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Lorentz symmetric dispersion relation from a random Hamiltonian

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We match the density of energy eigenstates of a local field theory with that of a random Hamiltonian order by order in a Taylor expansion. In our previous work we assumed Lorentz symmetry of the field theory, which entered through the dispersion relation. Here we extend that work to consider a generalized dispersion relation and show that the Lorentz symmetric case is preferred, in that the Lorentz symmetric dispersion relation give a better approximation to a random Hamiltonian than the other local dispersion relations we considered.

1. INTRODUCTION

We consider the question "can a random Hamiltonian be interpreted as a field theory to a sufficiently good approximation to describe the observed physical world?". We posed this question in earlier work [1, 2] from the point of view of a comparison of the energy eigenvalue spectrum, and found an affirmative answer. In that work we considered only field theories with a dispersion relation given by Lorentz symmetry. The dispersion relation is the only way the presumed Lorentz invariance of the field theory entered into our calculations. In this work we extend our previous analysis to include a one parameter family of possible dispersion relations. We also include the dimension of space d (as was already analyzed in [2]). Our analysis shows that choosing a Lorentz invariant dispersion relation makes it easier to approximate the field theory density of states by that of a random Hamiltonian. We thus conclude that in a physical picture where local field theory is just an approximate interpretation of a physical world which is fundamentally described by a random Hamiltonian, field theories with a Lorentz invariant dispersion relation are favored.

As emphasized in our earlier work, the comparison at the level of the density of energy eigenstates is only one check one can make on this set of ideas. To more fully develop our line of thinking one must explore the mapping between energy eigenstates and the fields representing observable particles or their constituents. This analysis could lead in principle to some symmetries and representations being favored over others. If such predictions are sharp enough one may even have a chance to falsify these ideas. In this paper we limit ourselves to a comparison at the level of energy spectra. We find it intriguing that already at this level the Lorentz invariant dispersion relation is favored. Still, our results are only sensitive to Lorentz invariance through the dispersion relation, therefore, our approach is unable to discern a Lorentz invariant theory from a non-invariant one that has the same dispersion relation – such as field theories in noncommutative spacetime [3] which generically break Lorentz symmetry but preserve P^2, unaltered, as a Casimir operator. To discriminate such theories other quantities beyond the densities of states should be probed involving other quantum numbers such as spin, for example, to specify the field content and the invariances unambiguously.

Our past work on this topic was motivated by the "clock ambiguity" in quantum gravity. That motivation and how it takes us to the comparison of energy spectra is discussed at length in [1] and [2]. In this paper we focus more narrowly on extending our formalism to generalized dispersion relations and interpreting our results. We feel that this technical work will be of interest to a wider group that may bring different motivations than our own. In section 2 we lay out our basic technical approach in a manner that is fairly self-contained. The key new technical ingredient of this paper is our derivation of the density of states for field theories with a generalized dispersion relation. We present two such derivations, one using statistical mechanics (section 3) and one using scaling arguments (section 4), and discuss the relationship between these two approaches in section 5. In section 6 we integrate the generalized dispersion relation results into the framework presented in section 2 and show how this leads to our main conclusions. Section 7 reviews our conclusions and explores some interesting directions for further study.

2. OUR BASIC APPROACH

If the evolution of the world around us were most fundamentally described by a random Hamiltonian, could that Hamiltonian be approximated to a sufficient degree by a field theory to allow us to *interpret* the physical world in terms of a field theory? In [1] we addressed that question at the level of the spectrum of energy eigenvalues. We found that the spectrum of a sufficiently large random Hamiltonian could approximate that of a free field theory over a sufficiently large range of energies to

allow the former to be approximated by the latter on scales relevant to our data about the physical world. In this section we review that work with an eye toward the topic of this paper: The extension of that analysis to field theories with a generalized dispersion relation.

The distribution of eigenvalues for a random Hamiltonian, represented as an $N_H \times N_H$ Hermitian matrix, follows under quite general assumptions the Wigner semicircle rule in the large N_H limit, for example, the distribution of eigenvalues of a large Hermitian matrix with elements drawn from a Gaussian distribution.

On the other hand, the density of states for a free field theory grows at large energies like an exponential of a power of the energy. On the face of it, these two forms for dN/dE are dramatically different. In order to press forward with the comparison we introduced a general parametrization for the random Hamiltonian and field theory spectral densities respectively:

$$\frac{dN_R}{dE} = \begin{cases} a\frac{N_H}{E_M}\left(1-\left(\frac{E-E_S}{E_M}\right)^\beta\right)^\gamma & |E-E_S|<E_M \\ 0 & \text{otherwise}\ , \end{cases} \quad (1)$$

$$\frac{dN_F}{dE} = \frac{B}{E}\exp\left\{b\left(\frac{E}{k_V}\right)^\alpha\right\}\ . \quad (2)$$

The parameters E_M and E_S represent the maximum eigenvalue of the random Hamiltonian and an offset energy between the two descriptions, k_V ($\equiv 2\pi/L$) is the resolution in k-space set by putting the field theory in a box of size L and B, b, α and γ are dimensionless parameters. Since we have not yet explored the emergence of gravity in this picture we initially allow the energy offset E_S to be a free parameter. The value of k_V sets a scale at which continuum field theory breaks down. The standard expression

$$N = \exp S \quad (3)$$

leads to

$$B = \alpha S\ . \quad (4)$$

Which relates parameters in Eqn. 2 to the entropy S. Throughout this paper we we use units where $\hbar = c = k_B = 1$.

We expand both Eqns. 1 and 2 in a Taylor series around a given central energy $E_0 = \rho_U V_U = 10^{80} GeV$ (using the energy density ρ_U and volume V_U of the observed Universe today). We explore the discrepancies at each order in $(E-E_0)$ to find the level of agreement between the two descriptions.

Requiring exact equality at zeroth order sets the size of the space of the random Hamiltonian to be exponentially large:

$$N_H = \frac{B}{a}\frac{E_M}{E_0}\left[1-\left(\frac{E_0-E_S}{E_M}\right)^\beta\right]^{-\gamma}\exp\left[b\left(\frac{E_0}{k_V}\right)^\alpha\right]\ . \quad (5)$$

More generally, this expression should be seen to only give a lower bound on N_H, since we currently only know upper bounds on k_V (which gives a scale of breakdown of continuum field theory).

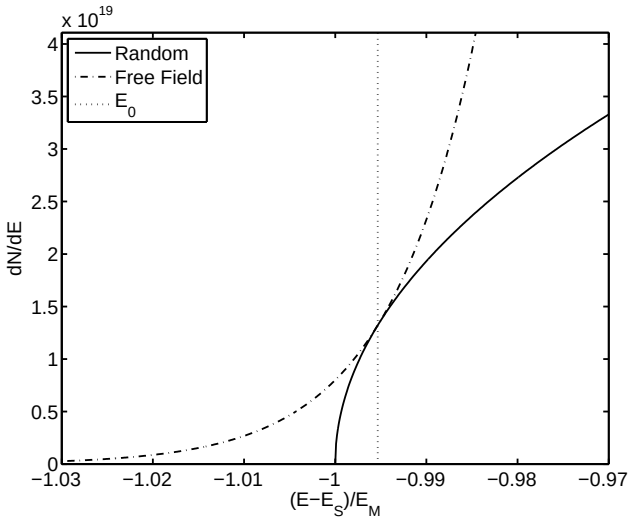

FIG. 1: A plot of the density of energy eigenvalues for a random Hamiltonian using Eqn. 1 and a field theory using Eqn. 2 matching the zeroth and first order terms in a Taylor expansion around E_0 (the vertical line).

Requiring equality at first order (as well as equality at zeroth order) sets the offset energy E_S in terms of the energy of the Universe E_0 by the following implicit expression:

$$-\beta\gamma\frac{E_0}{E_0-E_S}\frac{\left(\frac{E_0-E_S}{E_M}\right)^\beta}{1-\left(\frac{E_0-E_S}{E_M}\right)^\beta} = \alpha b\left(\frac{E_0}{k_V}\right)^\alpha\ . \quad (6)$$

Figure 1 gives examples of the two density of states curves equated at zeroth and first order according to the above analysis. Assuming equality and 0th and 1st order, the relative difference at second order is fixed and given by

$$\Delta_2 \equiv \frac{\Delta\frac{dN}{dE}}{\frac{dN}{dE}|_{E_0}} \approx \frac{\alpha^2 b^2}{\gamma}\left(\frac{E_0}{k_V}\right)^{2\alpha}\frac{(E-E_0)^2}{E_0^2}\ . \quad (7)$$

In [1] we considered specific values for the parameters in Eqn. 7 based on properties of the observed physical world and found that Δ_2 is small for realistic values. We concluded that the hypothesis that the laws of physics could most fundamentally be given by a random Hamiltonian, and just interpreted to a good approximation as a field theory had passed our "energy spectrum test".

The goal of this paper is to repeat the above analysis with a generalized dispersion relation used to get the field theory density of states. Since Lorentz invariance enters only though the dispersion relation, we hope to learn to what extent Lorentz invariant field theories are preferred when a random Hamiltonian is approximated by a field theory. To this end we introduce a general dispersion relation with two parameters, m and g, of the following

form:

$$\mathcal{E}\left(p\right) = p\Big(\frac{p}{m}\Big)^{g} \ . \tag{8}$$

Here $\mathcal{E}$ represents the energy of a single particle with momentum p, and should not be confused with E, the energy at which we evaluate the density of states $N(E)$. In general E represents the energy of multi-particle states (generally comprised of particles with with more than one energy). When the power g vanishes we recover the dispersion relation of relativistic massless particles and deviations from $g = 0$ break Lorentz symmetry. The energy scale m is needed to keep the dimensions correct for nonzero values of g. A discussion of possible values of m, which is critical for the interpretation of this work, takes place in Section 6.

There are a number of arguments to discard dispersion relations with negative values of g (see [4] and references therein for related studies); a very direct one is as follows: If we allow $g < 0$, the group velocity, $\partial\mathcal{E}/\partial p$ (proportional to p^g), becomes divergently large for packets with components of low momentum. Thus, if we require signals to propagate at a finite speed (below a certain cutoff of accessible momenta) we are restricted to consider non-negative values of g.

We conclude this section with some comments about additional assumptions and points of view implied by our framework. The formulas for the field theory densities of states quoted here (and generalized in the rest of this paper using Eqn. 8) assume flat space (Minkowski spacetime for g=0). Following the approach developed in earlier work [1, 2] and similar in spirit to what we have discussed here for field theory, we consider the possibility that general relativity gives only an approximate description of physics that need only be valid around the observed state of the Universe. Thus the field theory in Minkowski space and its extensions which we consider here, without any explicit account made for gravity, seem like a reasonable place to start. To the extent that we can argue here that Lorentz invariant theories are favored we will be led to consider fields in different representations of the Lorentz group, including spin-2. This can lead to the emergence of general relativity, as discussed briefly in section 7.

3. THE DENSITY OF STATES FOR A GENERALIZED DISPERSION RELATION: STAT MECH APPROACH

3.1. Outline

For a thermodynamic system at temperature T in a volume V, the density of states is given by

$$\frac{dN(E)}{dE} = \frac{d}{dE}\mathrm{e}^{S(E)} \ , \tag{9}$$

where

$$S \equiv Vs = V\frac{\rho + P}{T} \ , \tag{10}$$

gives the (extensive) entropy S in terms of the intensive quantities pressure P and energy density ρ. In this section we construct statistical mechanical expressions for $S(T)$ and $E(T)$ and then invert the latter to allow us to write

$$S(E) = S(T(E)) \ . \tag{11}$$

Although such a manipulation is only valid in the thermodynamic limit, we expect this to be a good approximation for a system the size of the observable Universe.

3.2. Energy density

For a homogeneous statistical system of degrees of freedom of degeneracy n in d space dimensions the energy density is given by

$$\rho(T) = \frac{n}{(2\pi)^d}\int \mathcal{E}(p) f(p) d^d p \ . \tag{12}$$

Throughout this paper we take degeneracy $n = 1$ for simplicity, without impacting our main points. For a free field theory in states of no net charge (and thus zero chemical potential, characteristic of the observed Universe) the momentum distribution $f(p)$ is given by

$$f(p) = \frac{1}{\mathrm{e}^{\mathcal{E}(p)/T} \pm 1} \tag{13}$$

where the upper/lower signs are for Fermi/Bose statistics respectively.

Using Eqn. 8 in Eqn. 12 gives

$$\begin{aligned} \rho(T) \ &= \ \frac{n}{(2\pi)^d}\int \frac{p\left(\frac{p}{m}\right)^g}{\mathrm{e}^{p\left(\frac{p}{m}\right)^g/T} \pm 1} d^d p \\ &= \ T\left(Tm^g\right)^{d/(g+1)} F(g,d) \ . \end{aligned} \tag{14}$$

Here F is $O(1)$ and is given by

$$\begin{aligned} F(g,d) \ &\equiv \ \frac{n}{(2\pi)^d}\int \frac{z^{g+1}}{\mathrm{e}^{z^{g+1}} \pm 1} d^d z \\ &= \ \frac{n}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{g+1}\int_0^\infty \frac{y^{d/(g+1)}}{\mathrm{e}^y \pm 1} dy \ . \end{aligned} \tag{15}$$

To arrive at Eqn. 15 we changed variables in Eqn. 14 using $z \equiv p\left(Tm^g\right)^{-1/(g+1)}$ and then $y \equiv z^{g+1}$. We show explicitly in section 6 that $F(g,d)$ is sufficiently slowly varying and close to unity that it can be safely neglected when building intuition about the expressions we are calculating. (Still, for completeness we keep it in for our final analysis.)

3.3. Pressure

There are some additional subtleties in defining pressure for a generalized dispersion relation. We start by rewriting the thermodynamic relation $P = -\partial E/\partial V$ in the following way

$$P = -\frac{\partial E}{\partial V} = -\frac{n}{(2\pi)^d}\int \frac{\partial p}{\partial V}\frac{d\mathcal{E}(p)}{dp} f(p) d^d p \qquad (16)$$

Next, as in standard kinetic theory, we associate a volume expansion $\delta V = A\delta x$ to the pressure on the area A due to elastic reflection of a particle with momentum p_x, that results in $\delta p_x = -2p_x A\delta x$. For a expansion in d dimensions this generalizes to

$$\frac{\partial p}{\partial V} = -\frac{2}{d}p \ , \qquad (17)$$

since $\delta x \sim \delta r/\sqrt{d}$ and $p_x \sim p/\sqrt{d}$. Thus,

$$P = \frac{2}{d}\frac{n}{(2\pi)^d}\int p\frac{d\mathcal{E}}{dp} f(p) d^d p \ , \qquad (18)$$

Using the dispersion relation of Eqn. 8, namely,

$$p\frac{d\mathcal{E}(p)}{dp} = (g+1)\mathcal{E}(p) \ , \qquad (19)$$

we find

$$P = \frac{2g+2}{d}\rho \ . \qquad (20)$$

3.4. The statistical result

We find it convenient to write the volume V in terms of the inverse length given by

$$k_V \equiv V^{-1/d} \ . \qquad (21)$$

We are now in a position to complete the approach outlined in section 3.1 using the expressions for $P(T)$ and $\rho(T)$ derived in the previous two sections. The result is

$$S(E,g,d) = \frac{d+2g+2}{d} F^{\frac{g+1}{d+g+1}} \left(\frac{E}{k_V}\right)^{\frac{d}{d+g+1}} \left(\frac{m}{k_V}\right)^{\frac{gd}{d+g+1}} . \qquad (22)$$

Note that both of the exponentiated factors in Eqn. 22 are greater than unity. The quantity k_V sets the energy scale for the breakdown of field theory, and it should certainly be smaller than m for the generalized dispersion relation to have any meaning, leading to $m/k_V > 1$. The factor with E is discussed explicitly in section 2.

4. THE DENSITY OF STATES FOR A GENERALIZED DISPERSION RELATION: SCALING APPROACH

4.1. Outline

In this section we arrive at a result equivalent to Eqn. 22 via an alternative approach. We consider a thermodynamic system characterized by its total energy, entropy, volume and energy scale m. We then make two assumptions:

1. The energy is an *extensive* function of volume and entropy.

   i.e., under a scaling of V and S by a factor λ,

   $$E(\lambda V,\ \lambda S) \ = \ \lambda\ E(V,\ S) \ . \qquad (23)$$

2. The total energy is parametrized by the following ansatz:

   $$E = E_{ns}\ \left(\frac{E_{ns}}{m}\right)^{\tilde{g}} \ , \qquad (24)$$

   where E_{ns} stands for the energy in the case where there are no other scales (such as m) entering in the description of the system. This *no-scale* energy, by dimensional analysis, must be proportional to $V^{-1/d}$, *i.e.*,

   $$V^{1/d}E_{ns}(V,\ S) \equiv h(S) \ , \qquad (25)$$

   is a function of entropy only.

Note that in Eqn. 24 we parametrized the dependence on the extra scale m as a power law in analogy with Eqn. 8.

4.2. The scaling result

Rescaling volume and entropy by λ we get from Eqn. 23 and Eqn. 25

$$\begin{aligned} E(V,\ S) \ &= \ \frac{1}{\lambda}E(\lambda V,\ \lambda S) \\ &= \ \frac{h(\lambda S)}{\lambda^{1+1/d}V^{1/d}}\left(\frac{h(\lambda S)}{m\lambda^{1/d}V^{1/d}}\right)^{\tilde{g}} \\ &= \ \left[\frac{(h(\lambda S))^{\tilde{g}+1}}{(\lambda S)^{(d+\tilde{g}+1)/d}}\right]\frac{S^{(d+\tilde{g}+1)/d}}{V^{(\tilde{g}+1)/d}m^{\tilde{g}}} \ , \end{aligned} \qquad (26)$$

that is consistent only if the quantity in square brackets is independent of S. This implies

$$S(E,\tilde{g},d) = S_0(\tilde{g},d)\left(\frac{E}{k_V}\right)^{\frac{d}{d+\tilde{g}+1}}\left(\frac{m}{k_V}\right)^{\frac{\tilde{g}d}{d+\tilde{g}+1}} \ . \qquad (27)$$

The similarity with Eqn. 22 is suggestive of a close relation to the statistical approach that will be explained in the next section[14].

5. COMPARISON OF THE TWO APPROACHES

The two approaches to obtain the density of states are designed to reflect different physical properties. The first one deals with variations of the energy density and pressure due to an altered dispersion relation in a way proper to statistical mechanics; using the representation of those quantities in terms of momentum distribution functions. The change in the dispersion relation is taken as the measure of breaking of Lorentz invariance. The second approach, in contrast, is centered in the response of the system to changes of scale, that is, in the breaking of conformality in the system. The results show, however a tight interplay of the two approaches. Indeed, if we consider a single particle state, we obtain a particular case of Eqn. 24, namely,

$$\mathcal{E} = \mathcal{E}_{ns} \left(\frac{\mathcal{E}_{ns}}{m} \right)^{\tilde{g}} \ , \tag{28}$$

where $\mathcal{E}_{ns}$ is the no-scale energy for a single particle. But, since we assumed the total energy to be extensive, we can sum all single particle states (with their corresponding density, for example, in momentum space) to obtain the total energy, namely,

$$E = V \frac{n}{(2\pi)^d} \int f(p) \mathcal{E}(p) \ d^d p \ , \tag{29}$$

where $\mathcal{E}(p)$ should be replaced with the expression on the rhs of Eqn. 28. Thus, by making the identifications

$$\mathcal{E}_{ns}(p) \ \equiv \ p \ , \tag{30}$$

$$\tilde{g} \ \equiv \ g \ , \tag{31}$$

we see that Eqn. 28 coincides with the dispersion relation Eqn. 8 and the energy density becomes equivalent to the one derived in the statistical approach. Furthermore, setting the undetermined factor $S_0(g,d)$ in Eqn. 27 to $(d+2g+2)F^{\frac{g+1}{d+g+1}}/d$, makes the entropy identical to the statistical counterpart in Eqn. 22.

We note though that this last step is done simply by construction, and is not a derivation. The price to pay for the simplicity and generality of the scaling argument is that the coefficient S_0 remains undetermined, while its numerical value is explicit in the statistical mechanics approach.

6. ANALYSIS

We are now ready to take up our original goal: The comparison of the field theory density of states with the Wigner semicircle. We have generalized the field theory density of states to include the parameter g which determines the dispersion relation according to Eqn. 8. Since our results preserve the general form taken in Eqn. 2 for the field theory density of states, we can continue to use the formalism reviewed in section 2.

Specifically, comparing Eqn. 22 with Eqns. 2 and 3 leads to

$$b \ \equiv \ \frac{2g+2+d}{d} F^{\frac{g+1}{d+g+1}} \left(\frac{m}{k_V} \right)^{\frac{gd}{d+g+1}} \ , \tag{32}$$

$$\alpha \ \equiv \ \frac{d}{d+g+1} \ , \tag{33}$$

$$\Delta_2 \ \approx \ \frac{\alpha^2 b^2}{\gamma} \left(\frac{E_0}{k_V} \right)^{2\alpha} \frac{\Delta E^2}{E_0^2} = \frac{\alpha^2}{\gamma} S(E_0)^2 \frac{\Delta E^2}{E_0^2} \ . \tag{34}$$

where

$$S\left(E_0, g, d\right) = \\ \frac{d+2g+2}{d} F^{\frac{g+1}{d+g+1}} \left(\frac{E_0}{k_V} \right)^{\frac{d}{d+g+1}} \left(\frac{m}{k_V} \right)^{\frac{gd}{d+g+1}} \ . \tag{35}$$

Recall that Δ_2 measures the fractional discrepancy between the field theory and Wigner densities of states when evaluated over an energy range ΔE centered at energy E_0. The main new ingredient here is that Δ_2 now has a dependence on the dispersion relation via the parameter g. We start with an analysis of this new g dependence and then place this in the context of a full analysis of all the parameters in Δ_2, the rest of which have already been analyzed in [1] and [2].

The smaller the value of Δ_2, the better the field theory can be approximated by the Wigner semicircle (and thus by a random Hamiltonian). We therefor expect values of g that lead to smaller Δ_2 to be strongly favored in our scheme, because these values will lead much more easily to a field theoretic interpretation of a random Hamiltonian.

The dominant contribution to the g dependence of Δ_2 comes from the $S(E_0)^2$ factor in Eqn. 34. In turn, the dominant contributions to the g dependence of $S(E_0)$ come from the appearance of g in the exponents of E_0 and m in Eqn. 35. Figure 2 illustrates that the exponent of the E factor in Eqn. 35 decreases with increasing g, while the exponent of the m factor grows with g. The values of g that lead to smaller values of Δ_2 thus depend on which factor dominates. Since (as discussed near Eqn. 22) both exponentiated factors in Eqn. 35 are always greater than unity, the question of which behavior dominates is related in a simple way to the relative sizes of m, E_0 and k_V. A helpful quantity is

$$m_c \equiv \left(E_0 k_V^d \right)^{1/(d+1)} \ . \tag{36}$$

Evaluating Eqn. 35 at $m = m_c$ gives

$$S\left(m = m_c, g, d\right) = \frac{d+2g+2}{d} F^{\frac{g+1}{d+g+1}} \left(\frac{m_c}{k_V} \right)^d \ . \tag{37}$$

Equation 37 shows that when m is set to the critical value m_c the two exponentiated factors in Eqn. 35 can be collected into a single factor which has no g dependence, as illustrated in Fig. 3.

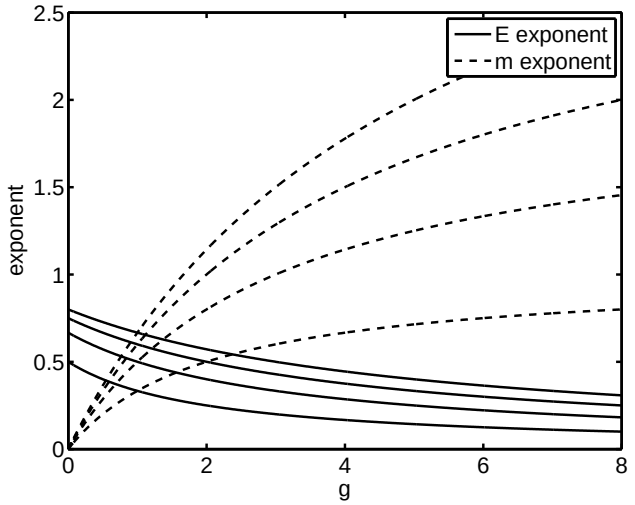

FIG. 2: The exponents of the E factor (solid) m factor (dashed) in Eqn. 35 evaluated at $d \in \{1, 2, 3, 4\}$ (height of the curve increases with with increasing d). Note that the E exponent decreases with increasing g while the m exponent grows.

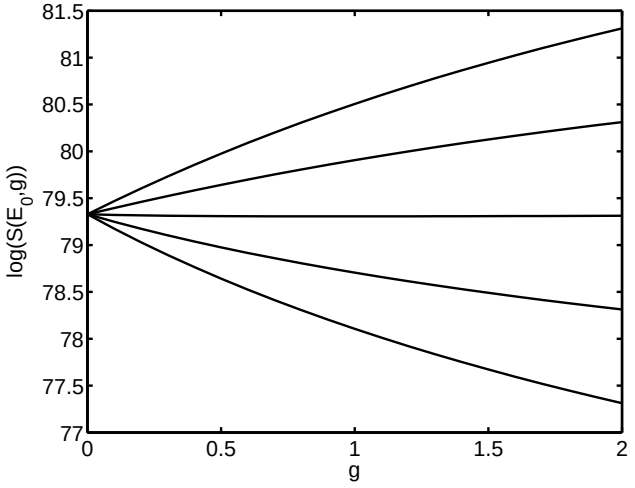

FIG. 3: The entropy $S(E_0, g)$ for $d = 3$ for $m/m_c = 0.01$, 0.1, 1, 10 and 100 in order of lowest to highest curves. We see that taking $m = m_c$ removes the g-dependence of S. For $m > m_c$, S increases with g, whereas S decreases for $m < m_c$. (Here $E_0 = 10^{80} GeV$ and $k_V = 10^{25} GeV$)

Figure 3 also shows that values of the dispersion relation scale m less than m_c make S decrease as a function of g and thus favor divergently large values of g, whereas $m > m_c$ causes $g = 0$ to be favored. Since large g corresponds to a non-local field theory, we conclude that to make a local field theoretic approximation to a random Hamiltonian one must choose $m > m_c$ in the dispersion relation. Having made that choice, the Lorentz symmetric case, $g = 0$ is favored, in that $g = 0$ minimizes Δ_2 and thus can be best approximated by the random Hamiltonian.

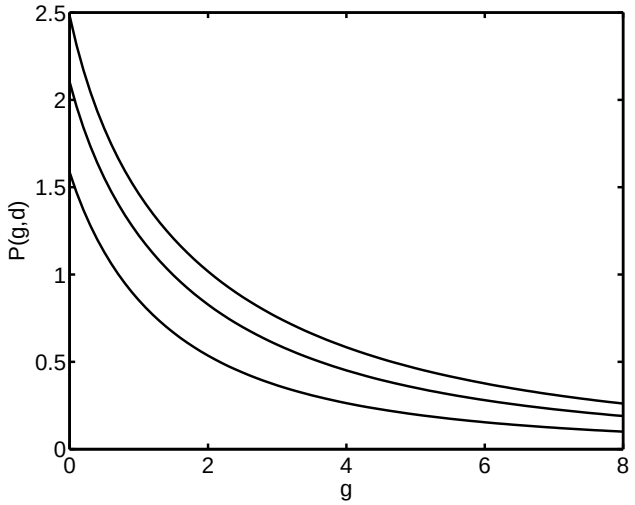

FIG. 4: The value of the prefactor P (defined in Eqn. 38) as a function of g. The three curves show $d = 2$, $d = 3$ and $d = 4$, lowest to highest. The fact that $P(g, d) = O(1)$ for a wide range of values is why we can neglect P when building intuition about the dependence of Δ_2 on g.

For completeness we now examine the dependence of Δ_2 on g from contributions other than the dominant E and m factors in S. We define the "prefactor" P from Eqns. 34 and 35 as

$$P(g, d) \equiv \frac{d + 2g + 2}{d} \alpha^2 F^{\frac{g+1}{d+g+1}} \ . \tag{38}$$

which contains additional g-dependence not accounted for by the exponents of the E and m factors in Eqn. 35. Figure 4 shows explicitly that P is $O(1)$ for a range of parameters as claimed. Still, for m very close to m_c the shape of $P(g)$ can have an impact, as illustrated in Fig. 5.

Since we have argued that the g dependence of Δ_2 drives us to the the Lorentz symmetric case already considered in earlier work, our analysis of the dependence of Δ_2 on the other parameters carries over unchanged. In particular, in [2] we considered the same d dependence used here and came to the intriguing conclusion that the $d = 3$ case put Δ_2 right at the edge of values allowed by current data. To achieve that one had to take $k_V \approx m_\gamma = 10^{25} GeV$, where m_γ is the upper bound on the photon mass. Since k_V denotes the scale where continuum field theory breaks down (not necessarily the scale of any actual volume V), this seems to be a very reasonable choice. Note also that in [2] we simply took $b = 1$ in Eqn. 2. Here we have derived an explicit formula for b (given by Eqn. 32), which indeed is $O(1)$ for the $g = 0$ case.

As already discussed, Δ_2 quantifies the degree to which the density of states curves for a field theory and a random Hamiltonian can over lap over an energy range ΔE. Figure 6 gives an explicit illustration of what is going on, where one can see how the two curves match more

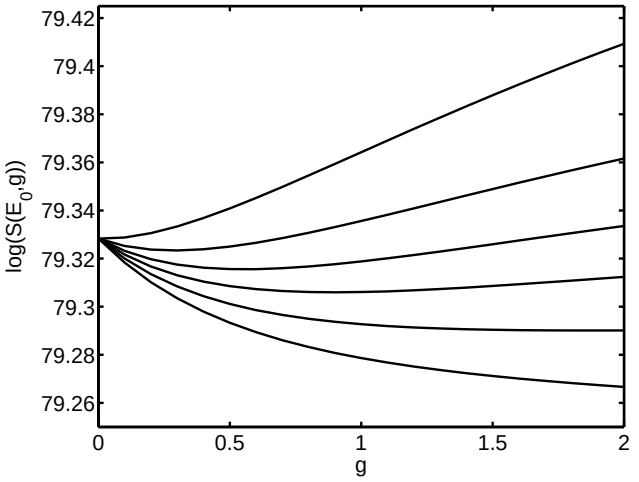

FIG. 5: This figure is similar to Fig. 3 but shows values of m much closer to m_c ($m/m_c = 0.9$, 0.95, 1, 1.05 and 1.25). For m so close to m_c the non-trivial shape of the prefactor shown in Fig. 4 becomes visible and can lead to local minimum somewhat away from $g = 0$. That shifts the value of m for which $g = 0$ is preferred to a value slightly higher than m_c.

closely for the $g = 0$ case. In this figure, some differences between the curves from one panel to the next reflect the different impacts of enforcing equality at zeroth and first order (Eqns. 5 and 6) for different values of g.

7. DISCUSSION AND CONCLUSIONS

We have calculated the density of states for a free field theory with an arbitrary power law dispersion relation. This allowed us to extend our earlier work examining the degree to which a random Hamiltonian could be approximated by a local field theory. Our conclusion is that choosing the Lorentz invariant $\mathcal{E}(p) = p$ dispersion relation reduces the discrepancy between the energy eigenvalue spectrum of a local field theory and that of the random Hamiltonian, as compared with the other dispersion relations considered. In this sense, Lorentz symmetry is favored by our analysis. This is the concrete result from our work.

As discussed in [1, 2], quite a few steps still need to be taken to check the viability of the "random Hamiltonian" picture. The work presented here offers additional hints about specific areas for further investigation. As emphasized in the introduction, by focusing on the dispersion relation we are unable to directly draw conclusions about the full Lorentz symmetry of the theory, although it is possible that preferences (for or against) could emerge from a deeper analysis that connects the eigenstates to observable particles. In any case, imposing Lorentz invariance on the field theory will be a quite generic way to ensure the preferred dispersion relation. One then would expect massless spin-1 and spin-2 representation could turn up in the interpretation of a random Hamiltonian

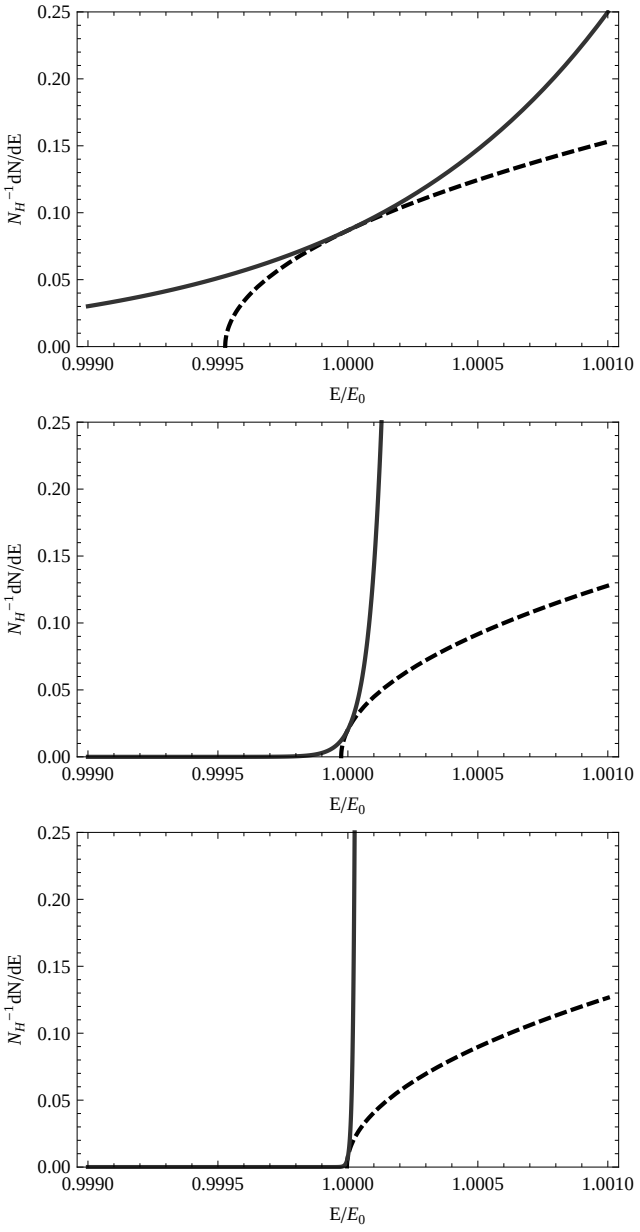

FIG. 6: $N_H^{-1} dN/dE$ for random Hamiltonians (dashed lines) and field theory (solid lines) matching the zeroth and first order terms in an expansion around E_0 as a function of E/E_0 for $g = 0$ (upper panel), $g = 1$ (center) and $g = 2$ (lower). The Lorentz symmetric ($g = 0$) case gives the broadest overlap, illustrating the main conclusion of this paper. The plots correspond to $d = 3$, $\beta = 2$, $\gamma = 1/2$, $k = 10{-}3E_M$, $E_0 = 8E_M$ and $m = E_M$. The parameters N_H and E_S were solved for as explained in section 2.

as a field theory. According to general arguments [5–11] these fields would lead to gauge forces and general relativity respectively. If we assume that the random Hamiltonian is the correct underlying theory, this picture should provide a systematic and hopefully testable way in which local field theory, gauge theory and gravity will break down.

Looking more specifically at the technical results in this paper, we chose $m > m_c$ (with m_c defined in Eqn. 36) to avoid favoring arbitrarily large powers of momentum p in the dispersion relation

$$\mathcal{E}(p) = p\Big(\frac{p}{m}\Big)^g \tag{39}$$

which would give a nonlocal field theory. This led to a preference for $g = 0$ which seems to eliminate m from the picture altogether. However, m_c may show up in some higher order way: The two values considered in [1] and [2] for k_V are $k_V = (V_U)^{1/d}$ (where V_U is the Hubble volume) and $k_V = m_\gamma$. Continuing to use $E_0 = \rho_U V_U = 10^{80} GeV$ as we have here and in our earlier work, these give $m_c = \rho^{1/d}$ and $m_c \approx 1GeV$ respectively. Both are interesting scales for physics in our world. Note the scale $\rho^{1/d}$ is associated with both the dark energy and the neutrino mass[15]. One way m_c could enter is through the fact (illustrated in Fig. 5) that local Lorentz symmetry breaking theories are preferred for m very close to m_c. We find it intriguing that other authors have associated neutrino mass with Lorentz-violation[12], while our scheme appears to associate Lorentz-violation with the neutrino mass scale.

The idea that physics at its most fundamental is simply described by a random Hamiltonian is a curious idea which we motivated in previous work on the clock ambiguity. There is much still to be explored before such an idea could look seriously viable. Still, we find it interesting that this idea has passed as many tests as it has so far. In particular, this paper shows that the random Hamiltonian picture predicts that we would interpret the physical world using theories with Lorentz symmetric dispersion relations. We find this result quite remarkable.

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[14] The choices made above are orthogonal to those of [13]. There, assumption 1 was dropped to include a *sub*-extensive Casimir energy component E_C while keeping the no-scale property ($\tilde{g} = 0$ in assumption 2). This leads to the generalized Cardy formula $S \sim V^{1/d}\sqrt{E_C(E - E_C)}$. We could generalize further our approach to include such a non-extensive component to arrive at the analog expression when the no-scale property is lost.
[15] We found in [2] that the value $k_V = (V_U)^{1/d}$ only passed our energy spectrum test when an alternate form of the density of states was employed. This form was based on [13] (as we mentioned briefly in section 4.2). One would need to extend results of [13] to include generalized dispersion relations to fully explore the $k_V = (V_U)^{1/d}$ case.