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Kirill Tuchin

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Kirill Tuchin[1]

[1] *Department of Physics and Astronomy, Iowa State University, Ames, IA 50011*

Magnetic fields produced in collisions of electrically charged particles at relativistic energies are strong enough to affect the dynamics of the strong interactions. In particular, it induces radiation of vector bosons by relativistic fermions. I calculate the corresponding spectrum in constant magnetic field and analyze its angular distribution and mass dependence. As an application, synchrotron radiation of vector bosons by relativistic plasmas is considered.

I. INTRODUCTION

It has been known since the pioneering paper of Ambjorn and Olesen [1] that extremely strong electromagnetic fields are produced in high energy collisions of charged particles. In recent years it was realized that these fields have an important impact on the dynamics of the strong interactions, though their precise structure and dynamics is being debated [2–7]. In this paper we focus on vector boson radiation by relativistic particles in an external magnetic field. In particular, we are interested in real and virtual photon production, which has important applications to the phenomenology of heavy-ion collisions [8–10], astrophysics [11] and the physics of intense laser pulses [12].

The real photon radiation rate in vacuum was calculated in [13] and is given by an infinite sum over the Landau levels. Based on this result synchrotron radiation from electromagnetic plasmas was calculated in [14–16]. Pair production by a photon in an external magnetic field is a cross channel of the synchrotron radiation. The most general expression for the pair production probability by a virtual photon in vacuum is derived in [17]. The results of [13, 17] are especially useful in very strong fields (defined below) when only a few lowest Landau levels contribute to the radiation rate. At not so strong fields and at ultra-relativistic energies, summation over the Landau levels is slowly convergent and is not convenient to deal with (in the context of heavy-ion physics see [8] for a detailed discussion of this issue). An alternative efficient method to calculate the scattering matrix in the ultra-relativistic approximation was developed by Baier and Katkov (see e.g. [18] and references therein) and is described in [19]. It is based on the quasi-classical approximation and allows one to perform explicit summation over the Landau levels yielding rather simple formulas that are convenient in numerical and analytical calculations, see e.g. [16]. While synchrotron radiation of real photons is of great interest in astrophysics, radiation of massive vectors bosons is of interest in heavy-ion collisions and in high intensity laser physics. Thus, in view of possible applications, it is very useful to have a compact expression for the synchrotron radiation of vector bosons. The goal of this paper is to feel the gap in the literature by calculating the synchrotron radiation of massive vector bosons and in particular virtual photons using the quasi-classical method.

In order to calculate the vector meson production rate we need to know their coupling to quarks. A simple model inspired by the Vector Meson Dominance is to assume that coupling of different vector mesons to quarks has the same structure as the coupling of the photon. The corresponding terms in the Lagrangian are

$$\mathcal{L}_\gamma = e\bar{q}\gamma^\mu Q q\, A_\mu\,, \quad \mathcal{L}_\rho = g_\rho \bar{q}\gamma^\mu \boldsymbol{\tau} q\cdot \boldsymbol{\rho}_\mu\,, \quad \mathcal{L}_\omega = g_\omega \bar{q}\gamma^\mu q\,\omega_\mu\,, \tag{1}$$

where q is the SU(2) doublet of u and d quarks, $\boldsymbol{\tau}$ are symmetry generators and $Q = \mathrm{diag}(q_u, q_d)$. Eqs. (1) constitute a part of the quark–meson coupling model [20, 21], which is used to describe the nuclear matter. Similar approach is successfully used for calculation of the vector meson production at high energy in perturbative QCD [22, 23].

Throughout the paper we employ the ultra-relativistic approximation that requires fermion and the vector boson to be relativistic and assume that magnetic field is adiabatic. Let $p = (\varepsilon, \boldsymbol{p})$ be the initial fermion four-momentum and $k = (\omega, \boldsymbol{k})$ the vector boson four-momentum, m and M their respective masses. Ultra-relativistic approximation requires that fermion energy before and after the vector boson emission satisfy $\varepsilon \gg m$ and $\varepsilon' = \varepsilon - \omega \gg m$. This implies that $\varepsilon'/\varepsilon \gg m/\varepsilon$ meaning that the vector boson does not carry away all the fermion energy. Another implication of the ultra-relativistic approximation, which is instrumental for the spectrum derivation in the next section, is that the angular distribution of the vector boson spectrum is concentrated inside a narrow solid angle with the opening angle θ around the fermion direction. This can be seen by examining the denominator of the outgoing fermion propagator

$$(p-k)^2 - m^2 \approx -\varepsilon\omega\left(\frac{m^2}{\varepsilon^2} + \frac{M^2}{\omega^2}\frac{\varepsilon'}{\varepsilon} + \theta^2\right)\,. \tag{2}$$

The same expression appears in the argument of the Airy function in the formulas for the spectrum (27),(28). Thus, the radiation cone is determined by the largest among the small ratios m/ε and $\sqrt{M^2\varepsilon'}/\sqrt{\omega^2\varepsilon} < M/\omega$.

The distance between the energy levels of a fermion in magnetic field is of the order of eB/ε. If $eB \ll \varepsilon^2$ the spectrum can be considered as approximately continuous. This is always true in fields weaker than the Schwinger field $B_S = m^2/e$. In the following I will assume that the magnetic field strength is such that the quasi-classical approximation holds, i.e. $eB \ll \varepsilon^2$ (but not necessarily $B < B_S$).

The paper is structured as follows: In Sec. II A I derive the vector boson spectrum radiated by a fast fermion moving in a plane perpendicular to the direction of magnetic field and in Sec. II B I analyze its mass dependence. In Sec. III the spectrum is boosted to an arbitrary frame. Sec. III is dedicated to synchrotron radiation from plasma. Conclusions are presented in Sec. V.

II. VECTOR BOSON RADIATION IN THE REACTION PLANE $\boldsymbol{p}\cdot\boldsymbol{B} = 0$.

A. Calculation of the spectrum

For the calculation of the vector boson spectrum I employ the method described in [18, 19]. I follow the notations of [19] apart from minor changes. The calculation is convenient to do in the frame K_0 where the fermion's momentum is perpendicular to the direction of magnetic field. The emission probability per unit time reads [19]

$$d\dot{w} = \frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\int_{-\infty}^{\infty} d\tau \, \langle R_2^* R_1\rangle \, e^{i\Phi} \,, \tag{3}$$

where $\alpha = g^2/4\pi$ (g stands for e, g_ρ, or g_ω), $\langle R_2^* R_1\rangle$ denotes the average over the initial fermion polarization and summation over the final fermion and boson polarization and

$$\Phi = \frac{\varepsilon}{\varepsilon'}[\boldsymbol{k}\cdot\boldsymbol{r}_2 - \boldsymbol{k}\cdot\boldsymbol{r}_1 - \omega\tau] + \frac{M^2\tau}{2\varepsilon'} \,, \tag{4}$$

$$R = -\frac{\bar{u}(p')}{\sqrt{2\varepsilon'}}\,\gamma\cdot\epsilon^*\,\frac{u(p)}{\sqrt{2\varepsilon}} \,. \tag{5}$$

Indexes 1 and 2 is a shorthand notation meaning that the corresponding quantity is taken at time $t_1 = t + \tau/2$ or $t_2 = t - \tau/2$. The bi-spinor is normalized as follows:

$$u(p) = \frac{1}{\sqrt{\varepsilon + m}}\begin{pmatrix} (\varepsilon + m)\varphi_p \\ (\boldsymbol{p}\cdot\boldsymbol{\sigma})\varphi_p \end{pmatrix} \,, \tag{6}$$

where φ_p is a two-component spinor and $\boldsymbol{\sigma}$ are Pauli matrices. The four-momentum of the incident fermion can be written as $p = \varepsilon(1, \boldsymbol{v})$. Similarly, I denote

$$s = \sqrt{1 - \frac{M^2}{\omega^2}} \,, \tag{7}$$

so that the vector boson four-momentum can be written as $k = (\omega, \boldsymbol{k}) = \omega(1, s\boldsymbol{n})$, where $\boldsymbol{n}$ is a unit vector. Substituting (6) into (5) I obtain for transversely polarized boson

$$R_T = \varphi^*_{p'}\boldsymbol{\epsilon}^*_T\cdot(\boldsymbol{A} + i\boldsymbol{B}\times\boldsymbol{\sigma})\varphi_p \,, \tag{8}$$

where the following auxiliary vectors are introduced:

$$\boldsymbol{A} = \left(\sqrt{\frac{\varepsilon' + m}{\varepsilon + m}} + \sqrt{\frac{\varepsilon + m}{\varepsilon' + m}}\right) \frac{\sqrt{\varepsilon}}{\sqrt{2\varepsilon'}} \boldsymbol{v} \,, \tag{9}$$

$$\boldsymbol{B} = \left[\left(\sqrt{\frac{\varepsilon' + m}{\varepsilon + m}} - \sqrt{\frac{\varepsilon + m}{\varepsilon' + m}}\right) \boldsymbol{p} + \sqrt{\frac{\varepsilon + m}{\varepsilon' + m}}\, \boldsymbol{k}\right] \frac{1}{2\sqrt{\varepsilon\varepsilon'}} \,, \tag{10}$$

and $\varepsilon' = \varepsilon - \omega$. Multiplying (8) by its complex conjugate and averaging using the formula $\langle \epsilon_{T,j}\epsilon_{T,k}\rangle = (\delta_{jk} - n_j n_k)/2$ we get

$$\left\langle R^*_{T,2} R_{T,1} \right\rangle = \boldsymbol{A}_1 \cdot \boldsymbol{A}_2 - (\boldsymbol{A}_1 \cdot \boldsymbol{n})(\boldsymbol{A}_2 \cdot \boldsymbol{n}) + \boldsymbol{B}_1 \cdot \boldsymbol{B}_2 + (\boldsymbol{B}_1 \cdot \boldsymbol{n})(\boldsymbol{B}_2 \cdot \boldsymbol{n}) \,. \tag{11}$$

Expanding (9),(10) in m^2/ε^2 and M^2/ω^2 yields

$$\boldsymbol{A} \approx \left(1 + \frac{\varepsilon}{\varepsilon'}\right) \frac{\boldsymbol{v}}{2} \,, \tag{12}$$

$$\boldsymbol{B} \approx \frac{\omega}{2\varepsilon'} \left(-\boldsymbol{v} + \boldsymbol{n} + \frac{m}{\varepsilon}\boldsymbol{n} + (s-1)\boldsymbol{n}\right) \,. \tag{13}$$

Terms like $\boldsymbol{v}_1 \cdot \boldsymbol{n}$ arising in (11) can be simplified using integration by parts in (3) as follows [19]

$$\boldsymbol{v}_1 \cdot \boldsymbol{n}\, e^{i\Phi} = \boldsymbol{v}_2 \cdot \boldsymbol{n}\, e^{i\Phi} = \left[1 + \frac{\omega(s^2-1)}{2s\varepsilon}\right] e^{i\Phi} \,, \tag{14}$$

where terms proportional to the total time derivative with respect to t_1, which vanish upon integration over time in (3), are dropped. Substituting (12),(13) into (11) I derive

$$\left\langle R^*_{T,2} R_{T,1} \right\rangle = \frac{\varepsilon'^2 + \varepsilon^2}{2\varepsilon'^2}(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 - 1) + \frac{M^2}{2\omega^2}\frac{\omega}{\varepsilon'}\left(\frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon}\right) + \frac{\omega^2 m^2}{2\varepsilon^2\varepsilon'^2} \,. \tag{15}$$

Explicit expression for the fermion trajectory in a plane perpendicular to magnetic field yields at small τ:

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1 - \frac{m^2}{\varepsilon^2} - \frac{1}{2}\omega_B^2 \tau^2 \,, \tag{16}$$

where $\omega_B = eB/\varepsilon$ is the synchrotron frequency. Thus, (15) takes form

$$\left\langle R^*_{T,2} R_{T,1} \right\rangle = -\frac{\varepsilon'^2 + \varepsilon^2}{2\varepsilon'^2}\omega_B^2\tau^2 + \frac{M^2}{2\omega^2}\frac{\omega}{\varepsilon'}\left(\frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon}\right) - \frac{m^2}{\varepsilon\varepsilon'} \,. \tag{17}$$

The longitudinal polarization is described by the four-vector $\epsilon_L = (s, \hat{\boldsymbol{n}})/\sqrt{s^2-1}$, which satisfies $\epsilon \cdot k = 0$ and $\epsilon^2 = 1$. Writing $R = -j \cdot \epsilon$ and using the Ward identity $j \cdot k = 0$ we have $j^0 = s\boldsymbol{j} \cdot \boldsymbol{n}$ implying that

$$j \cdot \epsilon_L = \frac{j^0 s - \boldsymbol{j} \cdot \boldsymbol{n}}{\sqrt{s^2-1}} = \sqrt{s^2-1}\, \boldsymbol{j} \cdot \boldsymbol{n} \,. \tag{18}$$

Using (6) and (5) produces

$$R_L = i\sqrt{1-s^2}\,\varphi^*_{p'}(F + i\boldsymbol{\sigma}\cdot\boldsymbol{G})\varphi_p\,. \tag{19}$$

and

$$\left\langle R^*_{L,2}R_{L,1}\right\rangle = (1-s^2)(F_2F_1 + \boldsymbol{G}_2\cdot\boldsymbol{G}_1)\,, \tag{20}$$

where

$$F = \frac{1}{2\sqrt{\varepsilon\varepsilon'}\sqrt{\varepsilon+m}\sqrt{\varepsilon'+m}}[(\varepsilon'+m)(\boldsymbol{n}\cdot\boldsymbol{p}) + (\varepsilon+m)(\boldsymbol{p}'\cdot\boldsymbol{n})] \tag{21}$$

$$\boldsymbol{G} = \frac{1}{2\sqrt{\varepsilon\varepsilon'}\sqrt{\varepsilon+m}\sqrt{\varepsilon'+m}}[(\varepsilon'+m)(\boldsymbol{n}\times\boldsymbol{p}) + (\varepsilon+m)(\boldsymbol{n}\times\boldsymbol{p}')]\,, \tag{22}$$

with $\boldsymbol{p}' = \boldsymbol{p}-\boldsymbol{k}$. In view of a small factor $1-s^2$ in the right hand side of (20) we only need to keep terms of the order one in expansion of F and $\boldsymbol{G}$ in powers of m^2/ε^2 and M^2/ω^2. Thus, in view of (14) $\boldsymbol{p}\cdot\boldsymbol{n}\approx\varepsilon$, $\boldsymbol{p}'\cdot\boldsymbol{n}\approx\varepsilon'$ and we have $F\approx 1$, $\boldsymbol{G}\approx -\frac{\omega}{2\varepsilon'}\boldsymbol{n}\times\boldsymbol{v}$. This implies that $\boldsymbol{G}_1\cdot\boldsymbol{G}_2 \propto 1-\boldsymbol{v}_1\cdot\boldsymbol{v}_2 \sim m^2/\varepsilon^2$ can be neglected and we derive

$$\left\langle R^*_{L,2}R_{L,1}\right\rangle \approx \frac{M^2}{\omega^2}\,. \tag{23}$$

The expression in the exponent of (3) upon expansion in τ and then in M/ω becomes

$$\Phi = -\frac{\varepsilon}{\varepsilon'}\omega\tau\left[1 - s\boldsymbol{n}\cdot\boldsymbol{v} + \frac{\omega}{2\varepsilon}(s^2-1) + s\omega_B^2\frac{\tau^2}{24}\right] \approx -\frac{\varepsilon}{\varepsilon'}\omega\tau\left[1-\boldsymbol{n}\cdot\boldsymbol{v} + \frac{M^2}{2\omega^2}\frac{\varepsilon'}{\varepsilon} + \omega_B^2\frac{\tau^2}{24}\right]\,. \tag{24}$$

Substituting (17), (23) and (24) into (3) we obtain for the transverse and longitudinal vector boson production rates

$$\begin{aligned} d\dot{w}_T =& \frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\int_{-\infty}^{\infty}d\tau\exp\left\{-\frac{i\varepsilon}{\varepsilon'}\omega\tau\left[1-\boldsymbol{n}\cdot\boldsymbol{v}+\frac{M^2}{2\omega^2}\frac{\varepsilon'}{\varepsilon}+\omega_B^2\frac{\tau^2}{24}\right]\right\} \\ &\times\left[-\frac{\varepsilon'^2+\varepsilon^2}{4\varepsilon'^2}\omega_B^2\tau^2 + \frac{M^2}{2\omega^2}\frac{\omega}{\varepsilon'}\left(\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon}\right) - \frac{m^2}{\varepsilon\varepsilon'}\right]\,, \end{aligned} \tag{25}$$

$$d\dot{w}_L = \frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\frac{M^2}{\omega^2}\int_{-\infty}^{\infty}d\tau\exp\left\{-\frac{i\varepsilon}{\varepsilon'}\omega\tau\left[1-\boldsymbol{n}\cdot\boldsymbol{v}+\frac{M^2}{2\omega^2}\frac{\varepsilon'}{\varepsilon}+\omega_B^2\frac{\tau^2}{24}\right]\right\}\,. \tag{26}$$

One can do integrals over τ using equations (A1) and (A3) which yields the angular distribution of the spectrum

$$\begin{aligned} \frac{d\dot{w}_T}{d\omega d\Omega} =& \frac{\alpha}{\pi}\omega\left(\frac{\varepsilon'}{\varepsilon\omega\omega_B^2}\right)^{1/3}\left[\frac{M^2}{2\omega^2}\left(\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon}\right)\frac{\varepsilon+\varepsilon'}{\varepsilon'} - \frac{m^2}{\varepsilon\varepsilon'} + 2\left(1-\boldsymbol{n}\cdot\boldsymbol{v}\right)\frac{\varepsilon'^2+\varepsilon^2}{\varepsilon'^2}\right] \\ &\times\mathrm{Ai}\left(2\left(\frac{\varepsilon\omega}{\varepsilon'\omega_B}\right)^{2/3}\left(1-\boldsymbol{n}\cdot\boldsymbol{v}+\frac{M^2\varepsilon'}{2\omega^2\varepsilon}\right)\right)\,, \end{aligned} \tag{27}$$

$$\frac{d\dot{w}_L}{d\omega d\Omega} = \frac{\alpha}{\pi}\omega\left(\frac{\varepsilon'}{\varepsilon\omega\omega_B^2}\right)^{1/3}\frac{M^2}{\omega^2}\,\mathrm{Ai}\left(2\left(\frac{\varepsilon\omega}{\varepsilon'\omega_B}\right)^{2/3}\left(1-\boldsymbol{n}\cdot\boldsymbol{v}+\frac{M^2\varepsilon'}{2\omega^2\varepsilon}\right)\right)\,, \tag{28}$$

where we used $d^3k = s\omega^2 d\omega d\Omega \approx \omega^2 d\omega d\Omega$. Notice the follwing expression

$$2\left(1 - \boldsymbol{n}\cdot\boldsymbol{v} + \frac{M^2\varepsilon'}{2\omega^2\varepsilon}\right) \approx \theta^2 + \frac{m^2}{\omega^2} + \frac{M^2\varepsilon'}{2\omega^2\varepsilon}\,, \tag{29}$$

which appears in the argument of the Airy function. It is proportional to the denominator of the outgoing fermion propagator (2) and guarantees emission of vector boson into a narrow cone.

Integration over the photon directions is convenient to do in (25),(26) followed by integration over τ [19]. The result is

$$\frac{d\dot{w}_T}{d\omega} = -\frac{\alpha m^2}{\varepsilon^2}\left\{\left(1 - \frac{M^2}{2\omega\varepsilon'}\frac{\varepsilon^2+\varepsilon'^2}{m^2}\right)\int_z^\infty \mathrm{Ai}(z')dz' + \left(\frac{\varepsilon}{\varepsilon'}\right)^{1/3}\left(\frac{\omega_B}{\omega}\right)^{2/3}\frac{\varepsilon^2+\varepsilon'^2}{m^2}\mathrm{Ai}'(z)\right\}\,, \tag{30}$$

$$\frac{d\dot{w}_L}{d\omega} = \frac{\alpha M^2\varepsilon'}{\omega^2\varepsilon}\int_z^\infty \mathrm{Ai}(z')dz'\,, \tag{31}$$

where

$$z = \left(\frac{\varepsilon}{\varepsilon'}\right)^{2/3}\left(\frac{\omega}{\omega_B}\right)^{2/3}\left(\frac{m^2}{\varepsilon^2} + \frac{M^2}{\omega^2}\frac{\varepsilon'}{\varepsilon}\right)\,. \tag{32}$$

B. Analysis of the spectrum

Vector boson spectrum (30),(31) is a function of ω and ε. Instead, we can express the spectrum in terms of the boost-invariant dimensionless quantities X and ξ defined as follows:

$$X = \sqrt{-\frac{e^2}{m^6}\left(F_{\mu\nu}p^\nu\right)^2} \approx \frac{\omega_B\varepsilon^2}{m^3} = \frac{eB\varepsilon}{m^3} \tag{33}$$

and

$$\xi = \frac{\omega}{\omega_c}\,, \tag{34}$$

where

$$\omega_c = \frac{\varepsilon X}{\frac{2}{3}+X}\,, \tag{35}$$

is the characteristic frequency of the classical photon spectrum. Its is also convenient to denote $\mu = M/m$. In terms of these variables we can write

$$z = \frac{\xi^{2/3}}{[\frac{2}{3}+X(1-\xi)]^{2/3}} + \mu^2\frac{[\frac{2}{3}+X(1-\xi)]^{1/3}(\frac{2}{3}+X)}{X^2\xi^{4/3}}\,, \tag{36}$$

$$\varepsilon' = \frac{\omega_c\left[\frac{2}{3}+X(1-\xi)\right]}{X}\,. \tag{37}$$

Becasue $\varepsilon' \geq 0$, it follows form (37) that

$$\xi \leq \frac{2}{3X} + 1\,. \tag{38}$$

When multiplied by ω, (30),(31) yield the radiation power. Dividing it by 3/2 of the total classical photon radiation power $\alpha m^2 X^2$ we represent the spectrum in terms of the dimensionless quantities

$$J_\lambda(\xi, X, \mu) = \frac{\omega}{\alpha m^2 X^2} \frac{d\dot{w}_\lambda}{d\xi}\,, \quad \lambda = L, T\,. \tag{39}$$

Their explicit form reads as follows

$$\begin{aligned} J_T = &- \frac{\xi}{(\frac{2}{3}+X)^2} \left\{ \left[1 - \mu^2 \left(\frac{\frac{2}{3}+X}{\xi X} + \frac{\xi X}{\frac{4}{3}+2X(1-\xi)} \right) \right] \int_z^\infty \mathrm{Ai}(z')dz' \right. \\ &\left. + \frac{(\frac{2}{3}+X)^2 + [\frac{2}{3}+X(1-\xi)]^2}{\xi^{2/3}[\frac{2}{3}+X(1-\xi)]^{1/3}(\frac{2}{3}+X)} \mathrm{Ai}'(z) \right\}, \end{aligned} \tag{40}$$

$$J_L = \frac{\mu^2[\frac{2}{3}+X(1-\xi)]}{X^2\xi(\frac{2}{3}+X)} \int_z^\infty \mathrm{Ai}(z')dz'\,. \tag{41}$$

Airy function exponentially decays at large values of its argument, hence the spectrum is suppressed at $z \gg 1$. Variable z as a function of ξ has a minimum z_0 at ξ_0 that depends on the values of X and μ. The main contribution to the spectrum comes form the kinematic region $z < 1$ which exists only if $z_0 < 1$. To determine z_0 and ξ_0 it is convenient to use instead of ξ an auxiliary variable u:

$$u = \frac{1}{\xi}\left[\frac{2}{3} + X(1-\xi)\right]\,, \tag{42}$$

$$z = \frac{1}{u^{2/3}} + \frac{\mu^2 u^{1/3}(u+X)}{X^2}\,. \tag{43}$$

The minimum of z as a function of u is located at

$$u_0 = \frac{X}{8}\left(\sqrt{1+\frac{32}{\mu^2}} - 1\right)\,. \tag{44}$$

The corresponding value of ξ reads

$$\xi_0 = \frac{\frac{2}{3}+X}{u_0+X} = \frac{\frac{2}{3}+X}{X}\frac{8}{7+\sqrt{1+\frac{32}{\mu^2}}}\,. \tag{45}$$

At $\mu \ll 1$, corresponding to an almost real photon,

$$u_0 \approx \frac{X}{\sqrt{2}\mu}\,, \quad \xi_0 \approx \mu\sqrt{2}\,\frac{\frac{2}{3}+X}{X}\,, \quad \mu \ll 1\,. \tag{46}$$

Replacing $X = \sqrt{2}u_0\mu$ in (43) we get

$$z_0 \approx \frac{3}{2u_0^{2/3}}\,, \quad \mu \ll 1\,. \tag{47}$$

Thus, the condition $z_0 < 1$ is satisfied only if $X > 2.6\mu$. Otherwise, the spectrum is exponentially suppressed.

In the opposite case, which is realized e.g. in production of high invariant mass dileptons, $\mu^2 \gg 32$ we have

$$u_0 \approx \frac{2X}{\mu^2}\,, \quad \xi_0 \approx \frac{\frac{2}{3}+X}{X}\,, \quad \mu \gg 4\sqrt{2}\,, \tag{48}$$

Comparing with (38), we observe that in this case the minimum of z is very close to the upper cutoff of the boson spectrum (i.e. when the boson takes nearly all energy of the fermion). Using $X = u_0\mu^2/2$ in (43) we have

$$z_0 \approx \frac{3}{u_0^{2/3}}\,, \quad \mu \gg 4\sqrt{2}\,. \tag{49}$$

In this case $z_0 < 1$ is satisfied if $X > 2.6\mu^2$ which is a much stronger condition than in the previous case.

The main contribution to the spectrum arises from $z \sim 1$, which for X and μ satisfying the above constraints and taking (43) into account happens when $u \sim 1$ fairly independently from the value of μ. This statement has been verified numerically. In particular, according to (42) $u \sim 1$ means that $\xi \sim \frac{2}{3} + X(1-\xi)$. In weak fields $X \ll 1$, $\xi \sim 1$ and so $\omega \sim \omega_c \sim \varepsilon X$, while in strong fields $X \gg 1$, $\xi \sim X\varepsilon'/\omega_c$ (see (37)) implying that $\varepsilon' \sim \omega/X \sim \varepsilon/X \sim m^3/eB$ [19].

These features of the spectrum are seen in Figs. 1–3. In Fig. 1 the transverse vector boson spectrum as a function of ξ is shown at different values of X and μ. The transverse bosons are much more abundantly produced than the longitudinal ones, which can be seen by comparing Fig. 1(b) and Fig. 2. Therefore, Fig. 1 represents approximately the total spectrum. The general trend observed in all figures is that the spectrum decreases with increase of μ. At larger μ it tends to peak around $\xi = 1$. This is because with increase of μ, X also increases, see the text after (48),(49); it follows from (48) that once $X \gg 1$, the typical ξ is about 1.

III. VECTOR BOSON SPECTRUM IN AN ARBITRARY FRAME

Consider now a reference frame K where fermions have an arbitrary direction of momentum. It is convenient to change our notations. We will append a subscript 0 to all quantities pertaining to the reference frame K_0. Thus, for example, ε_0 and ω_0 are the fermion and vector boson energies in K_0, whereas ε and ω are the fermion and vector boson energies in K. Let the y-axis be in the magnetic field direction $\boldsymbol{B} = B\hat{\boldsymbol{y}}$ and $\boldsymbol{V} = V\hat{\boldsymbol{y}}$ be velocity of K with respect to K_0. Then the

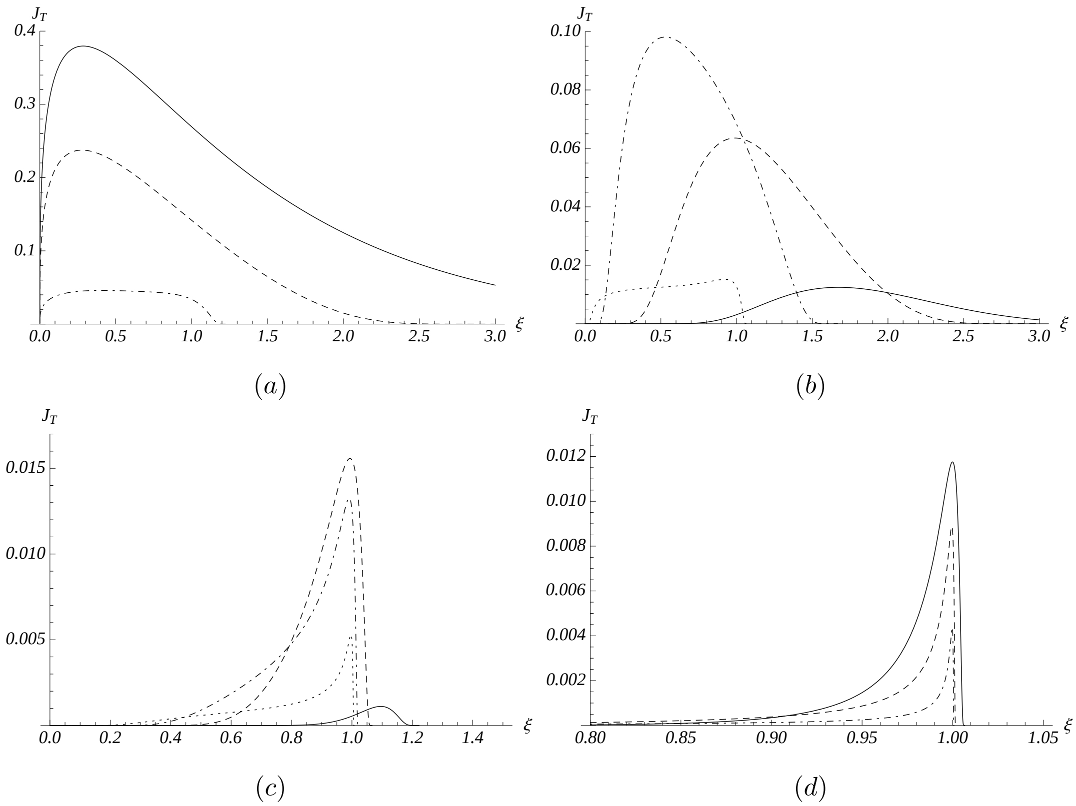

FIG. 1: Spectrum of transversely polarized vector bosons J_T as a function of ξ. (a) $\mu = 0$ and $X = 0$ (solid line), $X = 0.3$ (dashed line), $X = 3$ (dash-dotted line). (b) $\mu = 0.3$ and $X = 0.15$ (solid line), $X = 0.3$ (dashed line), $X = 1$ (dash-dotted line), $X = 10$ (dotted lines). (c) $\mu = 3$ and $X = 3$ (solid line), $X = 10$ (dashed line), $X = 30$ (dash-dotted line), $X = 100$ (dotted lines). (d) $\mu = 10$ and $X = 100$ (solid line), $X = 400$ (dashed line), $X = 1000$ (dash-dotted line). Notice different scales of the x and y axes.

Lorentz transformation reads

$$p_{x0} = p_x\,, \quad 0 = p_{y0} = \gamma(p_y + V\varepsilon)\,, \quad p_{z0} = p_z\,, \quad \varepsilon_0 = \gamma(\varepsilon + V p_y)\,. \tag{50}$$

$$k_{x0} = k_x\,, \quad k_{y0} = \gamma(k_y + V\omega)\,, \quad k_{z0} = k_z\,, \quad \omega_0 = \gamma(\omega + V k_y)\,. \tag{51}$$

$$\boldsymbol{B}_0 = \boldsymbol{B}\,, \tag{52}$$

where $\gamma = 1/\sqrt{1 - V^2}$. It follows from the second equation in (50) that

$$V = -\frac{p_y}{\varepsilon} \tag{53}$$

and

$$\varepsilon_0 = \sqrt{\varepsilon^2 - p_y^2}\,, \qquad \omega_0 = \frac{\omega\varepsilon - p_y k_y}{\sqrt{\varepsilon^2 - p_y^2}}\,. \tag{54}$$

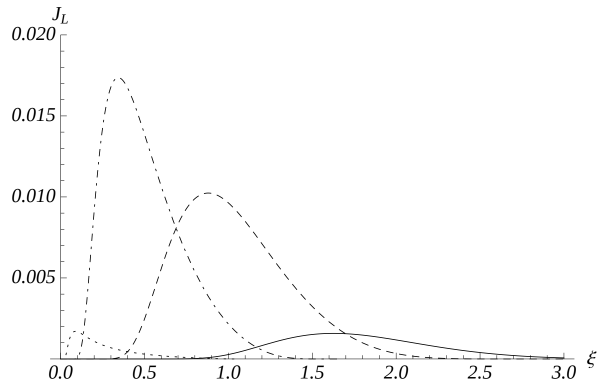

FIG. 2: Spectrum of longitudinaly polarized vector bosons J_L as a function of ξ at $\mu = 0.3$ and $X = 0.15$ (solid line), $X = 0.3$ (dashed line), $X = 1$ (dash-dotted line), $X = 10$ (dotted lines)

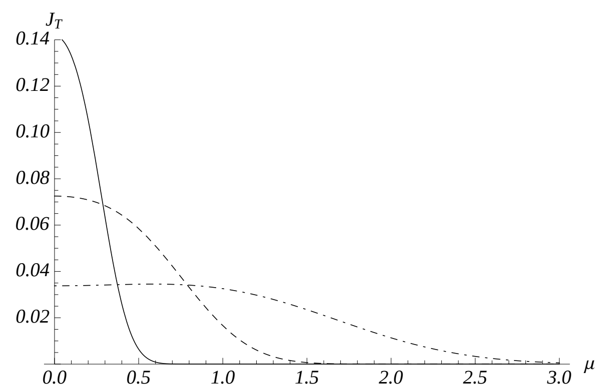

FIG. 3: Spectrum of transversely polarized vector bosons J_T as a function of μ at $\xi = 1$ and $X = 0.3$ (solid line), $X = 1$ (dashed line) and $X = 3$ (dashed-dotted line).

Using the boost invariance of $k \cdot p$ we get

$$1 - \boldsymbol{n}_0 \cdot \boldsymbol{v}_0 = \frac{\omega\varepsilon}{\omega_0\varepsilon_0}(1 - \boldsymbol{n} \cdot \boldsymbol{v})\,, \tag{55}$$

accurate up to the terms of the order m^2/ε^2 and M^2/ω^2.

Transformation of the photon emission rate reads [9]

$$\frac{d\dot{w}}{d\Omega d\omega} = \frac{1}{\gamma^2(1 + V\cos\theta)}\frac{d\dot{w}_0}{d\Omega_0 d\omega_0} = \frac{\omega\varepsilon_0}{\varepsilon\omega_0}\frac{d\dot{w}_0}{d\Omega_0 d\omega_0}\,, \tag{56}$$

where θ is angle between the photon momentum $\boldsymbol{k}$ and the magnetic field, i.e. $\cos\theta = n_y$. In the last step we used (53) and (54). $d\dot{w}_0$ in the right-hand-side of (56) is given by (27) and (28) with the replacements $\varepsilon \to \varepsilon_0$, $\omega \to \omega_0$ etc.

IV. VECTOR BOSON RADIATION BY A PLASMA

A system of electrically charged particles in thermal equilibrium in external magnetic field radiates vector bosons at the following rate per unit interval of vector boson energy $d\omega$ into a solid angle $d\Omega$:

$$\frac{dN}{dtd\Omega d\omega} = 2N_c \sum_f \int \frac{d\mathcal{V} d^3p}{(2\pi)^3} f(\varepsilon)[1-f(\varepsilon')]\frac{d\dot{w}}{d\Omega d\omega}\,, \tag{57}$$

where the sum runs over all charged particle species in plasma, and $f(\varepsilon)$ are their distribution functions. Integration over the fermion momentum can be done using a Cartesian reference frame span by three unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{n}$, such that vector $\boldsymbol{B}$ lies in plane span by $\boldsymbol{e}_1, \boldsymbol{n}$. In terms of the polar and azimuthal angles χ and ψ we can write

$$\boldsymbol{v} = v(\cos\chi\, \boldsymbol{n} + \sin\chi\cos\psi\, \boldsymbol{e}_1 + \sin\chi\sin\psi\, \boldsymbol{e}_2)\,, \tag{58}$$

$$\boldsymbol{B} = B(\cos\theta\, \boldsymbol{n}_1 + \sin\theta\, \boldsymbol{e}_1)\,. \tag{59}$$

Element of the solid angle is $do = d\cos\chi\, d\psi$. In this reference frame

$$p_y = \frac{\boldsymbol{p}\cdot\boldsymbol{B}}{B} = \varepsilon v(\cos\chi\cos\theta + \sin\chi\cos\psi\sin\theta)\,, \tag{60}$$

$$k_y = \frac{\boldsymbol{k}\cdot\boldsymbol{B}}{B} = k\cos\theta\,, \tag{61}$$

$$\boldsymbol{n}\cdot\boldsymbol{v} = v\cos\chi\,. \tag{62}$$

Fermions moving in plasma parallel to the magnetic field direction do not radiate due to the vanishing Lorentz force. Taking into account that at high energies fermions radiate mostly into a narrow cone with the opening angle $\chi \sim m/\varepsilon, M/\omega$ (see (2)), we conclude that vector boson radiation at angles $\theta \lesssim m/\varepsilon, M/\omega$ can be neglected. Thus, expanding at small χ we obtain from (54),(60)

$$\varepsilon_0 \approx \varepsilon\sin\theta\,, \quad \omega_0 \approx \omega\sin\theta\,, \qquad \theta > \frac{m}{\varepsilon}, \frac{M}{\omega}\,. \tag{63}$$

Omission of terms of order m/ε, $M\omega$ is consistent with the accuracy of (27),(28). Dependence of the integrand of (57) on the fermion direction specified by the angles χ, ψ comes only through (55), viz.

$$1 - \boldsymbol{n}_0\cdot\boldsymbol{v}_0 = \frac{1}{\sin^2\theta}\left(1 - \cos\chi + \frac{m^2}{2\varepsilon^2}\right)\,. \tag{64}$$

For this reason, integration over the quark momentum directions is similar to the one that led us from (25), (26) to (30), (31) (in the K_0 reference frame). Writing (57) as

$$\frac{dN}{dtd\Omega d\omega} = \frac{2N_c}{(2\pi)^3}\sum_f \int d\mathcal{V}\int_\omega^\infty d\varepsilon\, \varepsilon^2 f(\varepsilon)[1-f(\varepsilon')] \sum_{\lambda=L,T}\int do\, \frac{d\dot{w}_\lambda}{d\Omega d\omega} \tag{65}$$

and substituting (56), (25), (26) (with appropriate notation changes as described in Sec. III) and (63) we integrate first over do and then over τ with the following result

$$\begin{aligned}\int do\, \frac{d\dot{w}_T}{d\Omega d\omega} =& -\frac{\alpha m^2}{\varepsilon^2}\sin^2\theta \left\{ \left(1-\frac{M^2}{2\omega\varepsilon'}\frac{\varepsilon^2+\varepsilon'^2}{m^2}\right)\int_{z_\theta}^\infty \mathrm{Ai}(z')dz' \right. \\ & \left. +(\sin\theta)^{2/3}\left(\frac{\varepsilon}{\varepsilon'}\right)^{1/3}\left(\frac{\omega_B}{\omega}\right)^{2/3}\frac{\varepsilon^2+\varepsilon'^2}{m^2}\mathrm{Ai}'(z_\theta)\right\}\,, \end{aligned} \tag{66}$$

$$\int do\, \frac{d\dot{w}_L}{d\Omega d\omega} = \frac{\alpha M^2\varepsilon'}{\omega^2\varepsilon}\sin^2\theta \int_{z_\theta}^\infty \mathrm{Ai}(z')dz'\,, \tag{67}$$

where

$$z_\theta = (\sin\theta)^{-2/3}\left(\frac{\varepsilon}{\varepsilon'}\right)^{2/3}\left(\frac{\omega}{\omega_B}\right)^{2/3}\left(\frac{m^2}{\varepsilon^2\sin^2\theta}+\frac{M^2}{\omega^2}\frac{\varepsilon'}{\varepsilon}\right)\,. \tag{68}$$

If magnetic field is a slow function of time and/or coordinates one can adopt an adiabatic approximation and integrate (66) and (67) over the time and space which yields the total vector boson multiplicity spectrum radiated into a unit solid angle. This is the formula that has been recently employed in [26] for the calculation of the synchrotron radiation of real photons in heavy-ion collisions, which is one of the outstanding problems in the high energy nuclear physics [27–34].

For practical applications in relativistic heavy-ion phenomenology it is customary to represent the bosom spectra as functions of rapidity y and transverse momentum $k_\perp$ with respect to the collision axis z, in place of energy ω and emission angle θ with respect to the magnetic field. Let α and ϕ be the polar and azimuthal angles of boson with respect to the collision axis. They are related to ω and θ as follows [8]:

$$\omega = k_\perp \cosh y\,, \quad \cos\theta = \frac{\sin\phi}{\cosh y}\,. \tag{69}$$

The differential boson multiplicity can be represented as

$$\frac{dN}{dV dt\, d^2k_\perp dy} = \frac{dN}{dV dt\, \omega d\omega d\Omega}\,, \tag{70}$$

where one should substitute (69) in the right-hand-side of (70).

In deriving (65)–(68) we assumed that plasma is relativistic, i.e. that fermion energy satisfies $\varepsilon \sim T \gg m$. This condition must hold not only for the current mass m, but also for the temperature dependent contribution that fermions receive due to their interaction with the plasma. Evidently, this contribution must be small compared to the plasma temperature. This is true in a weakly coupled plasma, such as the electromagnetic plasma, because fermion mass receives a correction of order $gT \ll T$, where g is the coupling constant. As far as the quark-gluon plasma is concerned, the coupling g is not small at temperatures relevant in experiment. In practice, effective quark and gluon masses are treated as free parameters in models describing the quark-gluon plasma. Under such circumstances accuracy of the ultra-relativistic limit used to derive (65)–(68) depends on a particular model used to describe the plasma dynamics.

V. SUMMARY

In this paper we used the quasi-classical method to derive the synchrotron radiation rate of massive vector bosons including virtual photons. The main result is expressed in formulas (27)–(32) that give the vector boson radiation rate by a relativistic electrically charged fermion. They describe spectrum and the angular distribution of ultra-relativistic vector bosons. Our analysis of the mass dependence of the synchrotron spectra revealed that with increase of M, spectra become increasingly monochromatic with energy ω_c, given by (35). A more detailed structure is shown in Fig. 1 and Fig. 2.

Eqs. (27)–(32) can be directly applied to investigate the space-time structure of magnetic field and its dynamics in experiments with intense laser beams. In view of possible applications in high energy nuclear physics and in astrophysics, we derived vector boson spectrum (65)–(68) radiated by a relativistic plasma. These equations can be used, for example, to evaluate a contribution of synchrotron radiation to the dilepton spectrum produced in relativistic heavy-ion collisions at $k_\perp > M$ and $y = 0$ and compare with the experimental data reported in [35].

These and other applications deserve full consideration in separate publications.

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Appendix A: Some useful integrals involving the Airy function Ai(z)

In the following integrals a, b are real numbers and $z = a/(3b)^{1/3}$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i[a\tau+b\tau^3]}d\tau = \frac{1}{(3b)^{1/3}}\,\mathrm{Ai}(z)\,, \tag{A1}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \tau e^{-i[a\tau+b\tau^3]}d\tau = \frac{1}{(3b)^{2/3}}\,\mathrm{Ai}'(z)\,, \tag{A2}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tau^2 e^{-i[a\tau+b\tau^3]}d\tau = -\frac{z}{3b}\,\mathrm{Ai}(z)\,, \tag{A3}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\tau} e^{-i[a\tau+b\tau^3]}d\tau = \int_{z}^{\infty}\mathrm{Ai}(z')\,dz'\,, \tag{A4}$$

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