Standard model anatomy of WIMP dark matter direct detection. I. Weak-scale matching

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Phys. Rev. D **91**, 043504 — Published 5 February 2015

DOI: 10.1103/PhysRevD.91.043504

EFI Preprint 13-34

Standard Model anatomy of WIMP dark matter direct detection I: weak-scale matching

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Abstract

We present formalism necessary to determine weak-scale matching coefficients in the computation of scattering cross sections for putative dark matter candidates interacting with the Standard Model. Particular attention is paid to the heavy-particle limit. A consistent renormalization scheme in the presence of nontrivial residual masses is implemented. Two-loop diagrams appearing in the matching to gluon operators are evaluated. Details are given for the computation of matching coefficients in the universal limit of WIMP-nucleon scattering for pure states of arbitrary quantum numbers, and for singlet-doublet and doublet-triplet mixed states.

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1 Introduction

The compelling evidence for dark matter (DM) inconsistent with Standard Model (SM) particles has motivated many theoretical studies and experimental searches to elucidate its particle nature. In particular, the paradigm of Weakly Interacting Massive Particles (WIMPs) continues to play a prominent role, and experiments in the present decade should explore a significant region of remaining WIMP parameter space [1]. Given the multitude of WIMP candidates and search strategies, it is imperative to develop theoretical formalism to delineate the possible interactions of DM with known particles, making clear which uncertainties are inherently model dependent and which can, at least in principle, be improved by further SM analysis.

Even in many seemingly simple cases, determination of WIMP-nucleon cross sections demands an intricate analysis of competing amplitudes mediated by SM particles (see e.g., [2–7]). In this paper we set out the formalism for electroweak-scale matching computations for application both to theories with specified ultraviolet (UV) completion (e.g., supersymmetric models [8, 9]), and to the heavy WIMP limit where theoretical control is maintained in the absence of a specified UV completion [6]. We review relevant aspects of techniques such as the background field method for matching to gluon operators [3, 10], the extension of the onshell renormalization scheme for WIMP couplings to the electroweak SM, and the treatment of effective theory subtractions. Direct detection experimental constraints [11, 12], together with other phenomenological bounds such as LHC searches, may plausibly indicate that new particles must have mass somewhat above the mass of electroweak-scale particles ($M \gg m_W$). In this regime, the prospects for direct detection become more challenging, but in a precise sense more constrained due to heavy particle universality. Extending the particle content of the SM by one or a few electroweak multiplets, the heavy particle limit implies highly predictive cross sections with minimal parametric input beyond the SM. This limit is thus both physically interesting, as well as a useful pedagogical illustration. Within the heavy WIMP framework, we present a complete reduction of the required one- and two-loop amplitudes into a basis of heavy-particle loop integrals with nonzero residual mass.

Although we aim for generality, for definiteness throughout the paper, we illustrate these methods for the case where the lightest, electrically neutral particle of the new sector corresponds to a self-conjugate field (e.g., a Majorana fermion or real scalar) stabilized by a Z_2 symmetry, deriving from a theory consisting of one or two $SU(2)_W \times U(1)_Y$ multiplets beyond the SM particle content [2, 5–7, 13–26]. An important simplification occurs when a scale separation exists between SM masses ($\sim m_W$) and the lightest new particle mass ($\sim M$), allowing an expansion in m_W/M. We consider in detail the limit $M \gg m_W$ where universal behavior appears, and present the necessary heavy particle effective theory tools for such an analysis. For these SM extensions, we present details of the first complete computation of the matching at leading order in perturbation theory onto the full basis of operators at the electroweak scale [6].

The field of DM direct detection is by now a mature subject.[1] Early treatments of QCD effects in neutralino-nucleon scattering include the works of Drees and Nojiri [27]. Basic aspects of formalism may be found in the review of Jungman, Griest and Kamionkowski [8]. However, the last few years have witnessed the discovery and mass measurement for a SM-like Higgs boson [28, 29], new constraints on the mass scale of particles beyond the SM [30], and important computational advances in lattice QCD [31, 32]. A complete description of DM-SM interactions is now possible in many SM extensions but demands the systematic treatment of QCD effects and uncertainties, including the consideration of loop amplitudes that are typically neglected in the $m_W \sim M$ regime, but which

[1] A subset of recent work in the field may be found in the Snowmass review [1].

contribute at leading order in the general case.

In this work we extend some aspects of heavy particle formalism familiar from heavy quark effective theory [33] for DM applications, and we hope that a detailed treatment will serve as pedagogy for the DM practitioner unfamiliar with heavy particle tools. Both within and beyond the heavy particle limit, distinguishing between different DM candidates in direct detection experiments demands careful treatment of QCD corrections when passing from a theory renormalized at the electroweak scale to a low-energy theory of quarks and gluons where hadronic matrix elements are evaluated. A companion paper treats this separate problem for applications involving a range of dark matter candidates [34].

The remainder of the paper is structured as follows. In Sec. 2, we briefly review aspects of heavy particle effective theory relevant for DM applications. Section 3 specifies the operator basis for DM-SM interactions at the weak scale relevant for spin-independent, low-velocity scattering with nucleons. In Sec. 4, we construct the effective theory for one or two heavy electroweak multiplets interacting with SM Higgs and electroweak gauge fields, accounting for masses induced by electroweak symmetry breaking (EWSB), and presenting the lagrangian in terms of mass eigenstate fields from which the complete set of Feynman rules may be easily derived. In Sec. 5, we define an extension of the onshell renormalization scheme for the electroweak SM for a consistent loop-level evaluation of amplitudes. Section 6 presents the details of the matching calculation, including the systematic reduction of heavy-particle integrals, and the implementation of background field techniques for gluon operators. We present the bare matching coefficients in Sec. 7, and conclude with a summary in Sec. 8.

2 Heavy particle effective field theory for dark matter applications

Heavy particle methods may be used to efficiently describe the interactions of DM, of mass M, with much lighter degrees of freedom such as those of $n_f = 5$ flavor QCD (in the case $m_b \ll M$, where m_b is the bottom quark mass) or those of the SM electroweak sector (in the case $m_W \ll M$, where m_W is the $W^\pm$ boson mass). Let us briefly review a few aspects of heavy particle effective theory relevant for the DM applications in Secs. 3 and 4.

A heavy-particle field, h_v, is identified with a representation of the little group for massive particles, and carries a label v associated with the time-like unit vector v^μ that defines the little group [35]. The little group for massive particles is isomorphic to $SO(3)$, and therefore has field representations carrying spin $s = 0, 1/2, 1, \dots$. We may write such fields in covariant notation using a Dirac spinor-vector with appropriate constraints. For example, a spin-1/2 heavy-particle field has $2(1/2) + 1$ degrees of freedom and can be written as a Dirac spinor, h_v, obeying $\not{v} h_v = h_v$ as a projection constraint.[2] For integer spin we define $\bar{h}_v \equiv h_v^\dagger$, while for half-integer spin h_v carries spinor indices and we define $\bar{h}_v \equiv h_v^\dagger \gamma^0$. Additionally, for self-conjugate fields the simultaneous operations

$$v^\mu \to -v^\mu \,, \quad h_v \to h_v^c \equiv \mathcal{C} h_v^* \,, \tag{1}$$

where $\mathcal{C}$ is the charge conjugation matrix, implement a symmetry of the heavy-particle lagrangian.[3]

Having specified the building blocks, interactions with heavy-particle fields can be constructed in the usual way. We write down the most general set of gauge-invariant and Lorentz-covariant operators in terms of the heavy field h_v, the time-like unit vector v^μ, and other relativistic degrees of freedom up to a given order in the $1/M$ power counting. In the case of a self-conjugate heavy particle,

[2] The case of arbitrary spin is discussed in Appendix A.1 of Ref. [35].

[3] A discussion of this invariance is given in Appendix A.2 of Ref. [35]; for a heavy Majorana fermion, see also [36].

such as that derived from a Majorana fermion or a real scalar of the underlying UV completion, the invariance (1) is additionally imposed.

Lorentz invariance should also be implemented using the heavy-particle boost transformation rules that follow from the little group representation. The implementation of Lorentz invariance in heavy particle effective theories is formally interesting, and has important consequences for applications involving higher-order $1/M$ expansions [35, 37, 38]. In the present paper, we focus on the leading order in $1/M$.

3 Effective theory below the weak scale

Let us construct the effective theory of DM with mass $M \gtrsim m_W$ interacting with $n_f = 5$ flavor QCD. The hierarchy of scales between the DM mass and the relevant low-energy degrees of freedom, $\Lambda_{\rm QCD}, m_c, m_b \ll m_W$, allows us to use heavy particle effective theory to describe the DM field. The most general lagrangian relevant for spin-independent, low-velocity scattering with nucleons, is then given at energies $E \ll m_W$ by,[4]

$$\mathcal{L}_{\chi_v,\,\rm SM} = \bar{\chi}_v \chi_v \left\{ \sum_{q=u,d,s,c,b} \left[c_q^{(0)} O_q^{(0)} + c_q^{(2)} v_\mu v_\nu O_q^{(2)\mu\nu} \right] + c_g^{(0)} O_g^{(0)} + c_g^{(2)} v_\mu v_\nu O_g^{(2)\mu\nu} \right\} + \dots \,, \qquad (2)$$

where χ_v is the lightest, electrically neutral, self-conjugate WIMP of arbitrary spin. The ellipsis in the above equation includes higher-dimension operators suppressed by powers of $1/m_W$. The assumed self-conjugacy of χ_v implies that (2) is invariant under (1). The SM component of (2) is expressed in terms of quark and gluon fields as

$$\begin{aligned} O_q^{(0)} &= m_q \bar{q} q \,, \\ O_g^{(0)} &= (G^A_{\mu\nu})^2 \,, \\ O_q^{(2)\mu\nu} &= \frac{1}{2} \bar{q} \left(\gamma^{\{\mu} i D_-^{\nu\}} - \frac{1}{d} g^{\mu\nu} i \not{D}_- \right) q \,, \\ O_g^{(2)\mu\nu} &= -G^{A\mu\lambda} G^{A\nu}{}_\lambda + \frac{1}{d} g^{\mu\nu} (G^A_{\alpha\beta})^2 \,. \end{aligned} \qquad (3)$$

Here $D_- \equiv \overrightarrow{D} - \overleftarrow{D}$, and $A^{\{\mu}B^{\nu\}} \equiv (A^\mu B^\nu + A^\nu B^\mu)/2$ denotes symmetrization. The operators in (3) are expressed in terms of bare lagrangian fields, where we employ dimensional regularization with $d = 4 - 2\epsilon$ spacetime dimensions. We use the background field method for gluons in the effective theory thus ignoring gauge-variant operators, and assume that appropriate field redefinitions are employed to eliminate operators that vanish by leading order equations of motion. We ignore flavor non-diagonal operators, whose nucleon matrix elements have an additional weak-scale suppression relative to those considered. We will not be concerned here with leptonic interactions.

For a self-conjugate WIMP, χ_v, with mass $M \gtrsim m_W$ and arbitrary spin, Eq. (2) represents the most general effective lagrangian at leading order in $1/m_W$, relevant for spin-independent, low-velocity scattering with nucleons. Details of the UV completion are encoded in the twelve matching coefficients $c_q^{(0)}$, $c_q^{(2)}$, $c_g^{(0)}$ and $c_g^{(2)}$. Matching onto the effective theory (2) is in general dependent on the specific SM extension. Although much of the formalism applies more generally, for definiteness we focus on the heavy WIMP limit, $M \gg m_W$, where universal features appear [6].

[4]General bases including spin-dependent interactions, and non-self-conjugate WIMPs are presented in [34].

4 Effective theory for one or two heavy electroweak multiplets

In place of a specified UV theory for DM, let us use heavy particle effective theory to describe extensions of the SM consisting of one or two electroweak multiplets with masses large compared to the mass of electroweak-scale particles, $M, M' \gg m_W$. The extension to more than two multiplets is straightforward. We will construct the effective theory describing interactions of such heavy WIMPs with the SM in the regime $|M'-M|$, $m_W \ll M$, M'. In the case $|M'-M| \gg m_W$ the effects of the heavier multiplet appear as power corrections in the effective theory for the lighter multiplet. For notational clarity, below we omit the subscript v labeling a heavy-particle field.

Consider one or two multiplets of heavy-particle fields with arbitrary spin, transforming under irreducible representations of electroweak $SU(2)_W \times U(1)_Y$. Let us collect the heavy fields in a column vector h, and their masses in a diagonal matrix M. The precise specification of M beyond tree level is described in Sec. 5. At leading order in the $1/M$ expansion, the most general gauge- and Lorentz-invariant lagrangian, bilinear in h, and written in terms of the building-blocks h, v^μ, and SM fields, takes the form

$$\mathcal{L} = \bar{h}\left[iv\cdot D - \delta m - f(H)\right]h + \mathcal{O}(1/M)\,, \tag{4}$$

where $iD_\mu = i\partial_\mu + g_1 Y B_\mu + g_2 W^a_\mu T^a$, and $f(H)$ is a linear matrix function of H (and H^*). For pure states gauge invariance implies $f(H) = \mathbb{0}$, while for mixed states $f(H)$ describes the mixing of the pure-state constituents through the Higgs field. In terms of a reference mass $M_{\rm ref}$, the residual mass matrix is

$$\delta m = M - M_{\rm ref}\mathbb{1}. \tag{5}$$

Note that if the masses composing M are degenerate, as for a single "pure" electroweak multiplet, we may choose $M_{\rm ref}$ appropriately to set $\delta m = \mathbb{0}$. In the case of two "mixed" electroweak multiplets M will have non-degenerate entries in general.

Upon accounting for EWSB we may write (4) as

$$\begin{aligned}\mathcal{L} = \bar{h}\bigg[&iv\cdot\partial + eQv\cdot A + \frac{g_2}{c_W}v\cdot Z(T^3 - s_W^2 Q)\\ &+\frac{g_2}{\sqrt{2}}(v\cdot W^+T^+ + v\cdot W^-T^-) - \delta M(v_{\rm wk}) - f(\phi)\bigg]h + \mathcal{O}(1/M)\,,\end{aligned} \tag{6}$$

where $T^\pm = T^1 \pm iT^2$, the charge matrix is $Q = T^3 + Y$ in units of the proton charge, and ϕ denotes the fluctuation of the Higgs field about $\langle H\rangle$,

$$H = \frac{v_{\rm wk}}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}\phi_W^+\\ \frac{1}{\sqrt{2}}(h+i\phi_Z)\end{pmatrix}. \tag{7}$$

The residual mass matrix now includes EWSB contributions,

$$\delta M(v_{\rm wk}) = \delta m + f(\langle H\rangle), \tag{8}$$

and in the mass eigenstate basis for $\delta M(v_{\rm wk})$, we will set the residual mass of the lightest, (assumed) electrically neutral WIMP, χ, to zero by appropriate choice of $M_{\rm ref}$. Other states may have non-vanishing residual masses. In the following, we will suppress the subscript in $v_{\rm wk}$; the resulting v is not to be confused with the velocity v^μ.

The heavy-particle lagrangian (4) can also be obtained at tree level from a manifestly relativistic lagrangian by performing field redefinitions. We illustrate this for the singlet-doublet mixture in Appendix A.[5] Let us now have a detailed look at extensions with one (pure states) or two (mixed states) electroweak multiplets.

4.1 Pure states

The pure-state heavy-particle lagrangian is completely specified by electroweak quantum numbers since $\delta m = \mathbb{0}$ and $f(H) = \mathbb{0}$. We may proceed in generality, assuming a multiplet of fields in the isospin J representation of $SU(2)_W$ with hypercharge Y. The amplitudes for weak-scale matching in Sec. 6 will be given in terms of Y^2 and the Casimir $J(J+1)$. In particular, amplitudes with two $W^\pm$ bosons or two Z^0 bosons carry the respective factors

$$\mathcal{C}_W = J(J+1) - Y^2\,, \quad \mathcal{C}_Z = Y^2. \tag{9}$$

For extensions consisting of electroweak multiplets with non-zero hypercharge, we assume that higher-dimension operators cause the mass eigenstates after EWSB to be self-conjugate combinations. This forbids a phenomenologically disfavored tree-level vector coupling between the lightest, electrically neutral state, χ, and Z^0. We further assume that the mass difference between the self-conjugate eigenstates is large enough to suppress inelastic scattering, but small compared to m_W so that we may neglect its contribution to the residual mass δm.

As specific illustrations we consider the cases of an $SU(2)_W$ triplet ($J = 1$) with $Y = 0$, and a pair of $SU(2)_W$ doublets ($J = 1/2$) with opposite hypercharge $Y = \pm 1/2$. In supersymmetric extensions, these represent pure wino and pure higgsino states, respectively. Let us look at these cases in some detail.

4.1.1 Pure triplet

Let the column vector $h_T = (h^1, h^2, h^3)$, with subscript T for triplet, be a heavy, self-conjugate, $SU(2)_W$ triplet with $Y = 0$. The heavy-particle lagrangian for h_T is given by (4) with $(T^a)^{bc} = i\epsilon^{bac}$, $f(H) = \mathbb{0}$, and $\delta m = \mathbb{0}$. The electric charge eigenbasis is given by

$$\begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} \equiv \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_+ \\ h_- \end{pmatrix}. \tag{10}$$

In terms of the column vector $h = (h_0,\, h_+,\, h_-)$, where $h_0 \equiv \chi$, the lagrangian is given by (6) with

$$Q = T^3 = \mathrm{diag}(0, 1, -1)\,, \quad T^+ = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}. \tag{11}$$

[5] We remark that the consistency of an effective description for the one-heavy particle sector for a self-conjugate field follows from the identification of lowest-lying states odd under a Z_2 symmetry. In contrast, the one-heavy particle sector for a heavy field carrying $U(1)$ global symmetry (e.g., heavy-quark number in a heavy quark effective theory) is identified by this quantum number.

4.1.2 Pure doublet

Let h_ψ and h_{ψ^c} be heavy-particle doublets in the $(\mathbf{2}, 1/2)$ and $(\bar{\mathbf{2}}, -1/2)$ representations of $SU(2)_W \times U(1)_Y$.[6] Anticipating perturbations that cause the mass eigenstates to be self-conjugate fields, let us introduce the linear combinations

$$h_{D_1} = \frac{h_\psi + h_{\psi^c}}{\sqrt{2}} = \begin{pmatrix} h_1 \\ h_0 \end{pmatrix}, \quad h_{D_2} = \frac{i(h_\psi - h_{\psi^c})}{\sqrt{2}} = \begin{pmatrix} h_2 \\ h_0' \end{pmatrix}, \tag{12}$$

with subscript D for doublet. The heavy-particle lagrangian for the column vector $h = (h_{D_1}, h_{D_2})$ is given by (1), with $f(H) = \mathbb{0}$, and gauge couplings

$$T^a = \begin{pmatrix} \frac{\tau^a - \tau^{aT}}{4} & \frac{-i(\tau^a + \tau^{aT})}{4} \\ \frac{i(\tau^a + \tau^{aT})}{4} & \frac{\tau^a - \tau^{aT}}{4} \end{pmatrix}, \quad Y = \frac{i}{2}\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{13}$$

where τ^a are the Pauli isospin matrices. Neglecting the small mass perturbation mentioned above, the tree-level mass eigenstates are degenerate, and we may choose $\delta m = \mathbb{0}$. The charge eigenstates are given by

$$\begin{pmatrix} h_1 \\ h_0 \\ h_2 \\ h_0' \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_0' \\ h_+ \\ h_- \end{pmatrix}. \tag{14}$$

In terms of the column vector $h = (h_0, h_0', h_+, h_-)$, where $h_0 \equiv \chi$, the lagrangian is given by (6) with $Q = \mathrm{diag}(\mathbb{0}_2, 1, -1)$ and

$$T^3 = \begin{pmatrix} 0 & \frac{i}{2} & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad T^+ = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 0 \end{pmatrix}. \tag{15}$$

4.2 Admixtures

As an example of mixed states, let us consider in detail the singlet-doublet admixture. Results for the triplet-doublet admixture will also be given below.

4.2.1 Singlet-doublet admixture

Let h_S, with subscript S for singlet, be a heavy, self-conjugate, $SU(2)_W$ singlet with $Y = 0$ and mass M_S. Consider an admixture of h_S and the previously defined self-conjugate doublets h_{D_1} and h_{D_2}, with mass M_D. At leading order in the $1/M$ expansion, the gauge-invariant interactions of h_S, h_{D_1} and h_{D_2} involving the Higgs field are

$$\mathcal{L}_{H\bar{h}h} = -\bar{h}_S \left[yH^\dagger \frac{(h_{D_1} - ih_{D_2})}{\sqrt{2}} + y^* H^T \frac{(h_{D_1} + ih_{D_2})}{\sqrt{2}} \right] + \text{h.c.} = -\bar{h} f(H) h, \tag{16}$$

[6] This construction is analogous to that appearing in applications of heavy quark effective theory to processes where both a heavy quark and a heavy anti-quark are active degrees of freedom.

where we have imposed the invariance (1), and collected the heavy-particle fields in a column vector $h = (h_S, h_{D_1}, h_{D_2}) = (h_S, h_1, h_0, h_2, h_0')$. The Higgs coupling matrix is given by

$$f(H) = \frac{a_1}{\sqrt{2}} \begin{pmatrix} 0 & H^\dagger + H^T & i(H^T - H^\dagger) \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \\ i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} + \frac{a_2}{\sqrt{2}} \begin{pmatrix} 0 & -i(H^T - H^\dagger) & H^T + H^\dagger \\ -i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix}, \tag{17}$$

with real parameters $a_1 = \mathrm{Re}(y)$ and $a_2 = \mathrm{Im}(y)$. For comparison, the derivation in Appendix A obtains (16) at tree level starting from a manifestly relativistic lagrangian. The residual mass matrix is $\delta m = \mathrm{diag}\,(M_S,\, M_D \mathbb{1}_4) - M_{\mathrm{ref}} \mathbb{1}_5$, and we define M_S and M_D to be real and positive.[7] The gauge couplings are obtained by trivially extending (13) to include the singlet. This completely specifies the heavy-particle lagrangian given in (4).

The mass induced by EWSB is accounted for at tree level by including contributions from (17),

$$\delta M(v) = \delta M + v \begin{pmatrix} 0 & 0 & a_1 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

In the following, we use subscripts to denote the electric charge and bracketed superscripts to label the mass eigenstate. For neutral states we find the residual mass eigenvalues

$$\delta_0^{(0)} = M_D - M_{\mathrm{ref}}\,, \quad \delta_0^{(\pm)} = \frac{M_D + M_S}{2} \pm \sqrt{\Delta^2 + (av)^2} - M_{\mathrm{ref}}\,, \tag{19}$$

where we define

$$\Delta \equiv \frac{M_S - M_D}{2}\,, \quad a \equiv \sqrt{a_1^2 + a_2^2}\,. \tag{20}$$

By definition $a > 0$, and regardless of the sign of Δ, the smallest eigenvalue is $\delta_0^{(-)}$. Let us set this eigenvalue to zero by appropriately choosing the reference mass M_{ref}. The corresponding normalized eigenvectors in the (h_S, h_0, h_0') basis of electrically neutral states are then

$$\vec{v}_0^{(0)} = \frac{1}{a} \begin{pmatrix} 0 \\ a_2 \\ -a_1 \end{pmatrix}, \quad \vec{v}_0^{(\pm)} = \frac{1}{\left[\left(\Delta \pm \sqrt{\Delta^2 + (av)^2} \right)^2 + (av)^2 \right]^{\frac{1}{2}}} \begin{pmatrix} \Delta \pm \sqrt{\Delta^2 + (av)^2} \\ a_1 v \\ a_2 v \end{pmatrix}, \tag{21}$$

and we may construct the unitary matrix U_0 (on the three-dimensional neutral subspace) to translate to the mass eigenbasis,

$$U_0 = \left(\vec{v}_0^{(0)} \quad \vec{v}_0^{(+)} \quad \vec{v}_0^{(-)} \right), \quad \begin{pmatrix} h_S \\ h_0 \\ h_0' \end{pmatrix} = U_0 \begin{pmatrix} h_0^{(0)} \\ h_0^{(+)} \\ h_0^{(-)} \end{pmatrix}, \quad U_0^\dagger \delta M(v) U_0 = \mathrm{diag}\left(\delta_0^{(0)}, \delta_0^{(+)}, \delta_0^{(-)} \right). \tag{22}$$

[7] An additional phase redefinition of h_ψ, h_{ψ^c} could be used to enforce the vanishing of a_1 or a_2.

The tree-level masses for the electrically charged sector are unchanged by EWSB, given by $\delta_\pm^{(0)} = \delta_0^{(0)}$, and the corresponding charge eigenstates are given by

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} h_+^{(0)} \\ h_-^{(0)} \end{pmatrix} . \tag{23}$$

The basis of mass eigenstates is thus given by the column vector $h = \left(h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(0)}, h_-^{(0)} \right)$, where $h_0^{(-)} \equiv \chi$, and the lagrangian is given by (6) with

$$\begin{aligned}
\delta M(v) &= \mathrm{diag}\left(\delta_0^{(0)}, \delta_0^{(+)}, \delta_0^{(-)}, \delta_+^{(0)}, \delta_-^{(0)} \right) = av\, \mathrm{diag}\left(t_{\frac{\rho}{2}}, 2s_\rho^{-1}, 0, t_{\frac{\rho}{2}}, t_{\frac{\rho}{2}} \right) , \\
Q &= \mathrm{diag}(\mathbb{0}_3, 1, -1) , \\
T^3 - s_W^2 Q &= \begin{pmatrix} 0 & \frac{i}{2}|s_{\frac{\rho}{2}}| & \frac{i}{2}|c_{\frac{\rho}{2}}| & 0 & 0 \\ -\frac{i}{2}|s_{\frac{\rho}{2}}| & 0 & 0 & 0 & 0 \\ -\frac{i}{2}|c_{\frac{\rho}{2}}| & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} - s_W^2 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} + s_W^2 \end{pmatrix} ,
\end{aligned}$$

$$T^+ = \frac{e^{-i\xi}}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & -|s_{\frac{\rho}{2}}| \\ 0 & 0 & 0 & 0 & -|c_{\frac{\rho}{2}}| \\ i & |s_{\frac{\rho}{2}}| & |c_{\frac{\rho}{2}}| & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \quad T^- = \frac{e^{+i\xi}}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & |s_{\frac{\rho}{2}}| & 0 \\ 0 & 0 & 0 & |c_{\frac{\rho}{2}}| & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & -|s_{\frac{\rho}{2}}| & -|c_{\frac{\rho}{2}}| & 0 & 0 \end{pmatrix} ,$$

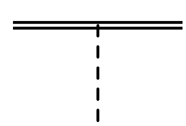
$$f(\phi) = a \begin{pmatrix} 0 & |c_{\frac{\rho}{2}}|\phi_Z & -|s_{\frac{\rho}{2}}|\phi_Z & 0 & 0 \\ |c_{\frac{\rho}{2}}|\phi_Z & s_\rho h & c_\rho h & |c_{\frac{\rho}{2}}|e^{+i\xi}\phi_W^- & |c_{\frac{\rho}{2}}|e^{-i\xi}\phi_W^+ \\ -|s_{\frac{\rho}{2}}|\phi_Z & c_\rho h & -s_\rho h & -|s_{\frac{\rho}{2}}|e^{+i\xi}\phi_W^- & -|s_{\frac{\rho}{2}}|e^{-i\xi}\phi_W^+ \\ 0 & |c_{\frac{\rho}{2}}|e^{-i\xi}\phi_W^+ & -|s_{\frac{\rho}{2}}|e^{-i\xi}\phi_W^+ & 0 & 0 \\ 0 & |c_{\frac{\rho}{2}}|e^{+i\xi}\phi_W^- & -|s_{\frac{\rho}{2}}|e^{+i\xi}\phi_W^- & 0 & 0 \end{pmatrix} , \tag{24}$$

where we have introduced

$$\sin\rho \equiv \frac{av}{\sqrt{(av)^2 + \Delta^2}} , \quad \cos\rho \equiv \frac{\Delta}{\sqrt{(av)^2 + \Delta^2}} , \quad e^{\pm i\xi} \equiv \frac{(a_1 \pm i a_2)}{a} . \tag{25}$$

The shorthand notation $c_x \equiv \cos x$, $s_x \equiv \sin x$, and $t_x \equiv \tan x$ is used throughout this paper. Note that s_ρ is positive, and that c_ρ can have either sign depending on the hierarchy between M_S and M_D. It is straightforward to extract Feynman rules from the lagrangian (6) and the matrices (24). For example, the propagator for χ, and its coupling to the physical Higgs boson, h, are

$$= \frac{i}{v \cdot k - \delta_0^{(-)} + i0} ,$$



$$= \quad i a s_\rho . \tag{26}$$

4.2.2 Triplet-doublet admixture

The construction for the triplet-doublet case follows closely that for the singlet-doublet case, with a heavy triplet h_T in place of the singlet h_S. Using $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$ and $\bar{\boldsymbol{\tau}} = -(\tau^{1T}, \tau^{2T}, \tau^{3T})$, the gauge-invariant interactions of h_T, h_{D_1} and h_{D_2} involving the Higgs field can be written in the form $\mathcal{L}_{H\bar{h}h} = -\bar{h} f(H) h$, where we collect fields in a seven-component column vector $h = (h_T, h_{D1}, h_{D2})$, and the matrix $f(H)$ is given by

$$f(H) = \frac{a_1}{\sqrt{2}} \begin{pmatrix} \mathbb{0}_3 & H^\dagger \boldsymbol{\tau} - H^T \bar{\boldsymbol{\tau}} & i(-H^T \bar{\boldsymbol{\tau}} - H^\dagger \boldsymbol{\tau}) \\ -\bar{\boldsymbol{\tau}} H^* + \boldsymbol{\tau} H & \mathbb{0}_2 & \mathbb{0}_2 \\ i(\boldsymbol{\tau} H + \bar{\boldsymbol{\tau}} H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} + \frac{a_2}{\sqrt{2}} \begin{pmatrix} \mathbb{0}_3 & i(H^T \bar{\boldsymbol{\tau}} + H^\dagger \boldsymbol{\tau}) & H^\dagger \boldsymbol{\tau} - H^T \bar{\boldsymbol{\tau}} \\ i(-\boldsymbol{\tau} H - \bar{\boldsymbol{\tau}} H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ -\bar{\boldsymbol{\tau}} H^* + \boldsymbol{\tau} H & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix}, \tag{27}$$

with real parameters a_1 and a_2. Upon accounting for mass contributions from EWSB, the basis of mass eigenstates is given by the column vector $h = \left(h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(+)}, h_+^{(-)}, h_-^{(+)}, h_-^{(-)}\right)$, where $h_0^{(-)} \equiv \chi$, and the lagrangian is given by (6) with

$$\delta M(v) = \mathrm{diag}\left(\delta_0^{(0)}, \delta_0^{(+)}, \delta_0^{(-)}, \delta_+^{(+)}, \delta_+^{(-)}, \delta_-^{(+)}, \delta_-^{(-)}\right) = av\,\mathrm{diag}\left(t_{\frac{\rho}{2}}, 2s_\rho^{-1}, 0, 2s_\rho^{-1}, 0, 2s_\rho^{-1}, 0\right),$$

$$Q = \mathrm{diag}(0,0,0,1,1,-1,-1),$$

$$T^3 = \begin{pmatrix} 0 & \frac{i}{2}|s_{\frac{\rho}{2}}| & \frac{i}{2}|c_{\frac{\rho}{2}}| & 0 & 0 & 0 & 0 \\ -\frac{i}{2}|s_{\frac{\rho}{2}}| & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{i}{2}|c_{\frac{\rho}{2}}| & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{1}{2}s^2_{\frac{\rho}{2}} & -\frac{1}{4}s_\rho & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4}s_\rho & 1-\frac{1}{2}c^2_{\frac{\rho}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1+\frac{1}{2}s^2_{\frac{\rho}{2}} & \frac{1}{4}s_\rho \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4}s_\rho & -1+\frac{1}{2}c^2_{\frac{\rho}{2}} \end{pmatrix},$$

$$T^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i|s_{\frac{\rho}{2}}| & i|c_{\frac{\rho}{2}}| \\ 0 & 0 & 0 & 0 & 0 & 1+c^2_{\frac{\rho}{2}} & -\frac{1}{2}s_\rho \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}s_\rho & 1+s^2_{\frac{\rho}{2}} \\ -i|s_{\frac{\rho}{2}}| & -1-c^2_{\frac{\rho}{2}} & \frac{1}{2}s_\rho & 0 & 0 & 0 & 0 \\ -i|c_{\frac{\rho}{2}}| & \frac{1}{2}s_\rho & -1-s^2_{\frac{\rho}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$
T^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i|s_{\frac{\rho}{2}}| & i|c_{\frac{\rho}{2}}| & 0 & 0 \\ 0 & 0 & 0 & -1-c^2_{\frac{\rho}{2}} & \frac{1}{2}s_\rho & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}s_\rho & -1-s^2_{\frac{\rho}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i|s_{\frac{\rho}{2}}| & 1+c^2_{\frac{\rho}{2}} & -\frac{1}{2}s_\rho & 0 & 0 & 0 & 0 \\ -i|c_{\frac{\rho}{2}}| & -\frac{1}{2}s_\rho & 1+s^2_{\frac{\rho}{2}} & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
f(\phi) = a \begin{pmatrix} 0 & |c_{\frac{\rho}{2}}|\phi_Z & -|s_{\frac{\rho}{2}}|\phi_Z & -i|c_{\frac{\rho}{2}}|\phi^-_W & i|s_{\frac{\rho}{2}}|\phi^-_W & i|c_{\frac{\rho}{2}}|\phi^+_W & -i|s_{\frac{\rho}{2}}|\phi^+_W \\ |c_{\frac{\rho}{2}}|\phi_Z & s_\rho h & c_\rho h & 0 & \phi^-_W & 0 & \phi^+_W \\ -|s_{\frac{\rho}{2}}|\phi_Z & c_\rho h & -s_\rho h & -\phi^-_W & 0 & -\phi^+_W & 0 \\ i|c_{\frac{\rho}{2}}|\phi^+_W & 0 & -\phi^+_W & s_\rho h & c_\rho h - i\phi_Z & 0 & 0 \\ -i|s_{\frac{\rho}{2}}|\phi^+_W & \phi^+_W & 0 & c_\rho h + i\phi_Z & -s_\rho h & 0 & 0 \\ -i|c_{\frac{\rho}{2}}|\phi^-_W & 0 & -\phi^-_W & 0 & 0 & s_\rho h & c_\rho h + i\phi_Z \\ i|s_{\frac{\rho}{2}}|\phi^-_W & \phi^-_W & 0 & 0 & 0 & c_\rho h - i\phi_Z & -s_\rho h \end{pmatrix}, \qquad (28)
$$

where s_ρ and c_ρ are as defined in (25), with $a = \sqrt{a_1^2 + a_2^2}$ and $\Delta = (M_T - M_D)/2$. Again, s_ρ is positive and c_ρ can have either sign depending on the hierarchy between M_T and M_D.

4.3 Pure-case limits

Appropriate parametric limits can be taken to decouple the pure state constituents of an admixture. This can be used to check the consistency of matching computations in Sec. 6. From the singlet-doublet admixture, we may recover the pure doublet (singlet) case by taking $a \to 0$ or $|\Delta| \to \infty$, with $\Delta > 0$ ($\Delta < 0$), or by taking $\rho \to 0$ ($\rho \to \pi$). Similarly, to recover the pure doublet (triplet) case from the triplet-doublet admixture, we decouple the triplet (doublet) component by taking $a \to 0$ or $|\Delta| \to \infty$, with $\Delta > 0$ ($\Delta < 0$), or by taking $\rho \to 0$ ($\rho \to \pi$).

5 Onshell renormalization scheme

A consistent evaluation of amplitudes beyond tree level demands renormalization of the Higgs-WIMP vertex, $h\bar{\chi}\chi$, that appears for admixtures. We define an extension of the onshell renormalization scheme for the electroweak SM (e.g., see [39]) by expressing the vertex amplitude in terms of physical masses in the SM and DM sectors. We begin by studying the singlet-doublet mixture, and will later quote the analogous results for the triplet-doublet mixture.

To avoid confusion with standard notation for counterterms, in this section (only) we denote a residual mass by μ, and a residual mass counterterm by $\delta\mu$. We keep the notation introduced in Sec. 4 for the residual mass eigenvalues, $\delta_0^{(0)}$, $\delta_0^{(\pm)}$, etc.

5.1 Singlet-doublet counterterm lagrangian

Let us write the bare lagrangian as the sum of renormalized and counterterm contributions

$$
\mathcal{L} = \bar{h}^{\text{bare}} \left[iv \cdot D - \mu^{\text{bare}} - f^{\text{bare}}(H^{\text{bare}}) \right] h^{\text{bare}}
$$

$$= \bar{h}\left[iv\cdot D + \delta Z_h iv\cdot D - \mu - \delta\mu - f(H^{\rm bare}) - \delta f(H^{\rm bare})\right] h\,, \tag{29}$$

where the bare quantities are given by

$$\begin{aligned}
\mu^{\rm bare} &= {\rm diag}(\mu_S^{\rm bare}, \mu_D^{\rm bare}, \mu_D^{\rm bare}, \mu_D^{\rm bare}, \mu_D^{\rm bare})\,, \\
f^{\rm bare}(H) &= \frac{a_1^{\rm bare}}{\sqrt{2}}\begin{pmatrix} 0 & H^\dagger + H^T & i(H^T - H^\dagger) \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \\ i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \\
&\quad + \frac{a_2^{\rm bare}}{\sqrt{2}}\begin{pmatrix} 0 & -i(H^T - H^\dagger) & H^T + H^\dagger \\ -i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \\
&\equiv a_1^{\rm bare} f_1(H) + a_2^{\rm bare} f_2(H)\,,
\end{aligned} \tag{30}$$

and the expression for $f^{\rm bare}(H)$ above is valid for arbitrary H (in particular, for $H^{\rm bare}$). The gauge symmetry preserving counterterms are given by

$$\begin{aligned}
Z_h &= 1 + \delta Z_h = 1 + {\rm diag}(\delta Z_S, \delta Z_D \mathbb{1}_4)\,, \\
\mu + \delta\mu &= Z_h^{\frac{1}{2}} \mu^{\rm bare} Z_h^{\frac{1}{2}} = {\rm diag}(\mu_S + \delta\mu_S, (\mu_D + \delta\mu_D)\mathbb{1}_4)\,, \\
f(H^{\rm bare}) + \delta f(H^{\rm bare}) &= Z_h^{\frac{1}{2}} f^{\rm bare}(H^{\rm bare}) Z_h^{\frac{1}{2}} = (a_1 + \delta a_1) f_1(H') + (a_2 + \delta a_2) f_2(H')\,.
\end{aligned} \tag{31}$$

We have introduced H' to absorb the renormalization of v:

$$H^{\rm bare} = Z_H^{\frac{1}{2}} \begin{pmatrix} \phi_W^+ \\ \frac{1}{\sqrt{2}}(v - \delta v + h + i\phi_Z) \end{pmatrix} = Z_H^{\frac{1}{2}}\left(1 - \frac{\delta v}{v}\right) H'\,. \tag{32}$$

Note that the renormalization of v introduces a coupling $\sim \frac{\delta v}{v} h\bar{\chi}\chi$ through the $a_1 f_1(H') + a_2 f_2(H')$ term in (31). We will fix the counterterms by enforcing renormalization conditions on the residual mass matrix (two point functions). Three point functions involving the Higgs interaction will then be determined.

5.2 Propagator corrections

Anticipating renormalization conditions that preserve the basis $h = \left(h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(0)}, h_-^{(0)}\right)$ of mass eigenstates introduced in Sec. 4.2.1, let us express the counterterms in this basis,

$$\delta\mu = \delta\mu_D \mathbb{1}_5 + \begin{pmatrix} 0 & |c_{\frac{\rho}{2}}|\frac{v}{a}(a_2\delta a_1 - a_1\delta a_2) & |s_{\frac{\rho}{2}}|\frac{v}{a}(a_1\delta a_2 - a_2\delta a_1) & 0 & 0 \\ \cdot & 2c_{\frac{\rho}{2}}^2(\delta\Delta) + s_\rho\frac{v}{a}(a_1\delta a_1 + a_2\delta a_2) & -s_\rho(\delta\Delta) + c_\rho\frac{v}{a}(a_1\delta a_1 + a_2\delta a_2) & 0 & 0 \\ \cdot & \cdot & 2s_{\frac{\rho}{2}}^2(\delta\Delta) - s_\rho\frac{v}{a}(a_1\delta a_1 + a_2\delta a_2) & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix},$$

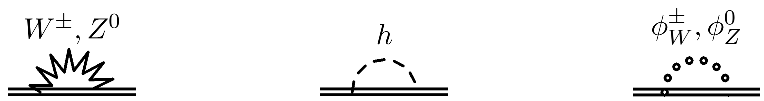


Figure 1: One-loop corrections to two-point functions. Double lines denote heavy WIMPs, zigzag lines denote gauge bosons, $W^\pm$ or Z^0, dashed lines denote the physical Higgs boson, h, and dotted lines denote Goldstone bosons, $\phi_W^\pm$ or ϕ_Z^0.

$$\delta Z_h = \delta Z_D \mathbb{1}_5 + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \cdot & c_{\frac{\rho}{2}}^2(\delta Z_S - \delta Z_D) & -\frac{1}{2}s_\rho(\delta Z_S - \delta Z_D) & 0 & 0 \\ \cdot & \cdot & s_{\frac{\rho}{2}}^2(\delta Z_S - \delta Z_D) & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \tag{33}$$

where the above matrices are symmetric, and $(\delta\Delta) = (\delta\mu_S - \delta\mu_D)/2$. Due to the masslessness of the photon, the onshell renormalization factor for the electrically charged state, δZ_D, is infrared (IR) divergent. To avoid the associated complications, we may turn off δZ_D, corresponding to an additional overall renormalization of the fields with $\delta Z_S = \delta Z_D$. This overall renormalization will not impact the determination of physical masses or mass eigenstates. However, we will of course need to include additional wavefunction renormalization factors when computing physical amplitudes. In the following, we allow for arbitrary δZ_D.

We compute the one-loop corrections to the amputated two-point function, Σ_2, from virtual Z^0, $W^\pm$, h, ϕ_Z^0 and $\phi_W^\pm$ exchange, as illustrated in Fig. 1. In the following results, we set the external momentum to zero (i.e., we compute $\Sigma_2(0)$), and the first (second) subscript denotes the final (initial) state, with values $(1, 2, 3, 4, 5)$ corresponding to the mass eigenstates $\left(h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(0)}, h_-^{(0)}\right)$. Using Feynman-t'Hooft gauge, and expressing results in terms of the basic integral $I_3(\delta, m)$ of Appendix B, we find

$$\begin{aligned}
-i[\Sigma_2(0)]_{11} &= -\frac{g_2^2}{4c_W^2} c_{\frac{\rho}{2}}^2 I_3(\delta_0^{(-)}, m_Z) - \frac{g_2^2}{4c_W^2} s_{\frac{\rho}{2}}^2 I_3(\delta_0^{(+)}, m_Z) - \frac{g_2^2}{2} I_3(\delta_\pm^{(0)}, m_W) \\
&\quad + a^2 c_{\frac{\rho}{2}}^2 I_3(\delta_0^{(+)}, m_Z) + a^2 s_{\frac{\rho}{2}}^2 I_3(\delta_0^{(-)}, m_Z)\,, \\
-i[\Sigma_2(0)]_{22} &= -\frac{g_2^2}{4c_W^2} s_{\frac{\rho}{2}}^2 I_3(\delta_0^{(0)}, m_Z) - \frac{g_2^2}{2} s_{\frac{\rho}{2}}^2 I_3(\delta_\pm^{(0)}, m_W) + a^2 s_\rho^2 I_3(\delta_0^{(+)}, m_h) \\
&\quad + a^2 c_\rho^2 I_3(\delta_0^{(-)}, m_h) + a^2 c_{\frac{\rho}{2}}^2 I_3(\delta_0^{(0)}, m_Z) + 2a^2 c_{\frac{\rho}{2}}^2 I_3(\delta_\pm^{(0)}, m_W)\,, \\
-i[\Sigma_2(0)]_{23} &= -i[\Sigma_2(0)]_{32} \\
&= -\frac{g_2^2}{8c_W^2} s_\rho I_3(\delta_0^{(0)}, m_Z) - \frac{g_2^2}{4} s_\rho I_3(\delta_\pm^{(0)}, m_W) + a^2 s_\rho c_\rho I_3(\delta_0^{(+)}, m_h) \\
&\quad - a^2 s_\rho c_\rho I_3(\delta_0^{(-)}, m_h) - \frac{a^2}{2} s_\rho I_3(\delta_0^{(0)}, m_Z) - a^2 s_\rho I_3(\delta_\pm^{(0)}, m_W)\,, \\
-i[\Sigma_2(0)]_{33} &= -\frac{g_2^2}{4c_W^2} c_{\frac{\rho}{2}}^2 I_3(\delta_0^{(0)}, m_Z) - \frac{g_2^2}{2} c_{\frac{\rho}{2}}^2 I_3(\delta_\pm^{(0)}, m_W) + a^2 s_\rho^2 I_3(\delta_0^{(-)}, m_h)
\end{aligned}$$

$$+ a^2 c_\rho^2 I_3(\delta_0^{(+)}, m_h) + a^2 s_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(0)}, m_Z) + 2a^2 s_{\frac{\varrho}{2}}^2 I_3(\delta_\pm^{(0)}, m_W)\,,$$

$$\begin{aligned} -i[\Sigma_2(0)]_{44} &= -i[\Sigma_2(0)]_{55} \\ &= -e^2 I_3(\delta_\pm^{(0)}, \lambda) - \frac{g_2^2}{4c_W^2}(1-2s_W^2)^2 I_3(\delta_\pm^{(0)}, m_Z) - \frac{g_2^2}{4} I_3(\delta_0^{(0)}, m_W) \\ &\quad - \frac{g_2^2}{4} s_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(+)}, m_W) - \frac{g_2^2}{4} c_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(-)}, m_W) + a^2 c_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(+)}, m_W) \\ &\quad + a^2 s_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(-)}, m_W)\,, \end{aligned} \tag{34}$$

where λ is a fictitious photon mass, and the self-energy components not displayed above vanish. We may evaluate $\Sigma(v \cdot k)$ by the substitution $I_3(\delta, m) \to I_3(\delta - v \cdot k, m)$.

5.3 Renormalization conditions

Let us fix the counterterms δa_1, δa_2, $\delta\mu_S$, $\delta\mu_D$ and δZ_S by enforcing that the physical residual masses of the neutral states are given by the renormalized parameters of the lagrangian,

$$\begin{aligned} [\delta\mu]_{11} + \mathrm{Re}[\Sigma_2(\delta_\pm^{(0)})]_{11} - \delta_0^{(0)}[\delta Z_h]_{11} &= 0\,, \\ [\delta\mu]_{22} + \mathrm{Re}[\Sigma_2(\delta_0^{(+)})]_{22} - \delta_0^{(+)}[\delta Z_h]_{22} &= 0\,, \\ [\delta\mu]_{33} + \mathrm{Re}[\Sigma_2(0)]_{33} &= 0\,, \end{aligned} \tag{35}$$

and that the lightest mass eigenstate is proportional to the vector $(0, 0, 1, 0, 0)$,

$$\begin{aligned} [\delta\mu]_{13} + \mathrm{Re}[\Sigma_2(0)]_{13} &= 0\,, \\ [\delta\mu]_{23} + \mathrm{Re}[\Sigma_2(0)]_{23} &= 0\,. \end{aligned} \tag{36}$$

This scheme defines renormalized values for a and $t_{\frac{\varrho}{2}}$ through the physical mass differences between neutral states,

$$\begin{aligned} M_{h_0^{(+)}} - M_{h_0^{(-)}} &= 2avs_\rho^{-1}\,, \\ M_{h_0^{(0)}} - M_{h_0^{(-)}} &= avt_{\frac{\varrho}{2}}\,, \end{aligned} \tag{37}$$

where the mass of the neutral mass eigenstate $h_0^{(\cdot)}$ is denoted $M_{h_0^{(\cdot)}}$. Note also that the presence of $\delta Z_S \neq \delta Z_D$ is required to maintain the orientation of the lightest mass eigenstate under renormalization. Solving for the counterterms, we find from $[\delta\mu]_{13}$,

$$\frac{\delta a_1}{a_1} = \frac{\delta a_2}{a_2} \implies a_1\delta a_1 + a_2\delta a_2 = a^2\frac{\delta a_1}{a_1}\,. \tag{38}$$

The remaining system of equations involving $[\delta\mu]_{23}$, $[\delta\mu]_{11}$, $[\delta\mu]_{22}$ and $[\delta\mu]_{33}$ then yields

$$\begin{aligned} av\frac{\delta a_1}{a_1} &= -[\delta\mu]_{23} + t_{\frac{\varrho}{2}}^{-1}\left([\delta\mu]_{11} - [\delta\mu]_{33}\right) \\ &= [\Sigma_2(0)]_{23} + t_{\frac{\varrho}{2}}^{-1}\Big([\Sigma_2(0)]_{33} - [\Sigma_2(\delta_0^{(0)})]_{11} + \delta_0^{(0)}[\delta Z_h]_{11}\Big)\,, \end{aligned}$$

$$\delta Z_S = \delta Z_D + \frac{1}{av}\left\{ t_{\frac{\rho}{2}}[\Sigma_2(\delta_0^{(+)})]_{22} + 2[\Sigma_2(0)]_{23} + t_{\frac{\rho}{2}}^{-1}[\Sigma_2(0)]_{33} - 2s_\rho^{-1}[\Sigma_2(\delta_0^{(0)})]_{11} \right\}. \tag{39}$$

We focus here on the counterterms δa_1, δa_2, and δZ_S which enter in the calculation of amplitudes relevant for WIMP-nucleon scattering. Explicit expressions for the remaining counterterms $\delta\mu_S$ and $\delta\mu_D$ may be similarly obtained. We note that the degeneracy between the mass of the $h_0^{(0)}$ state and the $h_\pm^{(0)}$ states is lifted by a finite amount, predicted in terms of renormalized parameters as

$$M_{h_\pm^{(0)}} - M_{h_0^{(0)}} = [\Sigma_2(\delta_\pm^{(0)})]_{44} - [\Sigma_2(\delta_0^{(0)})]_{11}\,, \tag{40}$$

where we have used that $[\delta\mu]_{11} = [\delta\mu]_{44}$, $[\delta Z_h]_{11} = [\delta Z_h]_{44}$ and $\delta_0^{(0)} = \delta_\pm^{(0)}$.

5.4 Extension to triplet-doublet

The extension to the triplet-doublet case is straightforward. The counterterms δa_1, δa_2, $\delta\mu_T$, $\delta\mu_D$, δZ_T and δZ_D are introduced in an analogous manner. In terms of the mass eigenbasis $h = \left(h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(+)}, h_+^{(-)}, h_-^{(+)}, h_-^{(-)}\right)$ introduced in Sec. 4.2.2, the counterterms are given by the 7×7 matrices,

$$\delta\mu = \delta\mu_D \mathbb{1}_7 + \begin{pmatrix} \delta\mu_0 & 0 & 0 \\ 0 & \delta\mu_+ & 0 \\ 0 & 0 & \delta\mu_- \end{pmatrix}, \quad \delta Z_h = \delta Z_D \mathbb{1}_7 + \begin{pmatrix} \delta Z_0 & 0 & 0 \\ 0 & \delta Z_+ & 0 \\ 0 & 0 & \delta Z_- \end{pmatrix}, \tag{41}$$

where the submatrices for the neutral and charged sectors are specified by the following symmetric matrices,

$$\delta\mu_0 = \begin{pmatrix} 0 & |c_{\frac{\rho}{2}}|\frac{v}{a}(a_2\delta a_1 - a_1\delta a_2) & |s_{\frac{\rho}{2}}|\frac{v}{a}(a_1\delta a_2 - a_2\delta a_1) \\ \cdot & 2c_{\frac{\rho}{2}}^2(\delta\Delta) + s_\rho\frac{v}{a}(a_1\delta a_1 + a_2\delta a_2) & -s_\rho(\delta\Delta) + c_\rho\frac{v}{a}(a_1\delta a_1 + a_2\delta a_2) \\ \cdot & \cdot & 2s_{\frac{\rho}{2}}^2(\delta\Delta) - s_\rho\frac{v}{a}(a_1\delta a_1 + a_2\delta a_2) \end{pmatrix},$$

$$\delta\mu_\pm = \begin{pmatrix} 2c_{\frac{\rho}{2}}^2(\delta\Delta) + s_\rho\frac{v}{a}(a_1\delta a_1 + a_2\delta a_2) & -s_\rho(\delta\Delta) + c_\rho\frac{v}{a}(a_1\delta a_1 + a_2\delta a_2) \pm i\frac{v}{a}(a_1\delta a_2 - a_2\delta a_1) \\ \cdot & 2s_{\frac{\rho}{2}}^2(\delta\Delta) - s_\rho\frac{v}{a}(a_1\delta a_1 + a_2\delta a_2) \end{pmatrix},$$

$$\delta Z_0 = \begin{pmatrix} 0 & 0 & 0 \\ \cdot & c_{\frac{\rho}{2}}^2(\delta Z_T - \delta Z_D) & -\frac{1}{2}s_\rho(\delta Z_T - \delta Z_D) \\ \cdot & \cdot & s_{\frac{\rho}{2}}^2(\delta Z_T - \delta Z_D) \end{pmatrix},$$

$$\delta Z_\pm = \begin{pmatrix} c_{\frac{\rho}{2}}^2(\delta Z_T - \delta Z_D) & -\frac{1}{2}s_\rho(\delta Z_T - \delta Z_D) \\ \cdot & s_{\frac{\rho}{2}}^2(\delta Z_T - \delta Z_D) \end{pmatrix}, \tag{42}$$

with $(\delta\Delta) = (\delta\mu_T - \delta\mu_D)/2$. To fix counterterms, we impose the same renormalization conditions given in (35) and (36). We again require the one-loop corrections to the two-point function, Σ_2. In the following results, the first (second) subscript denotes the final (initial) state, with values $(1,2,3,4,5,6,7)$ corresponding to the mass eigenstates $\left(h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(+)}, h_+^{(-)}, h_-^{(+)}, h_-^{(-)}\right)$. Using

Feynman-t'Hooft gauge and expressing results in terms of the basic integral $I_3(\delta, m)$ of Appendix B, we find

$$
\begin{aligned}
-i[\Sigma_2(0)]_{11} &= -\frac{g_2^2}{4c_W^2} s_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(+)}, m_Z) - \frac{g_2^2}{4c_W^2} c_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(-)}, m_Z) - \frac{g_2^2}{2} s_{\frac{\varrho}{2}}^2 I_3(\delta_\pm^{(+)}, m_W) \\
&\quad - \frac{g_2^2}{2} c_{\frac{\varrho}{2}}^2 I_3(\delta_\pm^{(-)}, m_W) + a^2 c_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(+)}, m_Z) + a^2 s_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(-)}, m_Z) \\
&\quad + 2a^2 c_{\frac{\varrho}{2}}^2 I_3(\delta_\pm^{(+)}, m_W) + 2a^2 s_{\frac{\varrho}{2}}^2 I_3(\delta_\pm^{(-)}, m_W)\,, \\
-i[\Sigma_2(0)]_{22} &= -\frac{g_2^2}{4c_W^2} s_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(0)}, m_Z) - \frac{g_2^2}{2}\left(1 + c_{\frac{\varrho}{2}}^2\right)^2 I_3(\delta_\pm^{(+)}, m_W) - \frac{g_2^2}{8} s_\rho^2 I_3(\delta_\pm^{(-)}, m_W) \\
&\quad + a^2 c_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(0)}, m_Z) + 2a^2 I_3(\delta_\pm^{(-)}, m_W) + a^2 c_\rho^2 I_3(\delta_0^{(-)}, m_h) + a^2 s_\rho^2 I_3(\delta_0^{(+)}, m_h)\,, \\
-i[\Sigma_2(0)]_{23} &= -i[\Sigma_2(0)]_{32} \\
&= -\frac{g_2^2}{8c_W^2} s_\rho I_3(\delta_0^{(0)}, m_Z) + \frac{g_2^2}{4} s_\rho \left(1 + c_{\frac{\varrho}{2}}^2\right) I_3(\delta_\pm^{(+)}, m_W) + \frac{g_2^2}{4} s_\rho \left(1 + s_{\frac{\varrho}{2}}^2\right) I_3(\delta_\pm^{(-)}, m_W) \\
&\quad - \frac{a^2}{2} s_\rho I_3(\delta_0^{(0)}, m_Z) + a^2 c_\rho s_\rho I_3(\delta_0^{(+)}, m_h) - a^2 c_\rho s_\rho I_3(\delta_0^{(-)}, m_h)\,, \\
-i[\Sigma_2(0)]_{33} &= -\frac{g_2^2}{4c_W^2} c_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(0)}, m_Z) - \frac{g_2^2}{2}\left(1 + s_{\frac{\varrho}{2}}^2\right)^2 I_3(\delta_\pm^{(-)}, m_W) - \frac{g_2^2}{8} s_\rho^2 I_3(\delta_\pm^{(+)}, m_W) \\
&\quad + a^2 s_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(0)}, m_Z) + 2a^2 I_3(\delta_\pm^{(+)}, m_W) + a^2 c_\rho^2 I_3(\delta_0^{(+)}, m_h) + a^2 s_\rho^2 I_3(\delta_0^{(-)}, m_h)\,, \\
-i[\Sigma_2(0)]_{44} &= -i[\Sigma_2(0)]_{66} \\
&= -\frac{g_2^2}{c_W^2}\left(c_W^2 - \frac{1}{2} s_{\frac{\varrho}{2}}^2\right)^2 I_3(\delta_\pm^{(+)}, m_Z) - \frac{g_2^2}{16c_W^2} s_\rho^2 I_3(\delta_\pm^{(-)}, m_Z) - \frac{g_2^2}{4} s_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(0)}, m_W) \\
&\quad - e^2 I_3(\delta_\pm^{(+)}, \lambda) - \frac{g_2^2}{4}\left(1 + c_{\frac{\varrho}{2}}^2\right)^2 I_3(\delta_0^{(+)}, m_W) - \frac{g_2^2}{16} s_\rho^2 I_3(\delta_0^{(-)}, m_W) + a^2 I_3(\delta_\pm^{(-)}, m_Z) \\
&\quad + a^2 I_3(\delta_0^{(-)}, m_W) + a^2 c_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(0)}, m_W) + a^2 s_\rho^2 I_3(\delta_\pm^{(+)}, m_h) + a^2 c_\rho^2 I_3(\delta_\pm^{(-)}, m_h)\,, \\
-i[\Sigma_2(0)]_{45} &= -i[\Sigma_2(0)]_{54} = -i[\Sigma_2(0)]_{67} = -i[\Sigma_2(0)]_{76} \\
&= \frac{g_2^2}{4c_W^2} s_\rho \left(c_W^2 - \frac{1}{2} s_{\frac{\varrho}{2}}^2\right) I_3(\delta_\pm^{(+)}, m_Z) + \frac{g_2^2}{4c_W^2} s_\rho \left(c_W^2 - \frac{1}{2} c_{\frac{\varrho}{2}}^2\right) I_3(\delta_\pm^{(-)}, m_Z) \\
&\quad - \frac{g_2^2}{8} s_\rho I_3(\delta_0^{(0)}, m_W) + \frac{g_2^2}{8} s_\rho \left(1 + c_{\frac{\varrho}{2}}^2\right) I_3(\delta_0^{(+)}, m_W) + \frac{g_2^2}{8} s_\rho \left(1 + s_{\frac{\varrho}{2}}^2\right) I_3(\delta_0^{(-)}, m_W) \\
&\quad - \frac{a^2}{2} s_\rho I_3(\delta_0^{(0)}, m_W) + a^2 c_\rho s_\rho I_3(\delta_\pm^{(+)}, m_h) - a^2 c_\rho s_\rho I_3(\delta_\pm^{(-)}, m_h)\,, \\
-i[\Sigma_2(0)]_{55} &= -i[\Sigma_2(0)]_{77}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{g_2^2}{16c_W^2} s_\rho^2 I_3(\delta_\pm^{(+)}, m_Z) - \frac{g_2^2}{c_W^2}\left(c_W^2 - \frac{1}{2}c_{\frac{\varrho}{2}}^2\right)^2 I_3(\delta_\pm^{(-)}, m_Z) - \frac{g_2^2}{4}c_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(0)}, m_W) \\
&\quad - e^2 I_3(\delta_\pm^{(-)}, \lambda) - \frac{g_2^2}{16}s_\rho^2 I_3(\delta_0^{(+)}, m_W) - \frac{g_2^2}{4}\left(1 + s_{\frac{\varrho}{2}}^2\right)^2 I_3(\delta_0^{(-)}, m_W) + a^2 I_3(\delta_\pm^{(+)}, m_Z) \\
&\quad + a^2 I_3(\delta_0^{(+)}, m_W) + a^2 s_{\frac{\varrho}{2}}^2 I_3(\delta_0^{(0)}, m_W) + a^2 c_\rho^2 I_3(\delta_\pm^{(+)}, m_h) + a^2 s_\rho^2 I_3(\delta_\pm^{(-)}, m_h)\,, \qquad (43)
\end{aligned}
$$

where λ is a fictitious photon mass, and the self-energy components not displayed above vanish. The remainder of the renormalization program proceeds as for the singlet-doublet system. In particular, the similarity of the neutral sectors implies relations similar to (39),

$$
\begin{aligned}
av\frac{\delta a_1}{a_1} &= av\frac{\delta a_2}{a_2} = [\Sigma_2(0)]_{23} + t_{\frac{\varrho}{2}}^{-1}\left([\Sigma_2(0)]_{33} - [\Sigma_2(\delta_0^{(0)})]_{11} + \delta_0^{(0)}[\delta Z_h]_{11}\right)\,, \\
\delta Z_T &= \delta Z_D + \frac{1}{av}\left\{t_{\frac{\varrho}{2}}[\Sigma_2(\delta_0^{(+)})]_{22} + 2[\Sigma_2(0)]_{23} + t_{\frac{\varrho}{2}}^{-1}[\Sigma_2(0)]_{33} - 2s_\rho^{-1}[\Sigma_2(\delta_0^{(0)})]_{11}\right\}, \qquad (44)
\end{aligned}
$$

where the self-energy components are those of the triplet-doublet system given in (43).

6 Matching at the weak scale

This section describes the matching of the effective theory described by (6) onto the effective theory described by (2), through integrating out weak-scale particles, $W^\pm$, Z^0, h, ϕ_Z^0, $\phi_W^\pm$, and t. The complete basis of twelve bare matching coefficients, $c_q^{(0)}$, $c_q^{(2)}$, $c_g^{(0)}$, and $c_g^{(2)}$, are determined at leading order in perturbation theory.

We may write the quark and gluon matching coefficients in terms of contributions from one-boson exchange (1BE) and two-boson exchange (2BE) diagrams,

$$
\begin{aligned}
c_q^{(0)} &= c_q^{(0)}{}_{\rm 1BE} + c_q^{(0)}{}_{\rm 2BE} + \dots\,, \\
c_g^{(0)} &= c_g^{(0)}{}_{\rm 1BE} + c_g^{(0)}{}_{\rm 2BE} + \dots\,, \\
c_q^{(2)} &= c_q^{(2)}{}_{\rm 2BE} + \dots\,, \\
c_g^{(2)} &= c_g^{(2)}{}_{\rm 2BE} + \dots\,, \qquad (45)
\end{aligned}
$$

where the ellipses denote subleading contributions with more than two bosons exchanged. Note that spin-2 coefficients do not receive contributions from one-boson exchange amplitudes.

In the following analysis, we denote generic up- and down-type quarks by U and D, respectively, and an arbitrary quark flavor by q. We specify the contributions to the matching coefficients in terms of the constants

$$
c_V^{(U)} = 1 - \frac{8}{3}s_W^2\,, \quad c_V^{(D)} = -1 + \frac{4}{3}s_W^2\,, \quad c_A^{(U)} = -1\,, \quad c_A^{(D)} = 1\,. \qquad (46)
$$

We systematically neglect subleading corrections involving light quark masses, and use CKM unitarity to simplify sums over quark flavors. Together with $|V_{tb}| \approx 1$ (and hence $|V_{td}| \approx |V_{ts}| \approx 0$), these assumptions lead to $c_u^{(S)} = c_c^{(S)}$ and $c_d^{(S)} = c_s^{(S)}$ for both $S = 0, 2$, reducing the number of independent matching coefficients to eight. When the interactions are isospin-conserving, e.g., as in the pure triplet case, we furthermore have $c_u^{(S)} = c_d^{(S)}$ and $c_c^{(S)} = c_s^{(S)}$ for both $S = 0, 2$, leaving only six independent

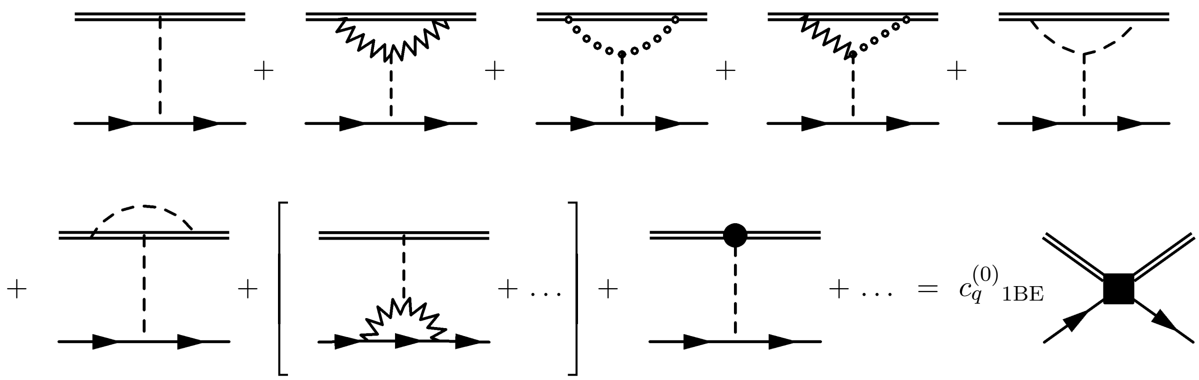


Figure 2: Matching condition for one-boson exchange contributions to quark operators. The full theory diagrams on the left-hand side illustrate the possible types of contributions to the $h\bar{\chi}\chi$ three-point function. Time-reversed diagrams are not shown. Double lines denote heavy WIMPs, zigzag lines denote gauge bosons, $W^\pm$ or Z^0, dotted lines denote Goldstone bosons, $\phi_W^\pm$ or ϕ_Z^0, dashed lines denote the physical Higgs boson, h, and single lines with arrows denote quarks. The solid circle denotes counterterm contributions. The solid square denotes effective theory vertices.

coefficients. We use Feynman-t'Hooft gauge for the electroweak sector, and neglect higher-order corrections to the tree-level relations between residual masses, $\delta_0^{(0)} = \delta_\pm^{(0)}$ for the singlet-doublet system, and $\delta_0^{(\pm)} = \delta_\pm^{(\pm)}$ for the triplet-doublet system. In Secs. 6.1 and 6.3, we match to quark operators using onshell external quarks, and thus use the equivalence of $m_q u_q(p)$ and $\not{p} u_q(p)$.

6.1 Quark matching: one-boson exchange

The matching condition for one-boson exchange is pictured in Fig. 2. The full-theory amplitude is given by

$$i\mathcal{M}_q = i\left(\hat{\mathcal{M}}_{\rm tree} + \hat{\mathcal{M}}_{\rm vertex,1} + \hat{\mathcal{M}}_{\rm vertex,2} + \hat{\mathcal{M}}_{\delta a_1} + \hat{\mathcal{M}}_{\delta Z} + \hat{\mathcal{M}}_{\delta v}\right) \frac{i}{-m_h^2} \frac{-ig_2 m_q}{2m_W} \bar{u}_q(p) u_q(p) \,, \tag{47}$$

where the $\hat{\mathcal{M}}_i$ are contributions to the $h\bar{\chi}\chi$ three-point function. These come from tree-level Higgs exchange ($\hat{\mathcal{M}}_{\rm tree}$), one-loop diagrams with Higgs coupling to $W^\pm$ or Z^0 ($\hat{\mathcal{M}}_{\rm vertex,1}$), one-loop vertex corrections with Higgs coupling to the heavy particle ($\hat{\mathcal{M}}_{\rm vertex,2}$), the δa_1 counterterm ($\hat{\mathcal{M}}_{\delta a_1}$), wave-function renormalization ($\hat{\mathcal{M}}_{\delta Z}$), and the renormalization of the Higgs vacuum expectation value ($\hat{\mathcal{M}}_{\delta v}$). Having included the counterterms, the sum of these contributions is finite. The one-boson exchange contribution to the spin-0 quark matching coefficient is thus

$$c_q^{(0)}{}_{\rm 1BE} = -\frac{g_2}{2m_h^2 m_W}\left(\hat{\mathcal{M}}_{\rm tree} + \hat{\mathcal{M}}_{\rm vertex,1} + \hat{\mathcal{M}}_{\rm vertex,2} + \hat{\mathcal{M}}_{\delta a_1} + \hat{\mathcal{M}}_{\delta Z} + \hat{\mathcal{M}}_{\delta v}\right) . \tag{48}$$

We neglect one-boson exchange contributions containing $\mathcal{O}(\alpha_2^1)$ corrections to the SM $h\bar{q}q$ coupling, shown in Fig. 2 within square brackets. This gauge-invariant class of diagrams is loop-suppressed relative to the tree-level diagram for any value of the $h\bar{\chi}\chi$ coupling. On the other hand, the remaining loop diagrams (including those in Fig. 4) may compete with, or even dominate, the tree-level contribution depending on the size of the $h\bar{\chi}\chi$ coupling. Let us proceed to specify the contributions, $\hat{\mathcal{M}}_i$, for each SM extension in terms of the integrals $I_1(\delta, m)$, $I_2(\delta, m)$, $I_3(\delta, m)$ and $I_4(\delta_1, \delta_2, m)$ of Appendix B.

6.1.1 Pure states

For pure states the only diagrams are those with Higgs coupling to $W^\pm$ and Z^0, and in terms of the constants $\mathcal{C}_W$ and $\mathcal{C}_Z$ specified in (9) the amplitude is given by

$$i\hat{\mathcal{M}}_{\rm vertex,1} = -\mathcal{C}_Z \frac{g_2^3}{c_W^3} m_Z I_1(0, m_Z) - \mathcal{C}_W g_2^3 m_W I_1(0, m_W) \, . \tag{49}$$

Using (48), we find the contribution to the spin-0 quark matching coefficient,

$$c_q^{(0)}{}_{\rm 1BE} = \frac{\pi\Gamma(1+\epsilon) g_2^4}{(4\pi)^{2-\epsilon}} \bigg\{ -\frac{m_W^{-3-2\epsilon}}{2x_h^2} \left(\mathcal{C}_W + \frac{\mathcal{C}_Z}{c_W^3} \right) + \mathcal{O}(\epsilon) \bigg\} \, , \tag{50}$$

where $x_h = m_h/m_W$. The pure triplet (doublet) result is obtained by setting $\mathcal{C}_W = 2$ and $\mathcal{C}_Z = 0$ ($\mathcal{C}_W = 1/2$ and $\mathcal{C}_Z = 1/4$) above.

6.1.2 Singlet-doublet admixture

For the singlet-doublet case, we have the following contributions to the $h\bar{\chi}\chi$ three-point function,

$$i\hat{\mathcal{M}}_{\rm tree} = ias_\rho \, , \quad i\hat{\mathcal{M}}_{\delta a_1} = ias_\rho \frac{\delta a_1}{a_1} \, , \quad i\hat{\mathcal{M}}_{\delta Z} = ias_\rho \delta Z_\chi \, , \quad i\hat{\mathcal{M}}_{\delta v} = ias_\rho \frac{\delta v}{v} \, ,$$

$$\begin{aligned} i\hat{\mathcal{M}}_{\rm vertex,1} &= -\frac{g_2^3}{4c_W^3} c_{\frac{\rho}{2}}^2 m_Z I_1(\delta_0^{(0)}, m_Z) + \frac{g_2^2 a}{4c_W^2} s_\rho I_2(\delta_0^{(0)}, m_Z) + \frac{g_2 a^2}{2} s_{\frac{\rho}{2}}^2 \frac{m_h^2}{m_W} I_1(\delta_0^{(0)}, m_Z) \\ &\quad + \frac{3g_2 a^2}{2} \frac{m_h^2}{m_W} \Big[s_\rho^2 I_1(\delta_0^{(-)}, m_h) + c_\rho^2 I_1(\delta_0^{(+)}, m_h) \Big] - \frac{g_2^3}{2} c_{\frac{\rho}{2}}^2 m_W I_1(\delta_0^{(0)}, m_W) \\ &\quad + \frac{g_2^2 a}{2} s_\rho I_2(\delta_0^{(0)}, m_W) + g_2 a^2 s_{\frac{\rho}{2}}^2 \frac{m_h^2}{m_W} I_1(\delta_0^{(0)}, m_W) \, , \\ i\hat{\mathcal{M}}_{\rm vertex,2} &= -a^3 s_\rho^3 I_4(\delta_0^{(-)}, \delta_0^{(-)}, m_h) + a^3 s_\rho c_\rho^2 I_4(\delta_0^{(+)}, \delta_0^{(+)}, m_h) - 2a^3 s_\rho c_\rho^2 I_4(\delta_0^{(-)}, \delta_0^{(+)}, m_h) \, , \end{aligned} \tag{51}$$

where δa_1 is given in (39), the onshell Z factor is given by

$$\begin{aligned} Z_\chi^{-1} - 1 = -\delta Z_\chi &= [\delta Z_h]_{33} - \frac{\partial}{\partial v\cdot k} [\Sigma_2(v\cdot k)]_{33} = \delta Z_D - [\Sigma_2'(0)]_{33} \\ &\quad + \frac{1}{av} s_{\frac{\rho}{2}}^2 \bigg\{ -2s_\rho^{-1} [\Sigma_2(\delta_0^{(0)})]_{11} + t_{\frac{\rho}{2}} [\Sigma_2(\delta_0^{(+)})]_{22} + 2[\Sigma_2(0)]_{23} + t_{\frac{\rho}{2}}^{-1} [\Sigma_2(0)]_{33} \bigg\} \, , \end{aligned} \tag{52}$$

and δv is determined by the SM result [40],

$$\frac{\delta v}{v} = \frac{1}{2}\Sigma^{AA\prime}(0) - \frac{s_W}{c_W}\frac{\Sigma^{AZ}(0)}{m_Z^2} - \frac{c_W^2}{2s_W^2}\frac{{\rm Re}[\Sigma^{ZZ}(m_Z^2)]}{m_Z^2} + \frac{c_W^2 - s_W^2}{2s_W^2}\frac{{\rm Re}[\Sigma^{WW}(m_W^2)]}{m_W^2} - \frac{1}{2}{\rm Re}[\Sigma^{HH\prime}(m_h^2)] \, . \tag{53}$$

The two-point functions required in (53) are specified in (121) of Appendix B.[8] The one-boson exchange quark matching coefficient is obtained by collecting the above amplitudes into (48). Upon taking the pure-case limits described in Sec. 4.3, we recover the results (49) and (50) for a pure doublet. In the pure singlet limit, the one-boson exchange amplitudes vanish.

[8] We are here neglecting contributions from states beyond the SM. Renormalization schemes relevant for WIMPs of mass $M \sim m_W$ are discussed in Refs. [41].

6.1.3 Triplet-doublet admixture

For the triplet-doublet case we have the following contributions to the $h\bar{\chi}\chi$ three-point function,

$$i\hat{\mathcal{M}}_{\rm tree} = ias_\rho \,, \quad i\hat{\mathcal{M}}_{\delta a_1} = ias_\rho \frac{\delta a_1}{a_1} \,, \quad i\hat{\mathcal{M}}_{\delta Z} = ias_\rho \delta Z_\chi \,, \quad i\hat{\mathcal{M}}_{\delta v} = ias_\rho \frac{\delta v}{v} \,,$$

$$\begin{aligned}
i\hat{\mathcal{M}}_{\rm vertex,1} &= -\frac{g_2^3}{4c_W^3} c_{\frac{\rho}{2}}^2 m_Z I_1(\delta_0^{(0)}, m_Z) + \frac{g_2^2 a}{4c_W^2} s_\rho I_2(\delta_0^{(0)}, m_Z) + \frac{g_2 a^2}{2} s_{\frac{\rho}{2}}^2 \frac{m_h^2}{m_W} I_1(\delta_0^{(0)}, m_Z) \\
&\quad + \frac{3g_2 a^2}{2} \frac{m_h^2}{m_W} \left[s_\rho^2 I_1(\delta_0^{(-)}, m_h) + c_\rho^2 I_1(\delta_0^{(+)}, m_h) \right] - \frac{g_2^3}{8} s_\rho^2 m_W I_1(\delta_0^{(+)}, m_W) \\
&\quad - \frac{g_2^3}{2} (1 + s_{\frac{\rho}{2}}^2)^2 m_W I_1(\delta_0^{(-)}, m_W) + \frac{g_2^2 a}{2} s_\rho I_2(\delta_0^{(+)}, m_W) + g_2 a^2 \frac{m_h^2}{m_W} I_1(\delta_0^{(+)}, m_W) \,, \\
i\hat{\mathcal{M}}_{\rm vertex,2} &= -\frac{g_2^2 a}{8} s_\rho^3 I_4(\delta_0^{(+)}, \delta_0^{(+)}, m_W) + \frac{g_2^2 a}{2} (1 + s_{\frac{\rho}{2}}^2) s_\rho c_\rho I_4(\delta_0^{(-)}, \delta_0^{(+)}, m_W) \\
&\quad + \frac{g_2^2 a}{2} (1 + s_{\frac{\rho}{2}}^2)^2 s_\rho I_4(\delta_0^{(-)}, \delta_0^{(-)}, m_W) + 2a^3 s_\rho I_4(\delta_0^{(+)}, \delta_0^{(+)}, m_W) \\
&\quad + a^3 c_\rho^2 s_\rho I_4(\delta_0^{(+)}, \delta_0^{(+)}, m_h) - 2a^3 c_\rho^2 s_\rho I_4(\delta_0^{(-)}, \delta_0^{(+)}, m_h) - a^3 s_\rho^3 I_4(\delta_0^{(-)}, \delta_0^{(-)}, m_h) \,, \qquad (54)
\end{aligned}$$

where δa_1 is specified in (44) and δv in (53). The onshell Z factor takes the same form as in (52), but uses the self-energy components for the triplet-doublet system given in (43). The one-boson exchange quark matching coefficient is obtained by collecting the above amplitudes into (48). Upon taking the pure case limits described in Sec. 4.3, we recover the results (49) and (50) for both pure triplet and pure doublet.

6.2 Gluon matching: one-boson exchange

One-boson exchange contributions to gluon matching are pictured in Fig. 3. The two-loop diagrams factorize into separate one-loop diagrams: the boson loop given by the amplitudes $\hat{\mathcal{M}}_i$ determined in the previous section, and the fermion loop familiar from, e.g., the top quark contribution to the effective $h(G^A_{\mu\nu})^2$ vertex (e.g., see [42]). In terms of quark matching coefficients from one-boson exchange, $c_q^{(0)}{}_{\rm 1BE}$, the leading contribution to the bare gluon matching coefficient is thus

$$c_g^{(0)}{}_{\rm 1BE} = -\frac{g^2}{(4\pi)^2} \frac{1}{3} c_q^{(0)}{}_{\rm 1BE} + \mathcal{O}(\epsilon) \,. \qquad (55)$$

For the same reason discussed after Eq. (48), we neglect the one-boson exchange contributions containing $\mathcal{O}(\alpha_2^1)$ corrections to the effective $h(G^A_{\mu\nu})^2$ coupling, shown within square brackets in Fig. 3. In the above result for $c_g^{(0)}{}_{\rm 1BE}$, the light quark contributions cancel between the full and effective theory amplitudes, leaving only contributions from the top quark. Further discussion of effective theory contributions can be found in Sec. 6.5.

6.3 Quark matching: two-boson exchange

Let us now consider quark matching from two-boson exchange, as displayed in Fig. 4. In covariant gauges, in particular Feynman-t'Hooft gauge employed here, the full theory contributions include

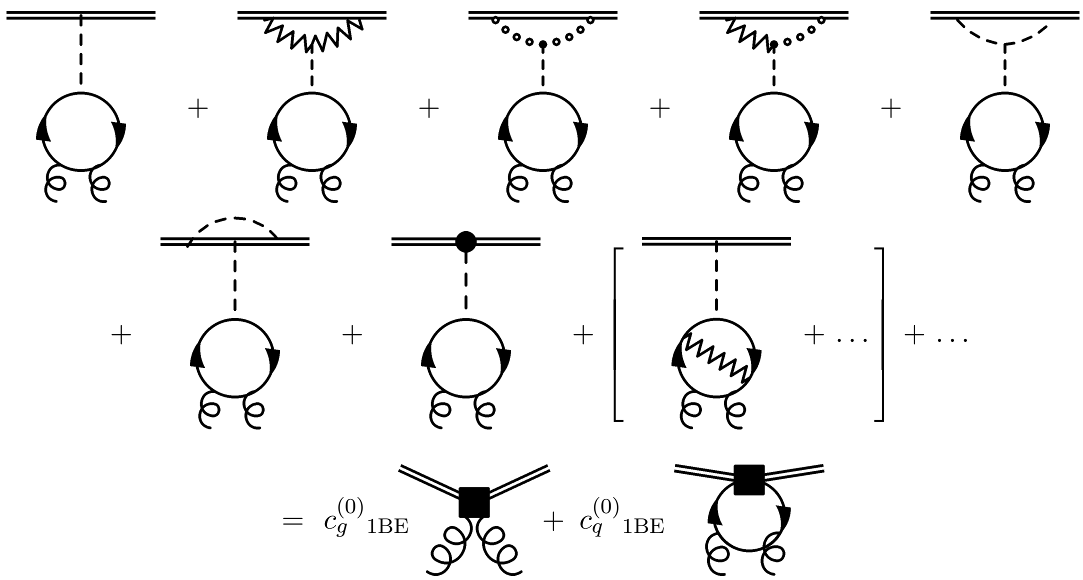


Figure 3: Matching condition for one-boson exchange contributions to gluon operators. The notation for the different lines and vertices is as in Fig. 2. All active quark flavors, such as the top quark in the full theory, are included in the loops.

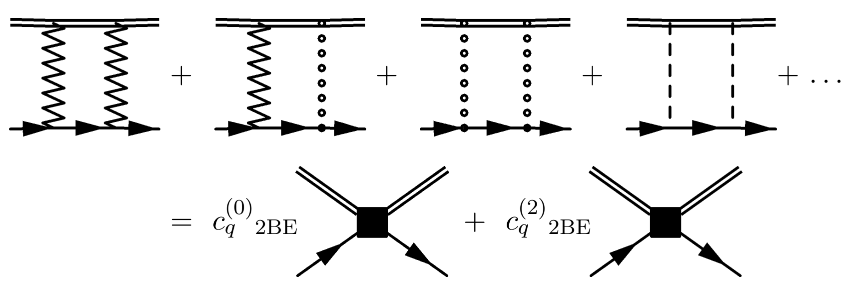


Figure 4: Matching condition for two-boson exchange contributions to quark operators. The notation for the different lines and vertices is as in Fig. 2. The full theory diagrams illustrate the possible types of two-boson exchange. Crossed diagrams and time-reversed diagrams are not shown.

diagrams with exchange of two gauge bosons ($W^\pm$ or Z^0), two Goldstone bosons (ϕ_Z^0 or $\phi_W^\pm$), one gauge and one Goldstone boson (Z^0 and ϕ_Z^0, or $W^\pm$ and $\phi_W^\pm$), or two Higgs bosons. In terms of these contributions the total amplitude is

$$\mathcal{M}_q = \mathcal{M}_q^{ZZ} + \mathcal{M}_q^{WW} + \mathcal{M}_q^{W\phi_W} + \mathcal{M}_q^{Z\phi_Z} + \mathcal{M}_q^{\phi_W\phi_W} + \mathcal{M}_q^{\phi_Z\phi_Z} + \mathcal{M}_q^{hh}\,, \tag{56}$$

where the superscripts denote which bosons are exchanged, and the contributions from crossed diagrams and time-reversed diagrams are included in each amplitude. Upon expressing the amplitudes in terms of the integrals $J(m_V, M, \delta)$, $J^\mu(p, m_V, M, \delta)$, $J_-(p, m_V, M, \delta)$ and $J_-^\mu(m_V, M, \delta)$ defined in Appendix C, we may write each amplitude in the form

$$\mathcal{M}_q^{BB'} = \bar{u}_q(p)\left[m_q\, c_q^{(0)BB'} + \left(\not{v}\, v\cdot p - \frac{\not{p}}{d}\right) c_q^{(2)BB'}\right] u_q(p)\,, \tag{57}$$

where the superscript BB' denotes the type of two-boson exchange. The contributions to spin-0 and spin-2 quark matching coefficients can then be read off as $c_q^{(0)BB'}$ and $c_q^{(2)BB'}$, respectively.

6.3.1 Pure states

For pure states the contributions come from diagrams with exchange of $W^\pm$ or Z^0 bosons. In terms of $\mathcal{C}_W$ and $\mathcal{C}_Z$ specified in (9), the amplitudes are

$$
\begin{aligned}
i\mathcal{M}_q^{ZZ} &= \frac{g_2^4 \mathcal{C}_Z}{16 c_W^4}\, \bar{u}_q(p)\Big[\big[c_V^{(q)2} + c_A^{(q)2}\big] \not{v}\big[\not{J}(p, m_Z, 0, 0) + \not{p} J(m_Z, 0, 0)\big]\not{v} \\
&\qquad + m_q \big[c_V^{(q)2} - c_A^{(q)2}\big] J(m_Z, 0, 0)\Big] u_q(p)\,, \\
i\mathcal{M}_U^{WW} &= \frac{g_2^4 \mathcal{C}_W}{8}\, \bar{u}_U(p) \not{v}\big[\not{J}(p, m_W, 0, 0) + \not{p} J(m_W, 0, 0)\big]\not{v} u_U(p)\,, \\
i\mathcal{M}_D^{WW} &= \sum_U \frac{g_2^4 \mathcal{C}_W}{8} |V_{UD}|^2\, \bar{u}_D(p) \not{v}\big[\not{J}(p, m_W, m_U, 0) + \not{p} J(m_W, m_U, 0)\big]\not{v} u_D(p)\,.
\end{aligned} \tag{58}
$$

Upon writing these amplitudes in the form of (57) and evaluating integrals, we find the contributions to the matching coefficients,

$$
\begin{aligned}
c^{(0)}_{U\ \mathrm{2BE}} &= \frac{\pi \Gamma(1+\epsilon) g_2^4}{(4\pi)^{2-\epsilon}} \left\{ \frac{m_Z^{-3-2\epsilon} \mathcal{C}_Z}{8 c_W^4} \big[c_V^{(U)2} - c_A^{(U)2}\big] + \mathcal{O}(\epsilon) \right\}, \\
c^{(0)}_{D\ \mathrm{2BE}} &= \frac{\pi \Gamma(1+\epsilon) g_2^4}{(4\pi)^{2-\epsilon}} \left\{ \frac{m_Z^{-3-2\epsilon} \mathcal{C}_Z}{8 c_W^4} \big[c_V^{(D)2} - c_A^{(D)2}\big] + \delta_{Db} \frac{m_W^{-3-2\epsilon} \mathcal{C}_W}{2} \left[- \frac{x_t}{4(x_t+1)^3} \right] + \mathcal{O}(\epsilon) \right\}, \\
c^{(2)}_{U\ \mathrm{2BE}} &= \frac{\pi \Gamma(1+\epsilon) g_2^4}{(4\pi)^{2-\epsilon}} \left\{ \left[m_W^{-3-2\epsilon} \mathcal{C}_W + \frac{m_Z^{-3-2\epsilon} \mathcal{C}_Z}{2 c_W^4} \big[c_V^{(U)2} + c_A^{(U)2}\big] \right] \left[\frac{1}{3} + \left(\frac{11}{9} - \frac{2}{3} \log 2 \right) \epsilon \right] + \mathcal{O}(\epsilon^2) \right\}, \\
c^{(2)}_{D\ \mathrm{2BE}} &= \frac{\pi \Gamma(1+\epsilon) g_2^4}{(4\pi)^{2-\epsilon}} \left\{ \left[m_W^{-3-2\epsilon} \mathcal{C}_W + \frac{m_Z^{-3-2\epsilon} \mathcal{C}_Z}{2 c_W^4} \big[c_V^{(D)2} + c_A^{(D)2}\big] \right] \left[\frac{1}{3} + \left(\frac{11}{9} - \frac{2}{3} \log 2 \right) \epsilon \right] \right. \\
&\qquad + \delta_{Db} \frac{m_W^{-3-2\epsilon} \mathcal{C}_W}{2} \left[\frac{(3x_t+2)}{3(x_t+1)^3} - \frac{2}{3} + \left(\frac{2x_t(7x_t^2-3)}{3(x_t^2-1)^3} \log x_t - \frac{2(3x_t+2)}{3(x_t+1)^3} \log 2 \right. \right. \\
&\qquad \left. \left. \left. - \frac{2(25x_t^2 - 2x_t - 11)}{9(x_t^2-1)^2(x_t+1)} - \frac{22}{9} + \frac{4}{3} \log 2 \right) \epsilon \right] + \mathcal{O}(\epsilon^2) \right\},
\end{aligned} \tag{59}
$$

where $x_t = m_t/m_W$, and the Kronecker delta, δ_{Db}, is equal to unity for $D = b$ and vanishes for $D = d, s$. We obtain the pure triplet (doublet) result upon setting $\mathcal{C}_W = 2$ and $\mathcal{C}_Z = 0$ ($\mathcal{C}_W = 1/2$ and $\mathcal{C}_Z = 1/4$) in (59).

6.3.2 Singlet-doublet admixture

For the singlet-doublet case the amplitudes for the different types of two-boson exchange are

$$
\begin{aligned}
i\mathcal{M}_q^{ZZ} &= \frac{g_2^4}{64 c_W^4} c^2_{\frac{\rho}{2}}\, \bar{u}_q(p)\Big[\big[c_V^{(q)2} + c_A^{(q)2}\big] \not{v}\big[\not{J}(p, m_Z, 0, \delta_0^{(0)}) + \not{p} J(m_Z, 0, \delta_0^{(0)})\big]\not{v} \\
&\qquad + m_q \big[c_V^{(q)2} - c_A^{(q)2}\big] J(m_Z, 0, \delta_0^{(0)})\Big] u_q(p)\,,
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M}_U^{WW} &= \frac{g_2^4}{16} c_{\frac{\rho}{2}}^2 \, \bar{u}_U(p) \not{v} \big[\not{J}(p, m_W, 0, \delta_0^{(0)}) + \not{p} J(m_W, 0, \delta_0^{(0)}) \big] \not{v} u_U(p) \,, \\
i\mathcal{M}_D^{WW} &= \sum_U \frac{g_2^4}{16} c_{\frac{\rho}{2}}^2 |V_{UD}|^2 \, \bar{u}_D(p) \not{v} \big[\not{J}(p, m_W, m_U, \delta_0^{(0)}) + \not{p} J(m_W, m_U, \delta_0^{(0)}) \big] \not{v} u_D(p) \,, \\
i\mathcal{M}_q^{Z\phi_Z} &= -\frac{g_2^3 a}{16 c_W^2} s_\rho \frac{m_q}{m_W} \bar{u}_q(p) \big[v \cdot J_-(m_Z, 0, \delta_0^{(0)}) \big] u_q(p) \,, \\
i\mathcal{M}_U^{W\phi_W} &= -\frac{g_2^3 a}{8} s_\rho \frac{m_U}{m_W} \bar{u}_U(p) \big[v \cdot J_-(m_W, 0, \delta_0^{(0)}) \big] u_U(p) \,, \\
i\mathcal{M}_D^{W\phi_W} &= \sum_U \frac{g_2^3 a}{8} s_\rho |V_{UD}|^2 \bar{u}_D(p) \bigg[-\frac{m_D}{m_W} v \cdot J_-(m_W, m_U, \delta_0^{(0)}) + \not{v} \frac{m_U^2}{m_W} J_-(p, m_W, m_U, \delta_0^{(0)}) \bigg] u_D(p) \,, \\
i\mathcal{M}_D^{\phi_W \phi_W} &= \frac{g_2^2 a^2}{4} s_{\frac{\rho}{2}}^2 \frac{m_t^2}{m_W^2} |V_{tD}|^2 \, \bar{u}_D(p) \big[-m_D J(m_W, m_t, \delta_0^{(0)}) + \not{J}(p, m_W, m_t, \delta_0^{(0)}) \big] u_D(p) \,, \\
i\mathcal{M}_U^{\phi_W \phi_W} &= 0 \,, \quad i\mathcal{M}_q^{\phi_Z \phi_Z} = 0 \,, \quad i\mathcal{M}_q^{hh} = 0 \,. \qquad (60)
\end{aligned}
$$

The amplitudes $\mathcal{M}_q^{hh}$, $\mathcal{M}_q^{\phi_Z \phi_Z}$, and $\mathcal{M}_U^{\phi_W \phi_W}$ are suppressed by light quark masses. Comparing each amplitude above with (57), we find the contributions to spin-0 and spin-2 quark matching coefficients,

$$
\begin{aligned}
c_q^{(0)ZZ} &= \frac{g_2^4}{64 c_W^4} c_{\frac{\rho}{2}}^2 \bigg\{ \big[c_V^{(q)2} - c_A^{(q)2} \big] J(m_Z, 0, \delta_0^{(0)}) \\
&\quad + \big[c_V^{(q)2} + c_A^{(q)2} \big] \bigg[-J(m_Z, 0, \delta_0^{(0)}) - J_2(m_Z, 0, \delta_0^{(0)}) + \frac{1}{d} \hat{J}(m_Z, 0, \delta_0^{(0)}) \bigg] \bigg\} \,, \\
c_q^{(2)ZZ} &= \frac{g_2^4}{64 c_W^4} c_{\frac{\rho}{2}}^2 \big[c_V^{(q)2} + c_A^{(q)2} \big] \hat{J}(m_Z, 0, \delta_0^{(0)}) \,, \\
c_U^{(0)WW} &= \frac{g_2^4}{16} c_{\frac{\rho}{2}}^2 \bigg[-J(m_W, 0, \delta_0^{(0)}) - J_2(m_W, 0, \delta_0^{(0)}) + \frac{1}{d} \hat{J}(m_W, 0, \delta_0^{(0)}) \bigg] \,, \\
c_U^{(2)WW} &= \frac{g_2^4}{16} c_{\frac{\rho}{2}}^2 \hat{J}(m_W, 0, \delta_0^{(0)}) \,, \\
c_D^{(0)WW} &= \sum_U \frac{g_2^4}{16} c_{\frac{\rho}{2}}^2 |V_{UD}|^2 \bigg[-J(m_W, m_U, \delta_0^{(0)}) - J_2(m_W, m_U, \delta_0^{(0)}) + \frac{1}{d} \hat{J}(m_W, m_U, \delta_0^{(0)}) \bigg] \,, \\
c_D^{(2)WW} &= \sum_U \frac{g_2^4}{16} c_{\frac{\rho}{2}}^2 |V_{UD}|^2 \hat{J}(m_W, m_U, \delta_0^{(0)}) \,, \\
c_q^{(0)Z\phi_Z} &= -\frac{g_2^3 a}{16 c_W^2} \frac{s_\rho}{m_W} J_{1-}(m_Z, 0, \delta_0^{(0)}) \,, \\
c_q^{(2)Z\phi_Z} &= 0 \,, \\
c_U^{(0)W\phi_W} &= -\frac{g_2^3 a}{8} \frac{s_\rho}{m_W} J_{1-}(m_W, 0, \delta_0^{(0)}) \,,
\end{aligned}
$$

$$
\begin{aligned}
c_U^{(2)W\phi_W} &= 0\,, \\
c_D^{(0)W\phi_W} &= \sum_U \frac{g_2^3 a}{8} s_\rho |V_{UD}|^2 \left[-\frac{1}{m_W} J_{1-}(m_W, m_U, \delta_0^{(0)}) + \frac{1}{d}\frac{m_U^2}{m_W} J_-(m_W, m_U, \delta_0^{(0)}) \right]\,, \\
c_D^{(2)W\phi_W} &= \sum_U \frac{g_2^3 a}{8} s_\rho |V_{UD}|^2 \frac{m_U^2}{m_W} J_-(m_W, m_U, \delta_0^{(0)})\,, \\
c_D^{(0)\phi_W\phi_W} &= \frac{g_2^2 a^2}{4} s_{\frac{\rho}{2}}^2 \frac{m_t^2}{m_W^2} |V_{tD}|^2 \left[J_2(m_W, m_t, \delta_0^{(0)}) - J(m_W, m_t, \delta_0^{(0)}) + \frac{1}{d} J_1(m_W, m_t, \delta_0^{(0)}) \right]\,, \\
c_D^{(2)\phi_W\phi_W} &= \frac{g_2^2 a^2}{4} s_{\frac{\rho}{2}}^2 \frac{m_t^2}{m_W^2} |V_{tD}|^2 J_1(m_W, m_t, \delta_0^{(0)})\,, \qquad (61)
\end{aligned}
$$

where we have defined

$$
\hat{J}(m_x, m_y, \delta_z) \equiv J_1(m_x, m_y, \delta_z) + 2J_2(m_x, m_y, \delta_z) + 2J(m_x, m_y, \delta_z). \qquad (62)
$$

The integrals $J(m_V, M, \delta)$, $J_1(m_V, M, \delta)$, $J_2(m_V, M, \delta)$, $J_-(m_V, M, \delta)$ and $J_{1-}(m_V, M, \delta)$ are given in Appendix C. The matching coefficients $c_q^{(0)}{}_{\rm 2BE}$ and $c_q^{(2)}{}_{\rm 2BE}$ for a given quark q are obtained by summing the nonvanishing contributions above,

$$
\begin{aligned}
c_U^{(0)}{}_{\rm 2BE} &= c_U^{(0)ZZ} + c_U^{(0)WW} + c_U^{(0)Z\phi_Z} + c_U^{(0)W\phi_W}\,, \\
c_D^{(0)}{}_{\rm 2BE} &= c_D^{(0)ZZ} + c_D^{(0)WW} + c_D^{(0)Z\phi_Z} + c_D^{(0)W\phi_W} + c_D^{(0)\phi_W\phi_W}\,, \\
c_U^{(2)}{}_{\rm 2BE} &= c_U^{(2)ZZ} + c_U^{(2)WW}\,, \\
c_D^{(2)}{}_{\rm 2BE} &= c_D^{(2)ZZ} + c_D^{(2)WW} + c_D^{(2)W\phi_W} + c_D^{(2)\phi_W\phi_W}. \qquad (63)
\end{aligned}
$$

Upon taking the pure-case limits described in Sec. 4.3, we recover the results (58) and (59) for a pure doublet. In the pure singlet limit, the two-boson exchange amplitudes vanish.

6.3.3 Triplet-doublet admixture

We may similarly compute the two-boson exchange amplitudes for the triplet-doublet system, and upon comparing with (57), we find the following contributions to spin-0 and spin-2 quark matching coefficients,

$$
\begin{aligned}
c_q^{(0)ZZ} &= \frac{g_2^4}{64 c_W^4} c_{\frac{\rho}{2}}^2 \Bigg\{ \big[c_V^{(q)2} - c_A^{(q)2}\big] J(m_Z, 0, \delta_0^{(0)}) \\
&\quad + \big[c_V^{(q)2} + c_A^{(q)2}\big] \left(-J(m_Z, 0, \delta_0^{(0)}) - J_2(m_Z, 0, \delta_0^{(0)}) + \frac{1}{d}\hat{J}(m_Z, 0, \delta_0^{(0)}) \right) \Bigg\}\,, \\
c_q^{(2)ZZ} &= \frac{g_2^4}{64 c_W^4} c_{\frac{\rho}{2}}^2 \big[c_V^{(q)2} + c_A^{(q)2}\big] \hat{J}(m_Z, 0, \delta_0^{(0)})\,, \\
c_U^{(0)WW} &= \frac{g_2^4}{16} \Bigg\{ (1 + s_{\frac{\rho}{2}}^2)^2 \left[-J(m_W, 0, \delta_0^{(-)}) - J_2(m_W, 0, \delta_0^{(-)}) + \frac{1}{d}\hat{J}(m_W, 0, \delta_0^{(-)}) \right]
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \frac{1}{4}s_\rho^2\left[-J(m_W,0,\delta_0^{(+)}) - J_2(m_W,0,\delta_0^{(+)}) + \frac{1}{d}\hat{J}(m_W,0,\delta_0^{(+)})\right]\Bigg\}\,,\\
c_U^{(2)WW} &= \frac{g_2^4}{16}\left[(1+s_{\frac{\rho}{2}}^2)^2\hat{J}(m_W,0,\delta_0^{(-)}) + \frac{1}{4}s_\rho^2\hat{J}(m_W,0,\delta_0^{(+)})\right]\,,\\
c_D^{(0)WW} &= \sum_U \frac{g_2^4}{16}|V_{UD}|^2\Bigg\{(1+s_{\frac{\rho}{2}}^2)^2\left[-J(m_W,m_U,\delta_0^{(-)}) - J_2(m_W,m_U,\delta_0^{(-)}) + \frac{1}{d}\hat{J}(m_W,m_U,\delta_0^{(-)})\right]\\
&\qquad + \frac{1}{4}s_\rho^2\left[-J(m_W,m_U,\delta_0^{(+)}) - J_2(m_W,m_U,\delta_0^{(+)}) + \frac{1}{d}\hat{J}(m_W,m_U,\delta_0^{(+)})\right]\Bigg\}\,,\\
c_D^{(2)WW} &= \sum_U \frac{g_2^4}{16}|V_{UD}|^2\left[(1+s_{\frac{\rho}{2}}^2)^2\hat{J}(m_W,m_U,\delta_0^{(-)}) + \frac{1}{4}s_\rho^2\hat{J}(m_W,m_U,\delta_0^{(+)})\right]\,,\\
c_q^{(0)Z\phi_Z} &= -\frac{g_2^3 a}{16c_W^2}\frac{s_\rho}{m_W}J_{1-}(m_Z,0,\delta_0^{(0)})\,,\\
c_q^{(2)Z\phi_Z} &= 0\,,\\
c_U^{(0)W\phi_W} &= -\frac{g_2^3 a}{8}\frac{s_\rho}{m_W}J_{1-}(m_W,0,\delta_0^{(+)})\,,\\
c_U^{(2)W\phi_W} &= 0\,,\\
c_D^{(0)W\phi_W} &= \sum_U \frac{g_2^3 a}{8}s_\rho|V_{UD}|^2\left[-\frac{1}{m_W}J_{1-}(m_W,m_U,\delta_0^{(+)}) + \frac{1}{d}\frac{m_U^2}{m_W}J_-(m_W,m_U,\delta_0^{(+)})\right]\,,\\
c_D^{(2)W\phi_W} &= \sum_U \frac{g_2^3 a}{8}s_\rho|V_{UD}|^2\frac{m_U^2}{m_W}J_-(m_W,m_U,\delta_0^{(+)})\,,\\
c_D^{(0)\phi_W\phi_W} &= \frac{g_2^2 a^2}{4}\frac{m_t^2}{m_W^2}|V_{tD}|^2\left[J_2(m_W,m_t,\delta_0^{(+)}) - J(m_W,m_t,\delta_0^{(+)}) + \frac{1}{d}J_1(m_W,m_t,\delta_0^{(+)})\right]\,,\\
c_D^{(2)\phi_W\phi_W} &= \frac{g_2^2 a^2}{4}\frac{m_t^2}{m_W^2}|V_{tD}|^2 J_1(m_W,m_t,\delta_0^{(+)})\,,
\end{aligned}
\tag{64}
$$

where $\hat{J}(m_x,m_y,\delta_z)$ is given in (62), and the relevant integrals can be found in Appendix C. The total matching coefficients $c_q^{(0)}{}_{\rm 2BE}$ and $c_q^{(2)}{}_{\rm 2BE}$ are obtained by summing the contributions above as in (63). Upon taking the pure-case limits described in Sec. 4.3, we recover the results (58) and (59) for both pure triplet and pure doublet.

6.4 Gluon matching: two-boson exchange

The gluon matching condition for two-boson exchange is pictured in Fig. 5. If we consider the external gluons as a background field [10], we may express the full theory diagrams in terms of electroweak polarization tensors induced by a loop of quarks. For example, using the Feynman rules for the

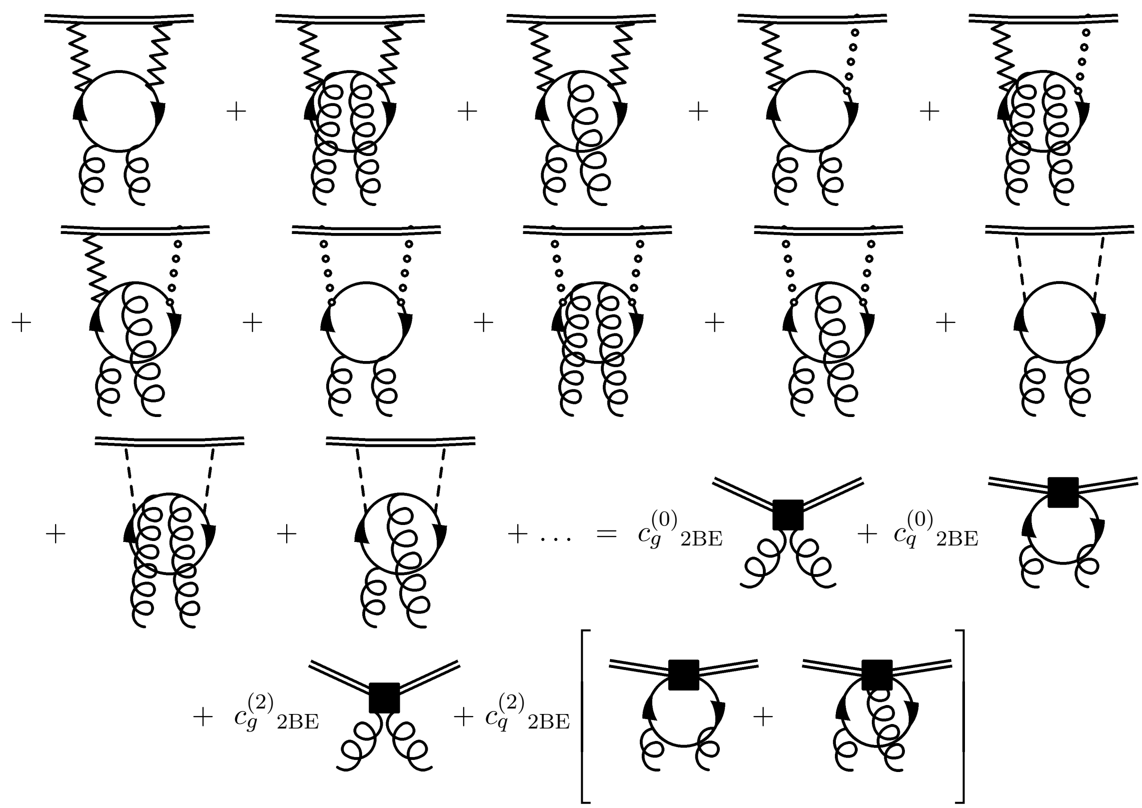


Figure 5: Matching condition for two-boson exchange contributions to gluon operators. The notation for the different lines and vertices is as in Fig. 2. The diagrams with a quark loop are obtained by closing the external legs of the corresponding diagrams in Fig. 4, and considering the possible attachments of two external gluons. All active quark flavors, such as the top quark in the full theory, are included in the loops.

WIMP-Z^0 coupling from (6), the contributions from exchanging two Z^0 bosons may be written as

$$\mathcal{M}^{ZZ} \sim \int (dL) \frac{1}{-v \cdot L - \delta + i0} \frac{1}{(L^2 - m_Z^2 + i0)^2} v_\mu v_\nu i \Pi^{\mu\nu}_{(ZZ)}(L) \,, \tag{65}$$

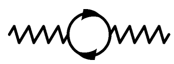
where $(dL) = d^dL/(2\pi)^d$ (this shorthand notation is used throughout this work), δ is a residual mass depending on the intermediate WIMP state, and $\Pi^{\mu\nu}_{(ZZ)}(L)$ is the two-gluon part of the Z^0 boson polarization tensor in a background gluon field. The amplitudes with exchange of one gauge and one Goldstone boson, two Goldstone bosons, or two Higgs bosons, have the same structure but with vector and scalar electroweak polarization tensors appearing. A similar analysis of gluon contributions to DM-nucleon scattering in Ref. [3] focused on the spin-0 operator. Here we perform a complete matching for both spin-0 and spin-2 gluon operators, and consider new contributions appearing in the case of mixed states.

The background field method presents the following strategy for evaluating the two-loop diagrams of the full theory. First, we determine the two-gluon part of the relevant polarization tensors. These amplitudes depend only on SM parameters, and can be used for gluon matching in general DM scenarios; in particular, this part of the computation is independent of whether the heavy-particle expansion is employed. Second, we insert the polarization tensors into the boson loop and perform the remaining integrals. We illustrate this second part by identifying a basis of heavy-particle integrals to compute the universal heavy WIMP limit.

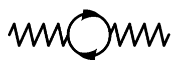
In our evaluation we neglect subleading corrections of $\mathcal{O}(m_q/m_W)$ for light quarks ($q \neq t$). The two-loop diagrams in the full theory (cf. Fig. 5) are UV finite, and may be evaluated in $d = 4$. However, we regulate the effective theory with dimensional regularization, and in performing the effective theory subtractions to determine Wilson coefficients it is convenient to also use dimensional regularization as IR regulator. Thus we choose to implement dimensional regularization as IR regulator also in the full theory. When considering only those terms contributing to the scalar operators appearing in (2), the relevant amplitudes do not involve γ_5 or $\epsilon^{\mu\nu\alpha\beta}$. In particular, the specification of γ_5 for $d \neq 4$ is unnecessary. Further discussion of effective theory contributions can be found in Sec. 6.5.

6.4.1 Electroweak polarization tensors in a background gluon field

Let us isolate the two-gluon amplitude of the relevant electroweak polarization tensors in a background gluon field. The generalized polarization tensors appearing in two-boson exchange contributions are

$$i\Pi^{\nu\mu}_{(W^+W^+)}(L) =$$


$$= -\sum_{U,D} \frac{g_2^2 |V_{UD}|^2}{8} \int d^dx\, e^{iL\cdot x} \langle T\{\bar{D}(x)\gamma^\nu(1-\gamma_5)U(x)\bar{U}(0)\gamma^\mu(1-\gamma_5)D(0)\}\rangle\,,$$

$$i\Pi^{\nu\mu}_{(ZZ)}(L) =$$


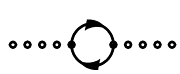
$$= -\sum_{q} \frac{g_2^2}{16c_W^2} \int d^dx\, e^{iL\cdot x} \langle T\{\bar{q}(x)\gamma^\nu(c_V^{(q)} + c_A^{(q)}\gamma_5)q(x)\bar{q}(0)\gamma^\mu(c_V^{(q)} + c_A^{(q)}\gamma_5)q(0)\}\rangle\,,$$

$$i\Pi^{\mu}_{(W^+\phi_W^+)}(L) =$$


$$= \sum_{U,D} \frac{g_2^2 |V_{UD}|^2}{8m_W} \int d^dx\, e^{iL\cdot x} \langle T\{\bar{D}(x)\big[-(m_U - m_D) - (m_U + m_D)\gamma_5\big]U(x)$$
$$\bar{U}(0)\gamma^\mu(1-\gamma_5)D(0)\}\rangle\,,$$

$$i\Pi^{\mu}_{(Z\phi_Z)}(L) =$$

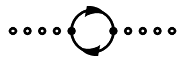

$$= \sum_{q} \frac{ig_2^2 m_q}{8c_W m_W} \int d^dx\, e^{iL\cdot x} \langle T\{\bar{q}(x)c_A^{(q)}\gamma_5 q(x)\bar{q}(0)\gamma^\mu(c_V^{(q)} + c_A^{(q)}\gamma_5)q(0)\}\rangle\,,$$

$$i\Pi_{(\phi_W^+\phi_W^+)}(L) =$$


$$= -\sum_{UD} \frac{g_2^2 |V_{UD}|^2}{8m_W^2} \int d^dx\, e^{iL\cdot x} \langle T\{\bar{D}(x)\big[-(m_U - m_D) - (m_U + m_D)\gamma_5\big]U(x)$$

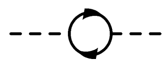
$$\bar{U}(0)\big[-(m_U - m_D) + (m_U + m_D)\gamma_5\big]D(0)\}\rangle\,,$$

$$i\Pi_{(\phi_Z\phi_Z)}(L) =$$



$$= \sum_q \frac{g_2^2 m_q^2}{4m_W^2}\int d^dx\, e^{iL\cdot x}\langle T\{\bar{q}(x)\gamma_5 q(x)\bar{q}(0)\gamma_5 q(0)\}\rangle\,,$$

$$i\Pi_{(hh)}(L) =$$



$$= -\sum_q \frac{g_2^2 m_q^2}{4m_W^2}\int d^dx\, e^{iL\cdot x}\langle T\{\bar{q}(x)q(x)\bar{q}(0)q(0)\}\rangle\,, \tag{66}$$

where the momentum L is flowing from left to right in the above diagrams. We also require the polarization tensors

$$\Pi^{\mu\nu}_{(W^-W^-)}(L)\,, \quad \Pi^{\mu}_{(W^-\phi_W^-)}(L)\,, \quad \Pi^{\mu}_{(\phi_W^\pm W^\pm)}(L)\,, \quad \Pi^{\mu}_{(\phi_Z Z)}(L)\,, \quad \Pi_{(\phi_W^-\phi_W^-)}(L)\,, \tag{67}$$

which are related to those we have specified in (66) through the identities in (84).

Let us now focus on the object,

$$i\tilde{\Pi}(L) \equiv \int d^dx\, e^{iL\cdot x}\langle T\{\bar{q}'(x)\Gamma q(x)\bar{q}(0)\Gamma' q'(0)\}\rangle\,, \tag{68}$$

where Γ and Γ' denote the possible Dirac structures whose indices we here suppress. The sum over quark mass eigenstates and other prefactors appearing in (66) are included in the final result for the polarization tensors. Let us write $\tilde{\Pi}$ in terms of momentum-space propagators in a background field,

$$i\tilde{\Pi}(L) = -\int d^dx\, e^{iL\cdot x}\mathrm{Tr}\big[\Gamma iS^{(q)}(x,0)\Gamma' iS^{(q')}(0,x)\big] = \int (dp)\mathrm{Tr}\big[\Gamma S^{(q)}(p)\Gamma'\tilde{S}^{(q')}(p-L)\big]\,, \tag{69}$$

where

$$iS^{(q)}(x,y) = \langle T\{q(x)\bar{q}(y)\}\rangle \tag{70}$$

and we have used

$$S^{(q)}(p) \equiv \int d^dx\, e^{ip\cdot x}S^{(q)}(x,0)\,, \quad \tilde{S}^{(q)}(p) \equiv \int d^dx\, e^{-ip\cdot x}S^{(q)}(0,x)\,. \tag{71}$$

We may expand the background field propagators at weak coupling,

$$\begin{aligned}
iS(p) &= \frac{i}{\not{p}-m} + g\int (dq)\frac{i}{\not{p}-m}i\not{A}(q)\frac{i}{\not{p}-\not{q}-m} \\
&\quad + g^2\int (dq_1)(dq_2)\frac{i}{\not{p}-m}i\not{A}(q_1)\frac{i}{\not{p}-\not{q}_1-m}i\not{A}(q_2)\frac{i}{\not{p}-\not{q}_1-\not{q}_2-m} + \dots\,, \\
i\tilde{S}(p) &= \frac{i}{\not{p}-m} + g\int (dq)\frac{i}{\not{p}+\not{q}-m}i\not{A}(q)\frac{i}{\not{p}-m} \\
&\quad + g^2\int (dq_1)(dq_2)\frac{i}{\not{p}+\not{q}_1+\not{q}_2-m}i\not{A}(q_1)\frac{i}{\not{p}+\not{q}_2-m}i\not{A}(q_2)\frac{i}{\not{p}-m} + \dots\,,
\end{aligned} \tag{72}$$

and upon insertion of these expressions into (69), the terms with two gluon fields are readily identified. Furthermore, in Fock-Schwinger gauge the gluon field can be written as

$$\begin{aligned} \not{A}(q) &= t^a \gamma^\alpha \int d^d x \, e^{iq\cdot x} A^a_\alpha(x) \\ &= t^a \gamma^\alpha \left[\frac{-i}{2} \frac{\partial}{\partial q_\rho} G^a_{\rho\alpha}(0) (2\pi)^d \delta^d(q) + \dots \right] , \end{aligned} \tag{73}$$

where the ellipsis denotes terms with derivatives acting on $G^a_{\mu\nu}$. Thus the amplitudes with gluon emission are given directly in terms of field-strengths, and intermediate steps involving gauge-variant combinations can be avoided.

In isolating the two-gluon amplitude, we may separately consider three cases depending on where the gluons are attached. Contributions with both gluons attached to the upper quark line in (66) are referred to as"*a*-type", those with both gluons attached to the lower quark line in (66) are referred to as "*b*-type", and those with one gluon attached to each of the upper and lower quark lines are referred to as "*c*-type". We thus have

$$\tilde{\Pi}(L) = \tilde{\Pi}_a(L) + \tilde{\Pi}_b(L) + \tilde{\Pi}_c(L) , \tag{74}$$

with

$$\begin{aligned} i\tilde{\Pi}_a(L) &= \frac{-g^2}{4} \mathrm{Tr}(t^a t^b) G^a_{\rho\alpha}(0) G^b_{\sigma\tau}(0) \int (dp) \frac{\partial}{\partial q_\rho} \frac{\partial}{\partial q'_\sigma} \\ &\quad \mathrm{Tr}\left[\Gamma \frac{1}{\not{p} - m_1} \gamma^\alpha \frac{1}{\not{p} - \not{q} - m_1} \gamma^\tau \frac{1}{\not{p} - \not{q} - \not{q}' - m_1} \Gamma' \frac{1}{\not{p} - \not{L} - m_2} \right]_{q=q'=0} , \\ i\tilde{\Pi}_b(L) &= \frac{-g^2}{4} \mathrm{Tr}(t^a t^b) G^a_{\rho\alpha}(0) G^b_{\sigma\tau}(0) \int (dp) \frac{\partial}{\partial q_\rho} \frac{\partial}{\partial q'_\sigma} \\ &\quad \mathrm{Tr}\left[\Gamma \frac{1}{\not{p} - m_1} \Gamma' \frac{1}{\not{p} - \not{L} + \not{q} + \not{q}' - m_2} \gamma^\alpha \frac{1}{\not{p} - \not{L} + \not{q}' - m_2} \gamma^\tau \frac{1}{\not{p} - \not{L} - m_2} \right]_{q=q'=0} , \\ i\tilde{\Pi}_c(L) &= \frac{-g^2}{4} \mathrm{Tr}(t^a t^b) G^a_{\rho\alpha}(0) G^b_{\sigma\tau}(0) \int (dp) \frac{\partial}{\partial q_\rho} \frac{\partial}{\partial q'_\sigma} \\ &\quad \mathrm{Tr}\left[\Gamma \frac{1}{\not{p} - m_1} \gamma^\alpha \frac{1}{\not{p} - \not{q} - m_1} \Gamma' \frac{1}{\not{p} - \not{L} + \not{q}' - m_2} \gamma^\tau \frac{1}{\not{p} - \not{L} - m_2} \right]_{q=q'=0} , \end{aligned} \tag{75}$$

where m_1 and m_2 are the masses of the quarks in the upper and lower lines in (66), respectively. To project these onto the spin-0 and spin-2 QCD gluon operators, $\mathcal{O}^{(0)}_g$ and $\mathcal{O}^{(2)}_g$ in (3), consider the four-index tensor $T_{\alpha\rho\gamma\delta} = G^A_{\alpha\rho} G^A_{\gamma\delta}$ with index symmetries $T_{\alpha\rho\gamma\delta} = T_{\gamma\delta\alpha\rho} = -T_{\rho\alpha\gamma\delta}$. We can decompose T into components $T = T^{(0)} + T^{(2)} + \Delta T$, where

$$\begin{aligned} T^{(0)}_{\alpha\rho\gamma\delta} &= \frac{1}{d(d-1)} \mathcal{O}^{(0)}_g (g_{\alpha\gamma} g_{\rho\delta} - g_{\alpha\delta} g_{\rho\gamma}) , \\ T^{(2)}_{\alpha\rho\gamma\delta} &= \frac{1}{d-2} \left(-g_{\alpha\gamma} \mathcal{O}^{(2)}_{g\,\rho\delta} + g_{\alpha\delta} \mathcal{O}^{(2)}_{g\,\rho\gamma} - g_{\rho\delta} \mathcal{O}^{(2)}_{g\,\alpha\gamma} + g_{\rho\gamma} \mathcal{O}^{(2)}_{g\,\alpha\delta} \right) , \end{aligned} \tag{76}$$

and ΔT, satisfying

$$g^{\alpha\gamma} g^{\rho\delta} (\Delta T)_{\alpha\rho\gamma\delta} = v^\alpha v^\gamma g^{\rho\delta} (\Delta T)_{\alpha\rho\gamma\delta} = 0 , \tag{77}$$

is not needed for the present analysis. The proportionality constants in $T^{(0)}$ and $T^{(2)}$ were obtained by contraction with $g^{\alpha\gamma}g^{\rho\delta}$ or $v^\alpha v^\gamma g^{\rho\delta}$. Upon applying the above decomposition to the expressions in (75), we obtain

$$i\tilde{\Pi}_k(L) \equiv \frac{-g^2}{8}\left[\frac{1}{d(d-1)}\mathcal{O}_g^{(0)}I_k^{(0)}(L) + \frac{1}{d-2}\mathcal{O}_g^{(2)\mu\nu}I^{(2)}_{k\,\mu\nu}(L) + \dots\right], \tag{78}$$

where $k = a, b, c$ and the ellipsis denotes irrelevant ΔT contributions.

Let us now determine $I_k^{(0)}(L)$ and $I^{(2)}_{k\,\mu\nu}(L)$ for the different cases of two-boson exchange. The trace and derivatives with respect to momenta q and q' in (75) are straightforward to evaluate, and the result is projected onto gluon operators of definite spin using (76). The quark-loop integral over momentum p is computed using standard methods, leaving an integral over a Feynman parameter, x, which will be evaluated after performing the boson-loop integral over momentum L. We may express the results in the form

$$I_k^{(S)}(L) \equiv \frac{i\Gamma[1+\epsilon]}{(4\pi)^{2-\epsilon}}\int_0^1 dx\; u_k(x)N_k^{(S)}(L)\,, \quad u_a(x) = \frac{(1-x)^3}{3!}\,, \quad u_b(x) = \frac{x^3}{3!}\,, \quad u_c(x) = x(1-x)\,, \tag{79}$$

where $S = 0, 2$ and for $S = 2$ the Lorentz indices are suppressed. Let us also introduce the parameters

$$z_n \equiv \frac{(-1)^n}{2^{3-n}}\frac{\Gamma[n+\epsilon]}{\Gamma[1+\epsilon]}\,, \quad \Delta \equiv (1-x)m_1^2 + xm_2^2 - x(1-x)L^2 - i0\,, \tag{80}$$

which appear in the expressions for $N_k^{(S)}(L)$ given below.

For the operators of interest in (2), the relevant projections of the a- and b-type amplitudes in (75) and (76) are related by CP transformation. This condition can be stated in terms of $N_k^{(S)}(L)$ as

$$N_b^{(S)}(L) = N_a^{(S)}(L)\bigg|_{x\leftrightarrow 1-x,\, m_1\leftrightarrow m_2}\,. \tag{81}$$

In the case of flavor-diagonal currents (Z^0, ϕ_Z^0, h) where we set $m_1 = m_2 = m_q$ in $\tilde{\Pi}_k(L)$, the above relation implies $I_b^{(S)}(L) = I_a^{(S)}(L)$. For flavor-changing currents $(W^\pm, \phi_W^\pm)$ we set the down-type quark mass to zero, $m_2 = m_D = 0$, but keep the up-type quark mass finite, $m_1 = m_U \neq 0$, to accommodate the top quark. This asymmetry in treating the masses does not allow us to systematically recover $N_b^{(S)}(L)$ from $N_a^{(S)}(L)$ using the relation (81). Below we provide $N_b^{(S)}(L)$ explicitly for flavor-changing currents.

To illustrate the explicit implementation of this program, we again focus on the heavy WIMP limit, retaining the leading order (in $1/M$) WIMP-SM couplings as in (65). Anticipating the insertion of polarization tensors into the boson loop with leading order heavy-particle Feynman rules, we thus contract the free Lorentz indices of Γ and Γ' in (66), (68) with v_μ's from the WIMP-vector boson vertices. It is straightforward to analyze the remaining components of $\Pi^{\mu\nu}(L)$ by the same methods. The following results are labelled by the bosons in the corresponding electroweak polarization tensor. For $N_k^{(0)}(L)$ we find,

$$N_a^{(0)}{}_{(W^+W^+)} = 64(3-2\epsilon)m_U^2\left\{2(1-\epsilon)\frac{z_2}{\Delta^{2+\epsilon}} + x(1-x)\left(2(v\cdot L)^2 - L^2\right)\frac{z_3}{\Delta^{3+\epsilon}}\right\},$$

$$N_b^{(0)}{}_{(W^+W^+)} = 0\,,$$

$$
\begin{aligned}
N_{c\,(W^+W^+)}^{(0)} &= 64(1-\epsilon)\bigg\{ -2(1+\epsilon)(3-2\epsilon)\frac{z_1}{\Delta^{1+\epsilon}} \\
&\quad + x(1-x)\big[2(1-2\epsilon)(v\cdot L)^2 + (1+2\epsilon)L^2\big]\frac{z_2}{\Delta^{2+\epsilon}}\bigg\}\,, \\
N_{a\,(ZZ)}^{(0)} &= 32(3-2\epsilon)m_q^2\bigg\{\big[c_V^{(q)2}+c_A^{(q)2}\big]\bigg[2(1-\epsilon)\frac{z_2}{\Delta^{2+\epsilon}} + x(1-x)(2(v\cdot L)^2 - L^2)\frac{z_3}{\Delta^{3+\epsilon}}\bigg] \\
&\quad - \big[c_V^{(q)2}-c_A^{(q)2}\big]\bigg[2(2-\epsilon)\frac{z_2}{\Delta^{2+\epsilon}} + x^2L^2\frac{z_3}{\Delta^{3+\epsilon}}\bigg]\bigg\}\,, \\
N_{c\,(ZZ)}^{(0)} &= 32\bigg\{\big[c_V^{(q)2}+c_A^{(q)2}\big](1-\epsilon)\bigg[-2(3-2\epsilon)(1+\epsilon)\frac{z_1}{\Delta^{1+\epsilon}} + x(1-x)\big[2(1-2\epsilon)(v\cdot L)^2 \\
&\quad + (1+2\epsilon)L^2\big]\frac{z_2}{\Delta^{2+\epsilon}}\bigg] + \big[c_V^{(q)2}-c_A^{(q)2}\big]\epsilon(3-2\epsilon)m_q^2\frac{z_2}{\Delta^{2+\epsilon}}\bigg\}\,, \\
N_{a\,(W^+\phi_W^+)}^{(0)} &= -64(3-2\epsilon)m_U^2 v\cdot L\bigg[2\big[2-3x-\epsilon(1-x)\big]\frac{z_2}{\Delta^{2+\epsilon}} + x^2(1-x)L^2\frac{z_3}{\Delta^{3+\epsilon}}\bigg]\,, \\
N_{b\,(W^+\phi_W^+)}^{(0)} &= 0\,, \\
N_{c\,(W^+\phi_W^+)}^{(0)} &= -64(3-2\epsilon)(1-\epsilon)(1-x)m_U^2 v\cdot L\frac{z_2}{\Delta^{2+\epsilon}}\,, \\
N_{a\,(Z\phi_Z)}^{(0)} &= -32(3-2\epsilon)c_A^{(q)2}m_q v\cdot L\bigg[2\big[2-3x-\epsilon(1-x)\big]\frac{z_2}{\Delta^{2+\epsilon}} + x\big[m_q^2 + x(1-x)L^2\big]\frac{z_3}{\Delta^{3+\epsilon}}\bigg]\,, \\
N_{c\,(Z\phi_Z)}^{(0)} &= -32(3-2\epsilon)(1-\epsilon)c_A^{(q)2}m_q v\cdot L\frac{z_2}{\Delta^{2+\epsilon}}\,, \\
N_{a\,(\phi_W^+\phi_W^+)}^{(0)} &= 64(3-2\epsilon)m_U^4\big[-2(2-\epsilon)\frac{z_2}{\Delta^{2+\epsilon}} + x(1-x)L^2\frac{z_3}{\Delta^{3+\epsilon}}\big]\,, \\
N_{b\,(\phi_W^+\phi_W^+)}^{(0)} &= 0\,, \\
N_{c\,(\phi_W^+\phi_W^+)}^{(0)} &= 64(1-\epsilon)(3-2\epsilon)m_U^2\big[-2(2-\epsilon)\frac{z_1}{\Delta^{1+\epsilon}} + x(1-x)L^2\frac{z_2}{\Delta^{2+\epsilon}}\big]\,, \\
N_{a\,(\phi_Z\phi_Z)}^{(0)} &= -32(3-2\epsilon)x m_q^2 L^2\frac{z_3}{\Delta^{3+\epsilon}}\,, \\
N_{c\,(\phi_Z\phi_Z)}^{(0)} &= 32(3-2\epsilon)\bigg[2(1-\epsilon)(2-\epsilon)\frac{z_1}{\Delta^{1+\epsilon}} - \big[(2-\epsilon)m_q^2 + (1-\epsilon)x(1-x)L^2\big]\frac{z_2}{\Delta^{2+\epsilon}}\bigg]\,, \\
N_{a\,(hh)}^{(0)} &= 32(3-2\epsilon)m_q^2\bigg[-4(2-\epsilon)\frac{z_2}{\Delta^{2+\epsilon}} + x(1-2x)L^2\frac{z_3}{\Delta^{3+\epsilon}}\bigg]\,, \\
N_{c\,(hh)}^{(0)} &= 32(3-2\epsilon)\bigg[-2(1-\epsilon)(2-\epsilon)\frac{z_1}{\Delta^{1+\epsilon}} + \big[(1-\epsilon)x(1-x)L^2 - (2-\epsilon)m_q^2\big]\frac{z_2}{\Delta^{2+\epsilon}}\bigg]\,. \qquad (82)
\end{aligned}
$$

For $N_{k\,\mu\nu}^{(2)}(L)$ the open indices are to be contracted with $\mathcal{O}_g^{(2)\,\mu\nu}$, which is symmetric in μ and ν and satisfies $g_{\mu\nu}\mathcal{O}_g^{(2)\,\mu\nu}=0$. The results are

$$N^{(2)}_{a\,\mu\nu(W^+W^+)} = 128(1-\epsilon)\bigg\{-4(2-\epsilon)v_\mu v_\nu\frac{z_1}{\Delta^{1+\epsilon}}$$
$$+2\Big[(m_U^2-x^2L^2)v_\mu v_\nu+2(2-\epsilon)x(1-x)v\cdot Lv_\mu L_\nu-x(2-x-\epsilon)L_\mu L_\nu\Big]\frac{z_2}{\Delta^{2+\epsilon}}$$
$$+x(1-x)\Big[(m_U^2-2x^2(v\cdot L)^2)L_\mu L_\nu-2(m_U^2-x^2L^2)v\cdot Lv_\mu L_\nu\Big]\frac{z_3}{\Delta^{3+\epsilon}}\bigg\}\,,$$

$$N^{(2)}_{b\,\mu\nu(W^+W^+)} = 128(1-\epsilon)\bigg\{-4(2-\epsilon)v_\mu v_\nu\frac{z_1}{\Delta^{1+\epsilon}}$$
$$-2(1-x)\Big[(1-x)L^2v_\mu v_\nu-2(2-\epsilon)xv\cdot Lv_\mu L_\nu+(1+x-\epsilon)L_\mu L_\nu\Big]\frac{z_2}{\Delta^{2+\epsilon}}$$
$$+2x(1-x)^3\Big[-(v\cdot L)^2L_\mu L_\nu+L^2v\cdot Lv_\mu L_\nu\Big]\frac{z_3}{\Delta^{3+\epsilon}}\bigg\}\,,$$

$$N^{(2)}_{c\,\mu\nu(W^+W^+)} = 128\bigg\{2(1-\epsilon)(1-2\epsilon)v_\mu v_\nu\frac{z_1}{\Delta^{1+\epsilon}}$$
$$+x(1-x)\big[\epsilon L_\mu L_\nu+2(1-2\epsilon)v\cdot Lv_\mu L_\nu-(1-2\epsilon)L^2v_\mu v_\nu\big]\frac{z_2}{\Delta^{2+\epsilon}}\bigg\}\,,$$

$$N^{(2)}_{a\,\mu\nu(ZZ)} = 64(1-\epsilon)\bigg\{\big[c_V^{(q)2}-c_A^{(q)2}\big]x^2m_q^2L_\mu L_\nu\frac{z_3}{\Delta^{3+\epsilon}}+\big[c_V^{(q)2}+c_A^{(q)2}\big]\Big[-4(2-\epsilon)v_\mu v_\nu\frac{z_1}{\Delta^{1+\epsilon}}$$
$$+2\big[(m_q^2-x^2L^2)v_\mu v_\nu+2(2-\epsilon)x(1-x)v\cdot Lv_\mu L_\nu+x(2-x-\epsilon)L_\mu L_\nu\big]\frac{z_2}{\Delta^{2+\epsilon}}$$
$$+x(1-x)\big[(m_q^2-2x^2(v\cdot L)^2L_\mu L_\nu-2(m_q^2-x^2L^2)v\cdot Lv_\mu L_\nu\big]\frac{z_3}{\Delta^{3+\epsilon}}\Big]\bigg\}\,,$$

$$N^{(2)}_{c\,\mu\nu(ZZ)} = 64\bigg\{-\big[c_V^{(q)2}-c_A^{(q)2}\big]2(1-\epsilon)m_q^2v_\mu v_\nu\frac{z_2}{\Delta^{2+\epsilon}}+\big[c_V^{(q)2}+c_A^{(q)2}\big]\Big[2(1-\epsilon)(1-2\epsilon)v_\mu v_\nu\frac{z_1}{\Delta^{1+\epsilon}}$$
$$+x(1-x)\big[\epsilon L_\mu L_\nu+2(1-2\epsilon)v\cdot Lv_\mu L_\nu-(1-2\epsilon)L^2v_\mu v_\nu\big]\frac{z_2}{\Delta^{2+\epsilon}}\Big]\bigg\}\,,$$

$$N^{(2)}_{a\,\mu\nu(W^+\phi_W^+)} = -128(1-\epsilon)xm_U^2\bigg\{2v_\mu L_\nu\frac{z_2}{\Delta^{2+\epsilon}}-x(1-x)v\cdot LL_\mu L_\nu\frac{z_3}{\Delta^{3+\epsilon}}\bigg\}\,,$$

$$N^{(2)}_{b\,\mu\nu(W^+\phi_W^+)} = -128(1-\epsilon)(1-x)m_U^2\bigg\{2(2-\epsilon)v_\mu L_\nu\frac{z_2}{\Delta^{2+\epsilon}}+(1-x)^2\big[L^2v_\mu L_\nu-v\cdot LL_\mu L_\nu\big]\frac{z_3}{\Delta^{3+\epsilon}}\bigg\}\,,$$

$$N^{(2)}_{c\,\mu\nu(W^+\phi_W^+)} = -128(1-\epsilon)(1-x)m_U^2v_\mu L_\nu\frac{z_2}{\Delta^{2+\epsilon}}\,,$$

$$N^{(2)}_{a\,\mu\nu(Z\phi_Z)} = \big[c_A^{(q)2}\big]64(1-\epsilon)xm_q\bigg\{-2(3-\epsilon)v_\mu L_\nu\frac{z_2}{\Delta^{2+\epsilon}}$$
$$+\big[(m_q^2-x^2L^2)v_\mu L_\nu+xv\cdot LL_\mu L_\nu\big]\frac{z_3}{\Delta^{3+\epsilon}}\bigg\}\,,$$

$$
\begin{aligned}
N^{(2)}_{c\,\mu\nu(Z\phi_Z)} &= -\big[c_A^{(q)2}\big]64(1-\epsilon)m_q v_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}\,, \\
N^{(2)}_{a\,\mu\nu(\phi_W^+\phi_W^+)} &= 128(1-\epsilon)x m_U^2 L_\mu L_\nu \bigg\{2(2-\epsilon)\frac{z_2}{\Delta^{2+\epsilon}} - (1-x)m_U^2\frac{z_3}{\Delta^{3+\epsilon}}\bigg\}\,, \\
N^{(2)}_{b\,\mu\nu(\phi_W^+\phi_W^+)} &= 256(1-\epsilon)(2-\epsilon)(1-x)m_U^2 L_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}\,, \\
N^{(2)}_{c\,\mu\nu(\phi_W^+\phi_W^+)} &= 128(1-\epsilon)x(1-x)m_U^2 L_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}\,, \\
N^{(2)}_{a\,\mu\nu(\phi_Z\phi_Z)} &= 64(1-\epsilon)x L_\mu L_\nu \bigg\{-2(2-\epsilon)\frac{z_2}{\Delta^{2+\epsilon}} + m_q^2\frac{z_3}{\Delta^{3+\epsilon}}\bigg\}\,, \\
N^{(2)}_{c\,\mu\nu(\phi_Z\phi_Z)} &= -64(1-\epsilon)x(1-x)L_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}\,, \\
N^{(2)}_{a\,\mu\nu(hh)} &= 64(1-\epsilon)x L_\mu L_\nu \bigg\{2(2-\epsilon)\frac{z_2}{\Delta^{2+\epsilon}} - (1-2x)m_q^2\frac{z_3}{\Delta^{3+\epsilon}}\bigg\}\,, \\
N^{(2)}_{c\,\mu\nu(hh)} &= 64(1-\epsilon)x(1-x)L_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}. \qquad (83)
\end{aligned}
$$

The results for $N_k^{(S)}(L)$ in (82) and (83) specify $I_a^{(S)}(L)$ through (79), and hence $\tilde\Pi_k(L)$ through (78), and $\tilde\Pi(L)$ through (74). This completes our determination of the polarization tensors in (66). The polarization tensors in (67) are obtained through the following relations

$$
\begin{aligned}
&\Pi^{\mu\nu}_{(W^-W^-)}(L) = \Pi^{\mu\nu}_{(W^+W^+)}(-L)\,, \quad \Pi^{\mu}_{(\phi_Z Z)}(L) = \Pi^{\mu}_{(Z\phi_Z)}(-L)\,, \quad \Pi_{(\phi_W^-\phi_W^-)}(L) = \Pi_{(\phi_W^+\phi_W^+)}(-L)\,, \\
&\Pi^{\mu}_{(\phi_W^- W^-)}(L) = \Pi^{\mu}_{(W^+\phi_W^+)}(-L)\,, \quad \Pi^{\mu}_{(\phi_W^+ W^+)}(L) = \Pi^{\mu}_{(W^-\phi_W^-)}(-L)\,, \\
&\Pi^{\mu}_{(W^-\phi_W^-)}(L) = \Pi^{\mu}_{(W^+\phi_W^+)}(-L)\,. \qquad (84)
\end{aligned}
$$

The identities in the first two lines are consequences of reversing the direction of momentum L in the diagrams in (66). The last relation follows from Hermitian conjugation and the identification $\overline{S(p)} \equiv \gamma^0 S(p)^\dagger \gamma^0 = \tilde S(p)$. We note that polarization tensors with one gauge and one Goldstone boson are odd in L, while all others are even in L. This property also holds for the corresponding $N_k^{(S)}(L)$, and we use it in the next section to systematically reduce the boson loop integrals into a convenient basis.

6.4.2 Basis reduction of the full theory boson loop

Having determined the generalized polarization tensors, we now proceed with the reduction of the remaining boson loop integrals. Upon insertion of the polarization tensors into the boson loop, we find the required set of basic loop integrals

$$
\begin{aligned}
&\int (dL)\bigg[\frac{1}{v\cdot L-\delta+i0} + \frac{1}{-v\cdot L-\delta+i0}\bigg]\frac{1}{(L^2-m_V^2+i0)^2}N_k^{(S)}(L) \equiv \mathcal{I}_{\rm even}(\delta,m_V)N_k^{(S)}(L)\,, \\
&\int (dL)\bigg[\frac{1}{v\cdot L-\delta+i0} - \frac{1}{-v\cdot L-\delta+i0}\bigg]\frac{1}{(L^2-m_V^2+i0)^2}N_k^{(S)}(L) \equiv \mathcal{I}_{\rm odd}(\delta,m_V)N_k^{(S)}(L)\,, \qquad (85)
\end{aligned}
$$

where δ is the residual mass of the intermediate WIMP state, and m_V is the mass of the exchanged bosons. We suppress the arguments, (δ, m_V), of these integral operators when making generic statements below. The integral operator $\mathcal{I}_{\rm even}$ requires that $N_k^{(S)}(L)$ be even in L as in those for polarization tensors with a single type of boson, while the integral operator $\mathcal{I}_{\rm odd}$ requires that $N_k^{(S)}(L)$ be odd in L as in those for polarization tensors with one gauge and one Goldstone boson. Let us denote even $N_k^{(S)}(L)$ by $N_{k\,\rm even}^{(S)}(L)$ and odd $N_k^{(S)}(L)$ by $N_{k\,\rm odd}^{(S)}(L)$. The subscripts even and odd may be dropped if we mean either type, or if the exchanged bosons are specified.

To reduce (85) to a set of basis integrals for evaluation, we begin by replacing factors of L^2 in $N_k^{(S)}(L)$ with

$$L^2 = -\frac{\Delta}{x(1-x)} + \frac{m_1^2}{x} + \frac{m_2^2}{(1-x)}\,, \tag{86}$$

which follows from the definition of Δ in (80). The $N_k^{(S)}(L)$ of (82) and (83) may then be written in terms of Δ and the vectors v_μ and L_μ. In $N_{k\,\rm even}^{(S)}(L)$ each term must have two or zero v_μ's, while in $N_{k\,\rm odd}^{(S)}(L)$ each term must have one v_μ. Organizing the result in powers of $(v\cdot L)$, we obtain the general expressions

$$\begin{aligned}
N_{k\,\rm even}^{(0)}(L) &= (v\cdot L)^0 \sum_n a_n^{(1)} \Delta^{-n-\epsilon} + (v\cdot L)^2 \sum_n a_n^{(2)} \Delta^{-n-\epsilon}\,,\\
N_{k\,\rm odd}^{(0)}(L) &= (v\cdot L)^1 \sum_n a_n^{(3)} \Delta^{-n-\epsilon}\,,\\
N_{k\,\rm even}^{(2)\,\mu\nu}(L) &= (v\cdot L)^0 \sum_n \left[v^\mu v^\nu a_n^{(4)} \Delta^{-n-\epsilon} + L^\mu L^\nu a_n^{(5)} \Delta^{-n-\epsilon}\right] + (v\cdot L)^1 \sum_n v^\mu L^\nu a_n^{(6)} \Delta^{-n-\epsilon}\\
&\quad + (v\cdot L)^2 \sum_n L^\mu L^\nu a_n^{(7)} \Delta^{-n-\epsilon}\,,\\
N_{k\,\rm odd}^{(2)\,\mu\nu}(L) &= (v\cdot L)^0 \sum_n v^\mu L^\nu a_n^{(8)} \Delta^{-n-\epsilon} + (v\cdot L)^1 \sum_n L^\mu L^\nu a_n^{(9)} \Delta^{-n-\epsilon}\,,
\end{aligned} \tag{87}$$

where the sums run over $n = 1, 2, \ldots$, and the coefficients $a_n^{(i)}$ are functions of x and ϵ. The above $N_k^{(S)}(L)$ structures require the set of integrals

$$\begin{aligned}
&H(n) = \mathcal{I}_{\rm even} \Delta^{-n-\epsilon}\,, \quad H^\mu(n) = \mathcal{I}_{\rm odd} \Delta^{-n-\epsilon} L^\mu\,, \quad H^{\mu\nu}(n) = \mathcal{I}_{\rm even} \Delta^{-n-\epsilon} L^\mu L^\nu\,,\\
&F(n) = \int (dL) \frac{1}{(L^2 - m_V^2 + i0)^2} \Delta^{-n-\epsilon}\,.
\end{aligned} \tag{88}$$

The integrals H^μ and $H^{\mu\nu}$ may be expressed in terms of $H(n)$ and $F(n)$ through standard reduction methods and the relation

$$\left[\frac{1}{v\cdot L - \delta + i0} \pm \frac{1}{-v\cdot L - \delta + i0}\right] v\cdot L = \delta \left[\frac{1}{v\cdot L - \delta + i0} \mp \frac{1}{-v\cdot L - \delta + i0}\right] + 1 \mp 1\,. \tag{89}$$

Furthermore, recursion relations in n may be derived by taking derivatives of parameters. A detailed discussion of these relations, as well as the evaluation of the above integrals, can be found in Appendix D. Note that the $(v\cdot L)^2$ term in $N_{k\,\rm even}^{(2)\,\mu\nu}(L)$ also requires the integral

$$\int (dL) \frac{1}{(L^2 - m_V^2 + i0)^2} \Delta^{-n-\epsilon} L^\mu L^\nu \sim g^{\mu\nu}\,, \tag{90}$$

however this does not contribute since it vanishes upon contraction with the traceless spin-2 gluon operator, $O_g^{(2)\mu\nu}$. Upon feeding the general expressions for $N_k^{(S)}(L)$ in (87) into the integrals in (85), we find the following decomposition in terms of basis integrals,

$$
\begin{aligned}
\mathcal{I}_{\rm even} N^{(0)}_{k\,{\rm even}}(L) &= \sum_n \left[a_n^{(1)} H(n) + a_n^{(2)} \left[\delta^2 H(n) + 2\delta F(n) \right] \right] , \\
\mathcal{I}_{\rm odd} N^{(0)}_{k\,{\rm odd}}(L) &= \sum_n a_n^{(3)} \left[\delta H(n) + 2F(n) \right] , \\
\mathcal{I}_{\rm even} N^{(2)\,\mu\nu}_{k\,{\rm even}}(L) &= v^\mu v^\nu \sum_n \left[a_n^{(4)} H(n) + a_n^{(5)} H_1(n) + a_n^{(6)} \left[\delta^2 H(n) + 2\delta F(n) \right] + a_n^{(7)} \delta^2 H_1(n) \right] , \\
\mathcal{I}_{\rm odd} N^{(2)\,\mu\nu}_{k\,{\rm odd}}(L) &= v^\mu v^\nu \sum_n \left[a_n^{(8)} \left[\delta H(n) + 2F(n) \right] + a_n^{(9)} \delta H_1(n) \right] , \qquad (91)
\end{aligned}
$$

where

$$
H_1(n) = \frac{1}{3-2\epsilon} \left\{ (4-2\epsilon) \left[\delta^2 H(n) + 2\delta F(n) \right] + \frac{H(n-1)}{x(1-x)} - \left[\frac{m_1^2}{x} + \frac{m_2^2}{1-x} \right] H(n) \right\} . \qquad (92)
$$

The above results apply generally to both pure and mixed states. Comparing with the explicit expressions for $N_k^{(S)}(L)$ in (82) and (83), we find that $H(n)$ for $n = 1, 2, 3$ and $F(n)$ for $n = 2, 3$ are required.

For pure states there is no residual mass, and $\mathcal{I}_{\rm odd}$ is irrelevant since the only contributions are from exchanges of $W^\pm$ and Z^0, involving $N^{(S)}_{k\,{\rm even}}(L)$. The vanishing of certain contributions in $\mathcal{I}_{\rm even} N^{(S)}_{k\,{\rm even}}(L)$ at $\delta = 0$ can be traced to the identity in (89).[9] Setting $\delta = 0$ in $\mathcal{I}_{\rm even} N^{(S)}_{k\,{\rm even}}(L)$ above and using the explicit expressions for $N_k^{(S)}(L)$ in (82) and (83), we find pure-state results that depend on $H(n)$ only,

$$
\begin{aligned}
\mathcal{I}(0, m_W) N^{(0)}_{a\,(W^+W^+)} &= 64(1+\epsilon)(3-2\epsilon) m_U^2 \Big\{ (2+\epsilon)(1-x) m_U^2 H(3) - (1+2\epsilon) H(2) \Big\} , \\
\mathcal{I}(0, m_W) N^{(0)}_{c\,(W^+W^+)} &= 32(1-\epsilon^2) \Big\{ (1+2\epsilon)(1-x) m_U^2 H(2) + 2(1-2\epsilon) H(1) \Big\} , \\
\mathcal{I}(0, m_Z) N^{(0)}_{a\,(ZZ)} &= \frac{32(1+\epsilon)(3-2\epsilon) m_q^2}{1-x} \Big\{ \left[c_V^{(q)2} + c_A^{(q)2} \right] (1-x) \left[(2+\epsilon) m_q^2 H(3) - (1+2\epsilon) H(2) \right] \\
&\quad + \left[c_V^{(q)2} - c_A^{(q)2} \right] \left[(2+\epsilon) x m_q^2 H(3) - (2 - \epsilon + 2\epsilon x) H(2) \right] \Big\} , \\
\mathcal{I}(0, m_Z) N^{(0)}_{c\,(ZZ)} &= 16(1+\epsilon) \Big\{ \left[c_V^{(q)2} + c_A^{(q)2} \right] (1-\epsilon) \left[(1+2\epsilon) m_q^2 H(2) + 2(1-2\epsilon) H(1) \right] \\
&\quad + \left[c_V^{(q)2} - c_A^{(q)2} \right] \epsilon (3-2\epsilon) m_q^2 H(2) \Big\} , \\
\mathcal{I}(0, m_W) N^{(2)}_{a\,\mu\nu(W^+W^+)} &= \frac{128(1-\epsilon) v_\mu v_\nu}{(3-2\epsilon)(1-x)} \Big\{ (2-\epsilon)(2 - x - 3\epsilon + 4\epsilon x) H(1)
\end{aligned}
$$

[9]In particular, this can be used to demonstrate gauge invariance for the electroweak part of the amplitudes since in a general R_ξ gauge the ξ-dependent terms carry a factor of $(v \cdot L)$.

$$+ (1+\epsilon)(1-x)m_U^2\big[(2+\epsilon)(1-x)m_U^2 H(3) + (3-4x-4\epsilon+2\epsilon x)H(2)\big]\Bigg\}\,,$$

$$\mathcal{I}(0,m_W)N^{(2)}_{c\,\mu\nu(W^+W^+)} = \frac{64(1-\epsilon)v_\mu v_\nu}{(3-2\epsilon)}\Bigg\{-(3-\epsilon-4\epsilon^2)(1-x)m_U^2 H(2) + \epsilon(7-8\epsilon)H(1)\Bigg\}\,,$$

$$\mathcal{I}(0,m_Z)N^{(2)}_{a\,\mu\nu(ZZ)} = \frac{64(1-\epsilon)v_\mu v_\nu}{(3-2\epsilon)(1-x)}\Bigg\{\big[c_V^{(q)2}+c_A^{(q)2}\big]\Big[(2-\epsilon)(2-x-3\epsilon+4\epsilon x)H(1)$$
$$+ m_q^2(1+\epsilon)\big[(2+\epsilon)(1-x)m_q^2 H(3) + (3-5x-4\epsilon+5\epsilon x)H(2)\big]\Big]$$
$$+ \big[c_V^{(q)2}-c_A^{(q)2}\big](1+\epsilon)(2+\epsilon)x m_q^2\big[m_q^2 H(3) - H(2)\big]\Bigg\}\,,$$

$$\mathcal{I}(0,m_Z)N^{(2)}_{c\,\mu\nu(ZZ)} = \frac{32(1-\epsilon)v_\mu v_\nu}{(3-2\epsilon)}\Bigg\{\big[c_V^{(q)2}+c_A^{(q)2}\big]\big[-(1+\epsilon)(3-4\epsilon)m_q^2 H(2) + \epsilon(7-8\epsilon)H(1)\big]$$
$$- \big[c_V^{(q)2}-c_A^{(q)2}\big]2(1+\epsilon)(3-2\epsilon)m_q^2 H(2)\Bigg\}\,, \tag{93}$$

where the subscript on $\mathcal{I}_{\text{even}}$ has been suppressed. The reduction for admixtures, where there are nonzero residual masses and the integral $\mathcal{I}_{\text{odd}}$ is relevant, is also straightforward to obtain.

We collect in Appendix D useful results for the remaining task of integrating over Feynman parameters. The singularity structure and evaluation of integrals can be classified into three cases corresponding to zero, one, or two heavy fermions contributing to the electroweak polarization tensor. The case of zero heavy fermions is for polarization tensors with no top quark in the loop. With subleading powers of light quark masses neglected, only polarization tensors of $W^\pm$ and Z^0 bosons are relevant in this case. The case of one heavy fermion is for polarization tensors of flavor-changing currents with one top quark and one down-type quark. The case of two heavy fermions is for polarization tensors of flavor-diagonal currents with a top quark loop.

6.4.3 Full theory contributions and matching coefficients for pure states

Let us now determine the full theory contributions to the matching using the generalized electroweak polarization tensors and the reduction method for the boson loop integral. For pure states, the total amplitude receives two-boson exchange contributions from $W^\pm$ and Z^0 bosons,

$$\mathcal{M} = \mathcal{M}^{WW} + \mathcal{M}^{ZZ}\,, \tag{94}$$

which may be written in terms of electroweak polarization tensors in a background field as

$$i\mathcal{M}^{WW} = \frac{ig_2^2\mathcal{C}_W}{2}\int(dL)\frac{1}{-v\cdot L+i0}\frac{1}{(L^2-m_W^2+i0)^2}v_\mu v_\nu\Big[i\Pi^{\mu\nu}_{(W^+W^+)}(L) + i\Pi^{\mu\nu}_{(W^-W^-)}(L)\Big]\,,$$
$$i\mathcal{M}^{ZZ} = \frac{ig_2^2\mathcal{C}_Z}{c_W^2}\int(dL)\frac{1}{-v\cdot L+i0}\frac{1}{(L^2-m_Z^2+i0)^2}v_\mu v_\nu i\Pi^{\mu\nu}_{(ZZ)}(L)\,, \tag{95}$$

with $\mathcal{C}_W$ and $\mathcal{C}_Z$ given in (9). The parity of the polarization tensors under $L\to -L$ and the identities in (84) allow us to write the above amplitudes in terms of the integrals defined in (85),

$$i\mathcal{M}^{WW} = \frac{ig_2^2\mathcal{C}_W}{2}\mathcal{I}_{\text{even}}(0,m_W)v_\mu v_\nu i\Pi^{\mu\nu}_{(W^+W^+)}(L)\,,$$

$$i\mathcal{M}^{ZZ} = \frac{ig_2^2\mathcal{C}_Z}{2c_W^2}\mathcal{I}_{\rm even}(0,m_Z)v_\mu v_\nu i\Pi^{\mu\nu}_{(ZZ)}(L)\,. \tag{96}$$

Upon inserting the explicit polarization tensors from (66) into the expressions above, we may employ the reduction of integrals given in (93) and write each contribution in terms of the gluon operators of definite spin,

$$\mathcal{M}^{BB'} = \mathcal{M}^{BB'(0)}\mathcal{O}_g^{(0)} + \mathcal{M}^{BB'(2)}\,v_\mu v_\nu \mathcal{O}_g^{(2)\mu\nu}\,, \tag{97}$$

where the superscript BB' denotes the different types of two-boson exchange. From the expression in (97), we may readily identify the contribution of each amplitude to $c_g^{(0)}{}_{\rm 2BE}$ and $c_g^{(2)}{}_{\rm 2BE}$ as $\mathcal{M}^{BB'(0)}$ and $\mathcal{M}^{BB'(2)}$, respectively. Let us decompose $\mathcal{M}^{WW(S)}$, for $S=0,2$, into contributions from each up-type quark flavor, and the a-, b-, and c-type gluon attachments,

$$\mathcal{M}^{WW(S)} = -\frac{[\Gamma(1+\epsilon)]^2}{(4\pi)^d}\frac{\pi g^2 g_2^4}{m_W^{3+4\epsilon}}\frac{\mathcal{C}_W}{16}\sum_{U=u,c,t}\,\sum_{k=a,b,c}\mathcal{M}^{WW(S)}_{U,k}\,. \tag{98}$$

Similarly, we decompose $\mathcal{M}^{ZZ(S)}$ into contributions from each quark flavor, and the a-, b-, and c-type gluon attachments,

$$\mathcal{M}^{ZZ(S)} = -\frac{[\Gamma(1+\epsilon)]^2}{(4\pi)^d}\frac{\pi g^2 g_2^4}{m_Z^{3+4\epsilon}}\frac{\mathcal{C}_Z}{64c_W^4}\sum_{q=u,c,t,d,s,b}\,\sum_{k=a,b,c}\mathcal{M}^{ZZ(S)}_{q,k}\,. \tag{99}$$

The results for $W^\pm$ exchange are as follows. The amplitudes with one top quark are

$$\begin{aligned}
\mathcal{M}^{WW(0)}_{t,a} &= 4x_t^2\log\frac{x_t+1}{x_t} - \frac{2x_t(6x_t^2+9x_t+2)}{3(x_t+1)^2}\,,\\
\mathcal{M}^{WW(0)}_{t,b} &= 0\,,\\
\mathcal{M}^{WW(0)}_{t,c} &= -4x_t^2\log\frac{x_t+1}{x_t} + \frac{2(6x_t^3+9x_t^2+2x_t-2)}{3(x_t+1)^2}\,,\\
\mathcal{M}^{WW(2)}_{t,a} &= \frac{16(30x_t^4-3x_t^2-4)}{9}\log\frac{x_t+1}{x_t} - \frac{8(60x_t^5+90x_t^4+14x_t^3-14x_t^2-8x_t-9)}{9(x_t+1)^2}\,,\\
\mathcal{M}^{WW(2)}_{t,b} &= \frac{16(3x_t+2)}{9(x_t+1)^3}\frac{1}{\epsilon} + \frac{32x_t(15x_t^9-48x_t^7+52x_t^5-15x_t^3+14x_t^2-6)}{9(x_t^2-1)^3}\log x_t\\
&\quad - \frac{32(15x_t^{10}-48x_t^8+52x_t^6-12x_t^4+3x_t^2-2)}{9(x_t^2-1)^3}\log(x_t+1)\\
&\quad - \frac{32(3x_t^4-21x_t^3+3x_t^2+9x_t-2)}{9(x_t^2-1)^3}\log 2\\
&\quad + \frac{8(180x_t^8+90x_t^7-426x_t^6-183x_t^5+285x_t^4+111x_t^3-220x_t^2-4x_t+71)}{27(x_t^2-1)^2(x_t+1)}\,,\\
\mathcal{M}^{WW(2)}_{t,c} &= -48x_t^2\log\frac{x_t+1}{x_t} + \frac{8x_t(6x_t^2+9x_t+2)}{(x_t+1)^2}\,,
\end{aligned} \tag{100}$$

where $x_t = m_t/m_W$. The amplitudes with only light quarks are

$$\mathcal{M}_{U,a}^{WW(0)} = \mathcal{M}_{U,b}^{WW(0)} = 0\,, \quad \mathcal{M}_{U,c}^{WW(0)} = -\frac{4}{3}\,,$$
$$\mathcal{M}_{U,a}^{WW(2)} = \mathcal{M}_{U,b}^{WW(2)} = \frac{32}{9\epsilon} + \frac{568}{27} - \frac{64}{9}\log 2\,, \quad \mathcal{M}_{U,c}^{WW(2)} = 0\,, \tag{101}$$

for $U = u, c$. The results for Z^0 exchange with a top quark loop are

$$\begin{aligned}
\mathcal{M}_{t,a}^{ZZ(0)} &= \mathcal{M}_{t,b}^{ZZ(0)} = \big[c_V^{(t)2} + c_A^{(t)2}\big]\bigg[\frac{4y_t^2(32y_t^6 - 28y_t^4 + 14y_t^2 - 1)}{(4y_t^2-1)^{7/2}}\arctan\big(\sqrt{4y_t^2-1}\big) - \frac{\pi y_t}{2} \\
&\quad + \frac{4y_t^2(y_t^2-1)(24y_t^2-1)}{3(4y_t^2-1)^3}\bigg] + \big[c_V^{(t)2} - c_A^{(t)2}\big]\bigg[\frac{16y_t^4(24y_t^4 - 21y_t^2 + 5)}{(4y_t^2-1)^{7/2}}\arctan\big(\sqrt{4y_t^2-1}\big) \\
&\quad + \frac{2(144y_t^6 - 70y_t^4 + 9y_t^2 - 2)}{3(4y_t^2-1)^3} - \frac{3\pi y_t}{2}\bigg]\,, \\
\mathcal{M}_{t,c}^{ZZ(0)} &= \big[c_V^{(t)2} + c_A^{(t)2}\big]\bigg[-\frac{8y_t^2(8y_t^2-1)(2y_t^2-1)}{(4y_t^2-1)^{5/2}}\arctan\big(\sqrt{4y_t^2-1}\big) - \frac{4(24y_t^4 - 7y_t^2 + 1)}{3(4y_t^2-1)^2} + 2\pi y_t\bigg]\,, \\
\mathcal{M}_{t,a}^{ZZ(2)} &= \mathcal{M}_{t,b}^{ZZ(2)} = \big[c_V^{(t)2} + c_A^{(t)2}\big]\bigg[\frac{16(480y_t^8 - 420y_t^6 + 214y_t^4 - 47y_t^2 + 4)}{9(4y_t^2-1)^{7/2}}\arctan\big(\sqrt{4y_t^2-1}\big) \\
&\quad + \frac{8(240y_t^6 - 314y_t^4 + 92y_t^2 - 9)}{9(4y_t^2-1)^3} - \frac{10\pi y_t}{3}\bigg] + \big[c_V^{(t)2} - c_A^{(t)2}\big]\bigg[-\frac{8y_t^2(48y_t^4 - 34y_t^2 + 13)}{9(4y_t^2-1)^3} \\
&\quad - \frac{32y_t^2(16y_t^6 - 14y_t^4 + 4y_t^2 - 1)}{3(4y_t^2-1)^{7/2}}\arctan\big(\sqrt{4y_t^2-1}\big) + \frac{2\pi y_t}{3}\bigg]\,, \\
\mathcal{M}_{t,c}^{ZZ(2)} &= \big\{\big[c_V^{(t)2} + c_A^{(t)2}\big] + 2\big[c_V^{(t)2} - c_A^{(t)2}\big]\big\}\bigg[-\frac{32y_t^2(16y_t^4 - 10y_t^2 + 3)}{(4y_t^2-1)^{5/2}}\arctan\big(\sqrt{4y_t^2-1}\big) \\
&\quad - \frac{16y_t^2(8y_t^2-5)}{(4y_t^2-1)^2} + 8\pi y_t\bigg]\,,
\end{aligned} \tag{102}$$

where $y_t = m_t/m_Z$. The amplitudes for Z^0 exchange with a light quark loop are

$$\mathcal{M}_{q,a}^{ZZ(0)} = \mathcal{M}_{q,b}^{ZZ(0)} = 0\,, \quad \mathcal{M}_{q,c}^{ZZ(0)} = \big[c_V^{(q)2} + c_A^{(q)2}\big]\bigg[-\frac{4}{3}\bigg]\,,$$
$$\mathcal{M}_{q,a}^{ZZ(2)} = \mathcal{M}_{q,b}^{ZZ(2)} = \big[c_V^{(q)2} + c_A^{(q)2}\big]\bigg[\frac{32}{9\epsilon} + \frac{568}{27} - \frac{64}{9}\log 2\bigg]\,, \quad \mathcal{M}_{q,c}^{ZZ(2)} = 0\,, \tag{103}$$

where $q = u, d, s, c, b$. The $\frac{1}{\epsilon}$ pieces in the above amplitudes are IR divergences that cancel upon subtraction of the effective theory contributions, $\mathcal{M}_{\rm EFT}^{(S)}$, discussed in Sec. 6.5. The bare coefficients are then given by

$$c_g^{(S)}{}_{\rm 2BE} = \mathcal{M}^{WW(S)} + \mathcal{M}^{ZZ(S)} - \mathcal{M}_{\rm EFT}^{(S)}\,, \tag{104}$$

where the remaining $\frac{1}{\epsilon}$ pieces are UV divergences.

6.4.4 Full theory contributions and matching coefficients for admixtures

For admixtures, the total amplitude receives contributions from other types of two-boson exchange beyond WW and ZZ,

$$\mathcal{M} = \mathcal{M}^{WW} + \mathcal{M}^{ZZ} + \mathcal{M}^{\phi_W \phi_W} + \mathcal{M}^{\phi_Z \phi_Z} + \mathcal{M}^{hh} + \mathcal{M}^{Z\phi_Z} + \mathcal{M}^{W\phi_W}. \tag{105}$$

Let us first consider the singlet-doublet case. In terms of the electroweak polarization tensors, we find integrals involving nonzero residual masses,

$$\begin{aligned}
i\mathcal{M}^{WW} &= \frac{ig_2^2}{4} c^2_{\frac{\rho}{2}} \int (dL) \frac{1}{-v\cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_W^2 + i0)^2} v_\mu v_\nu \left[i\Pi^{\mu\nu}_{(W^+W^+)}(L) + i\Pi^{\mu\nu}_{(W^-W^-)}(L) \right], \\
i\mathcal{M}^{ZZ} &= \frac{ig_2^2}{4c_W^2} c^2_{\frac{\rho}{2}} \int (dL) \frac{1}{-v\cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_Z^2 + i0)^2} v_\mu v_\nu i\Pi^{\mu\nu}_{(ZZ)}(L), \\
i\mathcal{M}^{hh} &= ia^2 \int (dL) \left[\frac{s_\rho^2}{-v\cdot L - \delta_0^{(-)} + i0} + \frac{c_\rho^2}{-v\cdot L - \delta_0^{(+)} + i0} \right] \frac{1}{(L^2 - m_h^2 + i0)^2} i\Pi_{(hh)}(L), \\
i\mathcal{M}^{\phi_Z\phi_Z} &= ia^2 s^2_{\frac{\rho}{2}} \int (dL) \frac{1}{-v\cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_Z^2 + i0)^2} i\Pi_{(\phi_Z\phi_Z)}(L), \\
i\mathcal{M}^{\phi_W\phi_W} &= ia^2 s^2_{\frac{\rho}{2}} \int (dL) \frac{1}{-v\cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_W^2 + i0)^2} \left[i\Pi_{(\phi_W^+\phi_W^+)}(L) + i\Pi_{(\phi_W^-\phi_W^-)}(L) \right], \\
i\mathcal{M}^{Z\phi_Z} &= \frac{g_2 a}{4c_W} s_\rho \int (dL) \frac{1}{-v\cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_Z^2 + i0)^2} v_\mu \left[i\Pi^\mu_{(Z\phi_Z)}(L) - i\Pi^\mu_{(\phi_Z Z)}(L) \right], \\
i\mathcal{M}^{W\phi_W} &= \frac{ig_2 a}{4} s_\rho \int (dL) \frac{1}{-v\cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_W^2 + i0)^2} v_\mu \Big[i\Pi^\mu_{(W^+\phi_W^+)}(L) - i\Pi^\mu_{(W^-\phi_W^-)}(L) \\
&\quad + i\Pi^\mu_{(\phi_W^+ W^+)}(L) - i\Pi^\mu_{(\phi_W^- W^-)}(L) \Big].
\end{aligned} \tag{106}$$

Using the behavior of the polarization tensors under $L \to -L$ and the identities in (84), we may write these amplitudes in terms of the integrals defined in (85),

$$\begin{aligned}
i\mathcal{M}^{WW} &= \frac{ig_2^2}{4} c^2_{\frac{\rho}{2}} \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_W) v_\mu v_\nu i\Pi^{\mu\nu}_{(W^+W^+)}(L), \\
i\mathcal{M}^{ZZ} &= \frac{ig_2^2}{8c_W^2} c^2_{\frac{\rho}{2}} \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_Z) v_\mu v_\nu i\Pi^{\mu\nu}_{(ZZ)}(L), \\
i\mathcal{M}^{hh} &= \frac{ia^2}{2} s_\rho^2 \mathcal{I}_{\text{even}}(\delta_0^{(-)}, m_h) i\Pi_{(hh)}(L) + \frac{ia^2}{2} c_\rho^2 \mathcal{I}_{\text{even}}(\delta_0^{(+)}, m_h) i\Pi_{(hh)}(L), \\
i\mathcal{M}^{\phi_Z\phi_Z} &= \frac{ia^2}{2} s^2_{\frac{\rho}{2}} \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_Z) i\Pi_{(\phi_Z\phi_Z)}(L), \\
i\mathcal{M}^{\phi_W\phi_W} &= ia^2 s^2_{\frac{\rho}{2}} \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_W) i\Pi_{(\phi_W^+\phi_W^+)}(L), \\
i\mathcal{M}^{Z\phi_Z} &= \frac{g_2 a}{4c_W} s_\rho \mathcal{I}_{\text{odd}}(\delta_0^{(0)}, m_Z) v_\mu i\Pi^\mu_{(Z\phi_Z)}(L),
\end{aligned}$$

$$i\mathcal{M}^{W\phi_W} = \frac{ig_2 a}{2} s_\rho \, \mathcal{I}_{\rm odd}(\delta_0^{(0)}, m_W) v_\mu i\Pi^\mu_{(W^+\phi_W^+)}(L) \,. \tag{107}$$

The required polarization tensors are specified in (66), and, in particular, the complete set of functions $N_k^{(S)}(L)$ are explicitly given in (82) and (83). Thus, the general result in (91) for reducing these integrals may be applied. Each amplitude may be written in the form of (97), i.e., in terms of its contributions to the gluon operators of definite spin. The bare coefficients are then given by

$$\begin{aligned} c_g^{(S)}{}_{\rm 2BE} &= \mathcal{M}^{(S)WW} + \mathcal{M}^{(S)ZZ} + \mathcal{M}^{(S)hh} + \mathcal{M}^{(S)\phi_Z\phi_Z} \\ &\quad + \mathcal{M}^{(S)\phi_W\phi_W} + \mathcal{M}^{(S)Z\phi_Z} + \mathcal{M}^{(S)W\phi_W} - \mathcal{M}^{(S)}_{\rm EFT} \,, \end{aligned} \tag{108}$$

where the remaining $\frac{1}{\epsilon}$ pieces are UV divergences. We may again organize each contribution in the previous equation in terms of the quark flavors in the loop, and the a-, b-, and c-type gluon attachments, as we have done in (98) and (99).

For the triplet-doublet case we find,

$$\begin{aligned} i\mathcal{M}^{WW} &= \frac{ig_2^2}{16} s_\rho^2 \, \mathcal{I}_{\rm even}(\delta_0^{(+)}, m_W) v_\mu v_\nu i\Pi^{\mu\nu}_{(W^+W^+)}(L) \\ &\quad + \frac{ig_2^2}{4} \big(1 + s^2_{\frac{\rho}{2}}\big)^2 \, \mathcal{I}_{\rm even}(\delta_0^{(-)}, m_W) v_\mu v_\nu i\Pi^{\mu\nu}_{(W^+W^+)}(L) \,, \\ i\mathcal{M}^{ZZ} &= \frac{ig_2^2}{8c_W^2} c^2_{\frac{\rho}{2}} \, \mathcal{I}_{\rm even}(\delta_0^{(0)}, m_Z) v_\mu v_\nu i\Pi^{\mu\nu}_{(ZZ)}(L) \,, \\ i\mathcal{M}^{hh} &= \frac{ia^2}{2} s_\rho^2 \, \mathcal{I}_{\rm even}(\delta_0^{(-)}, m_h) i\Pi_{(hh)}(L) + \frac{ia^2}{2} c_\rho^2 \, \mathcal{I}_{\rm even}(\delta_0^{(+)}, m_h) i\Pi_{(hh)}(L) \,, \\ i\mathcal{M}^{\phi_Z\phi_Z} &= \frac{ia^2}{2} s^2_{\frac{\rho}{2}} \, \mathcal{I}_{\rm even}(\delta_0^{(0)}, m_Z) i\Pi_{(\phi_Z\phi_Z)}(L) \,, \\ i\mathcal{M}^{\phi_W\phi_W} &= ia^2 \, \mathcal{I}_{\rm even}(\delta_0^{(+)}, m_W) i\Pi_{(\phi_W^+\phi_W^+)}(L) \,, \\ i\mathcal{M}^{Z\phi_Z} &= \frac{g_2 a}{4c_W} s_\rho \, \mathcal{I}_{\rm odd}(\delta_0^{(0)}, m_Z) v_\mu i\Pi^\mu_{(Z\phi_Z)}(L) \,, \\ i\mathcal{M}^{W\phi_W} &= \frac{ig_2 a}{2} s_\rho \, \mathcal{I}_{\rm odd}(\delta_0^{(+)}, m_W) v_\mu i\Pi^\mu_{(W^+\phi_W^+)}(L) \,. \end{aligned} \tag{109}$$

The rest of the analysis proceeds as above, using the same polarization tensors and integral reduction method. We check for both types of admixtures that the expected results are recovered upon taking the pure-case limits described in Sec. 4.3.

6.5 Effective theory amplitudes and infrared regulator

In the computation of both pure- and mixed-case amplitudes above, we have neglected subleading corrections of $\mathcal{O}(m_q/m_W)$ by Taylor expanding integrands about vanishing light quark masses.[10] This requires a regulator to control IR divergences (the full theory diagrams in Figs. 3 and 5 are UV finite but the projection onto the spin-2 operator $\mathcal{O}_g^{(2)}$ is IR divergent).

[10]For matching onto quark operators, we of course include the leading m_q factor appearing in $\mathcal{O}_q^{(0)}$ and $\mathcal{O}_q^{(2)}$. For matching onto gluon operators we may neglect light quark masses.

It is technically simplest to compute the full and effective theory amplitudes using dimensional regularization as IR regulator. Effective theory loop diagrams on the right hand sides of Figs. 3 and 5 then result in dimensionful but scaleless integrals that are required to vanish. Upon subtracting the effective theory amplitude, remaining $1/\epsilon$ pieces in matching coefficients are identified as UV divergences.

We have obtained identical renormalized matching coefficients by retaining light quark masses, $m_q \neq 0$, as an alternative IR regulator. In this scheme, the effective theory loop diagrams on the right-hand side of Figs. 3 and 5 yield nonvanishing contributions. The full theory diagrams on the left-hand side are correspondingly modified so that, upon subtracting the effective theory amplitude, consistent results are obtained.

7 Results for matching coefficients

We may now collect the results of the preceding analysis of quark and gluon matching to present the bare coefficients of the effective theory at the weak scale. We have analyzed the Wilson coefficients of the effective theory described by (2) in terms of contributions from exchanges of one or two electroweak bosons, as expressed in (45). The results for one-boson exchange matching to quark and gluon operators are given by (48) and (55), respectively. The results for two-boson exchange matching to quark and gluon operators are given by summing contributions of the form (57) and (97), respectively.

For pure cases, the results for the bare matching coefficients are as follows,

$$
\begin{aligned}
c_U^{(0)} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}}\Bigg\{ -\frac{m_W^{-3-2\epsilon}}{2x_h^2}\left[\mathcal{C}_W + \frac{\mathcal{C}_Z}{c_W^3}\right] + \frac{m_Z^{-3-2\epsilon}\mathcal{C}_Z}{8c_W^4}\left[c_V^{(U)2} - c_A^{(U)2}\right] + \mathcal{O}(\epsilon)\Bigg\}, \\
c_D^{(0)} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}}\Bigg\{ -\frac{m_W^{-3-2\epsilon}}{2x_h^2}\left[\mathcal{C}_W + \frac{\mathcal{C}_Z}{c_W^3}\right] + \frac{m_Z^{-3-2\epsilon}\mathcal{C}_Z}{8c_W^4}\left[c_V^{(D)2} - c_A^{(D)2}\right] \\
&\quad - \delta_{Db}\, m_W^{-3-2\epsilon}\mathcal{C}_W \frac{x_t}{8(x_t+1)^3} + \mathcal{O}(\epsilon)\Bigg\}, \\
c_g^{(0)} &= \frac{\pi[\Gamma(1+\epsilon)]^2 g_2^4 g^2}{(4\pi)^{4-2\epsilon}}\Bigg\{ \frac{m_W^{-3-4\epsilon}}{2}\left[\frac{1}{3x_h^2}\left[\mathcal{C}_W + \frac{\mathcal{C}_Z}{c_W^3}\right] + \mathcal{C}_W\left[\frac{1}{3} + \frac{1}{6(x_t+1)^2}\right]\right] \\
&\quad + \frac{m_Z^{-3-4\epsilon}\mathcal{C}_Z}{64c_W^4}\Bigg[4\left[c_V^{(D)2} + c_A^{(D)2}\right] + \left[c_V^{(U)2} + c_A^{(U)2}\right]\Bigg[\frac{8}{3} + \frac{32y_t^6(8y_t^2-7)}{(4y_t^2-1)^{7/2}}\arctan\left(\sqrt{4y_t^2-1}\right) \\
&\quad - \pi y_t + \frac{4(48y_t^6 - 2y_t^4 + 9y_t^2 - 1)}{3(4y_t^2-1)^3}\Bigg] + \left[c_V^{(U)2} - c_A^{(U)2}\right]\Bigg[3\pi y_t - \frac{4(144y_t^6 - 70y_t^4 + 9y_t^2 - 2)}{3(4y_t^2-1)^3} \\
&\quad - \frac{32y_t^4(24y_t^4 - 21y_t^2 + 5)}{(4y_t^2-1)^{7/2}}\arctan\left(\sqrt{4y_t^2-1}\right)\Bigg]\Bigg] + \mathcal{O}(\epsilon)\Bigg\}, \\
c_U^{(2)} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}}\Bigg\{\left[m_W^{-3-2\epsilon}\mathcal{C}_W + \frac{m_Z^{-3-2\epsilon}\mathcal{C}_Z}{2c_W^4}\left[c_V^{(U)2} + c_A^{(U)2}\right]\right]\left[\frac{1}{3} + \left(\frac{11}{9} - \frac{2}{3}\log 2\right)\epsilon\right] + \mathcal{O}(\epsilon^2)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
c_D^{(2)} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}}\Bigg\{\Bigg[m_W^{-3-2\epsilon}\mathcal{C}_W + \frac{m_Z^{-3-2\epsilon}\mathcal{C}_Z}{2c_W^4}\big[c_V^{(D)2}+c_A^{(D)2}\big]\Bigg]\Bigg[\frac{1}{3}+\left(\frac{11}{9}-\frac{2}{3}\log 2\right)\epsilon\Bigg] \\
&\quad + \delta_{Db}\,\frac{m_W^{-3-2\epsilon}\mathcal{C}_W}{2}\Bigg[\frac{(3x_t+2)}{3(x_t+1)^3}-\frac{2}{3}+\bigg(\frac{2x_t(7x_t^2-3)}{3(x_t^2-1)^3}\log x_t - \frac{2(3x_t+2)}{3(x_t+1)^3}\log 2 \\
&\quad - \frac{2(25x_t^2-2x_t-11)}{9(x_t^2-1)^2(x_t+1)}-\frac{22}{9}+\frac{4}{3}\log 2\bigg)\epsilon\Bigg]+\mathcal{O}(\epsilon^2)\Bigg\}\,, \\
c_g^{(2)} &= \frac{\pi[\Gamma(1+\epsilon)]^2 g_2^4 g^2}{(4\pi)^{4-2\epsilon}}\Bigg\{\frac{m_W^{-3-4\epsilon}\mathcal{C}_W}{2}\Bigg[-\frac{16}{9\epsilon}-\frac{284}{27}+\frac{32}{9}\log 2-\frac{2(3x_t+2)}{9(x_t+1)^3}\frac{1}{\epsilon} \\
&\quad + \frac{8(6x_t^8-18x_t^6+21x_t^4-3x_t^2-2)}{9(x_t^2-1)^3}\log(x_t+1)+\frac{4(3x_t^4-21x_t^3+3x_t^2+9x_t-2)}{9(x_t^2-1)^3}\log 2 \\
&\quad - \frac{4(12x_t^8-36x_t^6+39x_t^4+14x_t^3-9x_t^2-6x_t-2)}{9(x_t^2-1)^3}\log x_t \\
&\quad - \frac{144x_t^6+72x_t^5-312x_t^4-105x_t^3-40x_t^2+47x_t+98}{27(x_t^2-1)^2(x_t+1)}\Bigg] \\
&\quad + \frac{m_Z^{-3-4\epsilon}\mathcal{C}_Z}{64c_W^4}\Bigg[\Big[8\big[c_V^{(U)2}+c_A^{(U)2}\big]+12\big[c_V^{(D)2}+c_A^{(D)2}\big]\Big]\Big[-\frac{16}{9\epsilon}-\frac{284}{27}+\frac{32}{9}\log 2\Big] \\
&\quad + \big[c_V^{(U)2}+c_A^{(U)2}\big]\Bigg[\frac{128(24y_t^8-21y_t^6-4y_t^4+5y_t^2-1)}{9(4y_t^2-1)^{7/2}}\arctan\left(\sqrt{4y_t^2-1}\right)-\frac{4\pi y_t}{3} \\
&\quad + \frac{16(48y_t^6+62y_t^4-47y_t^2+9)}{9(4y_t^2-1)^3}\Bigg]+\big[c_V^{(U)2}-c_A^{(U)2}\big]\Bigg[\frac{16y_t^2(624y_t^4-538y_t^2+103)}{9(4y_t^2-1)^3}-\frac{52\pi y_t}{3} \\
&\quad + \frac{128y_t^2(104y_t^6-91y_t^4+35y_t^2-5)}{3(4y_t^2-1)^{7/2}}\arctan\left(\sqrt{4y_t^2-1}\right)\Bigg]\Bigg]+\mathcal{O}(\epsilon)\Bigg\}\,,
\end{aligned}
\tag{110}
$$

where, as before, $x_t = m_t/m_W$ and $y_t = m_t/m_Z$. Above, the Kronecker delta, δ_{Db}, is equal to unity for $D = b$, and vanishes for $D = d, s$. Note that beyond the specification of the WIMP quantum numbers J and Y (in $\mathcal{C}_W$ and $\mathcal{C}_Z$), the pure-state matching coefficients are completely given by SM parameters in the heavy WIMP limit. The pure triplet (doublet) results are given by setting $\mathcal{C}_W = 2$ and $\mathcal{C}_Z = 0$ ($\mathcal{C}_W = 1/2$ and $\mathcal{C}_Z = 1/4$). The renormalization of the theory involving these bare coefficients will be detailed in a forthcoming paper [34]. In particular, the relation between the bare coefficient $c_g^{(2)}$ given above and the renormalized coefficient $c_g^{(2)}(\mu)$ involves a nontrivial subtraction requiring the $\mathcal{O}(\epsilon)$ part of $c_q^{(2)}$ which we have retained.

The results for admixtures are similarly obtained by collecting contributions to the coefficients specified in (45). For example, the amplitudes in (51) for a singlet-doublet admixture, combined with the integrals defined in Appendix B, specify $c_q^{(0)}{}_{\rm 1BE}$ through (48), and $c_g^{(0)}{}_{\rm 1BE}$ through (55). The coefficients $c_q^{(S)}{}_{\rm 2BE}$ are specified in (63) in terms of the results in (61), which require the integrals in Appendix C. Finally, $c_g^{(S)}{}_{\rm 2BE}$ is specified in (108) in terms of the amplitudes in (107) which require the polarization tensors in (66), the basis reduction in (91), and the integrals in Appendix D.

The matching coefficients for admixtures are functions of the mass splitting between pure-state

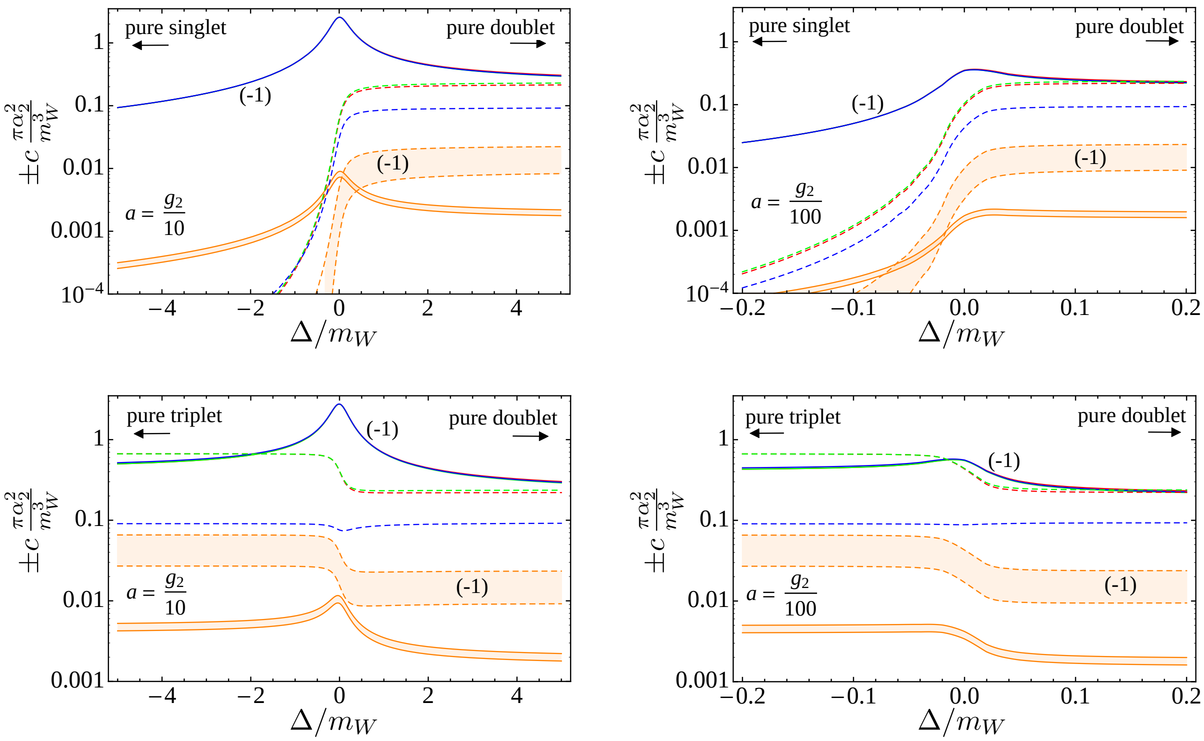


Figure 6: *Renormalized* coefficients (with $\pi\alpha_2^2/m_W^3$ extracted) for the singlet-doublet (upper panels) and triplet-doublet (lower panels) mixtures as a function of the respective mass splittings $\Delta = (M_S - M_D)/2$ and $\Delta = (M_T - M_D)/2$, in units of m_W. The panels on the left (right) use $a = g_2/10$ ($a = g_2/100$). The negative coefficients $c_q^{(0)}$ and $c_g^{(2)}$ are presented with opposite sign, as indicated by (-1). The solid red, green, and blue lines are respectively for $-c_{U=u,c}^{(0)}$, $-c_{D=d,s}^{(0)}$, and $-c_b^{(0)}$. The dashed red, green, and blue lines are respectively for $c_{U=u,c}^{(2)}$, $c_{D=d,s}^{(2)}$, and $c_b^{(2)}$. Some quark matching coefficients appear degenerate. The orange band with solid borders is $c_g^{(0)}$, and the orange band with dashed borders is $-c_g^{(2)}$. The band thickness represents renormalization scale variation, taking $m_W^2/2 < \mu_t^2 < 2m_t^2$ [6]. We indicate the pure-case limits at large $|\Delta|$.

components, Δ, and their coupling strength mediated by the Higgs field, a, as defined in (20) for the singlet-doublet mixture. We illustrate numerical values in Fig. 6 for both the singlet-doublet and triplet-doublet mixtures. Numerical inputs are collected in Table 1 of Appendix E. Depending on the value of a, the $\mathcal{O}(\alpha_2^1)$ tree-level Higgs exchange contribution to the spin-0 coefficients may dominate near $\Delta = 0$. However the $h\bar{\chi}\chi$ coupling may be suppressed by m_W/Δ, and thus for $\Delta \gg m_W$, or more precisely $\Delta \gg m_W(4\pi a/g_2)^2$, the $\mathcal{O}(\alpha_2^2)$ loop contributions dominate. The curves approach the correct pure-case values upon taking the limits described in Sec. 4.3. In particular, the coefficients vanish in the pure singlet limit.

The contributions of these coefficients to scattering cross sections depend on the detailed mapping onto the low-energy $n_f = 3$ flavor theory through renormalization group running and heavy quark threshold matching, and on the evaluation of nucleon matrix elements at a low scale, $\mu \sim 1\,\text{GeV}$. These effects enhance the contribution from certain coefficients, upsetting the α_s counting reflected in the relative magnitudes of the high-scale coefficients. One example is the enhancement of the spin-0 gluon contribution due both to a large anomalous dimension in the RG running, and to the large nucleon matrix element of the scalar gluon operator [43]. Another example is the enhanced impact of numerically subleading contributions due to a partial cancellation at leading order. The relative signs between high-scale coefficients in Fig. 6, combined with details of the mapping onto low-energy coefficients and evaluation of matrix elements, lead to a cancellation between the spin-0 and spin-2 amplitude contributions [5, 6]. Therefore, a robust determination of DM-nucleon scattering cross sections demands a careful analysis of the complete set of leading operators in (3).

The coefficient $c_g^{(2)}$ has been omitted in previous works [3, 5]. Due to a cancellation between spin-0 and spin-2 amplitude contributions to cross sections, the effect of neglecting $c_g^{(2)}$ ranges from a factor of a few to an order of magnitude difference in cross sections. For the pure-doublet and pure-triplet states, neglecting $c_g^{(2)}$ leads to an $\mathcal{O}(10-20\%)$ shift in the spin-2 amplitude, depending on the choice of renormalization scale, and an underestimation of its perturbative uncertainty by $\mathcal{O}(70\%)$. For comparison, neglecting $c_q^{(2)}$ for $q = b, c, s, d, u$ shifts the spin-2 amplitude by $\mathcal{O}(1\%)$, $\mathcal{O}(10\%)$, $\mathcal{O}(10\%)$, $\mathcal{O}(30\%)$, and $\mathcal{O}(50\%)$, respectively.

8 Summary

The present analysis focused on weak-scale matching conditions necessary for robustly computing WIMP-nucleon interactions, both in specified UV completions involving electroweak-charged DM, and in the model-independent heavy WIMP limit. Careful computation of competing Standard Model contributions is necessary to estimate the correct order of magnitude of scattering cross sections in many simple and motivated models of DM. For example, a simple dimensional estimate of the cross section for spin-independent, low-velocity scattering of a pure-state WIMP on a nucleon yields[11]

$$\sigma_{\rm SI} \sim \frac{\alpha_2^4 m_N^4}{m_W^2}\left(\frac{1}{m_W^2}, \frac{1}{m_h^2}\right)^2 \sim 10^{-45}\,{\rm cm}^2\,. \tag{111}$$

Cross sections of this order of magnitude are currently being probed by direct detection searches (e.g., see Refs. [22] for detection prospects computed using tree-level cross sections). However, a cancellation between spin-0 and spin-2 amplitude contributions leads to much smaller cross section values for motivated candidates such as the pure wino ($\sigma_{\rm SI} \sim 10^{-47}\,{\rm cm}^2$) and the pure higgsino ($\sigma_{\rm SI} \lesssim 10^{-48}\,{\rm cm}^2$) of supersymmetric SM extensions. This cancellation demands a careful analysis of perturbative contributions from weak-scale matching amplitudes presented here, e.g., the inclusion of the spin-2 gluon contribution, and of remaining theoretical and input uncertainties, which will be discussed in a companion paper [34]. Robust predictions for the cross sections of the pure triplet, pure doublet, singlet-doublet admixture, and triplet-doublet admixture can be found in Refs. [6]. Given the matching coefficients in (110), the cross sections for pure states with arbitrary electroweak quantum numbers can also be computed.

Although we find that cancellations are generic, their severity depends on SM parameters and on properties of DM such as its electroweak quantum numbers. The presence of additional low-lying states could also have impact, and the formalism for weak-scale matching presented here can be readily extended to investigate such scenarios. For example, including a second Higgs doublet in the pure-state analysis simply requires modification of the vertices in the amplitudes computed in Figs. 2 and 3. An extra Higgs boson modifies the spin-0 amplitude, and could potentially weaken the cancellation between spin-0 and spin-2 amplitudes. The case where the second Higgs-like doublet

[11] Cross sections of this magnitude were obtained in previous estimates that missed the cancellation between spin-0 and spin-2 amplitude contributions (and ignored gluon contributions) [2].

itself plays the role of DM (e.g., "inert Higgs DM" [7]) is related to the pure-doublet case in the heavy WIMP limit by heavy particle universality.

While we have focused here on the case of a heavy, self-conjugate WIMP, deriving from one or two electroweak multiplets, much of the formalism applies more generally. The construction of the heavy particle effective theories in Sec. 4 could be straightforwardly extended to include power corrections, and other light states within the context of specific UV completions. The generalized electroweak polarization tensors obtained through background field techniques depend only on SM parameters, and hence can be applied for gluon operator matching in general DM scenarios. Within the context of heavy particle effective theories, the new integral basis evaluated here may be applied to other processes such as low-energy lepton-nucleon scattering [38].

Separating improvable SM uncertainties from DM model dependence demands precise theoretical formalism. The focus of the present paper is on the systematic treatment of weak-scale matching calculations, with particular attention paid to the heavy WIMP limit. The remaining analysis below the weak scale may be applied to a broader class of theories, and is the subject of a forthcoming paper [34]. There, we present the necessary ingredients to systematically map high-scale matching coefficients onto the low-energy theory ($n_f = 3$ or $n_f = 4$ flavor QCD plus interactions with DM) where hadronic matrix elements are evaluated.

Acknowledgments

We acknowledge useful discussions with C.E.M. Wagner. M.S. also thanks T. Cohen and J. Kearney for useful discussions. This work was supported by the United States Department of Energy under Grant No. DE-FG02-13ER41958. M.S. is supported by a Bloomenthal Fellowship.

A The singlet-doublet mixture

The heavy-particle lagrangians in Sec. 4 may be obtained from a manifestly relativistic lagrangian by performing field redefinitions at tree level. Consider the case of a singlet-doublet mixture (see also [19]),

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \frac{1}{2}\bar{b}(i\partial\!\!\!/ - M_1)b + \bar{\psi}(iD\!\!\!\!/ - M_2)\psi - (y\,\bar{b}P_L H^\dagger \psi + y'\,\bar{b}P_L H^T \psi^c + \text{h.c.})\,, \tag{112}$$

where b is a gauge singlet (Majorana) fermion represented as a Dirac spinor with $b^c = b$, and ψ is a Dirac fermion in the $(\mathbf{2}, 1/2)$ representation of $SU(2)_W \times U(1)_Y$. In the above equation, $P_{\rm R,L} = (1 \pm \gamma_5)/2$, and we have included all renormalizable gauge-invariant interactions involving the SM Higgs field. Expressing the result in terms of Majorana combinations,

$$\lambda_1 = \frac{1}{\sqrt{2}}\left(\psi + \psi^c\right)\,, \quad \lambda_2 = \frac{i}{\sqrt{2}}\left(\psi - \psi^c\right)\,, \tag{113}$$

and collecting the fermions in the column vector $\lambda = (b, \lambda_1, \lambda_2)$, we may write the interactions with the Higgs field as

$$\begin{aligned}\mathcal{L}_{H\bar{\lambda}\lambda} &= -\frac{1}{\sqrt{2}}\bar{b}\frac{1-\gamma_5}{2}\Big[(yH^\dagger + y'H^T)\lambda_1 - i(yH^\dagger - y'H^T)\lambda_2\Big] + \text{h.c.} \\ &\equiv -\frac{1}{2}\bar{\lambda}\Big[f(H) + i\gamma_5 g(H)\Big]\lambda\,,\end{aligned} \tag{114}$$

with

$$f(H) = \frac{a_1}{\sqrt{2}} \begin{pmatrix} 0 & H^\dagger + H^T & i(H^T - H^\dagger) \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \\ i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} + \frac{a_2}{\sqrt{2}} \begin{pmatrix} 0 & -i(H^T - H^\dagger) & H^T + H^\dagger \\ -i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix},$$

$$g(H) = \frac{b_1}{\sqrt{2}} \begin{pmatrix} 0 & -i(H^T - H^\dagger) & H^T + H^\dagger \\ -i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} + \frac{b_2}{\sqrt{2}} \begin{pmatrix} 0 & H^\dagger + H^T & i(H^T - H^\dagger) \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \\ i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix}. \tag{115}$$

The real parameters a_i and b_i are given by

$$a_1 = \frac{1}{2}\text{Re}(y + y'), \quad a_2 = \frac{1}{2}\text{Im}(y - y'), \quad b_1 = \frac{1}{2}\text{Re}(y - y'), \quad b_2 = -\frac{1}{2}\text{Im}(y + y'). \tag{116}$$

We employ phase redefinitions of b, ψ_L and ψ_R to ensure that M_1 and M_2 are real and positive.[12] The gauge generators will be those given in (13), extended trivially to include the singlet. Upon performing the tree-level field redefinition

$$\lambda = \sqrt{2}e^{-i(M-\delta M)v\cdot x}(h_v + H_v), \tag{117}$$

where the fields h_v and H_v obey $\not{v}\, h_v = h_v$ and $\not{v}\, H_v = -H_v$, we obtain the heavy-particle lagrangian in (4). It follows from $\lambda^c = \lambda$ that the resulting lagrangian is invariant under the simultaneous transformations in (1). Note that $f(H)$ is the only term surviving the projection from the condition $\not{v}\, h_v = h_v$. The remaining analysis follows that of Sec. 4.2.1.

B Self energy integrals and Standard Model two-point functions

Here and in the following sections we use the notation

$$[c_\epsilon] = \frac{i\Gamma(1+\epsilon)}{(4\pi)^{2-\epsilon}}, \quad (dL) = \frac{d^dL}{(2\pi)^d}. \tag{118}$$

The self-energies in Sec. 5 and the $h\bar{\chi}\chi$ three-point functions in Sec. 6.1 require the following integrals,

$$\begin{aligned} I_1(\delta, m) &= \int (dL) \frac{1}{v\cdot L - \delta + i0} \frac{1}{(L^2 - m^2 + i0)^2} \\ &= \frac{\partial}{\partial m^2} I_3(\delta, m) \\ &= [c_\epsilon] m^{-2\epsilon} \left\{ \frac{2}{\sqrt{m^2 - \delta^2 - i0}} \left[\arctan\left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) - \frac{\pi}{2} \right] + \mathcal{O}(\epsilon) \right\}, \\ I_2(\delta, m) &= \int (dL) v\cdot L \frac{1}{v\cdot L - \delta + i0} \frac{1}{(L^2 - m^2 + i0)^2} \\ &= \delta I_1(\delta, m) + \frac{i}{(4\pi)^2} B_0(0, m, m) \end{aligned}$$

[12] An additional phase redefinition could be used to eliminate a_1, a_2, b_1 or b_2.

$$
\begin{aligned}
&= [c_\epsilon] m^{-2\epsilon} \left\{ \frac{1}{\epsilon} + \frac{2\delta}{\sqrt{m^2-\delta^2-i0}} \left[\arctan\left(\frac{\delta}{\sqrt{m^2-\delta^2-i0}} \right) - \frac{\pi}{2} \right] + \mathcal{O}(\epsilon) \right\}, \\
I_3(\delta, m) &= \int (dL) \frac{1}{v\cdot L - \delta + i0} \frac{1}{L^2 - m^2 + i0} \\
&= [c_\epsilon] m^{-2\epsilon} \left\{ -\frac{2\delta}{\epsilon} + 4\sqrt{m^2-\delta^2-i0} \left[\arctan\left(\frac{\delta}{\sqrt{m^2-\delta^2-i0}} \right) - \frac{\pi}{2} \right] - 4\delta + \mathcal{O}(\epsilon) \right\}, \\
I_4(\delta_1, \delta_2, m) &= \int (dL) \frac{1}{v\cdot L - \delta_1 + i0} \frac{1}{v\cdot L - \delta_2 + i0} \frac{1}{L^2 - m^2 + i0}\,. \qquad (119)
\end{aligned}
$$

For $I_4(\delta_1, \delta_2, m)$, let us specialize to $\delta_2 = 0$ or $\delta_1 = \delta_2$,

$$
\begin{aligned}
I_4(\delta, 0, m) &= \frac{1}{\delta} \left[I_3(\delta, m) - I_3(0, m) \right] \\
&= [c_\epsilon] m^{-2\epsilon} \left\{ -\frac{2}{\epsilon} + \frac{4\sqrt{m^2-\delta^2-i0}}{\delta} \left[\arctan\left(\frac{\delta}{\sqrt{m^2-\delta^2-i0}} \right) - \frac{\pi}{2} \right] \right. \\
&\quad \left. - 4 + \frac{2\pi m}{\delta} + \mathcal{O}(\epsilon) \right\}, \\
I_4(\delta, \delta, m) &= \frac{\partial}{\partial \delta} I_3(\delta, m) \\
&= [c_\epsilon] m^{-2\epsilon} \left\{ -\frac{2}{\epsilon} - \frac{4\delta}{\sqrt{m^2-\delta^2-i0}} \left[\arctan\left(\frac{\delta}{\sqrt{m^2-\delta^2-i0}} \right) - \frac{\pi}{2} \right] + \mathcal{O}(\epsilon) \right\}. \qquad (120)
\end{aligned}
$$

The two-point functions for the electroweak SM bosons appearing in (53) are obtained by summing the fermionic and bosonic contributions given below. Following Denner [40], we have

$$
\begin{aligned}
\Sigma^{AA\prime}(0) &= -\frac{\alpha}{4\pi} \left\{ 3B_0(0, m_W, m_W) + 4m_W^2 B_0'(0, m_W, m_W) \right. \\
&\quad \left. - \frac{4}{3} \sum_{f,i} \left[N_c^f Q_f^2 B_0(0, m_{f,i}, m_{f,i}) \right] \right\}, \\
\frac{\Sigma^{AZ}(0)}{m_Z^2} &= -\frac{\alpha}{4\pi} \left\{ -\frac{2c_W}{s_W} B_0(0, m_W, m_W) \right\}, \\
\frac{\Sigma^{ZZ}(m_Z^2)_{\text{fermion}}}{m_Z^2} &= -\frac{\alpha}{4\pi} \left\{ \frac{2}{3} \left[-B_0(m_Z, 0, 0) + \frac{1}{3} \right] \sum_{f,i} N_c^f [(g_f^+)^2 + (g_f^-)^2] \right. \\
&\quad + \frac{2}{3} N_c^t \left[[(g_t^+)^2 + (g_t^-)^2] \left[-\left(1 + \frac{2m_t^2}{m_Z^2} \right) B_0(m_Z, m_t, m_t) + B_0(m_Z, 0, 0) \right. \right. \\
&\quad \left. \left. \left. + \frac{2m_t^2}{m_Z^2} B_0(0, m_t, m_t) \right] + \frac{3}{4s_W^2 c_W^2} \frac{m_t^2}{m_Z^2} B_0(m_Z, m_t, m_t) \right] \right\}, \\
\frac{\Sigma^{ZZ}(m_Z^2)_{\text{boson}}}{m_Z^2} &= -\frac{\alpha}{4\pi} \frac{1}{s_W^2 c_W^2} \left\{ \frac{1}{12} (4c_W^2 - 1)(12c_W^4 + 20c_W^2 + 1) B_0(m_Z, m_W, m_W) \right. \\
&\quad - \frac{1}{3} c_W^2 (12c_W^4 - 4c_W^2 + 1) B_0(0, m_W, m_W) - \frac{1}{6} B_0(0, m_Z, m_Z)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{12}\left(\frac{m_h^4}{m_Z^4} - 4\frac{m_h^2}{m_Z^2} + 12\right) B_0(m_Z, m_Z, m_h) - \frac{1}{6}\frac{m_h^2}{m_Z^2} B_0(0, m_h, m_h) \\
&+ \frac{1}{12}\left(1 - \frac{m_h^2}{m_Z^2}\right)^2 B_0(0, m_Z, m_h) - \frac{1}{9}(1 - 2c_W^2)\bigg\}\,, \\
\frac{\Sigma^{WW}(m_W^2)_{\rm fermion}}{m_W^2} &= -\frac{\alpha}{4\pi}\frac{1}{2s_W^2}\bigg\{\frac{2}{3}\left[\frac{1}{3} - B_0(m_W, 0, 0)\right]\sum_{f,i}\frac{N_c^f}{2} \\
&+ \frac{2}{3}N_c^t\bigg[\frac{1}{2}\left(\frac{m_t^4}{m_W^4} + \frac{m_t^2}{m_W^2} - 2\right) B_0(m_W, m_t, 0) + B_0(m_W, 0, 0) \\
&+ \frac{m_t^2}{m_W^2} B_0(0, m_t, m_t) - \frac{m_t^4}{2m_W^4} B_0(0, m_t, 0)\bigg]\bigg\}\,, \\
\frac{\Sigma^{WW}(m_W^2)_{\rm boson}}{m_W^2} &= -\frac{\alpha}{4\pi}\bigg\{4B_0(m_W, m_W, \lambda) - \frac{4}{3}B_0(0, m_W, m_W) + \frac{2}{3}B_0(0, m_W, \lambda) + \frac{2}{9} \\
&+ \frac{1}{12s_W^2}\bigg[\frac{1}{c_W^4}(4c_W^2 - 1)(12c_W^4 + 20c_W^2 + 1)B_0(m_W, m_W, m_Z) \\
&- 2(8c_W^2 + 1)B_0(0, m_W, m_W) - \frac{2}{c_W^2}(8c_W^2 + 1)B_0(0, m_Z, m_Z) \\
&+ \frac{s_W^4}{c_W^4}(8c_W^2 + 1)B_0(0, m_W, m_Z) - \frac{2}{3}(1 - 4c_W^2)\bigg] \\
&+ \frac{1}{12s_W^2}\bigg[-\left(\frac{m_h^4}{m_W^4} - 4\frac{m_h^2}{m_W^2} + 12\right) B_0(m_W, m_W, m_h) - 2B_0(0, m_W, m_W) \\
&- 2\frac{m_h^2}{m_W^2} B_0(0, m_h, m_h) + \left(1 - \frac{m_h^2}{m_W^2}\right)^2 B_0(0, m_W, m_h) - \frac{2}{3}\bigg]\bigg\}\,, \\
\Sigma^{HH\prime}(m_h^2)_{\rm fermion} &= -\frac{\alpha}{4\pi}\frac{3m_t^2}{2s_W^2 m_W^2}\left[(4m_t^2 - m_h^2)B_0'(m_h, m_t, m_t) - B_0(m_h, m_t, m_t)\right], \\
\Sigma^{HH\prime}(m_h^2)_{\rm boson} &= -\frac{\alpha}{4\pi}\bigg\{-\frac{1}{2s_W^2}\bigg[\left(6m_W^2 - 2m_h^2 + \frac{m_h^4}{2m_W^2}\right) B_0'(m_h, m_W, m_W) \\
&- 2B_0(m_h, m_W, m_W)\bigg] - \frac{1}{4s_W^2 c_W^2}\bigg[\left(6m_Z^2 - 2m_h^2 + \frac{m_h^4}{2m_Z^2}\right) B_0'(m_h, m_Z, m_Z) \\
&- 2B_0(m_h, m_Z, m_Z)\bigg] - \frac{9m_h^4}{8s_W^2 m_W^2} B_0'(m_h, m_h, m_h)\bigg\}\,, \qquad (121)
\end{aligned}
$$

where the sums over indices f and i are for SM fermion flavors and generations, respectively. Above, N_c^f and Q_f respectively denote the number of colors and the electric charge of fermion f. We have also used

$$
\alpha = \frac{g_2^2 s_W^2}{4\pi}\,, \quad g_f^+ = \frac{1}{8s_W^2 c_W^2}\left[c_V^{(f)2} + c_A^{(f)2}\right], \quad g_f^- = \frac{1}{8s_W^2 c_W^2}\left[c_V^{(f)2} - c_A^{(f)2}\right], \qquad (122)
$$

where

$$c_V^{(\ell)} = -1 + 4s_W^2\,, \quad c_A^{(\ell)} = 1\,, \quad c_V^{(\nu)} = 1\,, \quad c_A^{(\nu)} = -1\,, \tag{123}$$

with ℓ and ν denoting charged lepton and neutrino, respectively. The coefficients $c_V^{(f)}$ and $c_A^{(f)}$ for quarks can be found in (46). The basic integral appearing above is

$$\begin{aligned}\frac{i}{(4\pi)^2}B_0(M,m_0,m_1) &= \int (dL)\frac{1}{L^2-m_0^2+i0}\frac{1}{(L+p)^2-m_1^2+i0} \\ &= [c_\epsilon]\left[\frac{1}{\epsilon}+2-\log(m_0m_1)+\frac{m_0^2-m_1^2}{M^2}\log\frac{m_1}{m_0}-\frac{m_0m_1}{M^2}\left(\frac{1}{r}-r\right)\log r+\mathcal{O}(\epsilon)\right],\end{aligned} \tag{124}$$

where $p^2 = M^2$ and

$$r = X + \sqrt{X^2-1}\,, \quad \frac{1}{r} = X - \sqrt{X^2-1}\,, \quad X = \frac{m_0^2+m_1^2-M^2-i0}{2m_0m_1}\,. \tag{125}$$

We find the following limits,

$$\begin{aligned}
B_0(0,m,m) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon}-2\log m+\mathcal{O}(\epsilon)\right], \\
B_0(0,m,0) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon}-2\log m+1+\mathcal{O}(\epsilon)\right], \\
B_0(0,m_0,m_1) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon}-\frac{m_0^2}{m_0^2-m_1^2}\log m_0^2+\frac{m_1^2}{m_0^2-m_1^2}\log m_1^2+1+\mathcal{O}(\epsilon)\right], \\
B_0(M,m,0) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon}+2-\frac{m^2}{M^2}\log m^2+\frac{m^2-M^2}{M^2}\log(m^2-M^2-i0)+\mathcal{O}(\epsilon)\right], \\
B_0(M,0,0) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon}+2-\log(-M^2-i0)+\mathcal{O}(\epsilon)\right], \\
\lim_{\lambda\to 0}B_0(m,m,\lambda) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{\epsilon}+2-\log m^2+\mathcal{O}(\epsilon)\right].
\end{aligned} \tag{126}$$

In the present application, only the real parts of the integrals are relevant. For the derivative of the integral we have,

$$\begin{aligned}B_0'(M,m,m) &\equiv \frac{\partial}{\partial p^2}B_0(M,m,m) \\ &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{m^2}{M^4}\left(\frac{1}{r}-r\right)\log r-\frac{1}{M^2}\left(1+\frac{r^2+1}{r^2-1}\log r\right)+\mathcal{O}(\epsilon)\right],\end{aligned} \tag{127}$$

which has the following limits,

$$\begin{aligned}
B_0'(0,m,m) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[\frac{1}{6m^2}+\mathcal{O}(\epsilon)\right], \\
B_0'(M,0,0) &= (4\pi)^\epsilon\Gamma(1+\epsilon)\left[-\frac{1}{M^2}+\mathcal{O}(\epsilon)\right].
\end{aligned} \tag{128}$$

C Box integrals

The integrals required for the two-boson exchange amplitudes in Sec. 6.3 may be written in terms of the integral operators $\mathcal{I}_{\rm even}$ and $\mathcal{I}_{\rm odd}$ defined in (85) as

$$\begin{aligned}
J(m_V,M,\delta) &= \mathcal{I}_{\rm even}(\delta,m_V)\frac{1}{L^2-M^2+i0}\,,\\
J^\mu(p,m_V,M,\delta) &= \mathcal{I}_{\rm even}(\delta,m_V)\frac{1}{L^2+2L\cdot p-M^2+i0}L^\mu\\
&= v\cdot p v^\mu J_1(m_V,M,\delta)+p^\mu J_2(m_V,M,\delta)+\mathcal{O}(p^3)\,,\\
J_-(p,m_V,M,\delta) &= -\mathcal{I}_{\rm odd}(\delta,m_V)\frac{1}{L^2+2L\cdot p-M^2+i0} = v\cdot p J_-(m_V,M,\delta)+\mathcal{O}(p^3)\,,\\
J_-^\mu(m_V,M,\delta) &= -\mathcal{I}_{\rm odd}(\delta,m_V)\frac{1}{L^2-M^2+i0}L^\mu = v^\mu J_{1-}(m_V,M,\delta)\,.
\end{aligned} \tag{129}$$

Note that $J^\mu(p,m_V,M,\delta)$ and $J_-(p,m_V,M,\delta)$ vanish when p^μ vanishes since the integrands are then odd in L^μ. By standard manipulations, we may express the integrals J_1, J_2, J, and J_-, as

$$\begin{aligned}
J_1(m_V,M,\delta) &= -8[c_\epsilon](1+\epsilon)\frac{\partial}{\partial m_V^2}\int_0^\infty d\rho\int_0^1 dx\rho^2(1-x)\\
&\quad \big[xm_V^2+(1-x)M^2+\rho^2+2\rho\delta-i0\big]^{-2-\epsilon}\,,\\
J_2(m_V,M,\delta) &= 4[c_\epsilon]\frac{\partial}{\partial m_V^2}\int_0^\infty d\rho\int_0^1 dx(1-x)\big[xm_V^2+(1-x)M^2+\rho^2+2\rho\delta-i0\big]^{-1-\epsilon}\,,\\
J(m_V,M,\delta) &= -4[c_\epsilon]\frac{\partial}{\partial m_V^2}\int_0^\infty d\rho\int_0^1 dx\big[xm_V^2+(1-x)M^2+\rho^2+2\rho\delta-i0\big]^{-1-\epsilon}\,,\\
J_-(m_V,M,\delta) &= 4[c_\epsilon]\frac{\partial}{\partial\delta}\frac{\partial}{\partial m_V^2}\int_0^\infty d\rho\int_0^1 dx(1-x)\big[xm_V^2+(1-x)M^2+\rho^2+2\rho\delta-i0\big]^{-1-\epsilon}\,.
\end{aligned} \tag{130}$$

Let us introduce the integral

$$\hat{J}(m_V,M,\delta) = [c_\epsilon]\int_0^\infty d\rho\int_0^1 dx(1-x)\big[xm_V^2+(1-x)M^2+\rho^2+2\rho\delta-i0\big]^{-1-\epsilon}\,, \tag{131}$$

and write the above integrals in terms of $\hat{J}(m_V,M,\delta)$ as

$$\begin{aligned}
J_2(m_V,M,\delta) &= 4\frac{\partial}{\partial m_V^2}\hat{J}(m_V,M,\delta)\,,\\
J_-(m_V,M,\delta) &= 4\frac{\partial}{\partial\delta}\frac{\partial}{\partial m_V^2}\hat{J}(m_V,M,\delta)\,,\\
J(m_V,M,\delta) &= -4\frac{\partial}{\partial m_V^2}\big[\hat{J}(m_V,M,\delta)+\hat{J}(M,m_V,\delta)\big]\,,\\
J_1(m_V,M,\delta) &= 4\frac{\partial}{\partial m_V^2}\bigg[-\hat{J}(m_V,M,\delta)+\frac{\partial}{\partial A}\hat{J}(m_V,M,\delta/A)\big|_{A=1}\bigg]\,.
\end{aligned} \tag{132}$$

For J_{1-}, we may use the identity (89) to write

$$
\begin{aligned}
J_{1-}(m_V,M,\delta) &= -2\int (dL)\frac{1}{(L^2-m_V^2+i0)^2}\frac{1}{L^2-M^2+i0} - \delta J(m_V,M,\delta) \\
&= \frac{2[c_\epsilon]m_V^{-2-2\epsilon}}{\epsilon(1-\epsilon)}\left(1-\frac{M^2}{m_V^2}\right)^{-2}\left[\epsilon+\frac{M^2}{m_V^2}\left(1-\epsilon-\frac{m_V^{2\epsilon}}{M^{2\epsilon}}\right)\right] - \delta\, J(m_V,M,\delta)\,. \qquad (133)
\end{aligned}
$$

Having determined the above integrals in terms of $\hat{J}(m_V,M,\delta)$, it remains to compute this function. Let us write

$$
\begin{aligned}
\hat{J}(m_V,M,\delta) &= -\frac{[c_\epsilon]}{\epsilon}\frac{\partial}{\partial M^2}\int_0^\infty d\rho\int_0^1 dx\big[xm_V^2+(1-x)M^2+\rho^2+2\rho\delta-i0\big]^{-\epsilon} \\
&= -\frac{[c_\epsilon]}{\epsilon}\frac{\partial}{\partial M^2}\int_0^\infty d\rho\frac{1}{m_V^2-M^2}\frac{1}{1-\epsilon} \\
&\qquad \left\{[m_V^2+\rho^2+2\rho\delta-i0]^{1-\epsilon}-[M^2+\rho^2+2\rho\delta-i0]^{1-\epsilon}\right\} \\
&= -\frac{[c_\epsilon]}{\epsilon}\frac{\partial}{\partial M^2}\frac{1}{m_V^2-M^2}\frac{1}{1-\epsilon}\left\{m_V^{3-2\epsilon}f_1(\delta/m_V,1-\epsilon)-M^{3-2\epsilon}f_1(\delta/M,1-\epsilon)\right\}, \qquad (134)
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(\delta,a) &= \int_0^\infty d\rho(1+\rho^2+2\rho\delta-i0)^a \\
&= (1-\delta^2-i0)^{a+\frac{1}{2}}\frac{\sqrt{\pi}}{2}\frac{\Gamma(-a-\frac{1}{2})}{\Gamma(-a)} - \delta^{2a+1}\int_0^1 dx\,\left[\delta^{-2}-1+x^2-i0\right]^a\,. \qquad (135)
\end{aligned}
$$

Although for the present application we require only $\delta>0$, the expression is for general sign of δ. We presently need $f_1(\delta,a)$ for $a=1-\epsilon$, and hence consider

$$
\begin{aligned}
&\frac{\sqrt{\pi}}{2}\frac{\Gamma(-\frac{3}{2}+\epsilon)}{\Gamma(-1+\epsilon)} = -\frac{2\pi}{3}\epsilon+\frac{2\pi}{9}(6\log 2-5)\epsilon^2+\mathcal{O}(\epsilon^3)\,, \\
&\int_0^1 dx\,\left[\delta^{-2}-1+x^2-i0\right]^{1-\epsilon} = B^2+\frac{1}{3}+\epsilon\left\{\frac{2}{9}+\frac{4}{3}B^2-\frac{4}{3}B^3\,\text{arccot}\,B-\left(B^2+\frac{1}{3}\right)\log(B^2+1)\right\} \\
&\quad +\epsilon^2\left\{\frac{4}{27}+\frac{20}{9}B^2+\frac{4}{9}B^3(6\log 2B-5)\,\text{arccot}\,B+\frac{4}{3}B^3 i\left[\text{Li}_2\left(\frac{1+iB}{1-iB}\right)-\text{arccot}^2 B+\frac{\pi^2}{12}\right]\right. \\
&\quad \left.+\frac{1}{2}\left(B^2+\frac{1}{3}\right)\log^2(B^2+1)-\left(\frac{4}{3}B^2+\frac{2}{9}\right)\log(B^2+1)\right\}+\mathcal{O}(\epsilon^3)\,, \qquad (136)
\end{aligned}
$$

where $B^2=1/\delta^2-1-i0$. For $B^2>0$, the bracket involving dilogarithm may be written

$$
i\left[\text{Li}_2\left(\frac{1+iB}{1-iB}\right)-\text{arccot}^2 B+\frac{\pi^2}{12}\right] = -\text{Im}\,\text{Li}_2\left(\frac{1+iB}{1-iB}\right) = -\text{Cl}_2\left[\arccos\left(\frac{1-B^2}{1+B^2}\right)\right]\,, \qquad (137)
$$

where Cl_2 is the Clausen function of order two. The general expression is required for continuing to arbitrary mass parameters. Having determined $f_1(\delta,1-\epsilon)$, we may proceed to compute $\hat{J}(m_V,M,\delta)$

using (134), and then $J_2(m_V, M, \delta)$, $J(m_V, M, \delta)$, $J_-(m_V, M, \delta)$ and $J_1(m_V, M, \delta)$ using (132), and $J_{1-}(m_V, M, \delta)$ using (133).

For $M = 0$, the expressions in (130), the expressions for $J_2(m_V, M, \delta)$, $J_1(m_V, M, \delta)$, and $J_-(m_V, M, \delta)$ in (132), and the expression for $J_{1-}(m_V, M, \delta)$ in (133), remain valid. The integral $J(m_V, 0, \delta)$ is now given by

$$J(m_V, 0, \delta) = -4[c_\epsilon]\frac{\partial}{\partial m_V^2}\left\{ -\frac{1}{\epsilon} m_V^{-1-2\epsilon}\left[f_1(\delta/m_V, -\epsilon) - f_0(\delta/m_V, -\epsilon)\right]\right\}, \tag{138}$$

and the integral $\hat{J}(m_V, 0, \delta)$ by

$$\begin{aligned}\hat{J}(m_V, 0, \delta) = \frac{[c_\epsilon] m_V^{-2}}{\epsilon}\int_0^\infty d\rho \Big\{ & (\rho^2 + 2\rho\delta - i0)^{-\epsilon} \\ & - \frac{m_V^{-2}}{1-\epsilon}\Big[(m_V^2 + \rho^2 + 2\rho\delta - i0)^{1-\epsilon} - (\rho^2 + 2\rho\delta - i0)^{1-\epsilon}\Big]\Big\} \\ = \frac{[c_\epsilon] m_V^{-1-2\epsilon}}{\epsilon}\Big\{ & f_0(\delta/m_V, -\epsilon) - \frac{1}{(1-\epsilon)}\big[f_1(\delta/m_V, 1-\epsilon) - f_0(\delta/m_V, 1-\epsilon)\big]\Big\}, \end{aligned} \tag{139}$$

where $f_1(\delta, a)$ is given by (135) and

$$f_0(\delta, a) = \int_0^\infty d\rho (\rho^2 + 2\rho\delta - i0)^a = \frac{\delta^{1+2a}\Gamma(1+a)\Gamma\left(-a-\frac{1}{2}\right)}{2\sqrt{\pi}}. \tag{140}$$

We also need $f_1(\delta/m_V, a)$ for $a = -\epsilon$, which we may write as

$$f_1(\delta/m_V, -\epsilon) = \frac{1}{1-\epsilon} m_V^{-1+2\epsilon}\frac{\partial}{\partial m_V^2}\left[m_V^{3-2\epsilon} f_1(\delta/m_V, 1-\epsilon)\right]. \tag{141}$$

At vanishing residual mass, $\delta = 0$, only the integrals $J(m_V, M, 0)$, $J_1(m_V, M, 0)$ and $J_2(m_V, M, 0)$ are required, and from (130) they can be easily represented in closed form,

$$\begin{aligned} J(m_V, M, 0) &= [c_\epsilon]\frac{2\sqrt{\pi}}{(1-2\epsilon)}\frac{\Gamma(\frac{1}{2}+\epsilon)}{\Gamma(1+\epsilon)}\frac{m_V^{1-2\epsilon}}{(M^2 - m_V^2)^2}\left[1 + 2\epsilon - 2\left(\frac{M}{m_V}\right)^{1-2\epsilon} + (1-2\epsilon)\left(\frac{M}{m_V}\right)^2\right], \\ J_2(m_V, M, 0) &= -J_1(m_V, M, 0) = [c_\epsilon]\frac{4\sqrt{\pi}}{(3-2\epsilon)(1-2\epsilon)}\frac{\Gamma(\frac{1}{2}+\epsilon)}{\Gamma(1+\epsilon)}\frac{m_V^{3-2\epsilon}}{(M^2 - m_V^2)^3} \\ & \left[1 + 2\epsilon - (3-2\epsilon)\left(\frac{M}{m_V}\right)^{1-2\epsilon} + (3-2\epsilon)\left(\frac{M}{m_V}\right)^2 - (1+2\epsilon)\left(\frac{M}{m_V}\right)^{3-2\epsilon}\right]. \end{aligned} \tag{142}$$

The result $J_2(m_V, M, 0) = -J_1(m_V, M, 0)$ follows from the observation that when $\delta = 0$ the identity in (89) implies $v_\mu J^\mu(p, m_V, M, 0) = 0$. The case $\delta = M = 0$ is simply obtained by substitution in (142).

D Heavy particle integrals with electroweak polarization tensor insertion

The two-boson exchange amplitudes for gluon matching require the integrals $H(n)$, $F(n)$, $H^{\mu\nu}(n)$, and $H^\mu(n)$ defined in (88). Let us parameterize the last two as

$$H^{\mu\nu}(n) = H_1(n) v^\mu v^\nu + H_2(n) g^{\mu\nu}, \quad H^\mu(n) = H_3(n) v^\mu. \tag{143}$$

Upon contracting the above expressions with v_μ and $g_{\mu\nu}$, we may solve for the relations

$$\begin{aligned} H_1(n) &= \frac{1}{3-2\epsilon}\left[(4-2\epsilon)v_\mu v_\nu H^{\mu\nu}(n) - H^\mu{}_\mu(n)\right], \\ H_2(n) &= \frac{1}{3-2\epsilon}\left[H^\mu{}_\mu(n) - v_\mu v_\nu H^{\mu\nu}(n)\right], \\ H_3(n) &= v_\mu H^\mu(n)\,. \end{aligned} \tag{144}$$

Using the identities in (86) and (89), we further obtain

$$\begin{aligned} v_\mu H^\mu(n) &= \delta H(n) + 2F(n)\,, \\ v_\mu v_\nu H^{\mu\nu}(n) &= \delta^2 H(n) + 2\delta F(n)\,, \\ H^\mu{}_\mu(n) &= \left[\frac{m_1^2}{x} + \frac{m_2^2}{(1-x)}\right] H(n) - \frac{H(n-1)}{x(1-x)}\,, \end{aligned} \tag{145}$$

and hence the boson loops are completely specified by $H(n)$ and $F(n)$. In evaluating these functions it may be advantageous to relate to more basic integrals by means of derivatives. Let us write,

$$\begin{aligned} H(n) &= 2\frac{\partial}{\partial m_V^2}\int (dL) \frac{1}{v\cdot L - \delta + i0}\frac{1}{L^2 - m_V^2 + i0}\Delta^{-n-\epsilon}\,, \\ F(n) &= \frac{\partial}{\partial m_V^2}\int (dL)\frac{1}{L^2 - m_V^2 + i0}\Delta^{-n-\epsilon}\,, \end{aligned} \tag{146}$$

with Δ as defined in (80). The singularity structure and evaluation of the above integrals can be classified into three cases, corresponding to zero, one, or two heavy fermions contributing to the electroweak polarization tensor. For pure states we obtain analytic expressions for all integrals, while for mixed states we encounter several integrals that require numerical evaluation of one Feynman parameter integral.

D.1 Case of zero heavy fermions

Upon setting $m_1 = m_2 = 0$ in Δ and performing the integration in $d = 4-2\epsilon$ dimensions, we obtain

$$\begin{aligned} F(n) &= [c_\epsilon]\frac{\Gamma(2-n-2\epsilon)\Gamma(n+2\epsilon)}{\Gamma(2-\epsilon)\Gamma(1+\epsilon)}[x(1-x)]^{-n-\epsilon} m_V^{-2n-4\epsilon}\,, \\ H(n) &= [c_\epsilon]\frac{4\Gamma(n+2\epsilon)}{\Gamma(n+\epsilon)\Gamma(1+\epsilon)}[x(1-x)]^{-n-\epsilon}\frac{\partial}{\partial m_V^2} I(n)\,, \end{aligned} \tag{147}$$

where

$$I(n) = \int_0^1 dy\,(1-y)^{n-1+\epsilon}\int_0^\infty d\rho(\rho^2 + 2\rho\delta + y m_V^2 - i0)^{-n-2\epsilon}\,. \tag{148}$$

We may reduce to the case of $I(1)$ by noticing that

$$I(n+1) = -\frac{m_V^{-2}}{n+2\epsilon}\int_0^1 dy(1-y)^{n+\epsilon}\frac{d}{dy}\int_0^\infty d\rho(\rho^2 + 2\rho\delta + y m_V^2 - i0)^{-n-2\epsilon}$$

$$= \frac{m_V^{-2}}{n+2\epsilon}\left[\int_0^\infty d\rho(\rho^2+2\rho\delta-i0)^{-n-2\epsilon}+(n+\epsilon)I(n)\right]$$

$$= \frac{m_V^{-2}}{n+2\epsilon}\left[\delta^{1-2n-4\epsilon}\frac{\Gamma(1-n-2\epsilon)\Gamma\left(n-\frac{1}{2}+2\epsilon\right)}{2\sqrt{\pi}}+(n+\epsilon)I(n)\right]. \tag{149}$$

Finally, for $I(1)$ we require

$$I(1) = \delta^{-1-4\epsilon}\int_0^1 dy(1+\epsilon\log(1-y)+\dots)\int_1^\infty d\rho(\rho^2+\alpha^2)^{-1}\left(1-2\epsilon\log(\rho^2+\alpha^2)+\dots\right), \tag{150}$$

where $\alpha = \left(ym_V^2/\delta^2-1-i0\right)^{\frac{1}{2}}$. The relevant integrals are

$$\int_1^\infty d\rho\frac{1}{\rho^2+\alpha^2} = \frac{1}{\alpha}\arctan\alpha\,,$$

$$\int_1^\infty d\rho\frac{\log(\rho^2+\alpha^2)}{\rho^2+\alpha^2} = \frac{1}{\alpha}\left[2\log(2\alpha)\arctan\alpha-\frac{1}{2i}\left(\mathrm{Li}_2\left(\frac{1-i\alpha}{1+i\alpha}\right)-\mathrm{Li}_2\left(\frac{1+i\alpha}{1-i\alpha}\right)\right)\right]. \tag{151}$$

We perform the remaining integral over Feynman parameter y numerically.

D.2 Case of one heavy fermion

Let us set $m_1 = M$ (not to be confused with heavy WIMP mass M used elsewhere in the paper) and $m_2 = 0$ in Δ, and consider separately the finite integrals for a- and c-type contributions, and the IR divergent integrals for b-type contributions.

D.2.1 Finite integrals for a- and c-type contributions

For the finite a- and c-type contributions we may take $d = 4$. Let us evaluate the required integrals $F(2)$ and $H(1)$, and obtain the remaining integrals by differentiating with respect to M. We find

$$F(2) = \frac{i}{(4\pi)^2}\frac{\partial}{\partial m_V^2}\left\{\left[x(1-x)m_V\left(1-\frac{M^2}{xm_V^2}\right)\right]^{-2}\left[-\log\frac{M^2}{xm_V^2}+\frac{M^2}{xm_V^2}-1\right]\right\},$$

$$H(1) = \frac{i}{(4\pi)^2}\frac{\partial}{\partial m_V^2}\left\{8\left[x(1-x)m_V^2\left(1-\frac{M^2}{xm_V^2}\right)\right]^{-1}\left[\sqrt{m_V^2-\delta^2}\arctan\left(\sqrt{\frac{m_V^2}{\delta^2}-1}\right)\right.\right.$$
$$\left.\left.-\sqrt{\frac{M^2}{x}-\delta^2}\arctan\left(\sqrt{\frac{M^2}{x\delta^2}-1}\right)-\frac{\delta}{2}\log\frac{xm_V^2}{M^2}\right]\right\}. \tag{152}$$

The integrals have been obtained by breaking an integration region into pieces, e.g.,

$$\int_\delta^\infty d\rho\left[\log(\rho^2+m_V^2-\delta^2)-\log\left(\rho^2+\frac{M^2}{x}-\delta^2\right)\right]$$
$$= \lim_{\varepsilon\to 0}\int_\delta^\infty d\rho\left[\log(\rho^2+m_V^2-\delta^2-i\varepsilon)-\log\left(\rho^2+\frac{M^2}{x}-\delta^2-i\varepsilon\right)\right]$$
$$= \delta\lim_{\varepsilon\to 0}\int_1^\infty d\rho\left[\log\left(\rho^2+\frac{m_V^2}{\delta^2}-1-i\varepsilon\right)-\log\left(\rho^2+\frac{M^2}{x\delta^2}-1-i\varepsilon\right)\right]$$

$$= \delta \lim_{\varepsilon\to 0} \Bigg\{ \int_0^\infty d\rho \left[\log\left(\rho^2 + \frac{m_V^2}{\delta^2} - 1 - i\varepsilon\right) - \log\left(\rho^2 + \frac{M^2}{x\delta^2} - 1 - i\varepsilon\right)\right]$$
$$- \int_0^1 d\rho \left[\log\left(\rho^2 + \frac{m_V^2}{\delta^2} - 1 - i\varepsilon\right) - \log\left(\rho^2 + \frac{M^2}{x\delta^2} - 1 - i\varepsilon\right)\right] \Bigg\} . \tag{153}$$

Since the original integral is independent of ε, either choice of $\mathrm{sgn}(\varepsilon)$ is correct provided it is used consistently in both terms. The continuation away from $\delta \to 0$ is thus obtained above by taking, e.g., $\delta \to \delta + i\varepsilon$ everywhere. For the evaluation of integrals over x involving $H(1)$, let us write

$$H(1) \equiv 2\frac{\partial}{\partial m_V^2} K(1) \equiv 2\frac{\partial}{\partial m_V^2}\left\{ \frac{M^2}{x m_V^2 - M^2} k(1) \right\} . \tag{154}$$

We then have

$$x^n K(1) = \left(\frac{M^2}{m_V^2}\right)^n K(1) + \frac{\left(\frac{M^2}{m_V^2}\right)^n - x^n}{\frac{M^2}{m_V^2} - x} \frac{M^2}{m_V^2} k(1) , \tag{155}$$

so that all powers $x^n K(1)$ can be reduced to the case $n = 0$, in addition to the remaining straightforward integral involving a polynomial in x times $k(1)$, which in practice is evaluated numerically. The remaining integrals involving $F(2)$ are similarly straightforward to evaluate.

D.2.2 Infrared divergent integrals for b-type contributions

Let us now turn to the integrals for b-type contributions, where we work in $d = 4 - 2\epsilon$ spacetime dimensions to account for singular behavior at the endpoints of the x integration. We find,

$$F(1) = [c_\epsilon][x(1-x)]^{-1-\epsilon} \frac{\Gamma(1+2\epsilon)}{[\Gamma(1+\epsilon)]^2} \Bigg\{ m_V^{-2-4\epsilon} \bigg[\left(r^2 - 1\right)^{-2} \left(r^2 \log r^2 - r^2 + 1 \right)$$
$$+ \epsilon \left(r^2 - 1\right)^{-2} \left(2r^2 \log r^2 - r^2 \log^2 r^2 - r^2 + 1 + r^2 \mathrm{Li}_2\left(1 - r^2\right) \right) \bigg]$$
$$+ m_V^{-2} \left[\left(\frac{r^2}{x} - 1\right)^{-2} \left(\frac{r^2}{x} \log\frac{r^2}{x} - \frac{r^2}{x} + 1 \right) - \left(r^2 - 1\right)^{-2} \left(r^2 \log r^2 - r^2 + 1 \right) \right] \Bigg\} , \tag{156}$$

where $r \equiv M/m_V$. The first term in curly braces is obtained by taking $x = 1$ inside the $\int dy$ integral, and the second term is the remainder having no singularity in the final $\int dx$ integral at $x = 1$.

Similarly we find,

$$H(1) = [c_\epsilon][x(1-x)]^{-1-\epsilon} \frac{4\Gamma(1+2\epsilon)}{[\Gamma(1+\epsilon)]^2} \frac{\partial}{\partial m_V^2} \left\{ \delta^{-1-4\epsilon} \left[Y_0(1) + \epsilon\Big(Y_1 + Y_2 \Big) \right] + \delta^{-1} \Big[Y_0(x) - Y_0(1) \Big] \right\} , \tag{157}$$

where

$$Y_0(x) = \frac{2}{r_V^2 - \frac{r_M^2}{x}} \left\{ \sqrt{r_V^2 - 1} \arctan\left(\sqrt{r_V^2 - 1}\right) - \sqrt{\frac{r_M^2}{x} - 1} \arctan\left(\sqrt{\frac{r_M^2}{x} - 1}\right) - \frac{1}{2}\log\frac{x m_V^2}{M^2} \right\} , \tag{158}$$

with $r_V \equiv m_V/\delta$ and $r_M \equiv M/\delta$. As in the discussion after (153), continuation away from $\delta = 0$ is given by taking $\delta \to \delta + i\varepsilon$ with arbitrary choice of $\mathrm{sgn}(\varepsilon)$. The remaining terms Y_1 and Y_2 are given by

$$
\begin{aligned}
Y_1 &= \int_0^1 dy \int_0^\infty d\beta \big(r_M^2 - r_V^2\big)^{-1} \frac{d}{dy} \log^2\left[\beta^2 + 2\beta + y r_V^2 + (1-y) r_M^2\right] \\
&= \big(r_M^2 - r_V^2\big)^{-1} \Big\{ -4\pi\sqrt{r_V^2 - 1}\Big[1 - \log\Big(2\sqrt{r_V^2-1}\Big)\Big] + 4\pi\sqrt{r_M^2-1}\Big[1 - \log\Big(2\sqrt{r_M^2-1}\Big)\Big] \\
&\quad - y_1\Big(\sqrt{r_V^2-1}\Big) + y_1\Big(\sqrt{r_M^2-1}\Big)\Big\}, \\
Y_2 &= \int_0^1 dy \, \log(1-y)\Big(y r_V^2 + (1-y) r_M^2 - 1\Big)^{-1} \arctan\left(\sqrt{y r_V^2 + (1-y) r_M^2 - 1}\right), \qquad (159)
\end{aligned}
$$

where

$$
y_1(A) \equiv \int_0^1 dx \log^2(x^2 + A^2). \qquad (160)
$$

For Y_2, we evaluate the remaining integral over Feynman parameter y numerically.

D.3 Case of two heavy fermions

Let us set $m_1 = m_2 = M$ (not to be confused with heavy WIMP mass M used elsewhere in the paper) in Δ, and work in $d = 4$ dimensions. Naming $x(1-x) \equiv z$, we find,

$$
\begin{aligned}
F(1) &= \frac{i}{(4\pi)^2}\left[z m_V^2 \left(1 - \frac{M^2}{z m_V^2}\right)^2\right]^{-1} \left[\frac{M^2}{z m_V^2}\log\frac{M^2}{z m_V^2} - \frac{M^2}{z m_V^2} + 1\right], \\
H(1) &= \frac{i}{(4\pi)^2}\frac{\partial}{\partial m_V^2}\Bigg\{ 8\left[z m_V^2\left(1 - \frac{M^2}{z m_V^2}\right)\right]^{-1}\Bigg[\sqrt{m_V^2 - \delta^2}\arctan\left(\sqrt{\frac{m_V^2}{\delta^2} - 1}\right) \\
&\quad - \sqrt{\frac{M^2}{z} - \delta^2}\arctan\left(\sqrt{\frac{M^2}{z\delta^2} - 1}\right) - \frac{\delta}{2}\log\frac{z m_V^2}{M^2}\Bigg]\Bigg\}. \qquad (161)
\end{aligned}
$$

The remaining integrals can be obtained by differentiating the above results with respect to M. In practice, we evaluate the remaining integral over Feynman parameter x (or z) numerically.

E Numerical inputs

We use the inputs of Table 1 in the numerical analysis of coefficients appearing in Fig. 6. Light fermion masses enter the analysis indirectly via the onshell renormalization scheme. The matching in (53) requires a limit of the photon two-point function which receives contributions from momentum regions of light (u, d and s) quark loops that are outside the domain of validity of QCD perturbation theory. A complete nonperturbative treatment of this function is not numerically relevant to the present analysis; for definiteness, we model these contributions using $\overline{\mathrm{MS}}$ light quark masses (cf. Table 1) in the one-loop evaluation of the two-point function. Varying these mass inputs by an order of magnitude in either direction does not appreciably change the numerical matching coefficients of Fig. 6.

Parameter	Value	Reference
$\|V_{td}\|, \|V_{ts}\|$	~ 0	-
$\|V_{tb}\|$	~ 1	-
m_e	0.511 MeV	[44]
m_μ	106 MeV	[44]
m_τ	1.78 GeV	[44]
m_h	126 GeV	[28, 29]
m_W	80.4 GeV	[44]
m_Z	91.188 GeV	[44]

Parameter	Value	Reference
m_t	172 GeV	[45]
m_b	4.75 GeV	[45]
m_c	1.4 GeV	[45]
m_s	93.5 MeV	[44]
m_d	4.70 MeV	[44]
m_u	2.15 MeV	[44]
c_W	m_W/m_Z	-
$\alpha_s(m_Z)$	0.118	[44]

Table 1: Inputs to the numerical analysis.

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