation of motion that the work-energy equation holds

$$\partial_0 \varepsilon^0_{\rm nc} + \partial_i \varepsilon^i_{\rm nc} = E_i j^i_{\rm nc}, \tag{24}$$

so it is clear that $\varepsilon^i_{\rm nc}$ is indeed the energy flux.

The energy current is also altered upon a change of the parity breaking parameters g and s. As before, translating from $\mathrm{g}' = s' = 0$ to $\mathrm{g} = 2$, $s = 1$ in the trivial background gives

$$\begin{aligned} \varepsilon^0_{\rm nc} &= \varepsilon'^0_{\rm nc} - \frac{1}{2} \epsilon^{ij} \partial_i j'_{j\,\rm nc} - \frac{1}{2m} B j'^0_{\rm nc} \\ \varepsilon^i_{\rm nc} &= \varepsilon'^i_{\rm nc} + \frac{1}{2} \epsilon^{ij} \partial_0 j'{}_{j\,\rm nc} - \frac{1}{2m} \left(B j'^i_{\rm nc} + \epsilon^{ij} E_j j'^0_{\rm nc} \right). \end{aligned} \tag{25}$$

We again refer the reader to appendix B for details as well as the case for general $\mathrm{g}', s', \mathrm{g}, s$ and a curved metric.

D. Spacetime Coordinate Invariance

Our placement of these new sources does much more than give a convenient definition of the energy current, it allows us to enlarge the group of spacetime symmetries by properly selecting the transformations of Φ and C_i. The action is invariant under arbitrary spacetime diffeomorphisms $\xi^\lambda(t, \mathbf{x})$

$$\begin{aligned} \delta\psi &= -\xi^\lambda \partial_\lambda \psi \\ \delta\Phi &= -\xi^\lambda \partial_\lambda \Phi + \partial_\lambda \xi^\lambda - \tilde{\partial}_i \xi^i \\ \delta C_i &= -\xi^\lambda \partial_\lambda C_i - C_j \tilde{\partial}_i \xi^j + \tilde{\partial}_i \xi^0 \\ \delta e^{ai} &= -\xi^\lambda \partial_\lambda e^{ai} + e^{ak} \tilde{\partial}_k \xi^i \\ \delta g^{ij} &= -\xi^\lambda \partial_\lambda g^{ij} + \tilde{\partial}^i \xi^j + \tilde{\partial}^j \xi^i \\ \delta \varepsilon^{ij} &= -\xi^\lambda \partial_\lambda \varepsilon^{ij} + \varepsilon^{ik} \tilde{\partial}_k \xi^j - \varepsilon^{jk} \tilde{\partial}_k \xi^i \\ \delta A_0 &= -\xi^\lambda \partial_\lambda A_0 - A_\lambda \partial_0 \xi^\lambda + \frac{\mathrm{g} - 2s}{4} \Big(\varepsilon^{ij} \tilde{\partial}_i (g_{jk} \dot{\xi}^k) + \varepsilon^i{}_j \dot{C}_i \dot{\xi}^j \Big) \\ \delta A_i &= -\xi^\lambda \partial_\lambda A_i - A_\lambda \partial_i \xi^\lambda - m e^\Phi g_{ij} \dot{\xi}^j - \frac{\mathrm{g} - 2s}{4} C_i \Big(\varepsilon^{jk} \tilde{\partial}_j (g_{kl} \dot{\xi}^l) + \varepsilon^j{}_k \dot{C}_j \dot{\xi}^k \Big) \end{aligned} \tag{26}$$

where λ now includes the temporal index 0. This is the full set of symmetries of a non-relativistic spacetime corresponding to infinitesimal coordinate transformations (see section VI B).

We stress once more that this represents full spacetime coordinate reparameterization invariance. From (18) it was clear that the theory was invariant under arbitrary time-dependent coordinate changes on spatial slices. Now we see that the theory is also unaffected by local time reparameterizations $\xi^0(t,\mathbf{x})$. In particular, we may choose a new spatial foliation of spacetime. This is another way to see that the new sources are not essential modifications to the physics. Even if C_i was zero initially, in these new slicings, it may not be (for example if the temporal shift is not constant – $\partial_i \xi^0 \neq 0$), just as the metric δ_{ij} may not look flat in some other randomly chosen curvilinear coordinate system. Hence we see allowing for nonzero C_i is necessary to make the full spacetime symmetry of the initial theory manifest, even if we are considering a trivial background that merely reduces the problem to the original action (3) in the end.

There is a minor complication to this story. One could imagine that given some background C_i that there is no slicing where it vanishes. It turns out that a $C_i = 0$ slicing exists if and only if

$$\varepsilon^{ij}\tilde{\partial}_i C_j = 0. \tag{27}$$

This is in fact a coordinate representation of the Frobenius' condition that a local spatial slicing exists. When $C_i = 0$, coordinates have been chosen so that constant time surfaces coincide with spatial slices. In the discussion below equation (78), we show from causal considerations that (27) must in general be satisfied. Coordinates where $C_i = 0$ are called *global time coordinates* (GTC). Since we shall always assume GTC exist, in what follows we will take $C_i = 0$, only restoring it when necessary to compute the energy current.

Gauge invariance and spatial diffeomorphisms are not the only local symmetries of the action (19). For each $\Omega(t,\mathbf{x})$, the theory also exhibits Weyl invariance

$$\begin{aligned}
&\delta\psi = \Omega\psi, \qquad \delta\Phi = 2\Omega, \qquad \delta g_{ij} = -2\Omega g_{ij}, \qquad \delta C_i = 0\\
&\delta A_0 = -\frac{1}{2m}\left(1-\frac{\mathrm{g}^2}{4}\right)\left(\frac{1}{\sqrt{g}}\tilde{\partial}_i\left(e^{-\Phi}\sqrt{g}\tilde{\partial}^i\Omega\right) + e^{-\Phi}\dot{C}_i\tilde{\partial}^i\Omega\right)\\
&\delta A_i = \frac{\mathrm{g}-2s}{2}\varepsilon_{ij}\tilde{\partial}^j\Omega.
\end{aligned} \tag{28}$$

This is of course specific to the point interaction $\lambda|\psi|^4$ where scale invariance is well known to be violated quantum mechanically [17]. In the massless limit however, λ does not run and the LLL theory is truly conformally invariant. Note that for $\mathrm{g} = 2$, $s = 1$, the vector potential does not transform.

This concludes the complete set of generalizations of the initial problem that are relevant for this paper. We are now considering particles of arbitrary spin and g-factor moving in the presence of an electromagnetic field A_μ, a curved metric g_{ij}, a dilaton Φ and a spatial vector C_i. g and s may be chosen at will so long as we remember to translate back to their physical values using (15), (16) and (25). In section VI we present a manifestly coordinate invariant treatment of this symmetry from which the anomalous transformation laws (26) follow naturally. This is the Newton-Cartan geometry first considered in Ref. [7] in the context of the FQH effect. There we shall find that properly defining the energy current requires a generalization of this formalism to include nonzero torsion.

III. THE MASSLESS LIMIT

We now perform the massless limit discussed earlier. In GTC the action is

$$S = \int d^3x \sqrt{g} e^{-\Phi}\left(\frac{i}{2}e^{\Phi}\psi^\dagger \overleftrightarrow{D}_0\psi - \frac{1}{2m}\big(g^{ij} + \frac{i\mathrm{g}}{2}\varepsilon^{ij}\big)D_i\psi^\dagger D_j\psi - \lambda|\psi|^4\right),$$

and the quantum partition function is given by

$$Z = \int \mathcal{D}\psi^\dagger\mathcal{D}\psi e^{iS}. \tag{29}$$

The matrix ε^{ij} has eigenvalues $\pm i$ and so the value $\mathrm{g} = 2$ is distinguished for the matrix $g^{ij} + i\varepsilon^{ij}$ is degenerate. In terms of the zweibein e_i^a we have

$$\varepsilon^{ij} = \epsilon_{ab}e^{ai}e^{bj}. \tag{30}$$

The eigenvectors of ε^{ij} are the chiral basis vectors

$$e_i = \frac{1}{\sqrt{2}}(e_i^1 + ie_i^2), \qquad \bar{e}_i = \frac{1}{\sqrt{2}}(e_i^1 - ie_i^2) \tag{31}$$

in terms of which we have the convenient formulas

$$g^{ij} = e^i\bar{e}^j + \bar{e}^i e^j \qquad \varepsilon^{ij} = i(e^i\bar{e}^j - \bar{e}^i e^j) \qquad g^{ij} + i\varepsilon^{ij} = 2\bar{e}^i e^j. \tag{32}$$

Hence the g = 2 action may be written as

$$S = \int d^3x\sqrt{g}e^{-\Phi}\left(\frac{i}{2}e^{\Phi}\psi^\dagger \overleftrightarrow{D}_0\psi - \frac{1}{m}(\bar{e}^i D_i\psi^\dagger)(e^j D_j\psi) - \lambda|\psi|^4\right). \tag{33}$$

In flat space, $e^i D_i\psi = D_{\bar{z}}\psi$ and we see the degeneracy direction corresponds precisely to particles in the LLL. Using a Hubbard-Stratonovich transformation, we write this as

$$S = \int d^3x\sqrt{g}e^{-\Phi}\left(\frac{i}{2}e^{\Phi}\psi^\dagger \overleftrightarrow{D}_0\psi - \chi(\bar{e}^i D_i\psi^\dagger) - \bar{\chi}(e^i D_i\psi) + m\bar{\chi}\chi - \lambda|\psi|^4\right). \tag{34}$$

The $m \to 0$ limit is manifestly regular and the higher Landau levels are now completely trivial to integrate out as χ and $\bar{\chi}$ simply become Lagrange multipliers enforcing the constraint

$$e^i D_i\psi = 0 \tag{35}$$

which is the curved space equation for the LLL wave function. The many-body problem of particles confined to the LLL thus can be understood as a system of interacting particles with no kinetic energy

$$S = \int d^3x\sqrt{g}\left(\frac{i}{2}\psi^\dagger \overleftrightarrow{D}_0\psi - e^{-\Phi}\lambda|\psi|^4\right) \tag{36}$$

for which path integration is only carried out subject to the holomorphic constraint (35). This theory inherits all the symmetries discussed above. In particular, one may check that both Eqs. (35) and (36) are preserved by spacetime diffeomorphisms and Weyl transformations.

We note briefly that for $s = 1$ the transformation laws are especially simple in the massless limit. In particular, A_μ is just a one-form

$$\delta A_\mu = -\xi^\lambda\partial_\lambda A_\mu - A_\lambda\partial_\mu\xi^\lambda \tag{37}$$

and is unchanged under Weyl transformations.

IV. NON-COVARIANT WARD IDENTITIES

In this section we derive the complete set of Ward identities that follow from the symmetries above. The Ward identities are a result of only the symmetries of the problem and are valid in arbitrary backgrounds so long as (26) is not anomalous. We begin however with a slight change of viewpoint however. In section II we used a model microscopic action S to motivate the introduction of sources and demonstrate the symmetry of the problem. The full quantum dynamics is however determined by the effective action W obtained from integrating out the field ψ and is a functional only of the external fields

$$e^{iW} = Z \qquad W = W[A_\mu, g_{ij}, \Phi, C_i]. \tag{38}$$

The currents defined from W are then the 1-point expectation values of the microscopic ones defined above. To simply notation we drop the brackets $\langle\rangle$ and simply denote them as

$$\delta W = \int d^3x\sqrt{g}e^{-\Phi}\left(\frac{1}{2}T^{ij}_{\text{nc}}\delta g_{ij} + j^\mu_{\text{nc}}\delta A_\mu + \varepsilon^0_{\text{nc}}\delta\Phi + \varepsilon^i_{\text{nc}}\delta C_i\right). \tag{39}$$

Let's begin with gauge invariance, which is the simplest of the symmetries considered above. The gauge variation of the electromagnetic potential is $\delta A_\mu = \partial_\mu\alpha$

$$\delta W = \int d^3x\sqrt{g}e^{-\Phi}j^\mu_{\text{nc}}\partial_\mu\alpha = 0 \qquad \Longrightarrow \qquad 0 = -\int d^3x\partial_\mu\left(\sqrt{g}e^{-\Phi}j^\mu_{\text{nc}}\right). \tag{40}$$

Since α is an arbitrary function of space and time, we conclude $\partial_\mu(\sqrt{g}e^{-\Phi}j^\mu_{\rm nc}) = 0$ or

$$\frac{1}{\sqrt{g}}\partial_0\left(\sqrt{g}e^{-\Phi}j^0_{\rm nc}\right) + \nabla_i\left(e^{-\Phi}j^i_{\rm nc}\right) = 0 \tag{41}$$

which is simply the continuity equation on a curved background with dilaton Φ.

The remaining Ward identities follow in like manner. Spatial diffeomorphisms ξ^k imply stress conservation

$$\begin{aligned}\frac{e^\Phi}{\sqrt{g}}\partial_0\left(\sqrt{g}\Big(mj_{i\rm nc} - \frac{\mathrm{g}-2s}{4}\varepsilon_{ij}\nabla^j(e^{-\Phi}j^0_{\rm nc})\Big)\right) + e^\Phi\nabla_j\left(e^{-\Phi}T_{i\rm nc}^{j}\right)&\\ = j^0_{\rm nc}E_i + \varepsilon_{ij}j^j_{\rm nc}B + \varepsilon^0_{\rm nc}\nabla_i\Phi.&\end{aligned} \tag{42}$$

Note that nonrelativistic diffeomorphism invariance completely determines the momentum current

$$p^i_{\rm nc} = mj^i_{\rm nc} - \frac{\mathrm{g}-2s}{4}\varepsilon^{ij}\nabla_j(e^{-\Phi}j^0_{\rm nc})$$

as mentioned previously. In particular, the mass leads to a momentum along the direction of charge flow while the parity breaking terms give rise to an intrinsic angular momentum density

$$l = -\frac{\mathrm{g}-2s}{2}e^{-\Phi}j^0_{\rm nc}$$

as may be seen by computing the flat space total angular momentum $L = \int d^2x\epsilon^{ij}x_ip_j$. Note that the dilaton exerts an external force on the system just as the electromagnetic field does.

Temporal diffeomorphisms ξ^0 result in the work-energy equation

$$\frac{1}{\sqrt{g}}\partial_0\big(\sqrt{g}\varepsilon^0_{\rm nc}\big) + e^\Phi\nabla_i(e^{-\Phi}\varepsilon^i_{\rm nc}) = E_ij^i_{\rm nc} - \frac{1}{2}T^{ij}_{\rm nc}\dot{g}_{ij}. \tag{43}$$

E_ij^i is the familiar work done by the electric field whereas the metric term corresponds to the work done on the walls of a volume element as it expands or contracts due to the internal forces of the system. The Ward identities of a system with a conserved particle number thus reproduce the full set of hydrodynamic equations of motion for a non-relativistic fluid [16].

Finally, Weyl invariance gives rise to a generalization of the tracelessness of the stress-energy tensor

$$\varepsilon^0_{\rm nc} = \frac{1}{2}T^{ij}_{\rm nc}g_{ij} + \frac{1}{4m}\left(1-\frac{\mathrm{g}^2}{4}\right)e^\Phi\nabla_i\left[e^{-\Phi}\nabla^i\left(e^{-\Phi}j^0_{\rm nc}\right)\right] - \frac{\mathrm{g}-2s}{4}e^\Phi\varepsilon^{ij}\nabla_i\left(e^{-\Phi}j_{j\rm nc}\right). \tag{44}$$

Note that the Ward identities take a particularly simple form for the LLL theory: the momentum vanishes and the energy is simply the trace of the stress tensor

$$e^\Phi\nabla_j\left(e^{-\Phi}T_{i\rm nc}^{j}\right) = j^0_{\rm nc}E_i + \varepsilon_{ij}j^j_{\rm nc}B + \varepsilon^0_{\rm nc}\nabla_i\Phi, \qquad \varepsilon^0_{\rm nc} = \frac{1}{2}T^{ij}_{\rm nc}g_{ij}. \tag{45}$$

It's worth pointing out that the quantum conservation laws in curved space contain the full information on Ward identities. By taking functional derivatives of these equations with respect to the sources one obtains higher order Ward identities which relate the n-point correlation functions to δ-function terms involving lower order correlators.

V. VISCOSITY-CONDUCTIVITY RELATION

As an illustration, here we will give two Ward identities for two- point functions and show how they can be used to extract the independent viscosity coefficients from the conductivities at all frequencies. Our work provides an alternative field-theory approach to the previous result [13] based on a microscopic Hamiltonian and generalize it to the nonvanishing g-factor and spin.

A. Ward identities on closed time path

To discuss the real-time response functions, let us invoke the closed time-path formalism [18–20] and double Eq. (42) on two time branches

$$\partial_t\left[g_{\pm ij}\left(mG^j_\pm-\frac{\mathrm{g}-2s}{4}\epsilon^{jk}\partial_k\frac{G^t_\pm}{\sqrt{g_\pm}}\right)\right]+2\partial_k(g_{\pm ij}G^{jk}_\pm)-\partial_i g_{\pm jk}G^{jk}_\pm-E_{\pm i}G^t_\pm-\varepsilon_{\pm ij}BG^j_\pm=0, \tag{46}$$

The non-equilibrium current and tensor (density) are defined by

$$G^\mu_\pm\equiv\frac{\delta W}{\delta A_{\pm\mu}},\ G^{ij}_\pm\equiv\frac{\delta W}{\delta g_{\pm ij}}.$$

The dilaton has been set to zero since it has no effect on the results of this section.

Taking variation of Eq. (46) with respect to sources A_l and g_{jk} on two branches, we obtain four identities. Their linear combinations give

$$0=(m\delta_{ij}\partial_t-\epsilon_{ij}B)\,G^{j,l}_{\mathrm{ra}}(x)-\frac{\mathrm{g}-2s}{4}\delta_{ij}\epsilon^{jk}\partial_t\partial_k G^{t,l}_{\mathrm{ra}}(x)+2\delta_{ij}\partial_k G^{jk,l}_{\mathrm{ra}}(x)+\delta^l_i G^t\partial_t\delta(x), \tag{47}$$

$$\begin{aligned}0\ =\ &\left(m\delta^i_l\partial_t-\epsilon^i{}_l B\right)G^{l,jk}_{\mathrm{ra}}(x)-\frac{\mathrm{g}-2s}{4}\epsilon^{in}\partial_t\partial_n G^{t,jk}_{\mathrm{ra}}(x)+2\partial_n G^{in,jk}_{\mathrm{ra}}(x)\\&+\frac{\mathrm{g}-2s}{8}\epsilon^{in}G^t\partial_t\partial_n\delta(x)\delta^{jk}+\left[\left(\delta^{ik}\delta^j_m+\delta^{ij}\delta^k_m\right)\partial_n\delta(x)-\frac{1}{2}\left(\delta^k_m\delta^j_n+\delta^j_m\delta^k_n\right)\partial^i\delta(x)\right]G^{mn},\end{aligned} \tag{48}$$

where the correlators $G_{\mathrm{ra}}(x)$ are defined by the second variation of the generating functional with respect to the sources in physical presentation [20]. It can be split as the retarded Green's function and the contact term

$$G^{B,A}_{\mathrm{ra}}(x)\equiv\frac{\delta^2 W}{\delta J^B_{\mathrm{a}}(x)\,\delta J^A_{\mathrm{r}}(0)}=i\theta(t)\langle\left[\varphi^B(x),\varphi^A(0)\right]\rangle+\langle\frac{\delta\varphi^B_{\mathrm{r}}(x)}{\delta J^A_{\mathrm{r}}(0)}\rangle, \tag{49}$$

where φ^A denotes the conjugate operator of the source J^A. One can also define the advanced correlators

$$G^{B,A}_{\mathrm{ar}}(x)\equiv\frac{\delta^2 W}{\delta J^B_{\mathrm{r}}(x)\,\delta J^A_{\mathrm{a}}(0)}=i\theta(-t)\langle\left[\varphi^A(0),\varphi^B(x)\right]\rangle+\langle\frac{\delta\varphi^B_{\mathrm{a}}(x)}{\delta J^A_{\mathrm{a}}(0)}\rangle.$$

Note that after the variation, we have put everything on the unperturbed background, which is assumed as a translationally invariant state with a uniform magnetic field and a vanishing electric field on the flat spacetime.

Keeping in mind the symmetry

$$G^{B,A}_{\mathrm{ra}}(x)=G^{A,B}_{\mathrm{ar}}(-x), \tag{50}$$

and the variation of the continuity equation

$$\partial_\mu G^{\mu,\nu}(x)=0,\ \partial_\mu G^{\mu,ij}(x)=0, \tag{51}$$

one can combine Eq. (47) and Eq. (48) as

$$\begin{aligned}&\left(m\delta^n_l\partial_t-\epsilon^n{}_l B+\frac{\mathrm{g}-2s}{4}\epsilon^{nm}\partial_m\partial_l\right)\left(m\delta_{ji}\partial_t-\epsilon_{ji}B-\frac{\mathrm{g}-2s}{4}\epsilon_i{}^k\partial_j\partial_k\right)G^{l,j}_{\mathrm{ra}}(x)\\=\ &4\delta_{ij}\partial_k\partial_m G^{nm,jk}_{\mathrm{ra}}(x)+2\delta^n_i\partial_k\partial_l\delta(x)G^{kl}-\left(m\delta^n_i\partial_t-\epsilon^n{}_i B\right)\partial_t\delta(x)G^t.\end{aligned} \tag{52}$$

In momentum space, this can be recast as a relation at all frequencies

$$\begin{aligned}&4G^{j(i,k)l}_{\mathrm{ra}}(\omega)+2\delta^{lj}G^{ik}\\=\ &\frac{1}{2}m^2 b^{jm}\left.\frac{\partial^2}{\partial q_i\partial q_k}G^{m,n}_{\mathrm{ra}}(q)\right|_{\vec{q}\to0}b^{nl}-\frac{\mathrm{g}-2s}{4}im\left[\epsilon^{j(k}G^{i),n}_{\mathrm{ra}}(x)b^{nl}+b^{jn}G^{n,(k}_{\mathrm{ra}}(x)\epsilon^{i)l}\right],\end{aligned} \tag{53}$$

where

$$b^{ij}=\omega\delta^{ij}-i\omega_c\epsilon^{ij},\ \omega_c=B/m.$$

B. Linear response tensor

In the following, we study the structure of the correlators, which allows us to transform Eq. (53) into a relation between the independent viscosity coefficients and the conductivity. Define the non-equilibrium current

$$\langle J^\mu(x)\rangle_{\mathrm{r}} \equiv \frac{1}{\sqrt{g}}\frac{\delta W}{\delta A_{\mathrm{a}\mu}(x)}, \tag{54}$$

from which we have

$$\frac{\delta\langle J^\mu(x)\rangle_{\mathrm{r}}}{\delta A_{\mathrm{r}\nu}(0)} = G^{\mu,\nu}_{\mathrm{ra}}(x). \tag{55}$$

The deviation in the current from its equilibrium value can be formally expanded in time derivatives

$$\delta\langle J^\mu(x)\rangle_{\mathrm{r}} = -\int d^3x'\sigma_1^{\mu\nu}(x-x')\delta A_{\mathrm{r}\nu}(x') - \int d^3x'\sigma_2^{\mu\nu}(x-x')\partial'_t\delta A_{\mathrm{r}\nu}(x') + \cdots . \tag{56}$$

In linear response theory, one usually is interested in the case

$$\delta\langle J^i(x)\rangle_{\mathrm{r}} = \int d^3x'\sigma^{ij}(x-x')\delta E_{\mathrm{r}j}(x'). \tag{57}$$

Its variation gives

$$\frac{\delta\langle J^i(x)\rangle_{\mathrm{r}}}{\delta A_{\mathrm{r}j}(0)} = -\partial_t\sigma^{ij}(x). \tag{58}$$

Combining Eqs. (55) and (58), the correlator of currents can be expressed as the conductivity tensor

$$G^{i,j}_{\mathrm{ra}}(x) = -\partial_t\sigma^{ij}(x). \tag{59}$$

Similarly, define the non-equilibrium stress tensor

$$\langle T^{ij}(x)\rangle_{\mathrm{r}} = \frac{2}{\sqrt{g}}\frac{\delta W}{\delta g_{\mathrm{a}ij}(x)}.$$

Then we have

$$\frac{\delta\langle T^{ij}(x)\rangle_{\mathrm{r}}}{\delta g_{\mathrm{r}kl}(0)} = 2G^{ij,kl}_{\mathrm{ra}}(x) - \frac{1}{2}\delta^{kl}\langle T^{ij}\rangle\delta(x).$$

Vary the spatial components of the metric and define the elastic modulus and viscosity tensors by the expansion

$$\delta\langle T^{ij}(x)\rangle_{\mathrm{r}} = -\frac{1}{2}\int d^3x'\lambda^{ijkl}(x-x')\delta g_{\mathrm{r}kl}(x') - \frac{1}{2}\int d^3x'\eta^{ijkl}(x-x')\partial'_t\delta g_{\mathrm{r}kl}(x') + \cdots .$$

In other words

$$\frac{\delta\langle T^{ij}(x)\rangle_{\mathrm{r}}}{\delta g_{\mathrm{r}kl}(0)} = -\frac{1}{2}\lambda^{ijkl}(x) - \frac{1}{2}\partial_t\eta^{ijkl}(x).$$

By comparison, we obtain

$$G^{ij,kl}_{\mathrm{ra}}(x) = \frac{1}{4}\delta^{kl}\langle T^{ij}\rangle\delta(x) - \frac{1}{4}\lambda^{ijkl}(x) - \frac{1}{4}\partial_t\eta^{ijkl}(x). \tag{60}$$

C. Elastic modulus

The tensor $\lambda^{ijkl}(x)$ is the stress response up to the zeroth-order in time derivatives. However, it is also enough to treat it at zeroth- order in space derivatives, since our final goal is to obtain the viscosity at all frequencies and zero wave number. In other words, we can use the approximation of the perfect fluid. For a system without magnetic field, the hydrodynamic expansion is given by Ref. [21]

$$\delta\langle T^{ij}(x)\rangle_{\mathrm{r}} = -\left(P\delta^{ik}\delta^{jl} + \frac{1}{2}\delta^{ij}\delta^{kl}\kappa^{-1}\right)\delta g_{kl}(x), \tag{61}$$

where $\kappa^{-1} \equiv -V(\partial P/\partial V)_{S,N}$ is the inverse compressibility. The case of the system with a magnetic field is similar. Since the stress density in a rotationally invariant system with volume V includes both the pressure P and magnetization M [13]

$$\langle T^{ij}\rangle = \delta^{ij}P_{\mathrm{int}},\ P_{\mathrm{int}} = P - \frac{MB}{V},$$

the constitutive equation at leading order can be written as

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P_{\mathrm{int}}\left(u^\mu u^\nu + g^{\mu\nu}\right).$$

This result is consistent with the one obtained in Refs. [22, 23] both for relativistic and non-relativistic systems, though B is taken there as first order in derivatives.

Consider the energy of the system as $E(N,V,B) = V\varepsilon(\nu,B)$, where ν is the filling factor. One can define the internal inverse compressibility

$$\kappa_{\mathrm{int}}^{-1} \equiv -V\left.\frac{\partial P_{\mathrm{int}}}{\partial V}\right|_{\nu,N} = B^2\left.\frac{\partial^2\varepsilon(\nu,B)}{\partial B^2}\right|_{\nu}. \tag{62}$$

Thus, the elastic modulus can be decomposed as

$$\lambda^{ijkl}(x) = \left[P_{\mathrm{int}}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right) + \delta^{ij}\delta^{kl}\kappa_{\mathrm{int}}^{-1}\right]\delta(x). \tag{63}$$

D. Irreducible decomposition of response tensors

Any rank-2 tensor can be decomposed as a symmetric trace, a symmetric traceless and an antisymmetric part, so we have

$$\sigma^{ij}(x) = \sigma_L(x)\delta^{ij} + \sigma_T^{ij}(x) + \sigma_H(x)\epsilon^{ij}, \tag{64}$$

in which the Hall and longitudinal conductivities

$$\sigma_H \equiv \frac{1}{2}\left(\sigma^{12} - \sigma^{21}\right),\ \sigma_L \equiv \frac{1}{2}\left(\sigma^{11} + \sigma^{22}\right)$$

are frequently used in references. Also consider the tensor $\eta^{ijkl}(x)$ divided into its symmetric and antisymmetric parts in the pairs of indices ij and kl

$$\eta^{ijkl}(x) = \eta_S^{ijkl}(x) + \eta_A^{ijkl}(x).$$

We restrict our interest to the systems with rotational invariance. Then the symmetric part has only two independent components

$$\eta_S^{ijkl}(x) = \zeta(x)\delta^{ij}\delta^{kl} + \eta_{sh}(x)\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \delta^{ij}\delta^{kl}\right), \tag{65}$$

and the antisymmetric part has one

$$\eta_A^{ijkl}(x) = \eta^H(x)\left(\delta^{jk}\epsilon^{il} - \delta^{il}\epsilon^{kj}\right). \tag{66}$$

E. Viscosity and conductivity

Now combing Eqs. (59), (60) and (63), we can recast Eq. (53) as

$$\begin{aligned}&\bar{\eta}^{ijkl}(\omega)-\frac{1}{2i\omega}\left(\delta^{ij}\delta^{kl}+\delta^{kj}\delta^{il}\right)\kappa_{\text{int}}^{-1}\\=&\ \frac{m^2}{2}b^{jm}\left.\frac{\partial^2\sigma^{mn}(q)}{\partial q_i\partial q_k}\right|_{\vec{q}\to 0}b^{nl}-\frac{\mathrm{g}-2s}{4}im\left[b^{nl}\epsilon^{j(i}\sigma^{k)n}(\omega)+b^{jn}\sigma^{n(i}(\omega)\epsilon^{k)l}\right].\end{aligned}\tag{67}$$

where the "−" denotes

$$\bar{\eta}^{ijkl}=\frac{1}{2}\left(\eta^{ijkl}+\eta^{kjil}\right).$$

Note that all the contact terms exactly cancel, up to the term with κ_{int}^{-1}.

Plugging Eqs. (64), (65) and (66) into Eq. (67), we can extract respectively the bulk, shear and Hall viscosities from the conductivities at all frequencies

$$\zeta-\frac{1}{i\omega}\kappa_{\text{int}}^{-1}=\frac{m^2}{2}\left(\omega^2-\omega_c^2\right)\frac{\partial^2}{\partial q_1^2}\left[\sigma^{11}(q)-\sigma^{22}(q)\right]\Big|_{\vec{q}\to 0}+\frac{\mathrm{g}-2s}{2}im\left[i\omega_c\sigma_L(\omega)-\omega\sigma_H(\omega)\right],\tag{68}$$

$$\eta_{sh}(\omega)=\frac{m^2}{2}\frac{\partial^2}{\partial q_1^2}\left[\omega^2\sigma^{22}(q)+\omega_c^2\sigma^{11}(q)+2i\omega_c\omega\sigma_H(q)\right]\Big|_{\vec{q}\to 0}-\frac{\mathrm{g}-2s}{2}im\left[i\omega_c\sigma_L(\omega)-\omega\sigma_H(\omega)\right],\tag{69}$$

$$\eta_H(\omega)=\frac{m^2}{2}\frac{\partial^2}{\partial q_1^2}\left[\left(\omega^2+\omega_c^2\right)\sigma_H(q)-2i\omega\omega_c\sigma_L(q)\right]\Big|_{\vec{q}\to 0}-\frac{\mathrm{g}-2s}{4}im\left[\omega\sigma_L(\omega)+i\omega_c\sigma_H(\omega)\right].\tag{70}$$

Note that the above four equations recover Eqs. (4.11-4.14) in Ref. [13] when $\mathrm{g}-2s=0$.

In the limit of $m\to 0$, we have the regular identities

$$\begin{aligned}&\bar{\eta}^{ijkl}(\omega)-\frac{1}{2i\omega}\left(\delta^{ij}\delta^{kl}+\delta^{kj}\delta^{il}\right)\kappa_{\text{int}}^{-1}\\=&\ -\frac{1}{2}B^2\epsilon^{jm}\epsilon^{nl}\left.\frac{\partial^2\sigma^{mn}(q)}{\partial q_i\partial q_k}\right|_{\vec{q}\to 0}-\frac{\mathrm{g}-2s}{4}B\left[\epsilon^{nl}\epsilon^{j(i}\sigma^{k)n}(\omega)+\epsilon^{jn}\sigma^{n(i}(\omega)\epsilon^{k)l}\right],\end{aligned}\tag{71}$$

and

$$\zeta-\frac{1}{i\omega}\kappa_{\text{int}}^{-1}=-\frac{1}{2}B^2\frac{\partial^2}{\partial q_1^2}\left[\sigma^{11}(q)-\sigma^{22}(q)\right]\Big|_{\vec{q}\to 0}-\frac{\mathrm{g}-2s}{2}B\sigma_L(\omega),\tag{72}$$

$$\eta_{sh}(\omega)=\frac{1}{2}B^2\left.\frac{\partial^2\sigma^{11}(q)}{\partial q_1^2}\right|_{\vec{q}\to 0}+\frac{\mathrm{g}-2s}{2}B\sigma_L(\omega),\tag{73}$$

$$\eta_H(\omega)=\frac{1}{2}B^2\left.\frac{\partial^2\sigma_H(q)}{\partial q_1^2}\right|_{\vec{q}\to 0}+\frac{\mathrm{g}-2s}{4}B\sigma_H(\omega).\tag{74}$$

A number of interesting identities of this type were recently found for nonzero g in Ref. [24].

VI. NEWTON-CARTAN GEOMETRY WITH TORSION

The derivation of the Ward identities in the previous sections is quite straightforward, but the diffeomorphism invariance of the resulting equations can be verified only by rather cumbersome direct calculation. We now develop a formalism in which the diffeomorphism invariance is explicit at each stage of the calculation. That formalism is a version of Newton-Cartan geometry, which has been previously applied to the quantum Hall problem was developed first in the context of non-relativistic gravity by Cartan [25, 26] and may be viewed as the natural structure preserved by a gauging of Galilean symmetry [27, 28].

This section differs from our previous work in that we consider torsionful backgrounds. Torsionful geometries are generally necessary in the presence of a nontrivial dilaton field and have for example been considered in Ref. [29], where it is shown that boundary theory corresponding to a $z = 2$ Lifschitz spacetime is set in a torsionful Newton-Cartan setting. We now describe this torsionful version of Newton-Cartan geometry.

A Newton-Cartan geometry is a manifold endowed with a one-form n_μ, a degenerate metric tensor with upper indices $g^{\mu\nu}$ for which n_μ is a zero eigenvector and a vector v^μ whose projection onto n_μ is 1

$$g^{\mu\nu} n_\mu = 0 \qquad\qquad n_\mu v^\mu = 1. \tag{75}$$

From (g, n, v) one can uniquely define a metric tensor with lower indices $g_{\mu\nu}$ by requiring

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu - v^\mu n_\nu, \quad g_{\mu\nu} v^\nu = 0. \tag{76}$$

We define a connection by

$$\Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_\mu n_\nu + \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{77}$$

It is easy to see that under coordinate reparameterizations $\Gamma^\lambda{}_{\mu\nu}$ transforms as required for a connection.

In the simplest version of the Newton-Cartan geometry, n_μ is assumed to be a closed one-form. In this case the connection (77) is torsionless: $\Gamma^\lambda{}_{[\mu\nu]} = 0$. We shall not assume that this is the case; instead, we only assume the weaker condition

$$n \wedge dn = 0. \tag{78}$$

By the Frobenius theorem, n_μ then locally defines a unique spatial slicing to which n_μ is normal, giving us a preferred notion of space. This condition was also imposed in Ref. [29] so that connection on these slices is the usual, torsionless Riemannian connection. However, we note that this is in fact generally required by the causality of a non-relativistic theory. One may show that if $n \wedge dn \neq 0$ at a point x, there is a neighborhood of x in which every point may be reached by a future directed curve (one in which the tangent u^μ satisfies $n_\mu u^\mu > 0$) [30]. In particular, an observer may with sufficient speed intersect his own past.

In the case that $dn \neq 0$ the connection has nonzero torsion

$$T^\lambda{}_{\mu\nu} \equiv 2\Gamma^\lambda{}_{[\mu\nu]} = 2v^\lambda \partial_{[\mu} n_{\nu]}. \tag{79}$$

The torsion has the following property: it vanishes when all indices are lowered or raised,

$$T_{\lambda\mu\nu} \equiv g_{\lambda\alpha} T^\alpha{}_{\mu\nu} = 0, \quad T^{\lambda\mu\nu} \equiv g^{\mu\alpha} g^{\nu\beta} T^\lambda{}_{\alpha\beta} = 0. \tag{80}$$

The first equation comes from $g_{\lambda\alpha} v^\alpha = 0$. To see the second equation, one can work in the coordinate system where $n_i = 0$. This condition assumes no spatial torsion, which in the condensed matter context corresponds to a nontrivial Burgers vector density, and may be relaxed if one wishes to study material defects. The torsion (79) on the other hand is temporal, and finds its origin in the presence of a nontrivial gravitational potential $-\Phi$ (see (89)).

The connection $\Gamma^\lambda{}_{\mu\nu}$ has some further interesting features. It is compatible with the metric $g^{\mu\nu}$ and with n_μ,

$$\nabla_\lambda g^{\mu\nu} = 0, \qquad \nabla_\nu n_\mu = 0. \tag{81}$$

On the other hand, the covariant derivatives of $g_{\mu\nu}$ and v^μ are nonzero. They can be expressed in terms of the Lie derivative of the metric along v^μ,

$$\nabla_\lambda g_{\mu\nu} = -\tau_{\lambda(\mu} n_{\nu)}, \qquad \nabla_\nu v^\mu = \frac{1}{2} \tau_{\nu\alpha} g^{\alpha\mu} \tag{82}$$

$$\tau_{\mu\nu} \equiv \pounds_v g_{\mu\nu} = v^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu} \partial_\mu v^\lambda + g_{\mu\lambda} \partial_\nu v^\lambda \tag{83}$$

Using $v^\mu \tau_{\mu\nu} = 0$ one can show that

$$\begin{aligned} &g_{\alpha[\mu} \nabla_{\nu]} v^\alpha = 0 \qquad \nabla_\alpha v^{[\mu} g^{\nu]\alpha} = 0 \qquad g^{\mu\alpha} g^{\nu\beta} \nabla_\lambda g_{\alpha\beta} = 0 \\ &v^\lambda \nabla_\lambda g_{\mu\nu} = 0 \qquad v^\lambda \nabla_\lambda v^\mu = 0. \end{aligned} \tag{84}$$

In fact, it is possible to show that the connection (77) is uniquely determined from equations (80), (81), and the first equation in (84). The connection of course also defines a unique volume element by

$$\nabla_\rho \varepsilon_{\mu\nu\lambda} = 0. \tag{85}$$

A. Conservation Laws with Torsion

The way the connection is defined introduces one subtlety which is important for our further discussion. Namely, in a Newton-Cartan theory current conservation

$$\partial_\mu(e^{-\Phi}\sqrt{g}\, j^\mu) = 0 \tag{86}$$

does not have the familiar form $\nabla_\mu j^\mu = 0$, but instead is

$$(\nabla_\mu - G_\mu) j^\mu = 0 \qquad \text{where} \qquad G_\mu = T^\nu{}_{\nu\mu}. \tag{87}$$

We will find this combination of ∇_μ and G_μ recurring often. This is because the usual formula for integration by parts is modified on a torsionful manifold. Because $\frac{1}{\sqrt{g}e^{-\Phi}}\partial_\mu\big(\sqrt{g}e^{-\Phi}\big) = \Gamma^\nu{}_{\nu\mu} - T^\nu{}_{\nu\mu}$, in addition to the usual minus sign, we must also take $\nabla_\mu \to \nabla_\mu - G_\mu$ upon an exchange of the derivative.

Furthermore, (87) is consistent with time independence of total charge on a torsionful manifold. By Stokes' theorem

$$\begin{aligned}\int_{\Sigma_1} n_\mu j^\mu - \int_{\Sigma_2} n_\mu j^\mu &= \int \varepsilon^{\mu\nu\lambda}\partial_\mu j_{\nu\lambda} \\ &= \int \left(\varepsilon^{\mu\nu\lambda}\nabla_\mu j_{\nu\lambda} - T^\nu{}_{\nu\mu} j^\mu\right) \\ &= \int (\nabla_\mu - G_\mu) j^\mu = 0,\end{aligned} \tag{88}$$

where Σ_1 and Σ_2 are spatial slices, $n_\mu j^\mu$ is the charge density and $j_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda} j^\lambda$ is the dual of j_μ.

B. Coordinate Expressions

To gain some intuition for the above objects and to connect this discussion with the non-covariant presentation of the previous sections, we introduce a parameterization of the geometry by going into coordinates. In some coordinate patch, we have without loss of generality

$$n_\mu = \left(e^{-\Phi},\ -e^{-\Phi}C_i\right) \qquad v^\mu = \begin{pmatrix} e^\Phi(1 + C_j v^j) \\ e^\Phi v^i \end{pmatrix}. \tag{89}$$

As we shall see, this is the same C_i introduced previously to couple to the energy current. Because $n \wedge dn = 0$, we may always choose coordinates where $C_i = 0$. Writing out $n \wedge dn = 0$ in coordinates gives (27), so such coordinates are indeed the global time coordinates discussed before. From the Newton-Cartan perspective, this condition is elegant and physically motivated.

However, it is often necessary to work outside of GTC, at least to first order, in order to calculate the energy current. Given (89), the following coordinate expressions follow straightforwardly

$$g^{\mu\nu} = \begin{pmatrix} C^2 & C^j \\ C^i & g^{ij} \end{pmatrix} \qquad g_{\mu\nu} = \begin{pmatrix} v^2 & -v_j - v^2 C_j \\ -v_i - v^2 C_i & g_{ij} + v_i C_j + v_j C_i + v^2 C_i C_j \end{pmatrix}. \tag{90}$$

The transformation laws (26) for Φ, C_i and g_{ij} can now be derived from the above expressions as natural consequences of the covariant transformations

$$\begin{aligned} &\delta n_\mu = -\xi^\lambda \partial_\lambda n_\mu - n_\lambda \partial_\mu \xi^\lambda \qquad\qquad \delta v^\mu = -\xi^\lambda \partial_\lambda v^\mu + v^\lambda \partial_\lambda \xi^\mu \\ &\delta g^{\mu\nu} = -\xi^\lambda \partial_\lambda g^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu. \end{aligned} \tag{91}$$

We also have

$$\varepsilon_{\mu\nu\lambda} = \sqrt{g}e^{-\Phi}\epsilon_{\mu\nu\lambda} \qquad \tau_{ij} = e^\Phi\left(\nabla_i v_j + \nabla_j v_i + \dot{g}_{ij}\right). \tag{92}$$

Here $\epsilon_{\mu\nu\lambda}$ is the antisymmetric symbol with $\epsilon_{012} = 1$. The remaining components of $\tau_{\mu\nu}$ are specified by the transverse condition $\tau_{\mu\nu} v^\nu = 0$. We see that $\tau_{\mu\nu}$ is a spacetime covariant form of the fluid shear.

C. The Velocity v^μ and the Covariant Vector Potential

In the above, $g^{\mu\nu}$ and n_μ play essential roles with clear physical interpretations. n_μ gives an absolute notion of space via it's integral submanifolds. $g^{\mu\nu}$ restricts to a Riemannian metric on space and supplies an invariant notion of distance. The "velocity" vector v^μ on the other hand is the odd man out, what is it supposed to represent? Indeed, for a general non-relativistic theory there can be no preferred vector field since it's integral curves would define a distinguished family of observers. Only in the presence of a background medium that breaks non-relativistic boost invariance (an "ether") would this be physically acceptable.

For us, v^μ is merely a convenience, an inessential structure that we use to help define a partial metric inverse and a connection. It may be selected at will only subject to the constraint $n_\mu v^\mu = 1$. In the presence of a fluid one useful choice is for v^i to simply be the fluid velocity. In this paper however we prefer not to assume anything about the system beyond diffeomorphism invariance, and so do not have any preferred notion for v^μ. We will rather be more concerned with demonstrating the v^μ independence of our treatment.

If we were to leave it at that v^μ may be more trouble than it's worth: whenever it appears we need to worry if the physics depends on an arbitrary choice. However, in the presence of a vector potential A_μ, having v^μ around is crucial. In GTC, the vector potential obeys an anomalous transformation law

$$\begin{aligned}\delta A_0 &= -\xi^\lambda\partial_\lambda A_0 - A_\lambda\partial_0\xi^\lambda + \frac{\mathrm{g}-2s}{4}\Big(\varepsilon^{ij}\tilde{\partial}_i(g_{jk}\dot{\xi}^k) + \varepsilon^i{}_j\dot{C}_i\dot{\xi}^j\Big)\\ \delta A_i &= -\xi^\lambda\partial_\lambda A_i - A_\lambda\partial_i\xi^\lambda - me^\Phi g_{ij}\dot{\xi}^j - \frac{\mathrm{g}-2s}{4}\varepsilon^{ij}C_i\partial_i(g_{jk}\dot{\xi}^k)\end{aligned} \tag{93}$$

to first order in C_i. Extending the discussion Ref. [7] to arbitrary g and s, we may use the components of v^μ to define a modified gauge field

$$\begin{aligned}\tilde{A}_0 &= A_0 - \frac{1}{2}me^\Phi v^2 - \frac{\mathrm{g}-2s}{4}\varepsilon^{ij}\left(\tilde{\partial}_i v_j + \dot{C}_i v_j\right)\\ \tilde{A}_i &= A_i + me^\Phi v_i + \frac{1}{2}me^\Phi v^2 C_i + \frac{\mathrm{g}-2s}{4}C_i\varepsilon^{jk}\partial_j v_k\end{aligned} \tag{94}$$

that transforms covariantly under diffeomorphisms

$$\delta\tilde{A}_\mu = -\xi^\lambda\partial_\lambda\tilde{A}_\mu - \tilde{A}_\lambda\partial_\mu\xi^\lambda. \tag{95}$$

All transformations (26) then follow by representing a Newton-Cartan background in a system of coordinates, except for the transformation of the non-covariant vector potential, which also relies on the decomposition (94).

Thus we may use v^μ to take any invariant effective action phrased in terms of the components g^{ij}, Φ, C_i and the vector potential A_μ and present it as a functional of only covariant objects

$$W[g^{ij}, \Phi, C_i, A_\mu] = W[g^{\mu\nu}, n_\mu, \tilde{A}_\mu, v^\mu]. \tag{96}$$

Since the original action carried no v^i dependence, the covariant version must have the following special property: it is invariant under changes to v^μ and $\tilde{A}_\mu$ that leave the physical vector potential A_μ unchanged. In section VII B, we use this to provide another demonstration that the momentum of a nonrelativistic system is determined by the charge flow.

It is easy to now write down our microscopic action in a manifestly covariant form using Newton-Cartan geometry

$$S = \int d^3x\sqrt{g}e^{-\Phi}\Big(\frac{i}{2}v^\mu\psi^\dagger\overleftrightarrow{D}_\mu\psi - \frac{1}{2m}\big(g^{\mu\nu} + \frac{i\mathrm{g}}{2}\varepsilon^{\mu\nu}\big)D_\mu\psi^\dagger D_\nu\psi - \lambda|\psi|^4\Big) \tag{97}$$

where the covariant derivative involves the modified vector potential and the Newton-Cartan spin connection

$$\omega_\mu = \frac{1}{2}\epsilon_{ab}e^{a\nu}\nabla_\mu e^b_\nu. \tag{98}$$

Plugging in the coordinate expressions of the geometry, we find this action reduces to the microscopic action considered previously with all sources present. Indeed, for $g = s = 0$, it was shown in Ref. [28] that one may generally promote a Galilean invariant theory to a diffeomorphism invariant one via the simple prescription

$$D_0 = v^\mu D_\mu \qquad D_a = e^\mu_a D_\mu \tag{99}$$

which is all that we've done here.

Note that in the LLL limit, the physical vector potential is already a one-form and need not be modified. In this simple case v^μ is truly unnecessary and can be discarded

$$W_{\mathrm{LLL}}[g^{ij}, \Phi, C_i, A_\mu] = W_{\mathrm{LLL}}[g^{\mu\nu}, n_\mu, A_\mu]. \tag{100}$$

VII. COVARIANT WARD IDENTITIES

In section IV we derived Ward identities by considering the variation of A_μ, Φ, C_i and g^{ij} under nonrelativistic diffeomorphisms. In this approach, the physical meaning of the currents $j^\mu_{\rm nc}$, $\varepsilon^\mu_{\rm nc}$, and $T^{ij}_{\rm nc}$ is clear, and the resulting Ward identities take the form of the fluid dynamical equations of motion [16]. However from the Newton-Cartan point of view, the above approach is somewhat unnatural. Φ, C_i, and g^{ij} are merely the components of covariant objects n_μ and $g^{\mu\nu}$ in some choice of coordinates and A_μ is not even a one-form. Similarly, the above currents do not form spacetime vectors and tensors in an obvious way.

In what follows, we reformulate the previous work in a fully geometric fashion. We begin with an effective action written as a functional of the geometry and the modified gauge field

$$W[n_\mu, g^{\mu\nu}, v^\mu, \tilde{A}_\mu] \tag{101}$$

and then define currents j^μ, ε^μ and $T^{\mu\nu}$ that transform as spacetime tensors. Covariant Ward identities are derived. In section VII B we impose the v^i independence of the action as well as demonstrate the relationship between the covariantly defined currents and the "nc" currents considered previously.

A. Variation of the Action

Defining covariant currents requires some care as not all components of the background fields n_μ, $g^{\mu\nu}$ and v^μ are independent. Since the geometry is constrained to satisfy $n_\mu v^\mu = 1$ and $g^{\mu\nu} n_\nu = 0$, an arbitrary variation is not allowed. Rather, the most general change may be parameterized in terms of an arbitrary δn_μ, a transverse velocity perturbation $\delta u^\mu n_\mu = 0$ and a transverse metric perturbation $\delta h^{\mu\nu} n_\nu = 0$

$$\delta n_\mu \qquad\qquad \delta v^\mu = -v^\mu v^\lambda \delta n_\lambda + \delta u^\mu \qquad\qquad \delta g^{\mu\nu} = -v^\mu \delta n^\nu - \delta n^\mu v^\nu - \delta h^{\mu\nu}. \tag{102}$$

δn_μ, δu^μ and $\delta h^{\mu\nu}$ are then completely independent and the currents defined by

$$\delta W = \int d^3x \sqrt{g} e^{-\Phi} \Big(\frac{1}{2} T^{\mu\nu} \delta h_{\mu\nu} + j^\mu \delta \tilde{A}_\mu - \varepsilon^\mu \delta n_\mu - p_\mu \delta u^\mu \Big) \tag{103}$$

where $T^{\mu\nu}$ and p_μ are fixed to be transverse

$$T^{\mu\nu} n_\nu = 0 \qquad p_\mu v^\mu = 0. \tag{104}$$

The only new current in this collection is p_μ. We shall find that it is related to the momentum density of the system and is completely fixed by the v^μ independence of the effective action.

Under spacetime diffeomorphisms, the background fields change as

$$\begin{aligned} \delta n_\mu &= -n_\lambda \nabla_\mu \xi^\lambda + T^\lambda{}_{\mu\nu} n_\lambda \xi^\nu & \qquad \delta g^{\mu\nu} &= \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu + \big(T^\mu{}_\lambda{}^\nu + T^\nu{}_\lambda{}^\mu \big) \xi^\lambda \\ \delta v^\mu &= -\xi^\lambda \nabla_\lambda v^\mu + v^\lambda \nabla_\lambda \xi^\mu - T^\mu{}_{\nu\lambda} v^\nu \xi^\lambda & \qquad \delta \tilde{A}_\mu &= -\xi^\lambda \nabla_\lambda \tilde{A}_\mu - \tilde{A}_\lambda \nabla_\mu \xi^\lambda + T^\lambda{}_{\mu\nu} \tilde{A}_\lambda \xi^\nu, \end{aligned} \tag{105}$$

where we have exchanged the coordinate derivatives appearing in (93) and (95) for covariant derivatives. These immediately give expressions for $\delta u^\mu = P^\mu{}_\nu \delta v^\nu$ and $\delta h^{\mu\nu} = -P^\mu{}_\lambda P^\nu{}_\rho \delta g^{\lambda\rho}$. Gauge transformations are of course unchanged

$$\delta \tilde{A}_\mu = \nabla_\mu \alpha. \tag{106}$$

Proceeding as before, we find the Ward identities corresponding to gauge invariance and diffeomorphism invariance are

$$\begin{gathered} (\nabla_\mu - G_\mu) j^\mu = 0 \\ \nabla_\nu (p_\mu v^\nu) + p_\nu \nabla_\mu v^\nu + (\nabla_\nu - G_\nu) T_\mu{}^\nu - n_\mu (\nabla_\nu - G_\nu) \varepsilon^\nu = \tilde{F}_{\mu\nu} j^\nu - G_{\mu\nu} \varepsilon^\nu \end{gathered} \tag{107}$$

where we have defined the following notation: $\tilde{F}_{\mu\nu} = (d\tilde{A})_{\mu\nu}$ is the Newton-Cartan analogue of the electromagnetic field strength and $G_{\mu\nu} = (dn)_{\mu\nu}$ is similarly a "torsional field strength."

Equations (107) are unfamiliar enough to deserve a few comments. We first observe that current conservation no longer takes the form $\nabla_\mu j^\mu = 0$, but rather $(\nabla_\mu - G_\mu) j^\mu = 0$ as discussed in section VI A. To bring the second equation into a more enlightening form, we first project it onto spatial slices by raising the index

$$\nabla_\nu (p^\mu v^\nu) + p_\nu \nabla^\mu v^\nu + (\nabla_\nu - G_\nu) T^{\mu\nu} = \tilde{F}^\mu{}_\nu j^\nu - G^\mu{}_\nu \varepsilon^\nu. \tag{108}$$

This simply expresses momentum conservation in the presence of external forces. $\tilde{F^\mu}_\nu j^\nu$ is of course the usual Lorentz force, but along with the p_μ terms, also makes contributions to the momentum current due to the modifications necessary to make $\tilde{A}_\mu$ covariant. For now, merely note that the torsion also exerts a "Lorentz force," but one that couples to the energy current rather than the charge current.

Finally, projecting (107) onto v^μ, we obtain the Newton-Cartan analogue of the work-energy equation

$$(\nabla_\mu - G_\mu)\varepsilon^\mu = -\tilde{F}_{\mu\nu} v^\mu j^\nu + G_{\mu\nu} v^\mu \varepsilon^\mu - \frac{1}{2}\tau^{\mu\nu} T_{\mu\nu}. \tag{109}$$

The first two terms on the right hand side represent the work done on the system by the external fields in a frame moving with velocity v^μ. In the case that v^μ represents a fluid velocity, the physics of the final term is relatively clear: it accounts for energy dissipated due to viscous forces.

B. Comparison with the Noncovariant Approach

Unfortunately, the currents defined in (103) differ from the standard currents $T^{ij}_{\rm nc}$, $\varepsilon^\mu_{\rm nc}$ and $j^\mu_{\rm nc}$ found in the non-covariant Ward identities. To see how, express δn_μ, $\delta h^{\mu\nu}$, δu^μ and $\delta \tilde{A}_\mu$, in terms of $\delta\Phi$, δC_i, δg^{ij}, δv^i and δA_μ and set (103) equal to

$$\delta W = \int d^3x \sqrt{g} e^{-\Phi}\Big(\frac{1}{2} T^{ij}_{\rm nc}\delta g_{ij} + j^\mu_{\rm nc}\delta A_\mu + \varepsilon^0_{\rm nc}\delta\Phi + \varepsilon^i_{\rm nc}\delta C_i\Big). \tag{110}$$

The absence of δv^i terms is equivalent to the v^μ independence of the original action. This procedure then completely fixes p_μ to be

$$p_\mu = m j_\mu - \frac{\mathrm{g}-2s}{4}\varepsilon_{\mu\nu}\nabla^\nu\left(n_\lambda j^\lambda\right). \tag{111}$$

So long as p_μ takes this value, the identities (107) are guaranteed to be independent of changes to v^μ and $\tilde{A}_\mu$ that leave A_μ fixed, despite appearances to the contrary. Note that since $j_\mu = g_{\mu\nu} j^\nu$, the ith component of j_μ is not $j_i = g_{ij} j^j$, but rather $j_i - v_i j^0$.

The remaining relationships are

$$\begin{aligned}
T^{ij}_{\rm nc} &= T^{ij} + m e^\Phi\big(v^i j^j + v^j j^i - j^0 v^i v^j\big) + \frac{\mathrm{g}-2s}{4} e^\Phi\big(j^0 \Omega g^{ij} - 2v^{(i}\varepsilon^{j)k} e^\Phi \partial_k(e^{-\Phi} j^0)\big)\\
\varepsilon^0_{\rm nc} &= e^{-\Phi}\varepsilon^0 + m e^\Phi\big(j_i v^i - \frac{1}{2} j^0 v^2\big)\\
\varepsilon^i_{\rm nc} &= e^{-\Phi}\varepsilon^i + T^{ij} v_j + e^\Phi v^i v^j p_j + \frac{1}{2} m e^\Phi j^i v^2 + \frac{\mathrm{g}-2s}{4}\big(\Omega j^i + \varepsilon^{ij} v_j e^\Phi \partial_0(e^{-\Phi} j^0)\big)\\
j^\mu_{\rm nc} &= j^\mu
\end{aligned} \tag{112}$$

where $\Omega = \varepsilon^{ij}\partial_i v_j$. Importantly, note that it is the non-covariant currents that are v^i independent. The covariant versions will change with the choice of v^i, but the above combinations will not.

For nonvanishing charge density, one convenient choice is $v^i = j^i/j^0$. We then have

$$\begin{aligned}
T^{ij}_{\rm nc} &= m e^\Phi j^0 v^i v^j + T^{ij} & j^\mu_{\rm nc} &= j^\mu\\
\varepsilon^0_{\rm nc} &= \frac{1}{2} m e^\Phi j^0 v^2 + e^{-\Phi}\varepsilon^0 & \varepsilon^i_{\rm nc} &= \frac{1}{2} m e^\Phi j^0 v^i v^2 + e^{-\Phi}\varepsilon^i + T^{ij} v_j
\end{aligned} \tag{113}$$

where we have taken $\mathrm{g} = 2$ $s = 1$ for simplicity. We thus see that in this frame the covariant currents have a clear physical interpretation: T^{ij} and ε^μ are the *internal* stress and energy currents of the system, that is, the currents that do not arise due to the motion of material from one place to another.

Let's express the covariant Ward identities in terms of the non- covariant currents to check their v^i independence. First decompose the Newton-Cartan field strength $\tilde{F}_{\mu\nu}$ into the usual electromagnetic field strength plus the modifications necessary to make a spacetime tensor.

$$\begin{aligned}
\tilde{F}_{\mu\nu} &= F_{\mu\nu}\\
&\quad + \begin{pmatrix} 0 & m\Big(\partial_0(e^\Phi v_j) + \frac{1}{2}\nabla_j(e^\Phi v^2)\Big) + \frac{\mathrm{g}-2s}{2}\nabla_j\Omega \\ -m\Big(\partial_0(e^\Phi v_i) + \frac{1}{2}\nabla_i(e^\Phi v^2)\Big) - \frac{\mathrm{g}-2s}{2}\nabla_i\Omega & m\big(\nabla_i(e^\Phi v_j) - \nabla_j(e^\Phi v_i)\big)\end{pmatrix}.
\end{aligned} \tag{114}$$

We also require the formula

$$T_\mu{}^\nu = \begin{pmatrix} 0 & -v_k T^{kj} \\ 0 & T_i^{\,j} \end{pmatrix}, \tag{115}$$

which follows from the transverseness of the stress tensor. Then expanding out the 0th and ith components of (107), we obtain

$$\begin{aligned} \frac{1}{\sqrt{g}}\partial_0(\sqrt{g}e^{-\Phi}j^0_{\rm nc}) + \nabla_i(e^{-\Phi}j^i_{\rm nc}) &= 0 \\ \frac{1}{\sqrt{g}}\partial_0\big(\sqrt{g}\varepsilon^0_{\rm nc}\big) + e^\Phi\nabla_i(e^{-\Phi}\varepsilon^i_{\rm nc}) &= E_i j^i_{\rm nc} - \frac{1}{2}T^{ij}_{\rm nc}\dot{g}_{ij} \\ \frac{e^\Phi}{\sqrt{g}}\partial_0\Big(\sqrt{g}\big(mj_i - \frac{{\rm g}-2s}{4}\varepsilon_{ij}\nabla^j(e^{-\Phi}j^0)\big)\Big) + e^\Phi\nabla_j(e^{-\Phi}T_{{\rm nc}i}{}^j) & \\ &= j^0_{\rm nc}E_i + \varepsilon_{ij}j^j_{\rm nc}B + \varepsilon^0_{\rm nc}\nabla_i\Phi. \end{aligned} \tag{116}$$

The result is independent of v^i and in perfect agreement with the non-covariant Ward identities found previously.

VIII. CONCLUSION

In this paper we have proposed a new approach to studying the FQH effect. The effort here has been essentially formal and will serve as the foundation of later work where physical consequences are addressed. We've shown that by a special choice of spin and gyromagnetic ratio, a smooth massless limit is obtained and we exactly integrate out all higher Landau levels. This choice can always be made by virtue of a translation formula that tells one how to convert results for one g and s to any other value.

Furthermore, we have derived the complete set of Ward identities that follow from spacetime symmetries in arbitrary backgrounds. These Ward identities are the usual fluid equations of motion: stress conservation and the work-energy equation, which can be viewed as the consequence of a spacetime symmetry as in relativity. Finally, a covariant treatment of these Ward identities is then developed that makes that symmetry manifest.

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Appendix A: From Relativistic to Non-Relativistic Conservation Equations

In this appendix, we motivate the conservation laws (41), (42) and (43) from the relativistic point of view. This also makes the physical significance of the dilaton field Φ clearer: it arises as the relativistic lapse function. We begin with the relativistic continuity equation and conservation of stress-energy

$$\nabla_\mu j^\mu = 0 \qquad \nabla_\mu T_\nu{}^\mu = F_{\nu\mu}j^\mu \tag{A1}$$

with the metric ansatz

$$g_{\mu\nu} = \begin{pmatrix} -e^{-2\Phi} & 0 \\ 0 & g_{ij} \end{pmatrix} \tag{A2}$$

The Christoffel symbol is then

$$\begin{aligned} \Gamma^0{}_{00} &= -\dot{\Phi}, & \Gamma^0{}_{ij} &= \frac{1}{2}e^{2\Phi}\dot{g}_{ij}, & \Gamma^k{}_{00} &= -e^{-2\Phi}g^{kl}\partial_l\Phi \\ \Gamma^0{}_{0i} &= -\partial_i\Phi, & \Gamma^k{}_{0i} &= \frac{1}{2}g^{kl}\dot{g}_{li}, & \Gamma^k{}_{ij} &= \frac{1}{2}g^{kl}\left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}\right). \end{aligned}$$

Plugging this in, we find the continuity equation reads

$$\partial_\mu \left(\sqrt{g} e^{-\Phi} j^\mu\right) = 0 \tag{A3}$$

whereas the time and space components of stress-energy conservation are

$$\frac{1}{\sqrt{g}}\partial_0\left(\sqrt{g}T_0{}^0\right) + \nabla_i T_0{}^i + e^{-2\Phi}T_0{}^i\partial_i\Phi - \frac{1}{2}T^{ij}\dot{g}_{ij} \;=\; F_{0\mu}j^\mu \tag{A4}$$

$$\frac{1}{\sqrt{g}e^{-\Phi}}\partial_0\left(\sqrt{g}e^{-\Phi}T_j{}^0\right) + e^{\Phi}\nabla_i\left(e^{-\Phi}T_j{}^i\right) + T_0{}^0\partial_j\Phi \;=\; F_{j\mu}j^\mu \tag{A5}$$

To bring this into a form closer to that which appears in the main text, we define the energy density $\varepsilon^0_{\mathrm{nc}}$, energy flux $\varepsilon^i_{\mathrm{nc}}$ and momentum density p^i_{nc} as

$$T_0{}^\mu = -\varepsilon^\mu \qquad T_j{}^0 = p_j.$$

The conservation equations (A4) and (A5) now read

$$\frac{1}{\sqrt{g}}\partial_0\left(\sqrt{g}\varepsilon^0\right) + e^{\Phi}\nabla_i\left(e^{-\Phi}\varepsilon^i\right) \;=\; -F_{0\mu}j^\mu - \frac{1}{2}T^{ij}\dot{g}_{ij} \tag{A6}$$

$$\frac{e^\Phi}{\sqrt{g}}\partial_0\left(\sqrt{g}e^{-\Phi}p_j\right) + e^{\Phi}\nabla_i\left(e^{-\Phi}T_j{}^i\right) \;=\; F_{j\mu}j^\mu + \varepsilon^0\nabla_j\Phi. \tag{A7}$$

matching our non-covariant ward identities. Of course, the momentum and energy currents are not independent in a relativistic theory, but they are in the non-relativistic case.

Appendix B: Current Redefinitions – Noncovariant Version

In section II we remarked that how we choose to couple the system to curved geometry is largely arbitrary for flat space physics. For example, one can imagine adding additional curvature terms to the microscopic action. In curved geometry, we would of course have different dynamics, but the flat space equations of motion would be unchanged. At the same time, non-minimal couplings would in general alter the definition of the stress tensor, even in flat space.

However, there is another class of modifications that do *not* affect the dynamics even in a curved background. This freedom has great utility: it allows us to choose the parity breaking couplings g and s at will. In particular, we may always choose $\mathrm{g} = 2$ and $s = 1$. The LLL limit then exists and upon taking $m \to 0$, the momentum density vanishes. We now demonstrate how this works in detail.

Let's begin with s. Consider, as above, a theory of a single field ψ with charge 1 and spin s so that the covariant derivative takes the form

$$D_\mu\psi = (\partial_\mu - iA_\mu + is\omega_\mu)\psi. \tag{B1}$$

Assuming that A_μ and ω_μ only appear in the action in this way, we may absorb part of ω_μ into A_μ

$$(\partial_\mu - iA_\mu + is\omega_\mu)\psi = (\partial_\mu - iA'_\mu + is'\omega_\mu)\psi$$
$$\text{where} \quad A'_\mu = A_\mu + (s'-s)\omega_\mu. \tag{B2}$$

The dynamics of the system is unchanged, but the point of view different; we now have a new spin and externally applied electromagnetic field.

For simplicity take $\Phi = 0$, $C_i = 0$. We have two effective actions, S_s and $S_{s'}$, satisfying

$$W_s[A_\mu, g_{ij}] = W_{s'}[A_\mu + (s'-s)\omega_\mu, g_{ij}]. \tag{B3}$$

Under a metric perturbation, choose a gauge where $\delta e^a_i = \frac{1}{2}\delta g_{ij}e^{aj}$ and $\delta e^{ai} = \frac{1}{2}\delta g^{ij}e^a_j$. The perturbed spin connection is then

$$\delta\omega_0 = \frac{1}{4}\varepsilon^j{}_k\dot{g}^{ki}\delta g_{ij} \qquad\qquad \delta\omega_i = -\frac{1}{2}\varepsilon^{jk}\nabla_j\delta g_{ik}. \tag{B4}$$

Setting

$$\int d^3x\sqrt{g}e^{-\Phi}\Big(\frac{1}{2}T^{ij}_{\rm nc}\delta g_{ij}+j^\mu_{\rm nc}\delta A_\mu\Big)=\int d^3x\sqrt{g}e^{-\Phi}\Big(\frac{1}{2}T'^{ij}_{\rm nc}\delta g_{ij}+j'^\mu_{\rm nc}\delta A'_\mu\Big), \tag{B5}$$

we find a relation between the stress tensors defined in the two different pictures

$$j^\mu_{\rm nc}=j'^\mu_{\rm nc} \qquad\qquad T^{ij}_{\rm nc}=T'^{ij}_{\rm nc}+(s'^{k(i}\nabla_k j'^{j)}_{\rm nc}+\frac{1}{2}(s'-s)\dot{g}^{k(i}\varepsilon^{j)}{}_k j'^0_{\rm nc}. \tag{B6}$$

We are free to choose the spin however we like, so long as we use this stress tensor and the modified electromagnetic field (B2).

The same procedure allows us to redefine g as well, though the formulas are more cumbersome. Recall the full microscopic action for arbitrary g and s is

$$S_{gs}=\int d^3x\sqrt{g}e^{-\Phi}\Big(\frac{i}{2}e^\Phi\psi^\dagger\overleftrightarrow{D}_0\psi-\frac{1}{2m}\big(g^{ij}+\frac{i\mathrm{g}}{2}\varepsilon^{ij}\big)(\tilde{D}_i\psi)^\dagger(\tilde{D}_j\psi)-\lambda|\psi|^4\Big). \tag{B7}$$

We must briefly work outside of GTC, at least to first order, since our modifications will affect the energy current. Explicitly accounting for all appearances of the vector potential in the microscopic action we have

$$\begin{aligned}S=\int d^3x\sqrt{g}e^{-\Phi}\Bigg(&\frac{i}{2}e^\Phi\psi^\dagger\overleftrightarrow{\tilde\partial}_0\psi-\frac{1}{2m}\tilde\partial_i\psi^\dagger\tilde\partial^i\psi-\frac{i}{2m}\big(\tilde{A}^i-s\tilde\omega^i+\frac{\mathrm{g}}{4}\varepsilon^{ij}(\dot{C}_j-\tilde\partial_j\Phi)\big)\psi^\dagger\overleftrightarrow{\tilde\partial}_i\psi\\&+e^\Phi\Big(A_0-s\omega^0+\frac{\mathrm{g}}{4m}e^{-\Phi}\big(F-\varepsilon^{ij}(\tilde{A}_i-s\tilde\omega_i)(\dot{C}_j-\tilde\partial_j\Phi)\big)\\&-\frac{1}{2m}e^{-\Phi}(\tilde{A}_i-s\tilde\omega_i)(\tilde{A}^i-s\tilde\omega^i)\Big)|\psi|^2-\lambda|\psi|^4\Bigg)\end{aligned} \tag{B8}$$

where for convenience we have defined $F=i\varepsilon^{ij}\tilde{D}_i\tilde{D}_j=\Big(B-\frac{s}{2}R+\varepsilon^{ij}\big(E_i-\frac{s}{2}R_i\big)C_j\Big)$. Here $\tilde{A}_i$ denotes not the modified vector potential but $A_i+C_iA_0$. R and R_i are the curvature equivalents of the magnetic and electric fields

$$2\left(\partial_\mu\omega_\nu-\partial_\nu\omega_\mu\right)=\begin{pmatrix}0 & -R_j\\ R_i & \varepsilon_{ij}R\end{pmatrix}. \tag{B9}$$

R is simply the spatial Ricci scalar and $R_i=\varepsilon^{jk}\nabla_j\dot{g}_{ik}$ measures change in the geometry with time.

We seek a transformation that sends the third and fourth terms of (B8) to themselves but with $\mathrm{g}\to\mathrm{g}'$ and $s\to s'$. The algebra is somewhat prohibitive, but is greatly simplified if we only work to leading order in C_i and Φ, which gives us enough information to access the currents at least for the torsionless case. The transformation

$$\begin{aligned}A'_0&=A_0+(s'-s)\omega_0+\frac{\mathrm{g}-\mathrm{g}'}{4m}e^{-\Phi}F-\frac{\mathrm{g}'(\mathrm{g}'-\mathrm{g})}{16m}e^{-\Phi}\left(\nabla^i\dot{C}_i-\nabla^2\Phi\right)\\A'_i&=A_i+(s'-s)\omega_i+\frac{\mathrm{g}-\mathrm{g}'}{4}\varepsilon_i{}^j(\dot{C}_j-\tilde\partial_j\Phi)-\frac{\mathrm{g}-\mathrm{g}'}{4m}e^{-\Phi}FC_i\end{aligned} \tag{B10}$$

does the trick. When $\Phi=0$, the electric and magnetic fields in the new picture are

$$\begin{aligned}B'&=B+\frac{1}{2}(s'-s)R\\E'_i&=E_i+\frac{1}{2}(s'-s)R_i+\frac{\mathrm{g}-\mathrm{g}'}{4m}\nabla_iB.\end{aligned} \tag{B11}$$

We have shown then that at the level of the effective action we have

$$W_{\mathrm{g}s}[g_{ij},\Phi,C_i,A_\mu]=W_{\mathrm{g}'s'}[g_{ij},\Phi,C_i,A'_\mu]. \tag{B12}$$

To relate the one-point correlators in the two conventions, proceed as before. Set

$$\begin{aligned}&\int d^3x\sqrt{g}e^{-\Phi}\Big(\frac{1}{2}T^{ij}_{\rm nc}\delta g_{ij}-\varepsilon^\mu_{\rm nc}\delta n_\mu+j^\mu_{\rm nc}\delta A_\mu\Big)\\&\qquad=\int d^3x\sqrt{g}e^{-\Phi}\Big(\frac{1}{2}T'^{ij}_{\rm nc}\delta g_{ij}-\varepsilon'^\mu_{\rm nc}\delta n_\mu+j'^\mu_{\rm nc}\delta A'_\mu\Big).\end{aligned} \tag{B13}$$

The resulting translation formulas for $\Phi = 0$ are

$$
\begin{aligned}
j^0_{\rm nc} &= j'^0_{\rm nc} \\
j^i_{\rm nc} &= j'^i_{\rm nc} + \frac{\mathrm{g}-\mathrm{g}'}{4m}\varepsilon^{ij}\nabla_j j'^0_{\rm nc} \\
\varepsilon^0_{\rm nc} &= \varepsilon'^0_{\rm nc} - \frac{\mathrm{g}-\mathrm{g}'}{4}\varepsilon^{ij}\nabla_i j'_{j\rm nc} - \frac{\mathrm{g}-\mathrm{g}'}{4m}Bj'^0_{\rm nc} + \frac{\mathrm{g}'(\mathrm{g}'-\mathrm{g})}{16m}\nabla^2 j'^0_{\rm nc} \\
\varepsilon^i_{\rm nc} &= \varepsilon'^i_{\rm nc} - \frac{1}{2}(s'^{ij}\dot{g}_{jk}j'^k_{\rm nc} + \frac{\mathrm{g}-\mathrm{g}'}{4}\varepsilon^{ij}\partial_0 j'_{j\rm nc} - \frac{\mathrm{g}-\mathrm{g}'}{4m}\left(Bj'^i_{\rm nc} + \varepsilon^{ij}\left(E_j - \frac{s}{2}R_i\right)j'^0_{\rm nc}\right) \\
&\quad - \frac{\mathrm{g}-\mathrm{g}'}{8m}s\left(g^{ij}g^{kl} - g^{il}g^{jl}\right)\dot{g}_{kl}\nabla_j j'^0_{\rm nc} - \frac{\mathrm{g}'(\mathrm{g}'-\mathrm{g})}{16m}\frac{1}{\sqrt{g}}\partial_0\left(\sqrt{g}\nabla^i j'^0_{\rm nc}\right) \\
T^{ij}_{\rm nc} &= T'^{ij}_{\rm nc} + (s'^{k(i}\nabla_k j'^{j)}_{\rm nc} + \frac{s'-s}{2}\dot{g}^{k(i}\varepsilon^{j)}{}_k j'^0_{\rm nc} \\
&\quad - \frac{\mathrm{g}-\mathrm{g}'}{4m}\left(Bj'^0_{\rm nc}g^{ij} + s\left(\nabla^i\nabla^j - g^{ij}\nabla^2\right)j'^0_{\rm nc}\right)
\end{aligned} \tag{B14}
$$

If we merely restricted ourselves to the microscopic action (19), the translation formulas are superfluous since we know the explicit form of the classical action for all g and s. Rather, their power derives from the equality of the full quantum partition functions for which we may not have this knowledge. One can imagine computing correlation functions for some convenient choice (such as g $= 2$, $s = 1$ for LLL physics). $W = W'$ then ensures that regardless of that choice we are actually describing the same physics as for the true values of g and s and there is a precise map that can be used to determine the physical correlation functions. (B14) is that map for one-point correlators. One may similarly derive a map for two-point correlators, etc. using the method above.

Appendix C: Current Redefinitions – Covariant Version

The same manipulations above may be carried out for the covariant currents as well. To begin, we recall that the microscopic action (19) may be written using Newton-Cartan geometry as

$$
S = \int d^3x\sqrt{g}e^{-\Phi}\left(\frac{i}{2}v^\mu\psi^\dagger\overleftrightarrow{D}_\mu\psi - \frac{1}{2m}\left(g^{\mu\nu} + \frac{i\mathrm{g}}{2}\varepsilon^{\mu\nu}\right)D_\mu\psi^\dagger D_\nu\psi - \lambda|\psi|^4\right) \tag{C1}
$$

where we have suppressed the volume element and $D_\mu = \nabla_\mu - i\tilde{A}_\mu + is\tilde{\omega}_\mu$. Here $\tilde{\omega}_\mu = \frac{1}{2}\epsilon_{ab}e^{a\nu}\nabla_\mu e^b_\nu$ is the spin connection associated to a transverse zweibein $g^{\mu\nu} = \delta_{ab}e^{a\mu}e^{b\nu}$ and $\tilde{A}_\mu$ is the modified vector potential.

By the same method as above, we find that the substitution

$$
\tilde{A}'_\mu = \tilde{A}_\mu + (s'-s)\tilde{\omega}_\mu + \frac{\mathrm{g}'-\mathrm{g}}{4}\varepsilon_{\mu\nu}G^\nu + \frac{\mathrm{g}-\mathrm{g}'}{4m}n_\mu\left(\tilde{F} + \frac{\mathrm{g}+\mathrm{g}'}{8}G_\nu G^\nu - \frac{\mathrm{g}'}{4}\nabla_\nu G^\nu\right) \tag{C2}
$$

sends the action to itself but with new parity breaking parameters g$'$ and s'. Here $\tilde{F} = \varepsilon^{\mu\nu}\nabla_\mu(\tilde{A}_\nu - s\tilde{\omega}_\nu)$. We may now derive the action with respect to δn_μ, $\delta h_{\mu\nu}$ and $\delta\tilde{A}_\mu$ to find how our field redefinition has affected the stress, energy and charge currents. For brevity, we cite the result only in the flat case $\Phi = 0$, $C_i = 0$, $g_{ij} = \delta_{ij}$.

$$
\begin{aligned}
j^\mu &= j'^\mu + \frac{\mathrm{g}-\mathrm{g}'}{4m}\varepsilon^{\mu\nu}\nabla_\nu n' \\
\varepsilon^\mu &= \varepsilon'^\mu - \frac{1}{4}(s'^{\mu\nu}\tau_{\nu\lambda}j'^\lambda - \frac{\mathrm{g}-\mathrm{g}'}{4}\tilde{B}j'^\mu + \frac{\mathrm{g}'-\mathrm{g}}{2}\varepsilon^{\lambda[\mu}\nabla_\nu(v^{\nu]}j'_\lambda) \\
&\quad - \frac{\mathrm{g}-\mathrm{g}'}{4m}\left(\varepsilon^{\mu\nu}\tilde{F}_{\nu\lambda}v^\lambda n' - \frac{1}{4}s\bar{\tau}^{\mu\nu}\nabla_\nu n'\right) - \frac{(\mathrm{g}-\mathrm{g}')\mathrm{g}'}{32m}\left(\bar{\tau}^{\mu\nu}\nabla_\nu n'^\nu(\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2)n'\right) \\
T^{\mu\nu} &= T'^{\mu\nu} + (s'^{\lambda(\mu}g^{\nu)\rho}\nabla_\lambda j'_\rho - \frac{\mathrm{g}-\mathrm{g}'}{4m}\left(\tilde{B}n'^{\mu\nu} + s(\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2)n'\right)
\end{aligned} \tag{C3}
$$

where $\bar{\tau}^{\mu\nu}$ is the trace reversed shear and $n = n_\mu j^\mu$.

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