This is the accepted manuscript made available via CHORUS. The article has been published as:

New approach to cosmological bulk viscosity

Marcelo M. Disconzi, Thomas W. Kephart, and Robert J. Scherrer

Phys. Rev. D **91**, 043532 — Published 27 February 2015

DOI: 10.1103/PhysRevD.91.043532

A New Approach to Cosmological Bulk Viscosity

Marcelo M. Disconzi
Department of Mathematics, Vanderbilt University, Nashville, TN 37235

Thomas W. Kephart and Robert J. Scherrer
Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235
(Dated: February 3, 2015)

We examine the cosmological consequences of an alternative to the standard expression for bulk viscosity, one which was proposed to avoid the propagation of superluminal signals without the necessity of extending the space of variables of the theory. The Friedmann equation is derived for this case, along with an expression for the effective pressure. We find solutions for the evolution of the density of a viscous component, which differs markedly from the case of conventional Eckart theory; our model evolves toward late-time phantom-like behavior with a future singularity. Entropy production is addressed, and some similarities and differences to approaches based on the Mueller-Israel-Stewart theory are discussed.

I. INTRODUCTION

The standard expression for relativistic viscosity was derived by Eckart in 1940 [1], and cosmological implications of the Eckart viscosity were examined by Treciokas and Ellis [2] and Weinberg [3, 4], and subsequently by several others [5–7]. Later the possibility of bulk viscosity was explored in the context of inflation [8–14] and as a source for the accelerated expansion of the universe [15–22], including the possibility that a single fluid with viscosity could account for both dark matter and accelerated expansion, although the latter idea faces severe difficulties [23, 24]. For a recent review of cosmological bulk viscosity, see Ref. [25].

It is well-known that the Eckart expression for viscosity has the flaw that it can yield superluminal signal propagation. Various proposals have been put forward to remedy this problem [26–30], with the most widely-studied being the Mueller-Israel-Stewart (MIS) theory of Refs. [26–28]. Cosmological implications of the latter are examined in Refs. [31–33], and further discussion can be found in Refs. [34–41].

Here we discuss cosmological aspects of a more recent proposal to evade the causality issue, namely, the model introduced by Disconzi [42], based on earlier work by Lichnerowicz [43, 44], and generalized in Ref. [45]. In the Eckart theory, the bulk viscosity is derived from the divergence of the four-velocity of the fluid, i.e., the stress-energy tensor is given by (we take $c = 8\pi G = 1$ throughout):

$$T^E_{\alpha\beta} = (p+\rho)u_\alpha u_\beta + p g_{\alpha\beta} - \zeta(g_{\alpha\beta} + u_\alpha u_\beta)\nabla_\mu u^\mu, \quad (1)$$

where p is the pressure, ρ is the density, u is the four-velocity, and g is the metric (with convention $-+++$). The viscosity coefficient, ζ, is not necessarily constant, and is frequently taken to vary as an unknown power of the fluid density. Since Eq. (1) can lead to superluminal signals, Disconzi [42] proposed instead that

$$T_{\alpha\beta} = (p+\rho)u_\alpha u_\beta + p g_{\alpha\beta} - \zeta(g_{\alpha\beta} + u_\alpha u_\beta)\nabla_\mu C^\mu, \quad (2)$$

where C is the dynamic velocity, defined by $C_\alpha = F u_\alpha$ [44]. F is called the index of the fluid (see below) and depends on the nature of the fluid (and thus on the equation of state). We can think of F as providing a suitable relativistic correction (see Sec. V) to the formulation of the velocity in the viscous case, as the very definition of the velocity four-vector is somewhat ambiguous when viscosity is present [46]. On the other hand, this ambiguity is absent when $\zeta = 0$; hence, it is plausible to introduce a modification only in the viscous part of Eq. (1), leading to Eq. (2).

The model for viscosity from Refs. [42, 45] leads to a well-posed theory without superluminal signals under many interesting conditions. While it has not been rigorously proven to be causual under all possible circumstances, there are no known systems in which it is non-causal, so it is plausible to conjecture that it is causal under all conditions. In comparison, the Eckart model for viscosity can be shown to be non-causal under some conditions. The MIS model has the same status as the model under consideration here, namely, it has not been rigorously proven to be causal, but there are no known systems for which it is non-causal. Hence, in terms of causality, the model of Refs. [42, 45] is at least an improvement on the Eckart theory, and it is no worse than the MIS model, while being much simpler than the latter.

The model under consideration here [42, 45] uses

$$F = \frac{p+\rho}{\mu}, \quad (3)$$

where μ is the rest mass density, although in principle other choices of F could be explored. Recall that μ is conserved along the flow lines, i.e.,

$$\nabla_\alpha(\mu u^\alpha) = 0. \quad (4)$$

Now we briefly turn our attention to some technical points. While it might appear from Eq. (2) that F can be rescaled by an arbitrary constant, with the constant absorbed into the definition of ζ, leaving $T_{\alpha\beta}$ unchanged, this is, in fact, not the case. The quantity F is not an arbitrary parameter, but is the specific enthalpy given by Eq. (3). Further, F is not, in general, constant except for certain special cases.

Although Eq. (2) reduces to Eq. (1) upon setting $F = 1$, which is useful for comparison, this holds only at a formal level, in that the limit $F \to 1$ is not well-behaved. Indeed, the hypotheses of the theorems in Refs. [42, 45] require $F > 1$ (compare with Eq. 23), as it should be, as those results ensure a causal dynamics, a feature not shared by Eckart's theory. In particular, the reader should be aware that, in light of Eq. (3), setting $F = 1$ in our equations corresponds to imposing the constraint $p + \rho = \mu$, a condition that will not hold in general in Eckart's theory. We return to this point in Sec. V.

In any case, despite the necessity of restrictions for the applicability of the theorems in Refs. [42, 45], we shall expand our study beyond the hypotheses of those theorems. We are justified in doing so because such theorems are sufficiently general (e.g., they make no symmetry assumption) as to encourage a detailed study of the physical implications of adopting Eq. (2). To expand a little more on this point, we notice that Eq. (2) is a particular case of

$$\begin{aligned} T_{\alpha\beta} = (p+\rho)u_\alpha u_\beta + p g_{\alpha\beta} - \zeta \pi_{\alpha\beta}\nabla_\mu C^\mu \\ - \vartheta \pi^\mu_\alpha \pi^\nu_\beta (\nabla_\mu C_\nu + \nabla_\mu C_\nu) \end{aligned}, \tag{5}$$

where $\pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ and ϑ is the shear-viscosity coefficient. The equations studied in Refs [42, 45] do not have bulk viscosity, but carry the shear term and make no symmetry assumption on the metric. From the point of view of the techniques of weakly-hyperbolic systems employed in those works, the shear term is the most problematic one due to the multiple characteristics that arise from $\pi^\mu_\alpha \pi^\nu_\beta$. Heuristically, therefore, when the full stress-energy tensor (5) is considered, one expects that causality would fail first due to the presence of the shear term. But, since causality in the presence of shear viscosity has been shown, under appropriate assumptions, in Refs. [42, 45], it becomes reasonable to suspect that the simpler case of $T_{\alpha\beta}$ with bulk viscosity only, i.e., Eq. (2), will also present a good causal behavior. With these considerations in hand, we point out that here we are concerned mainly with the applications of Eq. (2), with the question of the well-posedness and causality behavior of our equations left for future work, where these questions will be addressed in full detail.

We conclude with some comments on the parametrizations (2) and (3). While at first sight they seem arbitrary, they are in fact well motivated in light of known difficulties when introducing viscosity into General Relativity, difficulties which are ultimately traced to the lack of a Lagrangian formulation for viscous fluids. A full discussion is given in Refs. [42, 45] and, to a lesser extent, in Ref. [43]. In a nutshell, the adoption of Eq. (2) allows one to approach the problem via a traditional point of view, so successful in the study of other matter models in General Relativity, where information about the matter fields is essentially contained in the stress-energy tensor, which enters Einstein's equations. This is in contrast with the MIS theory, where many other aspects have to be incorporated, more or less arbitrarily, in the dynamics (see, e.g., Eq. (27) below and the discussion that follows).

II. MODIFIED FRIEDMANN EQUATIONS

The Friedmann-Robertson-Walker metric with spatially flat geometry (in accordance with observations) is

$$ds^2 = -dt^2 + a^2(t)\Big(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2\Big), \tag{6}$$

where a is the scale factor. In what follows, we will take $a = 1$ at the present.

It will be convenient to write $\nabla_\mu C^\mu$ more explicitly. For the Friedmann-Robertson-Walker metric,

$$\nabla_\mu u^\mu = 3\frac{\dot{a}}{a}, \tag{7}$$

and we find

$$\nabla_\mu C^\mu = \dot{F} + 3F\frac{\dot{a}}{a}. \tag{8}$$

The first Friedmann equation then becomes

$$\dot{H} + H^2 \equiv \frac{\ddot{a}}{a} = -\frac{1}{6}\Big(3p + \rho - 3\zeta\dot{F} - 9\zeta F\frac{\dot{a}}{a}\Big), \tag{9}$$

where, as usual, $H \equiv \dot{a}/a$ is the Hubble parameter, and we have absorbed a possible cosmological constant into the definition of ρ (with $\rho_\Lambda = -p_\Lambda$). The second Friedmann equation remains unchanged, i.e, $H^2 = \rho/3$. The evolution of ρ is given by

$$\dot{\rho} + 3(p+\rho)H - 3\zeta(\dot{F} + 3FH)H = 0. \tag{10}$$

Now we can define an effective pressure via

$$p_{eff} = p + \Pi, \tag{11}$$

where Π gives the effective change in the pressure due to viscosity. Then Eqs. (8) and (10) give

$$\Pi = -\zeta\nabla_\mu C^\mu = -\zeta\dot{F} - 3\zeta FH. \tag{12}$$

For the case of Eckart viscosity, one has $\Pi = -3\zeta H$, which formally agrees with Eq. (12) upon setting $F = 1$.

III. EVOLUTION OF VISCOUS FLUIDS

Consider a fluid with an equation of state

$$p = w\rho, \tag{13}$$

where, e.g., $w = 0$ corresponds to nonrelativistic matter, $w = 1/3$ to radiation, and $w = -1$ to vacuum energy. However, we take the most general possible case and allow w to vary with time.

From Eq. (4) and Eq. (7) one immediately gets

$$\mu = \mu_0 a^{-3}, \tag{14}$$

where μ_0 is the present-day value of μ. From Eqs. (3), (11), (12), (13), and (14) we find

$$p_{eff} = p - \frac{\zeta \dot{w}\rho}{\mu_0 a^{-3}} - \frac{\zeta(1+w)\dot{\rho}}{\mu_0 a^{-3}} - \frac{6\zeta(1+w)\rho H}{\mu_0 a^{-3}}. \tag{15}$$

Since $\dot{\rho} + 3H(\rho + p_{eff}) = 0$, we can eliminate $\dot{\rho}$ from Eq. (15):

$$p_{eff} = \left(1 - \frac{3H\zeta(1+w)}{\mu_0 a^{-3}}\right)^{-1} \times \left(p - \frac{\zeta \dot{w}\rho}{\mu_0 a^{-3}} - \frac{3H\zeta(1+w)\rho}{\mu_0 a^{-3}}\right).$$

Then the effective equation of state parameter, $w_{eff} \equiv p_{eff}/\rho$, which gives $\rho(a)$ via

$$\frac{d\ln\rho}{d\ln a} = -3(1 + w_{eff}), \tag{16}$$

is given by

$$w_{eff} = \frac{w\mu_0 a^{-3} - \zeta\dot{w} - 3H\zeta(1+w)}{\mu_0 a^{-3} - 3H\zeta(1+w)}. \tag{17}$$

Lacking detailed knowledge of the functional form of ζ, we will follow earlier treatments and simply take ζ to scale as an undetermined power of the density, namely, $\zeta = \zeta_0 \rho^\alpha$. Also, for simplicity, we will assume from now on that w is constant.

There are several special cases of interest. When $w = -1$ (vacuum energy), viscosity has no effect. This is clear from the definition of F (Eq. 3), which shows that $F = 0$ for $p + \rho = 0$, so there is no viscosity in our model. The opposite extreme, a stiff equation of state with $w = 1$, also yields no effect on the density evolution, as Eq. (17) gives $w_{eff} = w = 1$ in this case. This can be understood as follows. For an ideal fluid, it is possible to show that stiffness is equivalent to $\nabla_\alpha C^\alpha = 0$ [44]. In fact, this is one of the motivations to introduce F: in Newtonian fluids, the incompressibility condition is assured by the vanishing of the divergence of the velocity. A stiff fluid is the relativistic analogue of incompressible Newtonian fluids, and, therefore, we would like a similar divergence-free condition to hold, except that now we speak of a four-divergence. This is possible if we consider the dynamic velocity C instead of the ordinary four-velocity u. For perfect fluids, considering the stress-energy tensor uniquely in terms of u or C leads to the same results. However, as we have stressed, there is a fundamental difference in which of these quantities we take as defining the viscous part of $T_{\alpha\beta}$. On the other hand, the feature of a fluid being stiff should not depend on whether viscosity is present, exactly in the same way that incompressibility for Newtonian fluids is defined by a divergence-free condition in both the Euler and the Navier-Stokes equations. But under the assumption of a stiff fluid, i.e., $\nabla_\alpha C^\alpha = 0$, the bulk term drops out of Eq. (2), consistent with the previous behavior when $w = 1$. It is essential to stress, however, that this absence of viscosity effects is a consequence of the symmetry of the problem, and not of the general model [42] on which we base our equations. Indeed, in a Robertson-Walker space-time, the only allowed contribution to viscosity comes from the bulk term, but in a general space-time, shear viscosity will be present when the fluid is stiff.

Next, consider the case where the viscous fluid with constant w dominates the expansion, so that $H^2 = \rho/3$. Then Eqs. (16) and (17) give:

$$\frac{d\ln\rho}{d\ln a} = -3(1+w)\left(\frac{1 - 2\sqrt{3}(\zeta_0/\mu_0)a^3\rho^{\alpha+1/2}}{1 - \sqrt{3}(1+w)(\zeta_0/\mu_0)a^3\rho^{\alpha+1/2}}\right). \tag{18}$$

We can make some general qualitative arguments regarding the density evolution in this case. First assume that the viscosity is negligible at early times, so that the second terms in both the numerator and denominator of Eq. (18) are $\ll 1$ when $a \ll 1$. For negligible viscosity, ρ evolves as $a^{-3(1+w)}$, so both viscosity correction terms scale as $a^{3-3(1+w)(\alpha+1/2)}$. Thus, the viscosity correction to the equation of state will grow with time as long as $\alpha < \frac{1-w}{2(1+w)}$. The result will be (as expected) a value of $\rho(a)$ that decreases more slowly than in the standard non-viscous case. Then, when a reaches the value for which $\mu_0 a^{-3} < 2\sqrt{3}\zeta_0\rho^{\alpha+1/2}$, the value of w_{eff} drops below -1, and ρ begins to increase with a. This phantom evolution inevitably results in a future singularity [47]. We can see from Eq. (18) that this singularity is reached when $\mu_0 a^{-3} = \sqrt{3}(1+w)\zeta_0\rho^{\alpha+1/2}$, at which point $w_{eff} \to -\infty$. (For a discussion of future singularities with a different, more general set of viscosity-motivated modifications to the effective pressure, see Refs. [48–50]). Note that a value of $w_{eff} < -1$, while puzzling from a theoretical perspective, is not ruled out by current cosmological data and may even be observationally favored over $w_{eff} > -1$ [51–53]. Our model also provides an elegant way for the dark matter to "cross the phantom divide," evolving from $w_{eff} > -1$ to $w_{eff} < -1$, something which is notoriously difficult to achieve in more conventional dark energy models.

Explicit solutions to Eq. (18) can be obtained for several special cases. Consider first $\alpha = -1/2$. This value has no special significance, but the analytic solution illustrates some of these qualitative arguments. For this special case, the solution to Eq. (18) is

$$\rho \propto a^{-3(1+w)}(1 - Ba^3)^{w-1}, \tag{19}$$

where $B = \sqrt{3}(1+w)(\zeta_0/\mu_0)$. This solution exemplifies our earlier qualitative arguments. For $Ba^3 \ll 1$, we simply have standard non-viscous evolution. But when Ba^3 increases to $O(1)$, ρ decreases more slowly than in the non-viscous case. At $Ba^3 = (1+w)/2$, the value of w_{eff} decreases below -1, and the density begins to increase with the scale factor (phantom-like behavior), ultimately approaching $-\infty$ as $Ba^3 \to 1$.

A more complex implicit solution is obtained for the value $\alpha = -7/2$, namely

$$\rho^{\frac{1}{7}} a^{\frac{6}{7} - \frac{3(1-w)}{7(4+3w)}} \left[(4+3w)\rho^3 - \frac{7\sqrt{3}\zeta_0}{\mu_0}(1+w)a^3\right]^{\frac{1-w}{7(4+3w)}} = \text{constant}.$$

Another exact solution is found for $\alpha = -5/6$ and $w = -2/3$, for which

$$a\rho - \frac{\sqrt{3}\zeta_0}{2\mu_0} a^4 \rho^{\frac{2}{3}} = \text{constant}. \tag{20}$$

These solutions can be compared to the evolution in the case of conventional Eckhart theory. Repeating the above arguments with Eq. (1), so that $p_{eff} = p - 3\zeta H$, leads to a solution for $\rho(a)$ for arbitrary α and constant $w \neq -1$, namely

$$\rho = \left[\sqrt{3}\frac{\zeta_0}{1+w} + Ca^{3(1+w)(\alpha-1/2)}\right]^{2/(1-2\alpha)}, \tag{21}$$

for $\alpha \neq 1/2$. (The case $\alpha = 1/2$ simply yields a constant difference between w_{eff} and w). In Eq. (21), C is a constant that can be set to give $\rho = \rho_0$ at $a = 1$. For $\alpha < 1/2$, viscosity is subdominant at early times, so $\rho \propto a^{-3(1+w)}$, while ρ approaches a constant at late times. In contrast, as seen above, our model gives a density that evolves to phantom-like behavior at late times.

IV. ENTROPY PRODUCTION

Using Eqs. (2), (3), (4), and the conservation law $u^\beta \nabla^\alpha T_{\alpha\beta} = 0$, we find

$$\begin{aligned} &u^\alpha \partial_\alpha p - \mu u^\alpha \partial_\alpha F \\ &= -\zeta F (\nabla_\alpha u^\alpha)^2 - \zeta \nabla_\alpha u^\alpha u^\beta \partial_\beta F. \end{aligned} \tag{22}$$

Since $\rho = \mu(1+e)$, where e is the specific internal energy, we see that $p + \rho = \mu(1 + e + \frac{p}{\mu})$, and thus comparison with Eq. (3) yields

$$F = 1 + e + \frac{p}{\mu}. \tag{23}$$

In particular, we see that F is the relativistic specific enthalpy of the fluid, and furthermore, we can identify the dynamic velocity C^μ of Eq. (2) as the enthalpy current. (For a discussion, see e.g., Ref. [46].) From Eq. (23), $dF = de + pd\left(\frac{1}{\mu}\right) + \frac{1}{\mu}dp$, which combined with the first law, i.e., $Tds = de + pd\left(\frac{1}{\mu}\right)$, produces $-\mu T ds = -\mu dF + dp$, or yet $-\mu T \partial_\alpha s = -\mu \partial_\alpha F + \partial_\alpha p$. Here, T is the temperature and s the specific entropy. Contracting the last equality with u^α, combining with Eq. (22), and invoking Eq. (7), finally produces

$$\mu T \dot{s} = 3\zeta H(\dot{F} + 3FH). \tag{24}$$

For the sake of brevity, we shall restrict ourselves to the case $p = w\rho$, with w constant, as in the previous section, although our conclusions hold under other conditions. Let us also suppose in this section a general behavior of the form $\rho \propto a^\beta$ at lowest order, which is consistent with the discussion of Sec. III and much of the intuition drawn from standard cosmology. As $\mu \geq 0$, $T \geq 0$, and in the cases of interest we can assume $H > 0$ and $\zeta \geq 0$, we immediately see that, if $\beta \geq -6$, which covers a wide range of possible models, we obtain $\dot{s} \geq 0$, in accordance with the second law of thermodynamics. Notice that equality happens when $\zeta = 0$ (i.e., no viscosity).

We can also analyze the entropy current $S_\alpha \equiv s\mu u_\alpha$. Using Eqs. (4) and (24), we find at once that

$$T \nabla_\alpha S^\alpha = 3\zeta H(\dot{F} + 3FH), \tag{25}$$

and, once again, that the second law, $\nabla_\alpha S^\alpha \geq 0$, is satisfied under the same conditions as above.

These results should be contrasted with models based on the MIS theory [32], where $\nabla_\alpha S^\alpha \geq 0$ does not follow from the MIS stress-energy tensor and simple scaling arguments for the thermodynamic quantities, but rather is dynamically imposed along with a redefinition of S_α.

More precisely, let $\widetilde{S}^\alpha$ be the entropy current as in the MIS theory, i.e, $\widetilde{S}_\alpha = S^\alpha - \frac{\tau\widetilde{\Pi}^2}{2T\xi}u_\alpha$, where $\xi \geq 0$ is the bulk viscosity coefficient, $\tau \geq 0$ is the relaxation coefficient for transient bulk viscous effects, and $\widetilde{\Pi}$ is the MIS bulk viscous stress. From Eq. (25), it follows that

$$\begin{aligned} \nabla_\alpha \widetilde{S}^\alpha = &\frac{3H}{T}\Big(\frac{\zeta}{T}\dot{F} + 3\frac{\zeta}{T}FH - \frac{\tau\widetilde{\Pi}^2}{2T\xi}\Big) \\ &- \frac{\tau\widetilde{\Pi}^2}{2T\xi}\Big(2\frac{\dot{\widetilde{\Pi}}}{\widetilde{\Pi}} + \frac{\dot{\tau}}{\tau} - \frac{\dot{T}}{T} - \frac{\dot{\xi}}{\xi}\Big). \end{aligned} \tag{26}$$

In the MIS formulation, it is imposed that

$$\widetilde{\Pi} + \tau\dot{\widetilde{\Pi}} = -3\xi H - \frac{1}{2}\tau\widetilde{\Pi}\Big(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\Big). \tag{27}$$

Combining Eq. (27) with Eq. (26) gives

$$\nabla_\alpha \widetilde{S}^\alpha = \frac{3H\zeta}{T}(\dot{F} + 3FH) + \frac{3H\widetilde{\Pi}}{T} + \frac{\widetilde{\Pi}^2}{T\xi}. \tag{28}$$

Typically, $\widetilde{\Pi}$ is negative, and thus the sum of the last two terms in Eq. (28) will be non-negative, implying $\nabla_\alpha \widetilde{S}^\alpha \geq 0$, if $\widetilde{\Pi} \leq -3H\xi$, which will be satisfied for ξ sufficiently small. This can be relaxed since, under our assumptions, the first term in ζ gives a positive contribution.

Although this analysis remains very qualitative, the point here is that if we insist on defining the entropy current by $\widetilde{S}^\alpha$ and employ the assumptions of the MIS theory, while at the same time adopting Eq. (2), we still obtain that $\nabla_\alpha \widetilde{S}^\alpha \geq 0$ under reasonable conditions. In any case, it is important to remember that Eq. (27) is, in a sense, arbitrary. It is adopted in that it constitutes the simplest condition, linear in $\widetilde{\Pi}$, that enforces $\nabla_\alpha \widetilde{S}^\alpha \geq 0$, but other conditions can be imposed. For instance, in our context, $\nabla_\alpha \widetilde{S}^\alpha \geq 0$ will hold if $\widetilde{\Pi}$ satisfies, instead of Eq. (26), the equation $\dot{\Pi} + \frac{1}{2}(\frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} + 4H)\Pi = 0$. Yet other evolution equations for $\widetilde{\Pi}$ can be devised. For instance, one may set the entire right-hand side of Eq. (28) equal to a positive combination of the thermodynamic quantities.

The aim of these considerations is to emphasize the flexibility of our approach. While we showed above that $\nabla_\alpha S^\alpha \geq 0$ and $\dot{s} \geq 0$ follow naturally from the equations of motion under simple assumptions, one can still adopt a MIS-like point of view when desirable. In this sense, models based on Eq. (2) can be viewed as mixed between the traditional approach (on which Eckart's theory is based), and the Extended Irreversible Thermodynamics (on which MIS's theory is based).

V. THE LIMIT $F \to 1$

Here we explore in more detail the behavior, mentioned in the introduction, of solutions when $F \to 1$. We start by pointing out that if we restore the units in (23), we have $F = 1 + c^{-2}(e + p/\mu)$, where c is the speed of light. Thus, $F = 1 + O\left(\frac{1}{c^2}\right)$, which justifies the notion that F gives a relativistic correction to u, as initially remarked.

To explore the limit $F \to 1$, it is instructive to suppose that we are working in the traditional thermodynamic setting, where the pressure p and the specific energy are non-negative. In this case, setting $F = 1$ in (23) implies $p = 0 = e$. This corresponds to the special case of pressureless dust, for which $w = 0$. The first law of thermodynamics now gives $Tds = 0$, which combined with (24) leads to $\zeta = 0$, provided that $T > 0$ and $H > 0$. We can understand this in two ways. Generally, ζ is a function of the thermodynamic quantities, and hence a function of F, $\zeta = \zeta(F)$. Therefore, we see that $\zeta(F) \to 0$ as $F \to 1$. This is consistent with the idea that in the limit of zero pressure the interaction rate due to particle collisions should go to zero (for finite interaction lengths), so in that sense there should be no dissipation. We can, however, consider a second possibility, namely, the case where ζ is constant and non-zero. In this situation, passing to the limit $F \to 1$ gives a contradiction. Keeping in mind that (24) relies on the equations of motion, this means that, although we can obtain well-behaved solutions without superluminal signals for $F > 1$, these results do not pass to the limit; in other words, the limit of solutions is not, in general, a solution, when ζ is constant.

VI. FINAL COMMENTS

The model for relativistic viscosity introduced in Ref. [42] combines the advantages of Eckart's model with those of the MIS theory. It is nearly as simple as Eckart viscosity but does not have the causality problems of that model. It is much simpler than the MIS theory, and, like the MIS theory, it is plausible to conjecture that the model is causal for all physical systems of interest, although at present neither the model discussed here nor the MIS model can be rigorously *proven* to be causal under all possible circumstances. Acceptable thermodynamic behavior (i.e., agreement with the second law of thermodynamics) under reasonable circumstances is also achieved.

This model also yields a number of interesting results when applied to cosmological fluids. One appealing property is that it automatically reduces to zero viscosity for both vacuum energy and a stiff fluid. We find that constant-w fluids can produce a future singularity for a wide range of parameter choices.

All of these features make Eq. (2) a promising candidate for a viscous stress-energy tensor in cosmology, inviting further investigation of the model.

ACKNOWLEDGMENTS

We thank J. Barrow, S. Odintsov, and I. Brevik for helpful comments on the manuscript. T.W.K. and R.J.S. are supported in part by the DOE (DE-SC0011981). M.M.D. is supported in part by the NSF (grant 1305705).

[1] C. Eckart, Phys. Rev. **58**, 919 (1940).
[2] R. Treciokas and G.F.R. Ellis, Commun. Math. Phys. **23**, 1 (1971).
[3] S. Weinberg, Astrophys. J. **168**, 175 (1971).
[4] S. Weinberg, *Gravitation and Cosmology*, Wiley (1972).
[5] G.L. Murphy, Phys. Rev. D **8**, 4231 (1973).
[6] M. Heller, Z. Klimek, and L. Suszycki, Astrophys. Space Sci. **20**, 205 (1973).

[7] M. Heller and L. Suszycki, Acta Phys. Pol. B **5**, 345 (1974).
[8] J.D. Barrow, Phys. Lett. B **180**, 335 (1986).
[9] I. Waga, R.C. Falcao, and R. Chanda, Phys. Rev. D **33**, 1839 (1986).
[10] T. Padmanabhan and S.M. Chitre, Phys. Lett. A **120**, 433 (1987).
[11] J.D. Barrow, Phys. Lett. B **183**, 285 (1987).
[12] J.D. Barrow, Nucl. Phys. B **310**, 743 (1988).
[13] J.D. Barrow, in *The Formation and Evolution of Cosmic Strings*, eds. G. Gibbons, S. Hawking, and T. Vachaspati, Cambridge University Press (1990).
[14] O. Gron, Astrophys. Space Sci. **173**, 191 (1990).
[15] W. Zimdahl, D.J. Schwarz, A.B. Balakin, and D. Pavon, Phys. Rev. D **64**, 063501 (2001).
[16] A.B. Balakin, D. Pavon, D.J. Schwarz, and W. Zimdahl, New J. Phys. **5**, 85 (2003).
[17] R. Colistete Jr., J.C. Fabris, J. Tossa, and W. Zimdahl Phys. Rev. D **76**, 103516 (2007).
[18] A. Avelino and U. Nucamendi, JCAP **4**, 006 (2009).
[19] W.S. Hipolito-Ricaldi, H.E.S. Velten, and W. Zimdahl, JCAP **6**, 016 (2009).
[20] W.S. Hipolito-Ricaldi, H.E.S. Velten, and W. Zimdahl, Phys. Rev. D **82**, 063507 (2010).
[21] J.-S. Gagnon and J. Lesgourgues, JCAP **9**, 026 (2011).
[22] H. Velten, J. Wang, and X. Meng, Phys. Rev. D **88**, 123504 (2013).
[23] B. Li and J.D. Barrow, Phys. Rev. D **79**, 103521 (2009).
[24] H. Velten and D.J. Schwarz, JCAP **9**, 016 (2011).
[25] I. Brevik and O. Gron, in *Recent Advances in Cosmology*, eds. A. Travena and B. Soren, Nova Science Publishers (2013).
[26] I. Mueller, Zeit. fur Phys. **198**, 329 (1967).
[27] W. Israel, Ann. Phys. **100**, 310 (1976).
[28] W. Israel and J.M. Stewart, Phys. Lett. A **58**, 213 (1976).
[29] R. Geroch and L. Lindblom, Phys. Rev. D **41**, 1855 (1990).
[30] B. Carter, Proc. Roy. Soc. A **433**, 45 (1991).
[31] D. Pavon, J. Bafaluy, and D. Jou, Class. Quant. Grav. **8**, 347 (1991).
[32] R. Maartens, Class. Quant. Grav. **12**, 1455 (1995).
[33] O.F. Piattella, J.C. Fabris, and W. Zimdahl, JCAP **5**, 029 (2011).
[34] N. Banerjee and A. Beesham, Pramana Journal of Physics **46**, 213 (1996).
[35] I. Brevik and O. Gorbunova, Gen. Rel. Grav. **37**, 2039 (2005).
[36] S. Nojiri and S.D. Odintsov, Phys. Lett. B **639**, 144 (2006).
[37] F. Dosopoulou, F. D. Sordo, C. G. Tsagas, and A. Brandenburg, Phys. Rev. D **85**, 063514 (2012).
[38] F. Dosopoulou, F. and C.G. Tsagas, Phys. Rev. D **89**, 103519 (2014).
[39] M. Bastero-Gil, A. Berera, I. G. Moss and R. O. Ramos, JCAP **5**, 004 (2014).
[40] I. Pahwa, H. Nandan, and U. D. Goswami, arXiv:1404.1878.
[41] S. Kotambkar, G.P. Singh, and R. Kelkar, Int. J. Theor. Phys. **53**, 449 (2014).
[42] M. M. Disconzi, Nonlinearity **27**, 1915 (2014).
[43] Lichnerowicz, A. C. R. Acad. Sci. Paris 219, (1944). 270272.
[44] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics*, W.A. Benjamin (1967).
[45] M. M. Disconzi, and M. Czubak, arXiv:1407.6963.
[46] L. Rezzolla, and O. Zanotti, *Relativistic Hydrodynamics.* Oxford University Press (2013).
[47] R.R. Caldwell, Phys. Lett. B **545**, 23 (2002).
[48] S. Nojiri and S.D. Odintsov, Phys. Rev. D **72**, 023003 (2005).
[49] S. Capozziello, V.F. Cardone, E. Elizalde, S. Nojiri, and S.D. Odintsov, Phys. Rev. D **73**, 043512 (2006).
[50] I. Brevik, E. Elizalde, S. Nojiri, and S.D. Odintsov, Phys. Rev. D **84**, 103508 (2011).
[51] G. Hinshaw, et al., Ap.J. Suppl. **208**, 19 (2013).
[52] P.A.R. Ade, et al., [arXiv:1303.5076].
[53] M. Betoule, et al., [arXiv:1401.4064].