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High-dimensional Lifshitz-type spacetimes, universal horizons and black holes in Hořava-Lifshitz gravity

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In this paper, we present all $[(d+1)+1]$-dimensional static diagonal vacuum solutions of the non-projectable Hořava-Lifshitz gravity in the IR limit, and show that they give rise to very rich Lifshitz-type structures, depending on the choice of the free parameters of the solutions. These include the Lifshitz spacetimes with or without hyperscaling violation, Lifshitz solitons, and black holes. Remarkably, even the theory breaks explicitly the Lorentz symmetry and allows generically instantaneous propagations, universal horizons still exist, which serve as one-way membranes for signals with any large velocities. In particular, particles even with infinitely large velocities would just move around on these boundaries and cannot escape to infinity. Another remarkable feature appearing in the Lifshitz-type spacetimes is that the dynamical exponent z can take its values only in the ranges $1 \leq z < 2$ for $d \geq 3$ and $1 \leq z < \infty$ for $d = 2$, due to the stability and ghost-free conditions of the theory.

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I. INTRODUCTION

Lifshitz space-time has been extensively studied in the content of non-relativistic gauge/gravity duality [1, 2], after the seminal work of [3], which argued that nonrelativistic QFTs that describe multicritical points in certain magnetic materials and liquid crystals [4] may be dual to certain nonrelativistic gravitational theories in such a space-time background.

One of the remarkable feature of the Lifshitz space-time is its anisotropic scaling between space and time,

$$t \to b^z t, \quad x^i \to b x^i, \tag{1.1}$$

on a hypersurface $r =$ Constant, on which the nonrelativistic QFTs live, where z denotes the dynamical critical exponent, and in the relativistic scaling we have $z_{GR} = 1$. x^i denote the spatial coordinates tangential to the surfaces $t =$ Constant.

It is interesting to note that the anisotropic scaling (1.1) can be realized in two different levels. In Level one, the underlying theory itself is still relativistic-scaling invariant, but the space-time has the anisotropic scaling. This was precisely the case studied in [1, 3], where the theories of gravity is still of general covariance, but the metric of the space-time has the above anisotropic scaling. This is possible only when some matter fields are introduced to create a preferred direction, so that the anisotropic scaling (1.1) can be realized. In [3], this was done by two p-form gauge fields with $p = 1, 2$, and was soon generalized to different cases [1].

In Level two, not only the space-time has the above anisotropic scaling, but also the theory itself. In fact, starting with the anisotropic scaling (1.1), Hořava recently constructed a theory of quantum gravity at a Lifshitz fixed point, the so-called Hořava-Lifshitz (HL) theory [5], which is power-counting renormalizable, and lately has attracted lots of attention, due to its remarkable features when applied to cosmology and astrophysics [6]. Power-counting renomalizability requires $z \geq D$, where D denotes the number of spatial dimensions of the theory. Since the anisotropic scaling (1.1) is built in by construction in the HL gravity, it is natural to expect that the HL gravity provides a minimal holographic dual for non-relativistic Lifshitz-type field theories with the anisotropic scaling. Indeed, this was first showed in [7] that the Lifshitz spacetime,

$$ds^2 = -\left(\frac{r}{\ell}\right)^{2z} dt^2 + \left(\frac{r}{\ell}\right)^2 dx + \left(\frac{\ell}{r}\right)^2 dr^2, \tag{1.2}$$

is a vacuum solution of the HL gravity in (2+1) dimensions, and that the full structure of the $z = 2$ anisotropic Weyl anomaly can be reproduced in dual field theories, while its minimal relativistic gravity counterpart yields only one of two independent central charges in the anomaly.

Recently, we studied the HL gravity in (2+1) dimensions in detail [8], and found further evidence to support the above speculations. In particular, we found all the static (diagonal) solutions of the HL gravity in (2+1) dimensions, and showed that they give rise to very rich space-time structures: the corresponding spacetimes can

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represent the generalized BTZ black holes [9], the Lifshitz space-times, or Lifshitz solitons [10], in which the spacetimes are free of any kind of space-time singularities, depending on the choices of the free parameters of the solutions. Some space-times are not complete, and extensions beyond certain horizons are needed. In addition, it was shown recently that the Lifshitz space-time (1.2) is not only a solution of the HL gravity in the IR, but also a solution of the full theory, that is, even high-order operators are all included [11]. The only effects of these high-order operators are to shift z from one value to another, as longer as the spacetime itself is concerned.

In this paper, we shall generalize our above studies to any dimensions, and obtain all the static (diagonal) solutions of the vacuum HL gravity explicitly. With these exact vacuum solutions,we believe that the studies of the non-relativistic Lifshitz-type gauge/gravity duality will be simplified considerably, as so far most of such studies are numerical [1, 10, 12–14]. After studying each of these solutions in detail, we find that, similar to the (2+1) case, Lifshitz space-times and solitons can be all found in these solutions. Remarkably, the Lifshitz space-times with hyperscaling violation [13, 14],

$$ds^2 = r^{-\frac{2(d-\theta)}{d}}\left(-r^{-2(z-1)}dt^2 + dr^2 + d\vec{x}^2\right), \qquad (1.3)$$

can be also realized in the HL gravity as a vacuum solutions of the theory.

Moreover, some of the solutions to be presented in this paper also represent black holes, although the HL gravity explicitly breaks Lorentz symmetry and allows in principle propagations with any large velocities [5, 6]. This follows the recent discovery of the existence of the universal horizons in the khrononmetric theory of gravity [15], in which the khronon ϕ naturally defines a timelike foliations, parametrized by $\phi(x^{\mu}) =$ Constant. Among these leaves, there may exist a surface at which ϕ diverges, while physically nothing singular happens there, including the metric and the space-time. Given that ϕ defines an absolute time, any object crossing this surface from the interior would necessarily also move back in absolute time, which is something forbidden by the definition of the causality in the theory. Thus, even particles with superluminal velocities cannot penetrate this surface, once they are trapped inside it. In particular, particles even with infinitely large velocities would just move around on these boundaries and cannot escape to infinity. For more details, we refer readers to [15–17].

The rest of the paper is organized as follows: In Section II, we give a brief introduction to the (d+2)-dimensional HL gravity without the projectablity condition, while in Section III we first write down the corresponding field equations for static vacuum spacetimes, and then solve them for particular cases. In Section IV, we first obtain all the rest of the static (diagonal) vacuum (d+2)-dimensional solutions of the HL theory, and then study each of such solutions in detail. In Section V, following [17, 18] we study the black hole structures of solutions presented in Section III, and show explicitly that universal horizons exist in some of these solutions. Finally, in Section VI we present our main conclusions and provide some discussing remarks.

II. NON-PROJECTABLE HL THEORY IN D DIMENSIONS

In this paper, we shall take the Arnowitt-Deser-Misner (ADM) variables [19],

$$(N, N_i, g_{ij}),\ (i,\ j = 1, 2, \cdots, d+1), \qquad (2.1)$$

as the fundamental ones, which are all functions of both t and x^i, as in this paper we shall work in the version of the HL gravity without the projectability condition [5, 6]. Then, the general action of the HL theory in (d+2)-dimensions is given by

$$S = \zeta^2 \int dt d^{d+1}x N\sqrt{g}\Big(\mathcal{L}_K - \mathcal{L}_V + \zeta^{-2}\mathcal{L}_M\Big), \quad (2.2)$$

where $g = \det(g_{ij})$, $\zeta^2 = 1/(16\pi G)$, and

$$\begin{aligned} \mathcal{L}_K &= K_{ij}K^{ij} - \lambda K^2, \\ K_{ij} &= \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right). \end{aligned} \qquad (2.3)$$

Here λ is a dimensionless coupling constant, and ∇_i denotes the covariant derivative with respect to g_{ij}. $\mathcal{L}_M$ is the Lagrangian of matter fields. The potential $\mathcal{L}_V$ is constructed from R_{ij}, a_i and ∇_i, and formally can be written in the form,

$$\mathcal{L}_V = \gamma_0\zeta^2 + \gamma_1 R + \beta a_i a^i + \mathcal{L}_V^{z>2}\left(R_{ij}, a_i, \nabla_i\right), \quad (2.4)$$

where $\mathcal{L}_V^{z>2}$ denotes the part that includes all higher-order operators [20]. Power-counting renormalizability condition requires $z \geq (d+1)$ [5, 6]. R_{ij} denotes the Ricci tensor made of g_{ij}, and

$$a_i \equiv \frac{N_{,i}}{N}, \quad a_{ij} \equiv \nabla_i a_j. \qquad (2.5)$$

In the infrared (IR) limit, the higher-order operators are suppressed by M_*^{2-n}, so we can safely set them to zero,

$$\mathcal{L}_V^{z>2}\left(R_{ij}, a_i, \nabla_i\right) = 0, \qquad (2.6)$$

where $M_* \equiv 1/\sqrt{8\pi G}$ and n denotes the order of the operator. In this paper, we shall consider only the IR limit, so that Eq.(2.6) is always true.

A. Field Equations in IR Limit

Variation of the action (2.2) with respect to the lapse function N yields the Hamiltonian constraint

$$\mathcal{L}_K + \mathcal{L}_V^R + F_V = 8\pi G J^t, \qquad (2.7)$$

where

$$
\begin{aligned}
J^t &= 2\frac{\delta(N\mathcal{L}_M)}{\delta N}, \\
\mathcal{L}_V^R &= \gamma_0\zeta^2 + \gamma_1 R, \\
F_V &= -\beta\left(2a_i^i + a_i a^i\right).
\end{aligned} \tag{2.8}
$$

Variation with respect to the shift vector N_i yields the momentum constraint

$$
\nabla_j \pi^{ij} = 8\pi G J^i, \tag{2.9}
$$

where

$$
\pi^{ij} \equiv -K^{ij} + \lambda K g^{ij}, \quad J^i \equiv -\frac{\delta\left(N\mathcal{L}_M\right)}{\delta N_i}. \tag{2.10}
$$

The dynamical equations are obtained by varying the action with respect to g_{ij}, and are given by

$$
\begin{aligned}
&\frac{1}{\sqrt{g}N}\frac{\partial}{\partial t}\left(\sqrt{g}\pi^{ij}\right) + 2(K^{ik}K_k^j - \lambda K K^{ij}) \\
&-\frac{1}{2}g_{ij}\mathcal{L}_K + \frac{1}{N}\nabla_k(\pi^{ik}N^j + \pi^{kj}N^i - \pi^{ij}N^k) \\
&\qquad -F^{ij} - F_a^{ij} = 8\pi G \tau^{ij},
\end{aligned} \tag{2.11}
$$

where

$$
\begin{aligned}
\tau^{ij} &\equiv \frac{2}{\sqrt{g}N}\frac{\delta(\sqrt{g}N\mathcal{L}_M)}{\delta g_{ij}}, \\
F^{ij} &\equiv \frac{1}{\sqrt{g}N}\frac{\delta(-\sqrt{g}N\mathcal{L}_V^R)}{\delta g_{ij}} \\
&= -\Lambda g^{ij} + \gamma_1\left(R^{ij} - \frac{1}{2}R g^{ij}\right) \\
&\quad + \frac{\gamma_1}{N}\left(g^{ij}\nabla^2 N - \nabla^i\nabla^j N\right), \\
F_a^{ij} &\equiv \frac{1}{\sqrt{g}N}\frac{\delta(-\sqrt{g}N\mathcal{L}_V^a)}{\delta g_{ij}} \\
&= \beta\left(a^i a^j - \frac{1}{2}g^{ij}a^k a_k\right),
\end{aligned} \tag{2.12}
$$

with $\mathcal{L}_V^a \equiv \beta a_i a^i$.

In addition, the matter components (J^t, J^i, τ^{ij}) satisfy the conservation laws of energy and momentum,

$$
\begin{aligned}
&\int d^3x\sqrt{g}N\left[\dot{g}_{ij}\tau^{ij} - \frac{1}{\sqrt{g}}\partial_t(\sqrt{g}J^t)\right. \\
&\qquad \left. + \frac{2N_i}{\sqrt{g}N}\partial_t(\sqrt{g}J^i)\right] = 0,
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
&\frac{1}{N}\nabla^i(N\tau_{ik}) - \frac{1}{\sqrt{g}N}\partial_t(\sqrt{g}J_k) - \frac{J^t}{2N}\nabla_k N \\
&\quad - \frac{N_k}{N}\nabla_i J^i - \frac{J^i}{N}(\nabla_i N_k - \nabla_k N_i) = 0.
\end{aligned} \tag{2.14}
$$

B. Stability and Ghost-free Conditions

When $\gamma_0 = 0$, the above HL theory admits the Minkowski space-time

$$
\left(\bar{N}, \bar{N}_i, \bar{g}_{ij}\right) = (1, 0, \delta_{ij}), \tag{2.15}
$$

as a solution of the theory. Then, its linear perturbations reveals that the theory has two modes [7], one represents the spin-2 massless gravitons with a dispersion relation,

$$
\omega_T^2 = -\gamma_1 k^2, \tag{2.16}
$$

and the other represents the scalar mode with

$$
\omega_S^2 = -\frac{\gamma_1(\lambda-1)}{(d+1)\lambda - 1}\left[d\left(\frac{\gamma_1}{\beta} - 1\right) + 1\right]k^2. \tag{2.17}
$$

The stability conditions of these modes requires

$$
\omega_T^2 > 0, \quad \omega_S^2 > 0, \tag{2.18}
$$

for any given k.

On the other hand, the kinetic term of the scalar mode is proportional to $(\lambda-1)/[(d+1)\lambda-1]$ [7], so the ghost-free condition requires

$$
\frac{\lambda - 1}{(d+1)\lambda - 1} \geq 0, \tag{2.19}
$$

which is equivalent to

$$
i)\ \lambda \geq 1, \quad \text{or} \quad ii)\ \lambda \leq \frac{1}{d+1}. \tag{2.20}
$$

Then, Eq.(2.18) implies that [1]

$$
\gamma_1 < 0, \quad \frac{d\gamma_1}{d-1} < \beta < 0. \tag{2.21}
$$

III. STATIC VACUUM SOLUTIONS

In this paper, we consider static spacetimes given by,

$$
\begin{aligned}
N &= r^z f(r), \quad N^i = 0, \\
g_{ij}dx^i dx^j &= \frac{g^2(r)}{r^2}dr^2 + r^2 d\vec{x}^2,
\end{aligned} \tag{3.1}
$$

in the coordinates $\left(t, x^A, r\right), (A = 1, 2, \cdots, d)$, where $d\vec{x}^2 \equiv \delta_{AB}dx^A dx^B$. Note that in [8], the case $d = 1$ was studied in detail. So, in this paper we shall consider only the case where $d \geq 2$.

---

[1] It is interesting to note that in (2+1)-dimensions, the spin-2 gravitons do not exist, so the coupling constant γ_1 is free, while β is required to be negative, $\beta < 0$ [8].

Then, the $(d+1)-$dimensional Ricci scalar $R\ \left(\equiv g^{ij}R_{ij}\right)$ of the leaves t = Constant is given by

$$R=\frac{d}{g^{3}(r)}\left[2rg'(r)-(d+1)g(r)\right]. \tag{3.2}$$

On the other hand, since $N^i=0$ and that the space-times are static, so we must have $K_{ij}=0$. Then, the momentum constraint (2.9) is satisfied identically. The Hamiltonian constraint (2.7) and the rr-component of the dynamical equations (2.11) are non-trivial, while the AA-component of the dynamical equations can be derived from the Hamiltonian constraint and the rr component. Therefore, similar to the (2+1)-dimensional case, there are only two independent equations for two unknowns, $f(r)$ and $g(r)$, which can be cast in the forms,

$$\Lambda g^2-d\gamma_1 W-\frac{1}{2}\beta W^2-\frac{1}{2}d(d-1)\gamma_1=0, \tag{3.3}$$

$$\begin{aligned}\Lambda g^2-\beta\left[\left(\frac{rW}{g}\right)'+(d-1)W+\frac{W^2}{2}\right]\\ -d\gamma_1\left[\frac{d+1}{2}-r\frac{g'}{g}\right]=0,\end{aligned} \tag{3.4}$$

where

$$W\equiv z+r\frac{f'}{f},\quad \Lambda\equiv\frac{1}{2}\gamma_0\zeta^2. \tag{3.5}$$

From Eq.(3.3), we obtain

$$W_{\pm}=\frac{s[1\pm r_*(r)]}{1-s}, \tag{3.6}$$

where

$$\begin{aligned}s\ &\equiv\ \frac{d\gamma_1}{d\gamma_1-\beta},\\ r_*(r)\ &\equiv\ \sqrt{1+(1-d)\frac{\beta}{d\gamma_1}+\frac{2\beta\Lambda}{d^2\gamma_1^2}g(r)^2}.\end{aligned} \tag{3.7}$$

Then, from the stability conditions (2.21) we find that

$$1\le s<\frac{d-1}{d-2}, \tag{3.8}$$

where the equality holds only when $\beta=0$, which is possible when $\lambda=1$, as can be seen from Eq.(2.17).

Inserting the above expression into Eq.(3.4), we obtain a master equation for $r_*(r)$,

$$(s-1)rr_*'+\Delta\left(r_*^2-r_s^2\right)\left(r_*+\epsilon\mathcal{D}\right)=0, \tag{3.9}$$

where $\epsilon=\pm1$, and

$$\begin{aligned}r_s^2\ &\equiv\ 1-\frac{(d-1)\beta}{d\gamma_1},\quad \mathcal{D}\equiv\frac{d\gamma_1-(d-1)\beta}{d(\gamma_1-\beta)},\\ \Delta\ &\equiv\ \frac{d^2\gamma_1(\gamma_1-\beta)}{(d\gamma_1-\beta)[d\gamma_1-(d-1)\beta]}.\end{aligned} \tag{3.10}$$

Note that Eq.(3.9) with "$-$" sign can be always obtained from the one with "+" sign, by simply replacing r_* by $-r_*$. Therefore, although r_* defined by Eq.(3.7) is non-negative, we shall take the region $r_*<0$ as a natural extension, so that in the following we only need to consider the case with "+" sign.

From Eq.(3.7) we find that,

$$g^2(r)=\frac{d^2\gamma_1^2}{2\beta\Lambda}\left(r_*^2-r_s^2\right), \tag{3.11}$$

while from Eqs.(3.5) and (3.6), we obtain

$$\frac{df}{f}=\frac{s-z+zs+\epsilon sr_*}{1-s}\left(\frac{dr}{r}\right). \tag{3.12}$$

Therefore, once the master equation (3.9) is solved for $r=r(r_*)$, substituting it into Eq.(3.12) we can find $f(r_*)$. Then, in terms of r_*, the metric takes the form,

$$ds^2=-r^{2z}f^2dt^2+\frac{g^2}{r^2}\left(\frac{dr}{dr_*}\right)^2dr_*^2+r^2d\vec{x}^2. \tag{3.13}$$

In the rest of this section, we shall solve the above equations for some particular cases, and leave the one with $r_s^2>0$ to the next section.

A. Lifshitz Spacetime

A particular solution of Eq.(3.9) is $r_*=-\epsilon\mathcal{D}$. Then we obtain

$$g^2(r)\ =\ g_0^2,\quad f(r)=f_0r^{\frac{s}{d+s(1-d)}-z}, \tag{3.14}$$

where in terms of g_0, the cosmological constant is given by,

$$\Lambda=\gamma_1\frac{(\beta-d\beta+d\gamma_1)(\gamma_1-d\beta+d\gamma_1)}{2g_0^2(\gamma_1-\beta)^2}, \tag{3.15}$$

with f_0 and g_0 being the integration constants. Then, the corresponding line element takes the form,

$$\begin{aligned}ds^2\ =\ L^2\Bigg\{-\left(\frac{r}{\ell}\right)^{2z}dt^2+\left(\frac{\ell}{r}\right)^2dr^2\\ +\left(\frac{r}{\ell}\right)^2d\vec{x}^2\Bigg\},\end{aligned} \tag{3.16}$$

where $f_0\equiv L/\ell^z$, $g_0\equiv L\ell$, and

$$z=\frac{\gamma_1}{\gamma_1-\beta}, \tag{3.17}$$

which is independent of the space-time dimensions. On the other hand, from the stability and ghost-free condition (2.21), it can be shown that

$$1\le z<\frac{d-1}{d-2}. \tag{3.18}$$

Note that the above holds only for $d \geq 2$. In particular, we have

$$z = \begin{cases} < \infty, & d = 2, \\ < 1 + \frac{1}{d-2} \leq 2, & d \geq 3. \end{cases} \quad (3.19)$$

This is a unexpected result, but seems to agree with some numerical solutions found in other theories of gravity [1].

Rescaling the coordinates t, r, x^A, without loss of generality, one can always set $L = \ell = 1$. Then, we find that the corresponding curvature R is given by

$$R = -\frac{2d(d+1)\Lambda(\beta-\gamma_1)^2}{\gamma_1(\beta - d\beta + d\gamma_1)(\gamma_1 - d\beta + d\gamma_1)}, \quad (3.20)$$

which is a constant.

It is remarkable to note that when $r_s^2 > 0$, $r_* = \pm r_s$ is also a solution of Eq. (3.9). In this case we have the same Lifshitz solution (3.16) but z and Λ now are given by,

$$\begin{aligned} z &= \frac{s(1 \pm r_s)}{1-s} = -\frac{d\gamma_1}{\beta}\left\{1 \pm \left[1 - \frac{(d-1)\beta}{d\gamma_1}\right]^{1/2}\right\}, \\ \Lambda &= 0, \quad \left(r_* = \pm r_s,\ r_s^2 > 0\right). \end{aligned} \quad (3.21)$$

B. Generalized BTZ Black Holes

When $s = 1$, we find that $\beta = 0$. Then, from the stability conditions (2.21) we can see that this is possible only when $\lambda = 1$. Thus, we obtain

$$\begin{aligned} g^2(r) &= \frac{d(d+1)\gamma_1}{2}\frac{r^{d+1}}{M + \Lambda r^{d+1}}, \\ f(r) &= f_0 r^{\frac{1-d-2z}{2}}\sqrt{M + \Lambda r^{d+1}}, \end{aligned} \quad (3.22)$$

for which the metric takes the form,

$$\begin{aligned} ds^2 &= -N_0^2 r^{1-d}\left|M \pm \left(\frac{r}{\ell}\right)^{d+1}\right| dt^2 \\ &\quad + \frac{d(d+1)\gamma_1}{2}\left(\frac{r^{d-1}dr^2}{M \pm \left(\frac{r}{\ell}\right)^{d+1}}\right) + r^2 d\vec{x}^2, \end{aligned} \quad (3.23)$$

where "+" ("-") corresponds to $\Lambda > 0$ ($\Lambda < 0$), and $\ell \equiv |\Lambda|^{-\frac{1}{d+1}}$. Since $\gamma_1 < 0$ [cf. Eq.(2.21)], we find that, to have g_{rr} non-negative, we must require

$$M \pm \left(\frac{r}{\ell}\right)^{d+1} \leq 0. \quad (3.24)$$

For $M > 0$, the above is possible only when $\Lambda < 0$, for which, by rescaling t, r and x^A, the metric (3.23) can be cast in the form,

$$\begin{aligned} ds^2 &= L^2\left\{-\frac{\left(\frac{r}{\ell}\right)^{d+1} - M}{r^{d-1}}dt^2 \right. \\ &\quad \left. + \frac{r^{d-1}dr^2}{\left(\frac{r}{\ell}\right)^{d+1} - M} + r^2 d\vec{x}^2\right\}, (\Lambda < 0), \end{aligned} \quad (3.25)$$

which is nothing but the $(d+2)$-dimensional BTZ black holes [9] with the black hole mass given by M, where $L^2 \equiv d(d+1)|\gamma_1|/2$.

It should be noted that the original BTZ black hole was obtained in general relativity, for which we have $(\lambda,\ \gamma_1, \beta)_{GR} = (1, -1, 0)$. Clearly, the above solutions are valid for any given $\gamma_1 < 0$. In this sense we refer these black holes to as the generalized BTZ black holes.

Note that when $r_s^2 = 0$, we obtain $\beta = d\gamma_1/(d-1)$. Then, substituting it into the expression for s we obtain $s = (d-1)/(d-2)$. However, the condition (3.8) require $s < (d-1)/(d-2)$. Therefore, in the current case, r_s cannot vanish.

On the other hand, when $r_s^2 < 0$, from Eqs.(3.8) and (3.10), we find that this is possible only when $s < 0$, which is not allowed by Eq.(3.8). Therefore, $r_s^2 < 0$ is also impossible in the current case. Thus, in the rest of this paper, we only need to consider the case $r_s^2 > 0$, which will be studied in the next section.

IV. STATIC SPACETIMES FOR $r_s^2 > 0$

The condition $r_s^2 > 0$ implies,

$$0 < s < \frac{d-1}{d-2}. \quad (4.1)$$

However, Eq.(3.8) further exclude the region $0 < s < 1$. Therefore, in this section we need only to consider the case where

$$1 \leq s < \frac{d-1}{d-2}. \quad (4.2)$$

Then, Eq.(3.9) can be cast in the form,

$$\frac{dr}{r} = \left(\frac{r_s + \mathcal{D}}{r_* + r_s} + \frac{r_s - \mathcal{D}}{r_* - r_s} - \frac{2r_s}{r_* + \mathcal{D}}\right)\frac{dr_*}{2r_s\mathcal{P}}, \quad (4.3)$$

where

$$\mathcal{P} \equiv \frac{\mathcal{D}^2 - r_s^2}{s-1}\Delta. \quad (4.4)$$

Thus, from Eq.(4.3) we obtain

$$r(r_*) = r_H |r_* + r_s|^{\frac{r_s + \mathcal{D}}{2r_s\mathcal{P}}} |r_* - r_s|^{\frac{r_s - \mathcal{D}}{2r_s\mathcal{P}}} |r_* + \mathcal{D}|^{-\frac{1}{\mathcal{P}}}, \quad (4.5)$$

while from Eq.(3.12) we get

$$\frac{df}{f} = \left(\frac{\delta_1}{r_* - r_s} + \frac{\delta_2}{r_* + r_s} + \frac{\delta_3}{r_* + \mathcal{D}}\right) dr_*, \quad (4.6)$$

where

$$\begin{aligned} \delta_1 &\equiv \frac{s - z + sz + sr_s}{2r_s\Delta(r_s + \mathcal{D})}, \\ \delta_2 &\equiv \frac{s - z + sz - sr_s}{2r_s\Delta(r_s - \mathcal{D})}, \\ \delta_3 &\equiv \frac{z - s + s\mathcal{D} - zs}{\Delta(r_s^2 - \mathcal{D}^2)}. \end{aligned} \quad (4.7)$$

Thus, the general solution of f is given by

$$f = f_0|r_* - r_s|^{\delta_1}|r_* + r_s|^{\delta_2}|r_* + \mathcal{D}|^{\delta_3}. \tag{4.8}$$

Therefore, the metric can be rewritten in the form,

$$ds^2 = -N^2dt^2 + G^2dr_*^2 + r^2d\vec{x}^2, \tag{4.9}$$

where

$$\begin{aligned} N^2(r_*) &= N_0^2\left|\frac{r_* - r_s}{r_* + \mathcal{D}}\right|^{\frac{s(1+r_s)}{\Sigma_+}}\left|\frac{r_* + r_s}{r_* + \mathcal{D}}\right|^{\frac{s(1-r_s)}{\Sigma_-}}, \\ G^2(r_*) &= G_0^2\frac{(d-1)\beta + d\gamma_1(r_*^2 - 1)}{(r_*^2 - r_s^2)^2(r_* + \mathcal{D})^2}, \\ r^2(r_*) &= r_H^2\left|\frac{r_* - r_s}{r_* + \mathcal{D}}\right|^{\frac{1-s}{\Sigma_+}}\left|\frac{r_* + r_s}{r_* + \mathcal{D}}\right|^{\frac{1-s}{\Sigma_-}}, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} \Sigma_\pm &\equiv d(1-s) + s(1 \pm r_s), \\ G_0^2 &\equiv \frac{d\gamma_1(s-1)^2[d-1+(2-d)s]^2}{2\beta\Lambda s^2(s-sd+d)^2}. \end{aligned} \tag{4.11}$$

The corresponding Ricci scalar is given by

$$R = \frac{2\beta\Lambda[2\Delta(r_* + \mathcal{D})r_* - (d+1)(1-s)]}{d\gamma_1^2(1-s)(r_*^2 - r_s^2)}. \tag{4.12}$$

Therefore, the spacetime is singular at $r_* = \pm r_s$. In fact, near $r_* \simeq \pm r_s$ we find that

$$\begin{aligned} ds^2 &\simeq \left(\frac{r}{L_\pm}\right)^{\frac{2s(1\pm r_s)}{1-s}}\left[-d\hat{t}^2 + \hat{G}_0^2\left(\frac{r}{L_\pm}\right)^{2(d-1)}dr^2\right] \\ &\quad + r^2d\vec{x}^2, \end{aligned} \tag{4.13}$$

where $\hat{t} \equiv \tilde{L}_\pm t$, and

$$\begin{aligned} \epsilon^\pm &= \text{sign}(r_* \mp r_s), \\ L_\pm &= r_H|2r_s|^{\frac{r_s \pm \mathcal{D}}{2r_s\mathcal{P}}}|r_s \pm \mathcal{D}|^{-\frac{1}{\mathcal{P}}}, \\ \tilde{L}_\pm &= N_0|2r_s|^{\frac{s(1\mp r_s)}{2\Sigma_\mp}}|r_s \pm \mathcal{D}|^{\frac{s(s-1)}{(d+s-ds)^2 - s^2r_s^2}}, \\ \hat{G}_0^2 &= \frac{d^2\gamma_1^2\epsilon^\pm r_s}{\pm\beta\Lambda L_\pm^2}. \end{aligned} \tag{4.14}$$

On the other hand, as $r_* \to -\mathcal{D}$, we have

$$r \to \hat{r}_0|r_* + \mathcal{D}|^{-\frac{1}{\mathcal{P}}}, \tag{4.15}$$

where $\hat{r}_0 = r_H|r_s - \mathcal{D}|^{\frac{1-s}{2\Sigma_-}}|r_s + \mathcal{D}|^{\frac{1-s}{2\Sigma_+}}$. Thus, we find that the metric takes the asymptotical form

$$ds^2 \simeq -r^{2z}d\hat{t}^2 + \frac{dr^2}{r^2} + r^2d\vec{x}^2, \tag{4.16}$$

which is precisely the Lifshitz space-time (3.16) with

$$\begin{aligned} z &= \frac{s}{s(1-d)+d}, \\ \hat{t} &= N_0\hat{r}_0^{-\frac{s}{s-ds+d}}|r_s + \mathcal{D}|^{\frac{s(1+r_s)}{2\Sigma_+}}|r_s - \mathcal{D}|^{\frac{s(1-r_s)}{2\Sigma_-}}t. \end{aligned} \tag{4.17}$$

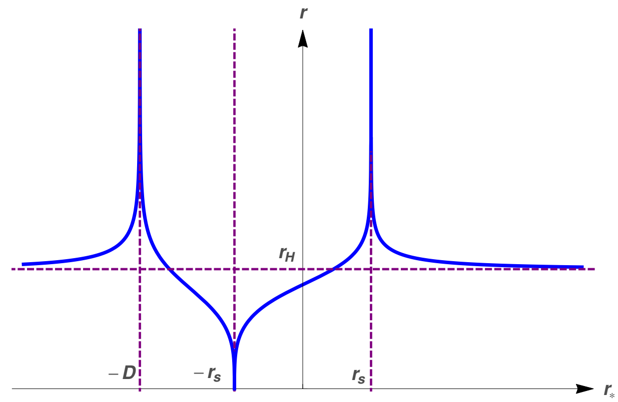

FIG. 1: The function $r \equiv r(r_*)$ for $r_s^2 > 0$ and $1 < s < \frac{d}{d-1}$, where $D \equiv \mathcal{D}$. The spacetime is singular at $r_* = \pm r_s$, and asymptotically Lifshitz as $r_* \to -\mathcal{D}$.

Note that in writing the above metric we had used the condition

$$d^2\gamma_1^2(\mathcal{D}^2 - r_s^2) = 2\beta\Lambda. \tag{4.18}$$

To study the above solutions further, let us consider the cases with different values of s separately.

A. $1 < s < \frac{d}{d-1}$

In this case, we have

$$r(r_*) = \begin{cases} r_H, & r_* \to -\infty, \\ \infty, & r_* = -\mathcal{D}, \\ 0, & r_* = -r_s, \\ \infty, & r_* = +r_s, \\ r_H, & r_* \to +\infty. \end{cases} \tag{4.19}$$

Fig. 1 shows the function $r(r_*)$ vs r_*, from which we can see that the region $r \in [0, \infty)$ is mapped into the region $r_* \in [-r_s, +r_s)$ or $r_* \in (-\mathcal{D}, -r_s]$. The region $r_* \in (-\infty, -\mathcal{D})$ or $r_* \in (r_s, +\infty)$ is mapped into the one $r \in (r_H, +\infty)$.

As shown before, the space-time is singular at $r_* = \pm r_s$, and as $r \to \infty$ (or $r_* \to -\mathcal{D}$), it is asymptotically approaching to the Lifshitz space-time (3.16) with $z = s(d+s-sd)^{-1}$.

To study the solutions further, let us rewrite Eq. (4.5) in the form

$$\left(\frac{r}{r_H}\right)^{\hat{s}} = \frac{(\mathcal{D} - r_s)\epsilon^-}{\mathcal{D} + r_s}\left(\epsilon^+\mathfrak{R}^{\frac{2r_s}{r_s - \mathcal{D}}} + \frac{2\epsilon^{\mathcal{D}}r_s}{\mathcal{D} - r_s}\mathfrak{R}\right), \tag{4.20}$$

where $\epsilon^{\mathcal{D}} \equiv \text{sign}(r_* + \mathcal{D})$ and

$$\mathfrak{R} \equiv \left|\frac{r_* - r_s}{r_* + \mathcal{D}}\right|^{\frac{r_s - \mathcal{D}}{r_s + \mathcal{D}}}, \quad \hat{s} \equiv \frac{2r_s\mathcal{P}}{r_s + \mathcal{D}}. \tag{4.21}$$

It should be noted that the above two equations are valid for any $1 \le s < \frac{d-1}{d-2}$. As a representative example, let us consider the case $\mathcal{D} = 3r_s$, which corresponds to

$$s = \frac{-1 - 17d + 18d^2 - \sqrt{1 + 34d + d^2}}{2(7 - 17d + 9d^2)}. \tag{4.22}$$

Thus, Eqs.(4.20) and (4.21) reduce to,

$$\begin{aligned} \left(\frac{r}{r_H}\right)^{\hat{s}} &= \frac{\epsilon^-}{2\mathfrak{R}}\left(\epsilon^+ + \epsilon^{\mathcal{D}}\mathfrak{R}^2\right), \\ \mathfrak{R} &= \left|\frac{\tilde{r}_* + 3}{\tilde{r}_* - 1}\right|^{1/2}. \end{aligned} \tag{4.23}$$

(a) $r_* \in (-\infty, -\mathcal{D}]$, we have $\epsilon^+ = \epsilon^- = \epsilon^{\mathcal{D}} = -1$. Then, from Eq.(4.23) we obtain

$$\begin{aligned} \mathfrak{R} &= \left(\frac{r}{r_H}\right)^{\hat{s}}\left(1 \pm \sqrt{1 - \left(\frac{r_H}{r}\right)^{2\hat{s}}}\right), \\ r_* &= \frac{\mathfrak{R}^2 + 3}{\mathfrak{R}^2 - 1}. \end{aligned} \tag{4.24}$$

Since $\mathfrak{R} \in [0, 1)$, as it can be seen from Eq.(4.23), we find that only the root $\mathfrak{R}_-$ satisfies this condition. On the other hand, from Eqs.(4.8) and (3.11) we find,

$$r^{2z}f^2 = \frac{N_0^2}{\mathfrak{R}_-^3}\left(\frac{r}{r_H}\right)^{\frac{3\hat{s}(r_s-1)}{2r_s}}, \tag{4.25}$$

$$g^2 = \frac{1 + \mathfrak{R}_-^2}{\left(\mathfrak{R}_-^2 - 1\right)^2}, \tag{4.26}$$

where

$$\mathfrak{R}_- = \frac{\left(\frac{r_H}{r}\right)^2}{1 + \sqrt{1 - \left(\frac{r_H}{r}\right)^4}} = \begin{cases} 1, & r = r_H, \\ 0, & r = \infty. \end{cases} \tag{4.27}$$

(b) $r_* \in (-\mathcal{D}, -r_s]$, we have $\epsilon^+ = \epsilon^- = -\epsilon^{\mathcal{D}} = -1$, and now $\mathfrak{R} \in (0, 1]$. Thus we find that

$$\begin{aligned} \mathfrak{R} &= \left(\frac{r}{r_H}\right)^{\hat{s}}\left(\sqrt{1 + \left(\frac{r_H}{r}\right)^{2\hat{s}}} - 1\right), \\ \tilde{r}_* &= \frac{\mathfrak{R}^2 - 3}{\mathfrak{R}^2 + 1}, \end{aligned} \tag{4.28}$$

are solutions to Eq. (4.20) in this region. This immediately leads to,

$$r^{2z}f^2 = \frac{N_0^2}{\mathfrak{R}^3}\left(\frac{r}{r_H}\right)^{\frac{3\hat{s}(r_s-1)}{2r_s}}, \tag{4.29}$$

$$g^2 = \frac{1 - \mathfrak{R}^2}{\left(\mathfrak{R}^2 + 1\right)^2}. \tag{4.30}$$

(c) $r_* \in (-r_s, r_s]$, we have $-\epsilon^+ = \epsilon^- = \epsilon^{\mathcal{D}} = 1$, implying $\mathfrak{R} \in (1, +\infty)$. Then, we find that

$$\mathfrak{R} = \left(\frac{r}{r_H}\right)^{\hat{s}}\left(\sqrt{1 + \left(\frac{r_H}{r}\right)^{2\hat{s}}} + 1\right), \tag{4.31}$$

are solutions to Eq. (4.20). We therefore obtain the same forms as region (b) for functions g and f

$$r^{2z}f^2 = \frac{N_0^2}{\mathfrak{R}^3}\left(\frac{r}{r_H}\right)^{\frac{3\hat{s}(r_s-1)}{2r_s}}, \tag{4.32}$$

$$g^2 = \frac{1 - \mathfrak{R}^2}{\left(\mathfrak{R}^2 + 1\right)^2}. \tag{4.33}$$

(d) $r_* \in [r_s, +\infty)$, we have $\epsilon^+ = \epsilon^- = \epsilon^{\mathcal{D}} = 1$, implying $\mathfrak{R} \in (1, +\infty)$. Then, we find that

$$\mathfrak{R} = \left(\frac{r}{r_H}\right)^{\hat{s}}\left(\sqrt{1 - \left(\frac{r_H}{r}\right)^{2\hat{s}}} + 1\right), \tag{4.34}$$

are solutions to Eq. (4.20). We therefore obtain the functions g and f

$$r^{2z}f^2 = \frac{N_0^2}{\mathfrak{R}^3}\left(\frac{r}{r_H}\right)^{\frac{3\hat{s}(r_s-1)}{2r_s}}, \tag{4.35}$$

$$g^2 = \frac{1 + \mathfrak{R}^2}{\left(\mathfrak{R}^2 - 1\right)^2}. \tag{4.36}$$

B. $s = \frac{d}{d-1}$

In this case, we find that $\beta = \gamma_1$. Then, we obtain

$$\begin{aligned} g^2(r) &= \frac{2d\gamma_1 g_0 r^{2\sqrt{d}}}{\Lambda\left(r^{2\sqrt{d}} - g_0\right)^2}, \\ f(r) &= \frac{f_0 r^{-d-z+\sqrt{d}}}{r^{2\sqrt{d}} - g_0}, \end{aligned} \tag{4.37}$$

where f_0 and g_0 are two integration constants. Then, the corresponding metric takes the form

$$\begin{aligned} ds^2 &= -f_0^2\frac{r^{2(\sqrt{d}-d)}dt^2}{\left(r^{2\sqrt{d}} - g_0\right)^2} + \left(\frac{2d\gamma_1 g_0}{\Lambda}\right)\frac{r^{2(\sqrt{d}-1)}dr^2}{\left(r^{2\sqrt{d}} - g_0\right)^2} \\ &\quad + r^2 d\vec{x}^2. \end{aligned} \tag{4.38}$$

Clearly, to have g_{rr} positive, we must assume that

$$\frac{\gamma_1 g_0}{\Lambda} > 0. \tag{4.39}$$

The corresponding Ricci scalar is given by

$$\begin{aligned} R &= \frac{\Lambda r^{-2\sqrt{d}}}{2g_0\gamma_1}\left(r^{2\sqrt{d}} - g_0\right)\Big[(1 - \sqrt{d})^2 g_0 \\ &\qquad - (1 + \sqrt{d})^2 r^{2\sqrt{d}}\Big]. \end{aligned} \tag{4.40}$$

which remains finite at the hypersurface $r = r_H$, and indicate that it might represent a horizon, where $r_H =$

$g_0^{1/(2\sqrt{d})}$. As $r \to \infty$, the metric takes the following asymptotical form,

$$ds^2 \simeq \left(\frac{\tilde{r}_0}{\tilde{r}}\right)^{\frac{2}{1+\sqrt{d}}} \left[-\left(\frac{\tilde{r}}{\tilde{r}_0}\right)^{\frac{2(\sqrt{d}+d+1)}{1+\sqrt{d}}} d\tilde{t}^2 + d\tilde{r}^2 + d\vec{x}^2\right], \tag{4.41}$$

where $\tilde{t} = f_0 t$, and

$$\tilde{r} = \tilde{r}_0 r^{-(1+\sqrt{d})}, \quad \tilde{r}_0 \equiv \frac{\sqrt{2d\gamma_1 g_0/\Lambda}}{\sqrt{d}+1}. \tag{4.42}$$

Rescaling $\tilde{t}, \tilde{r}$ and x^A, the above metric can be cast in the form,

$$ds^2 \simeq \hat{r}^{-\frac{2(d-\theta)}{d}} \left(-\hat{r}^{-2(z-1)} d\hat{t}^2 + d\hat{r}^2 + d\hat{\vec{x}}^2\right), \tag{4.43}$$

where

$$\theta = \frac{d\sqrt{d}}{1+\sqrt{d}}, \quad z = -\frac{d}{1+\sqrt{d}}. \tag{4.44}$$

The metric (4.43) is nothing but the space-time with non-relativistic scaling and hyperscaling violation. It was first constructed in Einstien-Maxwell-dilaton theories [13], and recently has been extensively studied in [14]. Under the anisotropic scaling (1.1), it is not invariant but rather scaling as $ds^2 \to b^{2\theta/d} ds^2$. This kind of non-relativistic scaling is closely related to the existence of Fermi surfaces, in which the entanglement entropy is logarithmically proportional to the erea, $S \simeq A \log A$.

C. $\frac{d}{d-1} < s < \frac{d^2}{d^2-d-1}$

In this case, we have

$$r(r_*) = \begin{cases} r_H, & r_* \to -\infty, \\ 0, & r_* = -r_s, \\ \infty, & r_* = +r_s, \\ 0, & r_* = -\mathcal{D}, \\ r_H, & r_* \to +\infty. \end{cases} \tag{4.45}$$

Note that in the current case we have $\mathcal{D} < -r_s < 0$. Fig. 2 shows the function $r(r_*)$ vs r_*, from which we can see that the region $r \in [0, \infty)$ is mapped into the region $r_* \in [-r_s, +r_s)$ or $r_* \in (r_s, -\mathcal{D}]$. The region $r_* \in (-\infty, -r_s)$ or $r_* \in (-\mathcal{D}, +\infty)$ is mapped into the one $r \in (r_H, +\infty)$.

Similar to the previous cases, let us consider the case with $\mathcal{D} = -3r_s$ in detail, which corresponds to

$$s = \frac{-1 - 17d + 18d^2 + \sqrt{1+34d+d^2}}{2(7-17d+9d^2)}. \tag{4.46}$$

Then, we find that

$$\left(\frac{r}{r_H}\right)^{\hat{s}} = 2\epsilon^{-}\left(\epsilon^{+}\mathfrak{R}^{\frac{1}{2}} - \frac{\epsilon^{\mathcal{D}}}{2}\mathfrak{R}\right),$$
$$\mathfrak{R} = \left(\frac{\tilde{r}_* - 3}{\tilde{r}_* - 1}\right)^2. \tag{4.47}$$

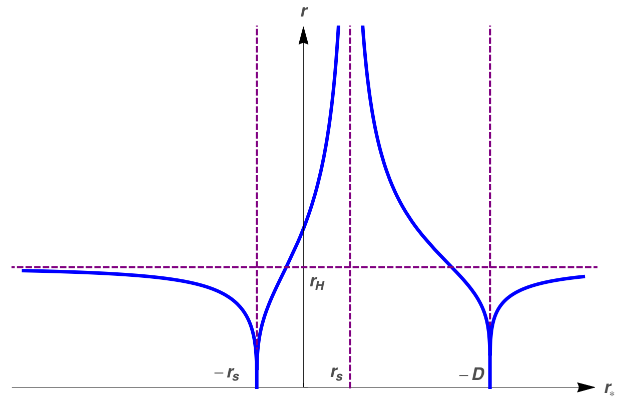

FIG. 2: The function $r \equiv r(r_*)$ for $r_s^2 > 0$ and $\frac{d}{d-1} < s < \frac{d^2}{d^2-d-1}$, where $D \equiv \mathcal{D} < -r_s$ in the present case. The spacetime is singular at $r_* = \pm r_s$.

Following what we did for the previous cases, one can solve it for $\mathfrak{R}$ in the following four regions.

(a) $r_* \in (-\infty, -r_s]$. In this region, we have the following solution

$$\mathfrak{R}^{\frac{1}{2}} = 1 + \sqrt{1 - \left(\frac{r}{r_H}\right)^{\hat{s}}}.$$

Then, the functions f and g are given by

$$f^2 = N_0^2 r^{-2z} \mathfrak{R}^{-\frac{3}{2}} \left(\frac{r}{r_H}\right)^{\frac{3(r_s-1)\hat{s}}{4r_s}}, \tag{4.48}$$

$$g^2 = \frac{2 - \mathfrak{R}^{\frac{1}{2}}}{2\left(1-\mathfrak{R}^{\frac{1}{2}}\right)^2}, \tag{4.49}$$

(b) $r_* \in (-r_s, r_s]$. In this region, we have the following solution

$$\mathfrak{R}^{\frac{1}{2}} = 1 + \sqrt{1 + \left(\frac{r}{r_H}\right)^{\hat{s}}}.$$

Then, the functions f and g are given by

$$f^2 = N_0^2 r^{-2z} \mathfrak{R}^{-\frac{3}{2}} \left(\frac{r}{r_H}\right)^{\frac{3(r_s-1)\hat{s}}{4r_s}}, \tag{4.50}$$

$$g^2 = \frac{2 - \mathfrak{R}^{\frac{1}{2}}}{2\left(1-\mathfrak{R}^{\frac{1}{2}}\right)^2}, \tag{4.51}$$

(c) $r_* \in (r_s, \mathcal{D}]$. In this region, we have the following solution

$$\mathfrak{R}^{\frac{1}{2}} = -1 + \sqrt{1 + \left(\frac{r}{r_H}\right)^{\hat{s}}}.$$

Then, the functions f and g are given by

$$f^2 = N_0^2 r^{-2z} \mathfrak{R}^{-\frac{3}{2}} \left(\frac{r}{r_H}\right)^{\frac{3(r_s-1)\hat{s}}{4r_s}}, \tag{4.52}$$

$$g^2 = \frac{2+\mathfrak{R}^{\frac{1}{2}}}{2\left(1+\mathfrak{R}^{\frac{1}{2}}\right)^2}, \quad (4.53)$$

(d) $r_* \in [\mathcal{D}, +\infty)$. In this region, we have the following solution

$$\mathfrak{R}^{\frac{1}{2}} = 1 - \sqrt{1-\left(\frac{r}{r_H}\right)^{\hat{s}}}.$$

Then, the functions f and g are given by

$$f^2 = N_0^2 r^{-2z}\mathfrak{R}^{-\frac{3}{2}}\left(\frac{r}{r_H}\right)^{\frac{3(r_s-1)\hat{s}}{4r_s}}, \quad (4.54)$$

$$g^2 = \frac{2-\mathfrak{R}^{\frac{1}{2}}}{2\left(1-\mathfrak{R}^{\frac{1}{2}}\right)^2}, \quad (4.55)$$

D. $s = \frac{d^2}{d^2-d-1}$

When

$$s = \frac{d^2}{d^2-d-1}, \quad (4.56)$$

we find that $\beta = \gamma_1 \frac{d+1}{d}$, and Eq.(3.9) becomes

$$r_*' = \frac{d^3(r_* - d^{-1})(r_*^2 - d^{-2})}{(d+1)r}. \quad (4.57)$$

To solve the above equation, we first write the above equation in the form,

$$\frac{dr}{r} = \frac{d+1}{2d^2}\left[\frac{1}{(r_*-d^{-2})^2} - \frac{d/2}{r_*-d^{-1}} + \frac{d/2}{r_*+d^{-1}}\right]dr_*, \quad (4.58)$$

which has the general solution,

$$r = r_H\left|\frac{r_*+d^{-1}}{r_*-d^{-1}}\right|^{\frac{d+1}{4d}} e^{-\frac{d+1}{2d^2(r_*-d^{-1})}}. \quad (4.59)$$

Thus, we have

$$r(r_*) = \begin{cases} r_H, & r_* \to -\infty, \\ \infty, & \left(r_* - d^{-1}\right) \to 0^-, \\ 0, & \left(r_* - d^{-1}\right) \to 0^+, \\ 0, & r_* = -d^{-1}, \\ r_H, & r_* \to +\infty. \end{cases} \quad (4.60)$$

Fig. 3 shows the curve of r vs r_*. From the definition of $W(r)$, on the other hand, we find that

$$\frac{df}{f} = \left[-\frac{d^2+d+z+dz}{2(dr_*-1)^2} + \frac{d^2-d+z+dz}{4(dr_*-1)} - \frac{d^2-d+z+dz}{4(dr_*+1)}\right]dr_*, \quad (4.61)$$

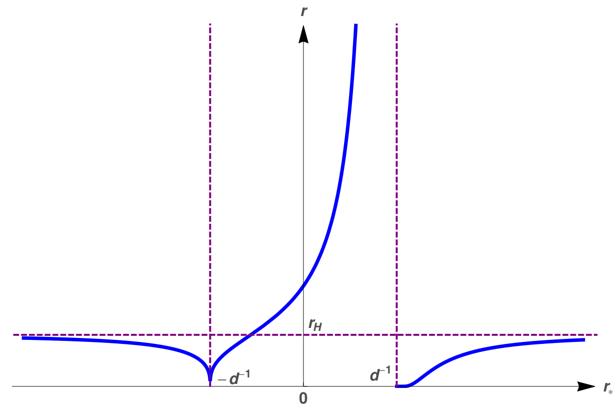

FIG. 3: The function $r \equiv r(r_*)$ for $s = \frac{d^2}{d^2-d-1}$. The space-time is singular at $r_* = \pm d^{-1}$, as can be seen from Eq.(4.65).

which has the general solution,

$$f = f_0\left|\frac{r_*-d^{-1}}{r_*+d^{-1}}\right|^{\frac{d^2-d+z+dz}{4d}} \exp\left[\frac{(d+1)(d+z)}{2d^2(r_*-d^{-1})}\right]. \quad (4.62)$$

Therefore, the corresponding metric takes the form,

$$ds^2 = -N^2(r_*)dt^2 + G^2(r_*)dr_*^2 + r^2(r_*)d\vec{x}^2, \quad (4.63)$$

where

$$N^2 = N_0^2\left|\frac{r_*-\frac{1}{d}}{r_*+\frac{1}{d}}\right|^{\frac{d-1}{2}} \exp\left[\frac{d+1}{d(r_*-d^{-1})}\right],$$
$$G^2 = \frac{(d+1)\gamma_1}{2d^3\Lambda(r_*-d^{-1})^3(r_*+d^{-1})}, \quad (4.64)$$

where $r(r_*)$ is given by Eq.(4.59). Then, the corresponding Ricci scalar is given by

$$R = \frac{4\Lambda d}{\gamma_1(r_*^2-d^{-2})}\left[r_*(r_*-d^{-1}) + \frac{(d+1)^2}{2d^3}\right], \quad (4.65)$$

from which it can be seen that the space-time is singular at $r_* = \pm d^{-1}$. Then, the physical interpretation of the solutions in the region $-d^{-1} \le r_* \le d^{-1}$ is not clear. On the other hand, to have a complete space-time in $r_* \in (-\infty, -d^{-1})$ or $r_* \in (d^{-1}, \infty)$, extensions beyond the hypersurfaces $r_* = \pm\infty$ are needed.

E. $\frac{d^2}{d^2-d-1} < s < \frac{d-1}{d-2}$

In this case, we have

$$r(r_*) = \begin{cases} r_H, & r_* \to -\infty, \\ 0, & r_* = -r_s, \\ \infty, & r_* = -\mathcal{D}, \\ 0, & r_* = +r_s, \\ r_H, & r_* \to +\infty. \end{cases} \quad (4.66)$$

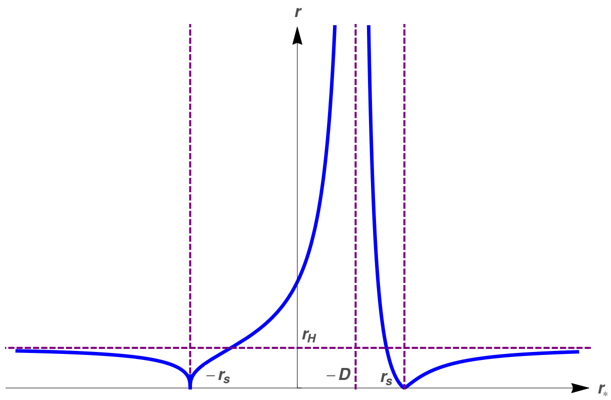

FIG. 4: The function $r \equiv r(r_*)$ for $\frac{d^2}{d^2-d-1} < s < \frac{d-1}{d-2}$, where now $-r_s < D \equiv \mathcal{D} < 0$. The spacetime is singular at $r_* = \pm r_s$, and asymptotically Lifshitz as $r_* \to -\mathcal{D}$.

Similar to the last case, now $\mathcal{D} < 0$ but with $\mathcal{D} > -r_s$. Fig. 4 shows the function $r(r_*)$ vs r_*, from which we can see that the region $r \in [0, \infty)$ is mapped into the region $r_* \in [-r_s, -\mathcal{D})$ or $r_* \in (-\mathcal{D}, r_s]$. The region $r_* \in (-\infty, -r_s)$ or $r_* \in (r_s, +\infty)$ is mapped into the one $r \in (r_H, +\infty)$.

Similar to the previous cases, let us consider the case with $r_s = -3\mathcal{D}$ in detail, which corresponds to

$$s = \frac{-9 + 7d + 2d^2 + 3\sqrt{9 - 14d + 9d^2}}{2(-17 + 7d + d^2)}. \tag{4.67}$$

Then, we find that

$$\begin{aligned} \left(\frac{r}{r_H}\right)^{\hat{s}} &= -2\epsilon^- \left(\epsilon^+ \mathfrak{R}^{\frac{3}{2}} - \frac{3\epsilon^{\mathcal{D}}}{2}\mathfrak{R}\right), \\ \mathfrak{R} &= \left(\frac{\tilde{r}_* - 1}{\tilde{r}_* - \frac{1}{3}}\right)^2. \end{aligned} \tag{4.68}$$

Following what we did for the previous cases, one can solve it for $\mathfrak{R}$ in the following four regions.

(a) $r_* \in (-\infty, -r_s]$. In this region, we have the following solution

$$\mathfrak{R} = \frac{1}{2} + \cos\frac{2\tilde{\theta}}{3} = \begin{cases} \frac{3}{2}, & r = r_H, \\ 1, & r = 0. \end{cases} \tag{4.69}$$

where $\tilde{\theta}$ is defined as

$$\cos\tilde{\theta} = \left(\frac{r}{r_H}\right)^{\frac{\hat{s}}{2}}, \quad \sin\tilde{\theta} = \sqrt{1 - \left(\frac{r}{r_H}\right)^{\hat{s}}}. \tag{4.70}$$

Since $\tilde{\theta} \in [0, \pi/2]$, we have $\mathfrak{R} \geq 1$ for $r \in [0, r_H]$. The functions f and g are also given by

$$f^2 = N_0^2 r^{-2z}\mathfrak{R}^{-\frac{1}{2}}\left(\frac{r}{r_H}\right)^{\frac{(r_s-1)\hat{s}}{4r_s}}, \tag{4.71}$$

$$g^2 = \frac{2\mathfrak{R} - 3\mathfrak{R}^{\frac{1}{2}}}{2\left(1 - \mathfrak{R}^{\frac{1}{2}}\right)^2}, \tag{4.72}$$

from which we can see that g becomes unbounded at $r = 0$ (or $\tilde{r}_* = \pm 1$). As shown above, this is a coordinate singularity.

To extend the above solution to the region $r > r_H$, one may simply assume that Eq.(4.70) hold also for $r > r_H$. In particular, setting $\tilde{\theta} = i\hat{\theta}$, we find that

$$\mathfrak{R} = \frac{1}{2} + \cosh\frac{2\hat{\theta}}{3} \geq \frac{3}{2}, \; (r \geq r_H), \tag{4.73}$$

where $\hat{\theta}$ is defined by

$$\cosh\hat{\theta} = \left(\frac{r}{r_H}\right)^{\frac{\hat{s}}{2}}, \quad \sinh\hat{\theta} = \sqrt{\left(\frac{r}{r_H}\right)^{\hat{s}} - 1}. \tag{4.74}$$

The above expression represents an extension of the solution originally defined only for $r \leq r_H$. Note that $\mathfrak{R} \simeq r^{4/3}$ as $r \to \infty$. Then, from Eq.(4.71) we find that

$$r^{2z}f^2 \sim r^{\frac{(r_s-1)\hat{s}}{4r_s} - \frac{2}{3}}, \quad g^2 \simeq 1, \tag{4.75}$$

as $r \to \infty$. That is, the space-time is asymptotically approaching to a Lifshitz space-time with its dynamical exponent now given by

$$z = \frac{(r_s - 1)\hat{s}}{8r_s} - \frac{1}{3}.$$

(b) $r_* \in (-r_s, \mathcal{D}]$. In this region, we have the following solution

$$\begin{aligned} \mathfrak{R}^{\frac{1}{2}} &= -\frac{1}{2} + \frac{1}{2}\left[\frac{r}{r_H} + \sqrt{1 + \left(\frac{r}{r_H}\right)^{\hat{s}}}\right]^{-\frac{2}{3}} \\ &\quad + \frac{1}{2}\left[\frac{r}{r_H} + \sqrt{1 + \left(\frac{r}{r_H}\right)^{\hat{s}}}\right]^{\frac{2}{3}}. \end{aligned} \tag{4.76}$$

Then, the functions f and g are given by

$$f^2 = N_0^2 r^{-2z}\mathfrak{R}^{-\frac{1}{2}}\left(\frac{r}{r_H}\right)^{\frac{(r_s-1)\hat{s}}{4r_s}}, \tag{4.77}$$

$$g^2 = \frac{2\mathfrak{R} - 3\mathfrak{R}^{\frac{1}{2}}}{2\left(1 - \mathfrak{R}^{\frac{1}{2}}\right)^2}. \tag{4.78}$$

(c) $r_* \in (\mathcal{D}, r_s]$. In this region, we have the following solution

$$\mathfrak{R}^{\frac{1}{2}} = \begin{cases} -\frac{1}{2} + \frac{1}{2}\mathcal{A}(r)^{-\frac{2}{3}} + \frac{1}{2}\mathcal{A}(r)^{\frac{2}{3}}, & r \geq r_H, \\ -\frac{1}{2} + \cos\frac{2\tilde{\theta}}{3}, & r < r_H, \end{cases} \tag{4.79}$$

where we have defined

$$\mathcal{A}(r) = \left(\frac{r}{r_H}\right)^{\frac{\hat{s}}{2}} + \sqrt{\left(\frac{r}{r_H}\right)^{\hat{s}} - 1}, \tag{4.80}$$

and $\tilde{\theta}$ is given by (4.70).

The functions f and g are given by

$$
\begin{aligned}
f^2 &= N_0^2 r^{-2z}\mathfrak{R}^{-\frac{1}{2}}\left(\frac{r}{r_H}\right)^{\frac{(r_s-1)\hat{s}}{4r_s}}, && (4.81)\\
g^2 &= \frac{2\mathfrak{R}+5\mathfrak{R}^{\frac{1}{2}}}{2\left(1+\mathfrak{R}^{\frac{1}{2}}\right)^2}. && (4.82)
\end{aligned}
$$

(d) $r_* \in (r_s, +\infty)$. In this region, we have the following solution

$$
\mathfrak{R} = \frac{1}{2}+\cos\frac{2\tilde{\theta}+\pi}{3} = \begin{cases} 1, & r = r_H,\\ 0, & r = 0,\end{cases} \qquad (4.83)
$$

where $\tilde{\theta}$ is defined by Eq.(4.70), so that $\mathfrak{R} \in (0, 1)$. Then, the functions f and g are given by

$$
\begin{aligned}
f^2 &= N_0^2 r^{-2z}\mathfrak{R}^{-\frac{1}{2}}\left(\frac{r}{r_H}\right)^{\frac{(r_s-1)\hat{s}}{4r_s}}, && (4.84)\\
g^2 &= \frac{2\mathfrak{R}-3\mathfrak{R}^{\frac{1}{2}}}{2\left(1-\mathfrak{R}^{\frac{1}{2}}\right)^2}. && (4.85)
\end{aligned}
$$

Clearly, the metric becomes singular at $r = r_H$. But this singularity is just a coordinate singularity and extension beyond this surface is needed. Simply assuming that Eq.(4.70) holds also for $r > r_H$ will lead to $\mathfrak{R}$ to be a complex function of r, and so are the functions f and g. Therefore, this will not represent a desirable extension.

V. UNIVERSAL HORIZONS AND BLACK HOLES

Remarkably, studying the behavior of a khronon field in the fixed Schwarzschild black hole background,

$$
ds^2 = -\left(1-\frac{r_s}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2, \qquad (5.1)
$$

where $r_s \equiv 2M$, Blas and Sibiryakov showed that a universal horizon exists inside the Killing horizon. But, in contrast to it, now the universal horizon is spacelike, and on which the time-translation Killing vector $\zeta^\mu [= \delta^\mu_v]$ becomes orthogonal to u^μ,

$$
u_\mu \zeta^\mu = 0, \qquad (5.2)
$$

where u_μ is the normal unit vector of the timelike foliations $\phi\left(x^\mu\right) =$ Constant,

$$
u_\mu = \frac{\phi_{,\mu}}{\sqrt{X}}, \qquad (5.3)
$$

with $X \equiv -g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$. Since u_μ is well-defined in the whole space-time, and remains timelike from the asymptotical infinity ($r = \infty$) all the way down to the space-time singularity ($r = 0$), Eq.(5.2) is possible only inside the Killing horizon, as only there ζ^μ becomes spacelike and can be possibly orthogonal to u_μ.

The above definition of the universal horizons can be easily generalized to any theory that breaks Lorentz symmetry either in the level of the action, such as the HL gravity studied in this paper, or spontaneously, such as the khrononmetric theory [15], ghost condensation [21], Einstein-aether theory [22] [2], and massive gravity [23]. The idea is simply to consider the khronon field as a probe field, and plays the same role as a Killing vector field for any given space-time [17, 18].

The equation that the khronon must satisfy in a given background $g_{\mu\nu}$ can be obtained form the action [17, 18],

$$
\begin{aligned}
S_\phi = \int d^{D+1}x\sqrt{|g|}\Big[&c_1\left(D_\mu u_\nu\right)^2 + c_2\left(D_\mu u^\mu\right)^2\\
&+ c_3\left(D^\mu u^\nu\right)\left(D_\nu u_\mu\right) - c_4 a^\mu a_\mu\Big], \qquad (5.4)
\end{aligned}
$$

where $a_\mu \equiv u^\alpha D_\alpha u_\mu$, and c_i's are arbitrary constants [3]. Then, the variation of S_ϕ with respect to ϕ yields,

$$
D_\mu \mathcal{A}^\mu = 0, \qquad (5.5)
$$

where,

$$
\begin{aligned}
\mathcal{A}^\mu &\equiv \frac{\left(\delta^\mu_\nu + u^\mu u_\nu\right)}{\sqrt{X}}\text{Æ}^\nu,\\
\text{Æ}^\nu &\equiv D_\gamma J^{\gamma\nu} + c_4 a_\gamma D^\nu u^\gamma,\\
J^\alpha{}_\mu &\equiv \big(c_1 g^{\alpha\beta}g_{\mu\nu} + c_2\delta^\alpha_\mu\delta^\beta_\nu + c_3\delta^\alpha_\nu\delta^\beta_\mu\\
&\qquad - c_4 u^\alpha u^\beta g_{\mu\nu}\big)D_\beta u^\nu. \qquad (5.6)
\end{aligned}
$$

To solve Eq.(5.5) in terms of ϕ directly, it is very complicated usually, as high-order spatial derivatives of ϕ are often involved, and the equation is highly nonlinear. So, often one divides the task into two steps: (i) One first solves it in terms of u_μ, so the corresponding equation becomes second-order, although it is still quite nonlinear. (ii) Once u_μ is given, one can find ϕ by integrating out Eq.(5.3). However, as far as the universal horizon is concerned, Eq.(5.2) shows that the second step is even not needed. Therefore, to find the location of the universal horizon now reduces first to solve Eq.(5.5) to obtain u_μ, subjected to the unit and hypersurface-orthoginal conditions,

$$
(i)\ u_\mu u^\mu = -1, \quad (ii)\ u_{[\nu}D_\alpha u_{\beta]} = 0, \qquad (5.7)
$$

---

[2] When the aether field u_μ is hypersurface-orthoginal, $u_{[\nu}D_\alpha u_{\beta]} = 0$, where D_μ denotes the covariant derivative with respect to the bulk metric $g_{\mu\nu}$, the the Einstein-aether theory is equivalent to the khrononmetric theory, as shown explicitly in [24] [See also [15, 25]].

[3] Because of the hypersurface-orthogonal condition, only three of them are independent [17, 18, 24]. But, here we shall leave this possibility open.

and then solve Eq.(5.2). Similar to the spherical case [24], the four-velocity $u_\mu = (u_t, u_r, 0, ..., 0)$ in the spacetimes,

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 dx^i dx^i, \tag{5.8}$$

is always hypersurface-orthoginal. Hence, the conditions given by Eq.(5.7) in the spacetimes of Eq.(5.8) simply reduces to $u_\mu u^\mu = -1$, which can be written as

$$u_t^2 - (u^r)^2 = F(r), \tag{5.9}$$

where $u^r \equiv F u_r$.

In review of the above, one can see that solving the khronon equation (5.5) now reduces to solve it in terms of u_μ, subjected to the constraint (5.9). As mentioned above, it is a second-order differential equation in terms of u_μ. Therefore, to determine uniquely u_μ, two boundary conditions are required, which can be [15, 17]: (i) The khronon vector is aligned asymptotically with the timelike Killing vector, $u^\mu \propto \zeta^\mu$. (ii) The khronon field has a regular future sound horizon.

Even with all the above simplification, it is found still very difficult to solve khronon equation (5.5) in the general case. But, when $c_1 + c_4 = 0$ we find that Eq.(5.5) has a simple solution $u^r = r_B/r^d$, where r_B is an integration constant. Then, from Eq.(5.9) we can get u_t, so finally we have,

$$u^\mu = \delta^\mu_t \frac{\sqrt{G(r)}}{F(r)} - \delta^\mu_r \frac{r_B}{r^d}, \tag{5.10}$$

where

$$G(r) \equiv \frac{r_B^2}{r^{2d}} + F(r). \tag{5.11}$$

Clearly, in order for the khronon field ϕ to be well-defined, we must assume

$$G(r) \geq 0, \tag{5.12}$$

in the whole space-time, including the internal region of a Killing horizon, in which we have $F(r) < 0$. In addition, $u^\mu \to u^t \delta^\mu_t \propto \zeta^\mu$, as $r \to \infty$, as longer as $F(r = \infty)$ remains positive. The latter is true for the case where spacetimes are either asymptotically flat or anti-de Sitter. Moreover, for the choice $c_1 + c_4 = 0$, the khronon has an infinitely large speed $c_\phi = \infty$ [18]. Then, by definition the universal horizon coincides with the sound horizon of the spin-0 khronon mode. So, the regularity of the khronon on the sound horizon now becomes the regularity on the universal horizon. On the other hand, from Eq.(5.2) we find that

$$u_\mu \zeta^\mu \;=\; \sqrt{G(r)} = 0, \tag{5.13}$$

at the universal horizons. Then, from the regular condition (5.12) we can see that the universal horizon located at $r = r_{UH}$ must be also a minimum of $G(r)$. Therefore, at the universal horizons we must have [17, 18, 26],

$$G(r)\big|_{r=r_{UH}} = 0 = G'(r)\big|_{r=r_{UH}}, \tag{5.14}$$

which are equivalent to

$$r_B^2 = -F(r_{UH}) r_{UH}^{2d}, \tag{5.15}$$
$$2dF(r_{UH}) + r_{UH} F'(r_{UH}) = 0. \tag{5.16}$$

The corresponding surface gravity is given by [27],

$$\begin{aligned} \kappa_{UH} &\equiv \frac{1}{2} u^\alpha D_\alpha \left(u_\lambda \zeta^\lambda\right) \\ &= \frac{r_B}{2\sqrt{2} r^d} \sqrt{G''(r)}\bigg|_{r=r_{UH}}. \end{aligned} \tag{5.17}$$

For the solutions found in Section III, we can see that only the generalized BTZ solutions have Killing horizons, and possibly have also universal horizons. For this class of solutions, we have

$$F(r) = -\frac{2m}{r^{d-1}} - \frac{2\Lambda_e r^2}{d(d+1)}, \tag{5.18}$$

for which the condition $F(r = \infty) \geq 0$ requires $\Lambda_e < 0$. Applying the above formulas to this class of solutions, we find that universal horizons indeed exist, and are given by,

$$r_{UH} \;=\; \left[-\frac{md(d+1)}{(d+2)\Lambda_e}\right]^{\frac{1}{d+1}}. \tag{5.19}$$

In addition, we also have

$$\begin{aligned} r_{EH} &= \left[-d(d+1)\frac{m}{\Lambda_e}\right]^{\frac{1}{d+1}}, \\ \kappa_{EH} &= -\frac{\Lambda_e}{d}\left[-\frac{d(d+1)m}{\Lambda_e}\right]^{\frac{1}{d+1}}, \\ \kappa_{UH} &= \frac{m(1+d)}{\sqrt{2(2+d)}}\left[-\frac{(2+d)\Lambda_e}{d(d+1)m}\right]^{\frac{3d}{2d+2}}, \end{aligned} \tag{5.20}$$

where r_{EH} and κ_{EH} denotes, respectively, the location of the Killing horizon and the corresponding surface gravity.

Figs. 5 - 7 show the locations of the universal and Killing horizons vs the mass parameter m in spacetimes with $d = 1, 2, 3$, respectively. In these figures, the corresponding surface gravities on the universal and Killing horizons are also given. From them we can see that the universal horizons are always inside the Killing horizons, as they should be [cf. the explanations given above]. On the other hand, when $m < m_c$, the surface gravity on the universal horizon is always granter than the surface gravity on the Killing horizon, where m_c is defined by $\kappa_{EH}(m_c) = \kappa_{UH}(m_c)$. But, for $m > m_c$, the opposite, i.e., $\kappa_{EH} > \kappa_{UH}$, always happens. It is interesting to note that κ_{UH} is independent of m in the case $d = 2$.

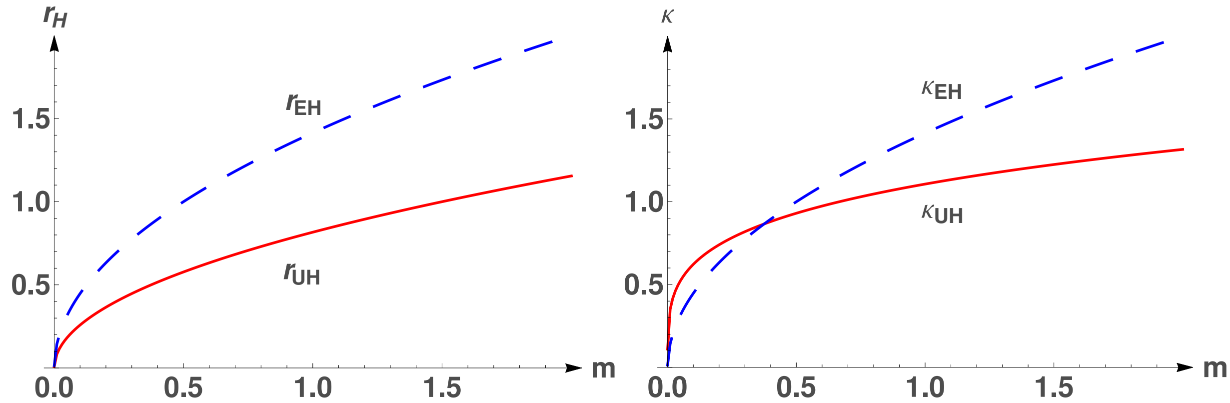

FIG. 5: The locations of the universal horizon $r = r_{UH}$ and Killing (event) horizon $r = r_{EH}$ and the corresponding surface gravities κ_{UH} and κ_{EH} on the universal and killing (event) horizon, respectively, for the solutions $d = 1$ and $\Lambda_e = -1$.

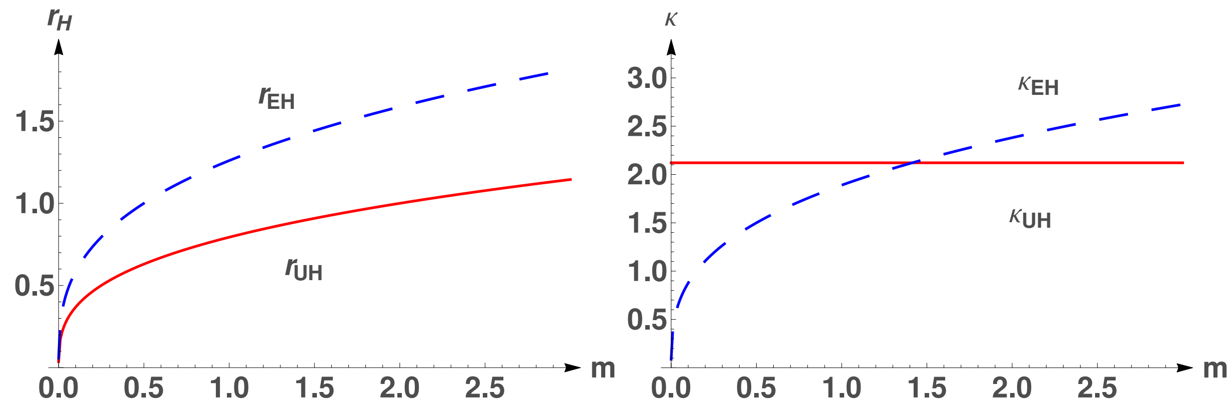

FIG. 6: The locations of the universal horizon $r = r_{UH}$ and Killing (event) horizon $r = r_{EH}$ and the corresponding surface gravities κ_{UH} and κ_{EH} on the universal and killing (event) horizon, respectively, for the solutions $d = 2$ and $\Lambda_e = -3$.

VI. CONCLUSIONS

In this paper, we have generalized our previous studies of Lifshitz-type spacetimes in the HL gravity from (2+1)-dimensions [8] to $(d + 2)$-dimensions with $d \geq 2$, and found explicitly all the static diagonal vacuum solutions of the HL gravity without the projectability condition in the IR limit.

After studying each of these solutions in detail (in Sections III - V), we have found that these solutions have very rich physics, and can give rise to almost all the structures of Lifshitz-type spacetimes found so far in other theories of gravity, including the Lifshitz spacetimes [1, 3], generalized BTZ black holes [9], Lifshitz solitons [10], and Lifshitz spacetimes with hyperscaling violation [13, 14], all depending on the free parameters of the solutions. Some solutions represent geodesically incomplete spacetimes, and extensions beyond certain horizons are needed. After the extension, it is expected that some of them may represent Lifshitz-type black holes [12].

A unexpected feature is that the dynamical exponent z in all the solutions can take values in the range $z \in [1, 2)$ for $d \geq 3$ and $z \in [1, \infty)$ for $d = 2$, because of the stability and ghost-free conditions given by Eqs.(2.20) and (2.21). Note that in (2+1)-dimensions the range of z takes its values from the range $z \in (-\infty, \infty)$, as shown explicitly in [8]. A up bound of z in high-dimensional spacetimes was also found in some numerical solutions in [1, 10, 12].

Another remarkable feature is the existence of black holes in the theory, considering the fact that the Lorentz symmetry is broken in this theory and propagations with instantaneous interactions exist. Similar to the Einstein-aether theory [15, 16], there exist regions that are causally disconnected from infinity by surfaces of finite areas — the universal horizons. Particles even with infinitely large velocity would just move around on these horizons and cannot escape to infinity. Such charged black holes have been also found recently in the HL gravity [17]. In addition, using the tunneling approach for Hawking radiation, it was shown that the universal horizon indeed radiates thermally, and a thermodynamical interpretation of the first law is possible [26]. Yet, only the surface gravity κ_{UH} defined by Eq.(5.17) is adopted, which was obtained after the nonrelativistic nature of

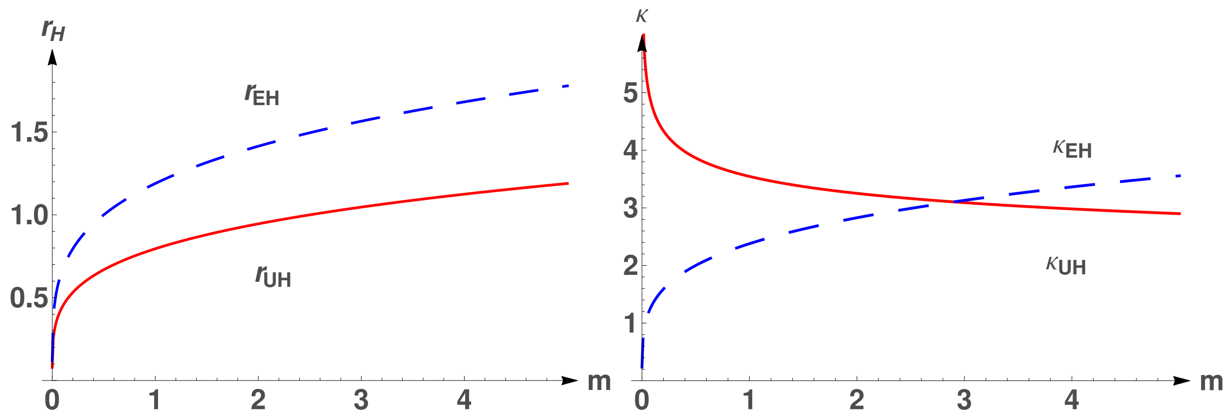

FIG. 7: The locations of the universal horizon $r = r_{UH}$ and Killing (event) horizon $r = r_{EH}$ and the corresponding surface gravities κ_{UH} and κ_{EH} on the universal and killing (event) horizon, respectively, for the solutions $d = 3$ and $\Lambda_e = -6$.

the particle dynamics was taken properly into account [27], can the standard relation $T_{UH} = \kappa_{UH}/2\pi$ between the Hawking temperature T_{UH} and the surface gravity κ_{UH} hold for the particular solutions of the Einstein-aether theory studied in [26]. The covariant form of the surface gravity Eq.(5.17) was further confirmed by considering the peeling behavior of the khronon at the universal horizons for the three well-known classical solutions, the Schwarzschild, Schwarzschild anti-de Sitter, and Reissner-Nordström [18]. It is not difficult to show that the black hole solutions presented in our current paper satisfy the first law of thermodynamics at the universal horizons, and the standard relation holds with the the surface gravity defined by Eq.(5.17).

Note that black holes defined by anisotropic horizons in the HL gravity were proposed recently in [28], and it would be very interesting to study space-time structures of the solutions presented in this paper in terms of these anisotropic horizons, not to mention the infinitely red-shifted horizons, proposed recently in [29].

With these exact vacuum solutions, it is expected that the studies of the non-relativistic Lifshitz-type gauge/gravity duality will be simplified considerably, and we wish to return to these issues soon. The stability of these structures is another important issue that must be addressed.

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