ANOMALY STRUCTURE OF REGULARIZED SUPERGRAVITY

Daniel Butter and Mary K. Gaillard

Phys. Rev. D **91**, 025015 — Published 23 January 2015

DOI: 10.1103/PhysRevD.91.025015

UCB-PTH-14/37
Nikhef-2014-044

THE ANOMALY STRUCTURE OF REGULARIZED SUPERGRAVITY*

Daniel Butter[1] and Mary K. Gaillard[2]

[1]*Nikhef Theory Group, Science Park 105, 1098 XG Amsterdam, The Netherlands*
[2]*Department of Physics and Theoretical Physics Group, Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720*

Abstract

On-shell Pauli-Villars regularization of the one-loop divergences of supergravity theories is used to study the anomaly structure of supergravity and the cancellation of field theory anomalies under a $U(1)$ gauge transformation and under the T-duality group of modular transformations in effective supergravity theories with three Kähler moduli T^i obtained from orbifold compactification of the weakly coupled heterotic string. This procedure requires constraints on the chiral matter representations of the gauge group that are consistent with known results from orbifold compactifications. Pauli-Villars regulator fields allow for the cancellation of all quadratic and logarithmic divergences, as well as most linear divergences. If all linear divergences were canceled, the theory would be anomaly free, with noninvariance of the action arising only from Pauli-Villars masses. However there are linear divergences associated with nonrenormalizable gravitino/gaugino interactions that cannot be canceled by PV fields. The resulting chiral anomaly forms a supermultiplet with the corresponding conformal anomaly, provided the ultraviolet cut-off has the appropriate field dependence, in which case total derivative terms, such as Gauss-Bonnet, do not drop out from the effective action. The anomalies can be partially canceled by the four-dimensional version of the Green-Schwarz mechanism, but additional counterterms, and/or a more elaborate set of Pauli-Villars fields and couplings, are needed to cancel the full anomaly, including D-term contributions to the conformal anomaly that are nonlinear in the parameters of the anomalous transformations.

*This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics, of the U.S. Department of Energy under Contracts DE-AC02-05CH11231 and DE-AC03-76SF00098, in part by the National Science Foundation under grants PHY-0457315, PHY-1002399 and PHY-0098840, in part by the Australian Research Council (grant No. DP1096372), in part by a UWA Research Development Award, by the ERC Advanced Grant no. 246974, *"Supersymmetry: a window to non-perturbative physics"* and by the European Commission Marie Curie International Incoming Fellowship grant no. PIIF-GA-2012-627976.

Disclaimer

1 Introduction

It has been shown [1]–[3] that on-shell Pauli-Villars (PV) regularization of one-loop quadratic and logarithmic ultraviolet divergences in general $N = 1$ supergravity [4, 5] is possible, subject to constraints on the matter representations of the gauge group that are consistent with the spectra found, for example in orbifold compactifications. Supergravity derived from weakly coupled string theory typically has classical symmetries that are broken at the quantum level by conformal and chiral anomalies which arise, respectively, from logarithmic and linear divergences in the light field loops. If they can be canceled by PV loops in the regulated theory, the remaining noninvariance under the classical symmetries arises from the noninvariance of the PV mass terms, provided all other PV couplings are invariant. In this paper we investigate the anomaly structure and anomaly cancellation in a class of Z_N orbifolds with just three "diagonal" Kähler moduli $T^I = T^{II}$ $(I = 1, 2, 3)$.

The anomalous symmetries that we consider are the target space duality transformations, hereafter referred to as modular transformations, and a gauge transformation under an Abelian gauge group, hereafter referred to as $U(1)_X$. These symmetries are perturbatively unbroken [6] in the underlying string theory and therefore must be canceled by some combination of loop contributions from heavy string and Kaluza-Klein modes ("string threshold corrections") and of counterterms, including a four-dimensional version of the Green-Schwarz (GS) mechanism [7]. The anomaly canceling contributions to the Yang-Mills (YM) Lagrangian have been determined for large classes of orbifolds (including the Z_N orbifolds considered here) by matching field theory and string loop calculations [8]–[13]. More recently, a string theory analysis [14] of a limited class of Z_N heterotic orbifolds found cancellation of all anomalies through a universal GS mechanism. Here we approach the same problem from the point of view of the effective four dimensional supergravity theory [15].

In the following section we define our notation and display the ultraviolet divergent part of the low energy effective Lagrangian obtained from light particle loops [16]–[18] in the form of superfield operators that will be convenient for the subsequent analysis. In Section 3 we will use the results of [2] and [3], hereafter referred to as I and II, respectively, to construct invariant couplings of PV supermultiplets needed to cancel the light loop divergences. Mechanisms for anomaly cancellation and constraints on PV masses will be discussed in Section 4, and the explicit form of the anomalies will be displayed in Section 5. Our results are summarized in Section 6.

Some calculational details are presented in a series of appendices. As discussed below, requiring the cancellation of quadratic and logarithmic UV divergences that were identified in [16]–[18] does not uniquely fix the couplings of the PV sector. In Appendix A we derive additional constraints that assure the cancellation of almost all linear divergences. While the constraints from the requirement of cancellation of (on-shell) quadratic divergences automatically assures the cancellation of the chiral Kähler anomaly arising from the fermion spin connection, there is a residual linear divergence associated with the affine connection of the gravitino. In addition there is an off-diagonal gravitino/gaugino connection that has no counterpart in the PV sector. The corresponding contributions to the chiral anomaly must have supersymmetric counterparts; these can be obtained by introducing a field-dependent UV cut-off

$$\Lambda = \mu_0 e^{K/4}, \tag{1.1}$$

where μ_0 is a constant parameter that may be set to infinity at the end of the one-loop calculation, and K is the Kähler potential of the light field theory. This has only the effect that total derivatives with nonvanishing coefficients of $\ln \Lambda$ do not drop out of the S-matrix elements of the regulated theory. Once these procedures have been implemented we recover the standard form of the anomaly coefficients of the Yang-Mills and curvature field strengths as well as agreement with string theory results. However the anomalous coefficients of operators that themselves depend on the modular weights of the light fields depend on details of the PV regularization procedure, and are not uniquely fixed by the cancellation of ultraviolet divergences.

In Appendix B we calculate the chiral anomaly for a general supergravity theory. As has been recently emphasized [19], in supergravity the fermion connections and corresponding field strengths contain many more operators than the Yang-Mills [8]–[13] and space-time curvature [20]-[22] terms that have been studied previously in the context of anomaly cancellation. These additional operators include [21, 16, 17] the Kähler $U(1)_K$ connection for all fermions, the reparameterization connection for chiral fermions, an axion coupling in the gaugino connection and a (matrix-valued) connection [17] linear in the Yang-Mills field strength in the gaugino-gravitino sector.[1] In addition, besides the spin connection common to all fermions, the gravitino connection includes a term proportional to the affine connection, and the gauginos have an additional connection that involves the dilaton and its axionic superpartner. The anomaly is ill defined in an unregulated theory. The authors of Ref. [19] study supergravity theories in which a subgroup of the invariance group of the Kähler metric (group of modular transformations) is gauged. They use consistency conditions analogous to those used to obtain the "consistent anomaly" [25]–[27] for Yang-Mills theories. They

[1] In supergravity without a GS term to cancel the $U(1)_X$ anomaly, there is an additional connection [23] which must also be included [19, 24].

also note that some of the operators in their expression can be removed by counter terms [19]. Here we are working with a regulated theory in which the ambiguity is removed, except for those terms for which a linear divergence remains. The squared field strength $\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}$ corresponding to the full fermion connection appears in the coefficient of $\ln\Lambda$. It combines with other contributions from both light and PV loops to cancel the UV divergence, up to a total derivative. Part of the conformal anomaly, including the standard Yang-Mills term and part of the curvature term, is determined by the field-dependence of the PV masses in such a way that it combines with the part of the chiral anomaly arising from PV masses to form a superfield. The remainder of the conformal anomaly arises from the residual total derivative logarithmic divergence mentioned above, and combines with the part of the chiral anomaly associated with the residual linear divergence to form a superfield, provided the cut-off has the correct field-dependence. Specifically, the choice (1.1) assures a supersymmetric result for the full anomaly coefficient of the curvature field strength term. When combined with the PV contribution we recover an anomaly coefficient that is consistent with string loop calculations [20]. Similarly, the chiral anomaly associated with the off-diagonal gaugino-gravitino connection, which depends on the Yang-Mills field strength, forms a supermultiplet with a contribution to the conformal anomaly from a total derivative in the coefficient of $\ln\Lambda$, when the field-dependence of the cut-off (1.1) is included. This contribution to the anomaly is canceled by a PV loop contribution for a particular choice of the relevant PV mass ratio.

In the absence of linear divergences, the chiral anomaly arises solely from the noninvariance of PV masses. In Appendix B.2 we sketch how this may be demonstrated by comparing the direct chiral anomaly calculation with an indirect method which assumes that no linear divergences are present. We also show that under appropriate assumptions the "consistent anomaly" [25]–[27] is recovered. As described in [1]–[3], and reviewed in Section 3 below, all the quadratic and logarithmic UV divergences of supergravity can be regulated with PV fields in chiral supermultiplets and Abelian vector supermultiplets. The Abelian vector fields acquire $U(1)_X$ and modular invariant masses through the superhiggs mechanism and do not contribute to the anomaly, except for a contribution, mentioned above, that cancels the gravitino-gaugino mixed loop contribution. As a consequence, the part of the PV sector relevant to the bulk of the anomaly does not contain any connections associated with the nonrenormalizable couplings of the gravitino-gaugino sector, and the only field strength bilinears that appear in their contribution to the anomaly coefficient are those associated with the spin connection, the YM gauge connection, the $U(1)_K$ connection and the scalar reparameterization connections associated with the Kähler metric for PV chiral superfields. As shown in Appendix C, the latter can be chosen such that PV fields with noninvariant masses have very simple reparameterization connections.

In the case of an anomalous Yang-Mills theory with constant PV masses, the anomaly is uniquely determined, as illustrated in Appendix B.2 by the standard result (B.58). However in PV regulated supergravity, some PV masses are necessarily field dependent, and there is considerable leeway in the choice of these masses, which can only be fixed by a detailed knowledge of string/Planck scale physics. As explained in Section 4.1, the QFT anomaly cannot be completely removed, even in the absence of linear divergences. Terms linear in gauge charges and modular weights are fixed by the requirement that quadratic divergences cancel; this uniquely fixes the PV contribution to the coefficient of $r_{\mu\nu\rho\sigma}\tilde{r}^{\mu\nu\rho\sigma}$ as given in (C.43), but terms cubic in these parameters are not fixed by the requirement of cancellation of quadratic (or logarithmic) UV divergences. However the expansion of the parity odd part of the fermion determinant [see (B.1)], given explicitly in [17], has, in addition to a linear divergence proportional to $\mathrm{Tr}A$ that generates an $r_{\mu\nu\rho\sigma}\tilde{r}^{\mu\nu\rho\sigma}$ term in the anomaly, a linear divergence proportional to $\mathrm{Tr}A^3$, where A is the axial current in the fermion connection. This contains the $U(1)_K$ connection as well as the (symmetric) YM gauge connections and the (axial part of the) scalar reparameterization connections for chiral fermions.[2] Requiring that this linear divergence vanish imposes cubic constraints, but as mentioned above, does not completely determine the anomaly.

In Appendix C we consider simple parameterizations of the PV sector that are motivated by physical considerations, and are consistent with perturbative modular invariance of the Kähler potential and cancellation of quadratic and logarithmic UV divergences, as well as cancellation of all linear divergences from chiral multiplet loops. Under these assumptions we calculate the bosonic part of the variation of the Lagrangian under a variation of the noninvariant PV masses. We then identify the corresponding anomaly superfields and compare them with operators that could potentially cancel anomalies in a generalized GS term; these include a generalization to Kähler superspace of the (F-term) operator found in Ref. [22] for the case of pure supergravity, as well as new D-term operators [15], that can be inferred [29] by first working in conformal supergravity and then gauge-fixing to Kähler superspace supergravity which we use in this paper.

In order to evaluate the anomaly coefficients for specific orbifold models, in Appendix D we construct simple examples of PV sectors that can be used to regularize the matter sector in such a way that those PV fields that contribute to the renormalization of the Kähler potential have

[2] As in any regularization procedure, the definition of γ_5 is *a priori* ambiguous; as discussed in [1]–[3] the separation into "vector" and "axial" currents is dictated by the finiteness requirement and nonrenormalization of the θ-parameter [28].

modular and $U(1)_X$ invariant masses. Implementation of the GS anomaly cancellation mechanism for the F-term imposes constraints on the modular weights and gauge charges. Cancellation of UV divergences assures that some of these constraints are satisfied, but terms nonlinear in the charges under the anomalous symmetries are not completely determined by finiteness conditions and depend on the specifics of the PV spectrum and couplings. The simple procedure adopted in Appendix C and Appendix D is not sufficient to assure the factorization needed for implementation of anomaly cancellation by a universal GS term.[3] In addition, some D-term anomaly operators are nonlinear in the parameters of the anomalous transformations and cannot obviously be canceled by a straightforward generalization of the standard four dimensional GS term; it is possible that the additional counterterms that may be needed could correspond to 4-d remnants of the conformal analogue of the 10-d GS [7] term that cancels the 10-d chiral anomaly.
The construction of the GS term is discussed in Appendix E, and our notations and conventions are summarized in Appendix F.

2 One-Loop On-Shell Ultraviolet Divergences

In this paper we consider supergravity theories with classical Kähler potential K, superpotential W and gauge kinetic function f given by

$$\begin{aligned} K(Z,\bar{Z}) &= -\ln(S+\bar{S}) + G(T+\bar{T},\Phi,\bar{\Phi}) = k + G, \qquad W(Z) = W(T,\Phi), \\ f_{ab}(Z) &= \delta_{ab}S, \qquad S| = \delta_{ab}(x+iy), \end{aligned} \tag{2.1}$$

which are the classical functions found in string compactifications with affine level one.[4] S is the dilaton superfield in the chiral formulation used here, T^i are the moduli chiral superfields, and Φ^a are superfields for gauge-charged matter.
The ultraviolet divergent part of the one-loop corrected bosonic supergravity Lagrangian was calculated in [16]-[18]. As shown in II, the result for the logarithmically divergent part can be interpreted as the bosonic part of a superfield expression. After a Weyl redefinition to put the Einstein term in canonical form, one obtains

$$\mathcal{L}_{eff} = \mathcal{L}\left(g, K_R\right) + \sqrt{g}\frac{\ln\Lambda^2}{32\pi^2}\left(L_D + L_F\right) + \mathcal{L}_Q, \tag{2.2}$$

[3] The difficulty of obtaining factorization in a direct field theory approach was noted in [14].

[4] The results can be generalized to higher affine level with $f_{ab} = \delta_{ab}k_a f$, $k_a =$ constant, by making the substitutions $F^a_{\mu\nu} \to k_a^{\frac{1}{2}} F^a_{\mu\nu}$, $A^a_\mu \to k_a^{\frac{1}{2}} A^a_\mu$, $T^a \to k_a^{-\frac{1}{2}} T^a$.

where $\mathcal{L}(g,K)$ is the standard Lagrangian [4, 5] for $N=1$ supergravity coupled to matter with space-time metric $g_{\mu\nu}$, Kähler potential K and superpotential W. If N_G is the dimension of the gauge group, the renormalized Kähler potential K_R is given by[5]

$$\begin{aligned} K_R &= K + \frac{\ln\Lambda^2}{32\pi^2}\left[e^{-K}A_{ij}\bar{A}^{ij} - 2\hat{V} + (N_G-10)M^2 - 4\mathcal{K}^a_a - 16\mathcal{D}\right], \\ \mathcal{K}^a_b &= \frac{1}{x}(T^a z)^i(T_b\bar{z})^{\bar{m}}K_{i\bar{m}}, \qquad A = e^K W = \bar{A}^\dagger, \qquad A_{ij} = D_i D_j A, \end{aligned} \tag{2.3}$$

where $V = \hat{V} + \mathcal{D}$ is the classical scalar potential with

$$\begin{aligned} \hat{V} &= e^{-K}A_i\bar{A}^i - 3M^2, \qquad A_i = D_i A, \qquad \mathcal{D} = (2x)^{-1}\mathcal{D}^a\mathcal{D}_a, \\ \mathcal{D}_a &= K_i(T_a z)^i, \qquad M^2 = e^{-K}A\bar{A}, \end{aligned} \tag{2.4}$$

where M^2 is the field-dependent squared gravitino mass, and D_i is the scalar field reparameterization covariant derivative:

$$A_i = \partial_i A, \qquad A_{ij} = \partial_i\partial_j A - \Gamma^k_{ij}\partial_k A, \qquad \ldots \tag{2.5}$$

Scalar indices are lowered and raised with the Kähler metric $K_{i\bar{m}}$ and its inverse $K^{i\bar{m}}$. In the Kähler $U(1)$ superspace formulation [5] of supergravity, which we use throughout, a general "F-term" Lagrangian takes the form

$$L_F = L(\Phi) = \frac{1}{2}\int d^4\theta \frac{E}{R}\Phi + \text{h.c.}, \tag{2.6}$$

where Φ is a chiral superfield of Kähler $U(1)$ weight $w(\Phi)=2$, and a general "D-term" Lagrangian has the form

$$L_D = L(\phi) = \int d^4\theta E\phi = -\frac{1}{16}\int d^4\theta\frac{E}{R}\left(\bar{\mathcal{D}}^2 - 8R\right)\phi + \text{h.c.}, \tag{2.7}$$

with ϕ a real superfield of Kähler weight $w(\phi)=0$. These contributions to (2.2) are given by[6]

$$\begin{aligned} \Phi &= \Phi_1 + \Phi_2 + \frac{3}{2}C_a\Phi^a_{YM} + \frac{1}{36}\Phi_X\left(N - 9N_G - 79\right) + \frac{1}{6}\Phi_W\left(41 + N - 3N_G\right) + N_G\Phi'_g, \\ \Phi_X &= X^\beta X_\beta, \qquad \Phi_W = W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}, \qquad \Phi^a_{YM} = W^\alpha_a W^a_\alpha, \\ \Phi_1 &= -\frac{1}{2}\left(C^M_a\Phi^a_{YM} + \Gamma^{i\alpha}_j\left[\Gamma^j_{i\alpha} + 2(T_a)^j_i W^a_\alpha\right]\right), \\ \Phi_2 &= \frac{1}{3}X^\alpha\left[\Gamma_\alpha + 2(T_a)^i_i W^a_\alpha\right], \qquad \Gamma_\alpha = \Gamma^i_{i\alpha}. \end{aligned} \tag{2.8}$$

---

[5]We denote by $z^i = Z^i\big|$ the lowest (scalar) components of the light chiral superfields Z^i.

[6]The signs of the Φ^a_{YM} terms were inadvertently flipped in transcribing the results of I to II. In addition, the sign of W_{ab} as defined in I is incorrect, as is the sign of (A12) (see Appendix E). Here we have normalized some of the operators differently from those in II: $\Phi_{\rm YM}|_{\rm here} = 4\,\Phi_{\rm YM}|_{\rm II}$, $\Phi_W|_{\rm here} = 6\,\Phi_W|_{\rm II}$, $\Phi_X|_{\rm here} = -2\,\Phi_\alpha|_{\rm II}$.

for the F-terms with N the number of chiral supermultiplets $Z^i = S, T, \Phi$, and

$$\begin{aligned}
\phi &= \phi_3 - 4\phi_{\mathcal{W}} - 4\phi_{\mathcal{W}k} + 8\hat{\phi}_0 + \phi'_0 + N_G\phi'_g + \frac{1}{2}\phi_\chi\,(41 + N + 9N_G)\,, \\
\phi_3 &= \frac{1}{2}R^{\alpha k}{}_{\alpha}{}^{l}R^{\dot\beta}{}_{k\dot\beta l} + \left(R^{\alpha k}{}_{\alpha}{}^{l}e^{-K/2}A_{kl} + \text{h.c.}\right), \qquad \phi'_0 = 3iW^\alpha_a W^{\dot\beta a}\mathcal{D}_{\alpha\dot\beta}(S - \bar{S}) \\
\phi_\chi &= \frac{1}{3}\left(G_{\dot\beta\alpha}G^{\alpha\dot\beta} - 4R\bar{R}\right), \qquad \hat{\phi}_0 = K^{\alpha\dot\beta}K_{\alpha\dot\beta} - 4R\bar{R}, \\
\phi_{\mathcal{W}^a_b} &= \frac{x^2}{4}W^\alpha_a W^b_\alpha W^a_{\dot\beta}W^{\dot\beta}_b, \quad \phi_{\mathcal{W}k} = \frac{1}{8x}W^\alpha_a\mathcal{D}_\alpha S W^a_{\dot\beta}\mathcal{D}^{\dot\beta}\bar{S},
\end{aligned} \tag{2.9}$$

for the D-terms, where we used the on-shell relation

$$G_{\alpha\dot\beta} = K_{\alpha\dot\beta} - \frac{x}{2}W_\alpha W_{\dot\beta} \tag{2.10}$$

to simplify the expression for ϕ_χ given in (2.15) of II, and rewrote the "F-term" contribution Φ'_0 of that paper as the "D-term" contribution ϕ'_0. Here x is understood as the superfield $\frac{1}{2}(S+\bar{S})$, and $G_{\alpha\dot\beta}$ and $R = \bar{R}^\dagger$ are auxiliary fields [5] of the supergravity supermultiplet. The chiral superfield W^a_α is the Yang-Mills superfield strength, with a a gauge index, and $L(\Phi_{YM})$ gives the standard gauge charge renormalization. C_a and C^M_a are the quadratic Casimirs in the adjoint and matter representations, respectively, of the gauge subgroup $\mathcal{G}_a$:

$$\mathrm{Tr}(T_aT_b)_{\mathrm{adj}} = \delta_{ab}C_a, \qquad \mathrm{Tr}(T_aT_b)_{\mathrm{matter}} = \delta_{ab}C^M_a, \tag{2.11}$$

with T_a a generator of $\mathcal{G}_a$ and T_b any generator. $W_{\alpha\beta\gamma}$ is the Weyl chiral superfield [5], and $L(\Phi_W)$ contains the Gauss-Bonnet (GB) term and the pseudoscalar operator that is bilinear in the Riemann tensor; its explicit expression is given by (E.4) (with $H(Z) = 1$). The other chiral superfields in (2.8) are constructed from superfields of the general form

$$T_\alpha = -\frac{1}{8}\left(\mathcal{D}_{\dot\alpha}\mathcal{D}^{\dot\alpha} - 8R\right)\hat{T}_\alpha, \qquad \hat{T}_\alpha = T_i\mathcal{D}_\alpha Z^i, \tag{2.12}$$

where $T_i(Z,\bar{Z})$ is any (tensor-valued) zero-weight function of the chiral and anti-chiral superfields Z^i and $\bar{Z}^{\bar\imath}$, respectively. In particular, the chiral superfield

$$K_\alpha = X_\alpha = -\frac{1}{8}\left(\mathcal{D}_{\dot\alpha}\mathcal{D}^{\dot\alpha} - 8R\right)\mathcal{D}_\alpha K, \tag{2.13}$$

was introduced in [5]; the lowest component of its spinorial derivative $-\frac{1}{2}\mathcal{D}^\alpha X_\alpha|$ is the kinetic term for matter fields in the classical Lagrangian. For the D-terms we further introduced the zero-weight real superfields

$$
\begin{aligned}
T^{\alpha\dot\beta}_{\alpha\dot\beta} &= \frac{1}{16}\mathcal{D}^\alpha Z^i\mathcal{D}_\alpha Z^j\mathcal{D}_{\dot\beta}\bar Z^{\bar m}\mathcal{D}^{\dot\beta}\bar Z^{\bar n}T_{ij\bar m\bar n}, \\
T_{\alpha\dot\beta} &= \frac{1}{4}\mathcal{D}_\alpha Z^i\mathcal{D}_{\dot\beta}\bar Z^{\bar m}T_{i\bar m}, \qquad T^\alpha_\alpha = \frac{1}{2}\mathcal{D}^\alpha Z^i\mathcal{D}_\alpha Z^j T_{ij} + \text{h.c.}
\end{aligned} \tag{2.14}
$$

Thus ϕ_3 is constructed from the Riemann tensor

$$
R^i_{jk\bar m} = K^{i\bar n}R_{\bar n jk\bar m} = D_{\bar m}\Gamma^i_{jk} \tag{2.15}
$$

associated with the Kähler metric. As discussed[7] in II, the superfields Φ'_g, ϕ'_g are equivalent–up to terms that vanish on shell–to linear combinations of the the generic operators introduced above and D-terms that involve supergravity superfields: $G_{\alpha\dot\beta}K_{s\bar s}\mathcal{D}^\alpha S\mathcal{D}^{\dot\beta}\bar S, \ldots$. As shown in Appendix C of II, these terms must be exactly canceled by PV Abelian gauge multiplets that couple to the dilaton. In the string-derived models considered here and in II, the masses of these PV fields are invariant under T-duality and the anomalous $U(1)$, and therefore do not contribute to the field theory anomalies.

The various terms in the bosonic part of (2.2) are given in component form in I and II, where total derivatives (such as the GB term) were dropped. These terms cannot be dropped if, after regularization, the constant cut-off is replaced by a field-dependent PV mass that plays the role of effective cut-off (or if the cut-off itself is field-dependent). All terms relevant to the anomaly structure and anomaly cancellation will be included in Sections 5.

The quadratically divergent part $\mathcal{L}_Q$ of (2.2)

$$
\begin{aligned}
\mathcal{L}_Q &= -\sqrt{g}\frac{\Lambda^2}{32\pi^2}\left[\frac{1}{2}(3+N_G-N)\;\mathcal{D}^\alpha X_\alpha| + \left(\hat V + M^2\right)(7+3N_G-N)\right. \\
&\quad \left. +N_G\;\mathcal{D}^\alpha k_\alpha| + \mathcal{D}^\alpha\Gamma_\alpha| + \frac{2}{x}\mathcal{D}_a\mathrm{Tr}T^a\right] + \text{fermion terms}
\end{aligned} \tag{2.16}
$$

can be interpreted as a further renormalization of the Kähler potential K_Q only after regularization [1, 2]; K_Q is governed by the PV squared masses.

3 Invariant PV Regularization

In this section we construct gauge and modular invariant PV couplings to light fields that are needed to cancel the ultraviolet divergences from light field loops in the theory defined by (2.1).

[7] In the expression (2.2) of II, $L'_g = L(\Phi'_g) + L(\phi'_g)$.

3.1 Chiral multiplet loops

To regulate the loops arising from dimension-four operators involving only chiral supermultiplets Z^i, we need to introduce at least one set of PV chiral superfields $\dot{Z}^I$ with negative metric and the same gauge charges, Kähler metric, and quadratic superpotential couplings as the light chiral multiplets. This assures cancellation of the ultraviolet divergences associated with Φ_1 and Φ_2 in (2.8), and $\mathcal{D}^\alpha\Gamma_\alpha|$ in (2.16). Since the dilaton does not have a classical superpotential, and its Kähler metric is easily reproduced, we concentrate in this subsection on the chiral supermultiplets $Z^p = T^i, \Phi^a$ and their PV counterparts $\dot{Z}^P = \dot{Z}^I, \dot{Z}^A$:

$$\begin{aligned} K_0^{\dot{Z}} &= \sum_{P,Q=p,q}\left\{K_{p\bar{q}}\dot{Z}^P\bar{\dot{Z}}^{\bar{Q}} + \frac{1}{2}\left[\left(\partial_p\partial_q K - K_pK_q\right)\dot{Z}^P\dot{Z}^Q + \text{h.c.}\right]\right\}, \\ W_{02}^{\dot{Z}} &= \frac{1}{2}\sum_{P,Q=p,q}\dot{Z}^P\dot{Z}^Q W_{pq}, \end{aligned} \tag{3.1}$$

where the subscript 2 on W_{02} signals the presence of a further contribution: we will denote by W_1 the part of the PV superpotential that gives PV fields order Planck scale masses and that will be constructed explicitly in Section 4.2. From (3.1) it is straightforward to determine that [2]

$$\begin{aligned} R^{\dot{Z}}_{I\bar{m}J\bar{n}} &= R_{i\bar{m}j\bar{n}} + K_{i\bar{m}}K_{j\bar{n}} + K_{i\bar{m}}K_{j\bar{n}}, \qquad A^{\dot{Z}}_{IJ} = A_{ij}, \qquad \bar{A}^{IJ}_{\dot{Z}} = \bar{A}^{ij}, \\ R^{\dot{Z}}_{I\bar{m}J\bar{n}}\left(R^{\dot{Z}}\right)^{I\ J}_{\ k\ \ell} &= R_{i\bar{m}j\bar{n}}R^{i\ j}_{\ k\ \ell} + 4R_{k\bar{m}\ell\bar{n}} + 2\left(K_{k\bar{m}}K_{\ell\bar{n}} + K_{\ell\bar{m}}K_{k\bar{n}}\right), \\ A^{\dot{Z}}_{IJ}\bar{A}^{IJ}_{\dot{Z}} &= A_{ij}\bar{A}^{ij}, \qquad R^{\dot{Z}}_{I\bar{m}J\bar{n}}\bar{A}^{IJ}_{\dot{Z}} = R_{i\bar{m}j\bar{n}}\bar{A}^{ij} + 2\bar{A}_{\bar{m}\bar{n}}, \end{aligned} \tag{3.2}$$

assuring [2] cancellation of the divergences arising from the first term in brackets in (2.3) and from ϕ_3 in (2.9). The couplings in (3.1) are gauge invariant, provided $\dot{Z}^P$ transforms the same way as $\dot{Z}^p$ under gauge transformations. Now consider a chiral field redefinition that effects a Kähler transformation

$$\begin{aligned} Z^p &\to Z'^p, \qquad K \to K' = K(Z',\bar{Z}') = K + F(Z) + \bar{F}(Z'), \\ W &\to W' = W(Z') = e^{-F}W, \qquad dZ'^p = M^p_q dZ^q, \qquad M^p_q = \frac{\partial Z'^p}{\partial Z^q}. \end{aligned} \tag{3.3}$$

The part of (3.1) involving the PV Kähler metric $K_{P\bar{Q}} = K_{p\bar{q}}$ is invariant under (3.3) provided

$$\dot{Z}^P \to \dot{Z}'^P = M^p_q\dot{Z}^Q, \tag{3.4}$$

but the other terms are not. Nevertheless the matrix elements (3.2) are covariant, depending only on the covariant scalar Riemann tensor and on the covariant holomorphic scalar derivatives of the

Kähler covariant operator

$$A = e^K W, \qquad A' = e^{\bar{F}} A, \qquad A'_p = e^{\bar{F}} A_p, \qquad etc. \tag{3.5}$$

This suggests that (3.1) can be made invariant without modifying (3.2). For example, if we introduce uncharged fields $\dot{Z}^N$ and add to (3.1) expressions of the form

$$\begin{aligned} \Delta K^{\dot{Z}} &= \sum_N \left[\rho_N \sum_{P=p} K_p \dot{Z}^P \dot{Z}^N + \frac{1}{2}\rho'_N \left(\dot{Z}^N\right)^2 + \text{h.c.}\right], \\ \Delta W_2^{\dot{Z}} &= \sum_N \left[\rho_N \sum_{P=p} W_p \dot{Z}^P \dot{Z}^N - \frac{1}{2}\rho'_N \left(\dot{Z}^N\right)^2 W\right], \end{aligned} \tag{3.6}$$

where ρ, ρ' are constants, the one-loop corrections to K_R and ϕ_3 are unchanged:

$$A_{PN} = R_{\bar{n}PN\bar{m}} = A_{NM} = R_{\bar{n}NM\bar{m}} = 0. \tag{3.7}$$

This trick was used in II to construct an explicitly invariant (covariant) expression for $K^{\dot{Z}}$ ($W_2^{\dot{Z}}$) in no-scale models with the special properties: $G_{pq} \propto G_p G_q$, $G^{p\bar{q}} G_p G_{\bar{q}} = 3$, $\phi^p W_p = 3W$. The general covariance of (3.2) suggests that this should be possible in general modular invariant models, *e.g.*, including the twisted sector in orbifold compactification. In general we have under (3.3)

$$\begin{aligned} K'_p &= N_p^q \left(K_q + F_q\right), \qquad K'_{p\bar{m}} = N_p^k N_{\bar{m}}^{\bar{n}} K_{k\bar{n}}, \qquad N = M^{-1}, \\ K'_{pq} &= N_p^n N_q^m [K_{nm} + F_{mn} - N_k^l \left(K_l + F_l\right) \partial_n M_m^k], \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} W'_p &= e^{-F} N_p^q \left(W_q - F_q W\right), \\ W'_{pq} &= e^{-F} N_p^k N_q^m [W_{km} - F_k W_m - F_m W_k - \left(F_{km} - F_k F_m\right) W \\ &\qquad - N_n^l \left(W_l - F_l\right) \partial_k M_m^n]. \end{aligned} \tag{3.9}$$

Restoring invariance/covariance of these expressions in the general case would be cumbersome at best. For this reason, we restrict our attention to effective supergravity theories with three moduli T^i, that have some promising features for a viable phenomenology (see for example [30]). These theories are classically invariant under the modular transformation:

$$\begin{aligned} T^i &\to T'^i = \frac{aT^i - ib}{icT^i + d}, \qquad ad - bc = 1, \\ \Phi^a &\to \Phi'^a = e^{-q_i^a F^i} \Phi^a = e^{-F^a} \Phi^a, \qquad F^i = \ln\left(icT^i + d\right), \end{aligned} \tag{3.10}$$

which effects the Kähler transformation (3.3) with

$$F(Z) = \sum_i F^i(T^i). \tag{3.11}$$

The parameters q_a^i are the modular weights of Φ^a. In the absence of a twisted sector potential ($W_i = 0$), the classical invariance is $SL(2,\mathcal{R}):\ a,b,c,d,\in\mathcal{R}$. In the presence of a twisted sector potential, this is broken to $SL(2,\mathcal{Z}):\ a,b,c,d,\in\mathcal{Z}$. We have

$$\begin{aligned} M_b^a &= e^{-F^a}\delta_b^a, \qquad M_j^i = \delta_j^i e^{-2F^i}, \qquad M_b^i = 0, \qquad M_j^a = -F_j^a e^{-F^a}\Phi^a, \\ N_b^a &= e^{F^a}\delta_b^a, \qquad N_j^i = \delta_j^i e^{2F^i}, \qquad N_b^i = 0, \qquad N_j^a = F_j^a e^{2F^j}\Phi^a, \\ F_i &\equiv F_{t^i} = F_{t^i}^i, \qquad F_{ij} = -\delta_{ij}F_i^2. \end{aligned} \tag{3.12}$$

Writing the superpotential W as a sum of monomials:

$$W = \sum_\alpha W^\alpha, \qquad W^\alpha = c_\alpha \prod_{n=1}^{N_\alpha}\Phi^a \prod_i [\eta(iT^i)]^{2\left(\sum_n q_i^q - 1\right)}, \tag{3.13}$$

where $\eta(iT)$ is the Dedekind function:

$$\eta(iT'^i) = e^{\frac{1}{2}F^i}\eta(iT^i), \tag{3.14}$$

the PV Kähler potential and superpotential can be made modular invariant and covariant, respectively, if we introduce three PV chiral superfields $\dot{Z}^N$ that transform under (3.10) as

$$\dot{Z}'^N = \dot{Z}^N - \dot{a}F_i^n\dot{Z}^I, \tag{3.15}$$

and modify (3.1) follows:

$$\begin{aligned} K^{\dot{Z}} &= K_0^{\dot{Z}} - \dot{a}^{-1}\Bigg[2\sum_{A,N=a,n}(1-q_n^a)\,\dot{Z}^N\dot{Z}^A K_a + 2\sum_{I,M=i,m}(1-\delta_{im})\,K_i\dot{Z}^I\dot{Z}^M \\ &\qquad + \dot{a}^{-1}\sum_{N,M=n,m}\left[\delta_{nm}(1+q_n) - 1 - q_{nm}\right]\dot{Z}^N\dot{Z}^M + \text{h.c.}\Bigg], \\ W_2^{\dot{Z}} &= W_{02}^{\dot{Z}} - 2\dot{a}^{-1}\sum_{\alpha;A,N=a,n}(1-q_n^a)\,\dot{Z}^N\dot{Z}^A W_a^\alpha - 2\dot{a}^{-1}\sum_{\alpha;I,M=i,m}(1-\delta_{im})\,W_i^\alpha\dot{Z}^I\dot{Z}^M \\ &\qquad + \dot{a}^{-2}\sum_{\alpha;N,M=n,m}\left[\delta_{nm}(1+q_n^\alpha) - 1 - q_{nm}^\alpha\right]W^\alpha\dot{Z}^N\dot{Z}^M, \end{aligned}$$

$$\begin{aligned} q_i &= \sum_a q_i^a \phi^a K_a, \qquad q_{ij} = \sum_a q_i^a q_j^a \phi^a K_a, \\ q_i^\alpha &= \sum_a q_i^a \phi^a \partial_a \ln W^\alpha, \qquad q_{ij}^\alpha = \sum_a q_i^a q_j^a \phi^a \partial_a \ln W^\alpha, \end{aligned} \tag{3.16}$$

which reduces to the result found in II for the untwisted sector: $\phi^a \to \phi^{ia}$, $q_j^{ia} = \delta_j^i$. Note that $q_i^\alpha, q_{ij}^\alpha$ are constants, whereas q_i, q_{ij} are not. Nevertheless we still get

$$A_{PN} = R_{\bar{n}PN\bar{m}} = A_{NM} = R_{\bar{n}NM\bar{m}} = 0. \tag{3.17}$$

so the one-loop corrections to K_R and ϕ_3 are unchanged, provided $K^{P\bar{Q}}$ does not change. We must also introduce a modular invariant Kähler metric for $\dot{Z}^N$; leaving $K^{P\bar{Q}}$ unchanged requires that it be of the form

$$K(\dot{Z}^N) = \sum_{n=N,m=M} \xi_{n\bar{m}} \left(\dot{Z}^N + \sum_{p=P} \chi_p^n \dot{Z}^P \right) \left(\dot{\bar{Z}}^{\bar{M}} + \sum_{q=Q} \bar{\chi}_{\bar{q}}^{\bar{m}} \dot{\bar{Z}}^{\bar{Q}} \right), \tag{3.18}$$

where the metric $\xi_{n\bar{m}}(Z^i, \bar{Z}^{\bar{\jmath}})$ is modular invariant and

$$\chi_p'^n(Z', \bar{Z}') = N_p^q \left(\chi_q^n(Z, \bar{Z}) + \dot{a} F_q \right). \tag{3.19}$$

As shown in Appendix D, for a general orbifold metric, this leads to unwanted contributions to $\mathcal{L}_Q$ and $\Phi_{1,2}^{PV}$ unless $\xi_{n\bar{m}} = \delta_{nm}$ and χ_p^n is holomorphic, *e.g.* $\chi_p^n = 2\dot{a}\partial_p \ln \eta(iT^n)$. Alternatively, most of the unwanted terms can be canceled by including several copies of the above: $\dot{Z} \to \dot{Z}_\alpha$, $\alpha = 1, \ldots, 2n_{\dot{Z}} + 1$ with signatures $\eta_\alpha^{\dot{Z}}$ and parameters $\dot{a}_\alpha$ such that

$$\sum_\alpha \eta_\alpha^{\dot{Z}} = -1, \qquad \sum_\alpha \eta_\alpha^{\dot{Z}} \dot{a}_\alpha^2 = \sum_\alpha \eta_\alpha^{\dot{Z}} \dot{a}_\alpha^4 = 0. \tag{3.20}$$

Only one of these with negative metric need have the superpotential and additional Kähler potential couplings in (3.16) required to regulate the Kähler potential corrections and ϕ_3. Then the only remaining unwanted contribution is from the curvature terms associated with the metric $\xi_{n\bar{m}}$, which vanish if this metric is flat, or can be canceled by the introduction of additional gauge-singlet chiral PV multiplets with positive signature and just the metric $\xi_{n\bar{m}}$. The correct procedure cannot be determined without further knowledge of the string loop corrections, and/or higher order terms in the classical twisted sector metric. We will consider only the case $\xi_{n\bar{m}} = \delta_{nm}$, with just a few simple choices for χ_p^n.

3.2 Gauge couplings

To regulate light gauge field loops, we must introduce at least three Pauli-Villars chiral supermultiplets $\varphi^a, \tilde{\varphi}^a, \hat{\varphi}^a$, with positive signature, that transform according to the adjoint representation of the gauge group. Their contributions cancel the term proportional to C_a in (2.8). To cancel the divergences associated with the term proportional to $\mathcal{K}^a_a$ in (2.3), we need at least one chiral supermultiplet $\widehat{Y}_P$ that transforms according to the representation of the gauge group conjugate to that of Z^P, with positive signature, and superpotential coupling [2]

$$W_2^\phi = 2\varphi^a \widehat{Y}_P (T_a Z)^p, \tag{3.21}$$

which is modular covariant if under (3.10) $\widehat{Y}_P$ and φ^a transforms as

$$\widehat{Y}'_P = N^q_p \widehat{Y}_Q, \qquad \varphi'^a = e^{-F}\varphi^a. \tag{3.22}$$

Then the Kähler potentials

$$K_0^{\widehat{Y}} = \sum_{P,Q=p,q} K^{p\bar{q}} \widehat{Y}_P \bar{\widehat{Y}}_{\bar{Q}}, \qquad K^\varphi = e^G \sum_a |\varphi^a|^2 \tag{3.23}$$

are gauge and modular invariant, and

$$A^{\widehat{Y}\varphi}_{Pa} = 2e^K (T_a z)^p, \qquad \bar{A}^{Pa}_{\widehat{Y}\varphi} = 2e^{-G}(T^a \bar{z})^{\bar{m}} K_{p\bar{m}}, \qquad 2\sum_{Pa} A^{\widehat{Y}\varphi}_{Pa} \bar{A}^{Pa}_{\widehat{Y}\varphi} = 4\mathcal{K}^a_a. \tag{3.24}$$

However $\widehat{Y}_P$ loops contribute terms proportional to Φ_1, Φ_2 and $C_a^M \Phi_{YM}$ in (2.8), and $\mathcal{D}^\alpha \Gamma_\alpha|$ in (2.16). Therefore we must introduce a set $\widehat{Y}^\alpha_P$ of such fields with signatures $\eta^{\widehat{Y}}_\alpha$ such that $\sum_\alpha \eta^{\widehat{Y}}_\alpha = 0$, with at least one of these, say $\widehat{Y}^1_P$, $\eta^{\widehat{Y}}_1 = +1$, participating in the coupling (3.21).

In order to introduce PV masses that are gauge invariant under the nonanomalous gauge group, we have to introduce equal numbers of PV fields that transform like Z^p and its conjugate representation. This requires that we also introduce additional charged PV fields U^A_β and U^β_A that transform according to the representation R^a_A and its conjugate, respectively, under the nonanomalous gauge group factor $\mathcal{G}_a$, and/or fields V^A_β that transform according to a (pseudo)real representation that is traceless and anomaly-free. Their gauge couplings must satisfy

$$\sum_{A\in U,V} \eta^A_\beta C^a_A = C^a_M, \tag{3.25}$$

where

$$\mathrm{Tr}_R \left(T^a T^b\right) = \delta_{ab} C^a_R, \tag{3.26}$$

which may imply a constraint on the matter representations of the gauge group in the light spectrum, as discussed in I. This constraint can be trivially satisfied for all $U(1)$'s. It is satisfied in any supersymmetric extension of the Standard Model (SM). In addition to two Higgs doublets, its minimal supersymmetric extension, the MSSM, has $2N_f$ fundamental representations (reps) $\mathbf{n}$ of each factor $\mathcal{G}_n = SU(n),\ n = 2,3$, where N_f is the number of quark flavors. Their Casimirs can be mimicked by N_f real PV reps $(\mathbf{n}+\bar{\mathbf{n}})$. Further extensions necessarily involve real reps of the SM gauge group, so that the additional states can get SM gauge invariant masses. The condition (3.25) is also satisfied in the hidden sectors [31] that can accompany the SM-like Z_N orbifolds found in [30]. These hidden sectors also come in even numbers of reps + antireps, except for one hidden sector that contain 3 **16**'s of $SO(10)$ that contribute $C_M^{SO(10)} = 6$, which can be mimicked by a real PV rep with 6 **10**'s, and another hidden sector with 3 $(\mathbf{5}+\mathbf{10})$'s and 6 $\bar{\mathbf{5}}$'s of $SU(5)$ with $C_M^{SU(5)} = 9$, that can be mimicked by 9 real PV reps $(\mathbf{5}+\bar{\mathbf{5}})$. Since the underlying theory is finite when all degrees of freedom are included, one would expect (3.25) to have a solution for general superstring compactifications.

3.3 Nonrenormalizable couplings

In order to cancel the term proportional to $A_p\bar{A}^p$ in (2.3) we need a PV superpotential coupling of the form [2]

$$W_2 \ni W_2^{\widehat{Y}} = \sum_\alpha \left[\hat{a}_\alpha W_p \widetilde{Z}^P_\alpha \widehat{Y}^\alpha_0 + W \widetilde{Z}^P_\alpha \widehat{Y}^\alpha_P\right], \tag{3.27}$$

where $\widetilde{Z}^P, \widehat{Y}_P$ transform like $Z^p, \bar{Z}^{\bar{p}}$, respectively, under the gauge group, and $(G_{p\bar{q}} \equiv K_{p\bar{q}},\ Z^q \neq S)$

$$\begin{aligned}
K^{\widetilde{Z}}_\alpha &= \sum_{P,Q=p,q} G_{p\bar{q}} \widetilde{Z}^P_\alpha \bar{\widetilde{Z}}^{\bar{Q}}_\alpha, \\
K^{\widehat{Y}}_\alpha &= \sum_{P,J=p,j} G^{p\bar{q}} \widehat{Y}^\alpha_P \bar{\widehat{Y}}^\alpha_{\bar{Q}} - \hat{a}_\alpha \sum_{P=p}\left(\widehat{Y}^\alpha_P \bar{\widehat{Y}}^\alpha_0 G^p + \text{h.c.}\right) + |\widehat{Y}^\alpha_0|^2\left[1 + \hat{a}^2_\alpha G^p G_p\right], \\
\eta^{\widehat{Y}}_\alpha &= \eta^{\widetilde{Z}}_\alpha, \qquad \hat{N} = \sum_\alpha \eta^{\widehat{Y}}_\alpha = 0, \qquad G^p = G^{p\bar{q}} G_{\bar{q}}, \qquad G^{p\bar{q}} G_{\bar{q}k} = \delta^p_k.
\end{aligned} \tag{3.28}$$

The superpotential (3.27) is covariant and the Kähler potential (3.28) are invariant under modular transformations provided that under (3.3)

$$\widehat{Y}'^\alpha_P = N^q_p\left(\widehat{Y}^\alpha_Q + a_\alpha F_q \widehat{Y}_0\right), \qquad \widehat{Y}'_0 = \widehat{Y}_0, \qquad \widetilde{Z}'^Q_\alpha = M^q_p \widetilde{Z}^P_\alpha. \tag{3.29}$$

The same form of the Kähler potential is in fact needed for the PV fields $\widehat{Y}_P$ that couple to φ^a in (3.21) in order to cancel [2, 3] the term proportional to $\mathcal{D}$ in (2.3). We therefore generalize these to a subset of the $\widehat{Y}_\alpha$'s with the same signatures as a subset of φ^a_α, with the full set satisfying $\sum_\alpha \eta^\varphi_\alpha = 1$, and take for the full superpotential W_2

$$W_2 \quad = \quad W_2^{\dot{Z}} + W_2^{\widehat{Y}} + 2\sum_\alpha g_\alpha \varphi^a_{1+\alpha} \widehat{Y}^\alpha_P (T_a Z)^p + c_\alpha \phi^S_\alpha \phi^\alpha_S W, \tag{3.30}$$

where ϕ^S, ϕ_S are gauge singlets needed to cancel dilaton loop contributions, with a Kähler potential

$$K^S = \sum_\alpha \left[e^{-2k}|\phi^\alpha_S|^2 + 2|\phi^\alpha_0|^2 - e^{-k}\left(\bar{\phi}^\alpha_{\bar{S}}\phi^\alpha_0 + \text{h.c.}\right) + e^{2k}|\phi^S_\alpha|^2 \right], \tag{3.31}$$

that involves an additional chiral superfield ϕ_0, analogous to $\widehat{Y}_0$; this field is also needed [see (3.33)] to regulate dilaton couplings. The superpotential (3.30) is modular covariant and the Kähler potential (3.31) is modular invariant if the superfields $\phi^r = \phi^S, \phi_S, \phi_0$ are modular invariant. Note that because $F(Z)$ is holomorphic and gauge invariant it satisfies

$$(T^a Z)^p F_p = 0, \tag{3.32}$$

so (3.21) remains modular covariant with the modification (3.28) of the Kähler potential for $\widehat{Y}_I$. To cancel the remaining divergences arising from nonrenormalizable couplings we introduce gauge singlets ϕ^γ, as well as $U(1)$ vector supermultiplets $V_\gamma = V^0_\gamma, V^s_\gamma$, with signatures $\eta^0_\gamma, \eta^s_\gamma$, respectively, that form massive vector supermultiplets with chiral multiplets $\Phi^{0,s}_\gamma = e^{\theta^{0,s}_\gamma}$ of the same signature and $U(1)_\beta$ charges $q_\gamma\delta_{\gamma\beta}$. We also need additional chiral multiplets $\tilde{\varphi}^\alpha, \hat{\varphi}^\alpha$ in the adjoint representation of the low energy gauge group; one of these sets, $\hat{\varphi}^\alpha$, participates in the matrix-valued gauge kinetic function that couples the PV superfield strengths W^0_γ, W^s_γ and the light gauge fields W^a to one another in the Yang-Mills kinetic term:

$$\begin{aligned} f^{ab} &= \delta^{ab}\left(S + \sum_\alpha h^S_\alpha \phi^S_\alpha \phi^\alpha_0\right), \qquad f^{a\gamma}_s = 0, \\ f^0_{\gamma\beta} &= \delta_{\gamma\beta}, \qquad f^s_{\gamma\beta} = \delta_{\gamma\beta} S, \qquad f^{a\gamma}_0 = \sum_\beta e^{\gamma\beta}\hat{\varphi}^a_\beta. \end{aligned} \tag{3.33}$$

These couplings are modular invariant provided θ_γ and $\hat{\varphi}^a_\alpha$ are modular invariant. The matrices $d_{\alpha\beta}, e_{\alpha\beta}$, are nonvanishing only when they couple fields of the same signature. Including the fields U, V, as well as additional chiral supermultiplets ϕ_γ needed to regulate gravity loops [1], the full Kähler potential takes the form

$$\begin{aligned} K_{PV} \quad = \quad & \sum_\gamma \left[\frac{1}{2} f_{\phi_\gamma}\phi^\gamma\bar{\phi}_\gamma + \frac{1}{2}\nu_\gamma(\theta_\gamma + \bar{\theta}_\gamma)^2 + \sum_A \left(f_{U^A_\gamma}|U^A_\gamma|^2 + f_{U^\gamma_A}|U^\gamma_A|^2 + f_{V^A_\gamma}|V^A_\gamma|^2 \right) \right] \\ & + \sum_{\alpha,a}\left(e^G \varphi^a_\alpha \bar{\varphi}^\alpha_a + e^k \hat{\varphi}^a_\alpha \bar{\hat{\varphi}}^\alpha_a + \tilde{\varphi}^a_\alpha \bar{\tilde{\varphi}}^\alpha_a \right) + \sum_\alpha \left(K^{\widetilde{Z}}_\alpha + K^{\widehat{Y}}_\alpha \right) + K^{\dot{Z}} + K^S. \end{aligned} \tag{3.34}$$

As shown in I and II, the ultraviolet divergences from light loops are canceled if the PV coupling constants satisfy[8]

$$\begin{aligned}
\sum_\alpha \eta_\alpha^{\widehat{Y}} \hat{a}_\alpha^2 &= -2, \qquad \sum_\alpha \eta_\alpha^{\widehat{Y}} \hat{a}_\alpha^4 = +2, \\
\sum_\alpha \eta_\alpha^{\phi^S} c_\alpha^2 &= 5, \qquad \sum_\alpha \eta_\alpha^{\widehat{Y}} g_\alpha^2 \hat{a}_\alpha^2 = -1, \qquad \sum_\alpha \eta_\alpha^{\widehat{Y}} g_\alpha^2 = 1, \\
\frac{1}{2}\sum_{\alpha,\beta} \eta_\alpha^{\hat{\varphi}} e_{\alpha\beta}^2 &= -4 = \frac{3}{4}\sum_{\alpha\beta\gamma\delta} \eta_\gamma^{\hat{\varphi}} e_\alpha^\beta e_\beta^\gamma e_\gamma^\delta e_\delta^\alpha, \qquad \sum_\alpha \eta_\alpha^{\phi^S} h_\alpha^S c_\alpha = 1, \\
\sum_\alpha \eta_\alpha^{\phi^S} (h_\alpha^S)^2 &= 2, \qquad \sum_\gamma \eta_\gamma^s = -N_G,
\end{aligned} \tag{3.35}$$

and the PV signatures satisfy

$$\begin{aligned}
\sum_\alpha \eta_\alpha^{\varphi^a} &= \sum_\alpha \eta_\alpha^{\hat{\varphi}^a} = \sum_\alpha \eta_\alpha^{\tilde{\varphi}^a} = 1, \qquad \eta_{1+\alpha}^{\varphi^a} = \eta_\alpha^{\widehat{Y}}, \qquad \eta_1^{\varphi^a} = +1, \qquad \eta_\alpha^{U^A} = \eta_\alpha^{U_A}, \\
\sum \eta_\alpha^{\dot{Z}} &= \sum_\alpha \eta_\alpha^{\phi^S} = -1, \qquad \sum_\alpha \eta_\alpha^{\widetilde{Z}} = 0, \qquad \eta_\alpha^{\widetilde{Z}} = \eta_\alpha^{\widehat{Y}}, \qquad \eta_\alpha^{\phi^S} = \eta_\alpha^{\phi_S} = \eta_\alpha^{\phi_0}, \\
\sum_\gamma \eta_\gamma^\theta &= \sum_\gamma \eta_\gamma^s + \sum_\gamma \eta_\gamma^0 = -12 - N_G = N'_G, \qquad \sum_P \eta^C = -29 - N = N',
\end{aligned} \tag{3.36}$$

where C is any PV chiral superfield and $N(N_G)$ is the number of chiral (gauge) superfields in the light sector. Cancellation of quadratic divergences requires that the pre-factors f_C in (3.34) satisfy:

$$\sum_C \eta_C \partial_i \ln f_C = -10K_i, \qquad \sum_C \eta_C \partial_{\bar{\imath}} \ln f_C = -10K_{\bar{\imath}}, \tag{3.37}$$

and cancellation of logarithmic divergences requires

$$\sum_C \eta_C \partial_i \ln f_C \partial_{\bar{\jmath}} \ln f_C = -4K_i K_{\bar{\jmath}} + 2k_i k_{\bar{\jmath}}. \tag{3.38}$$

where in (3.37) ϕ^C is any PV chiral superfield except $\dot{Z}, \widetilde{Z}, \widehat{Y}, \phi^S, \phi_{S,0}$; for example

$$f_\varphi = e^{K-k}, \qquad f_{\hat{\varphi}} = e^k, \qquad f_{\tilde{\varphi}} = 1. \tag{3.39}$$

The Kähler potentials for φ_α^a and $\hat{\varphi}_\alpha^a$ are determined by their couplings so as to cancel light field divergences, while the choice for $\tilde{\varphi}_\alpha^a$ assures the Kähler anomaly matching condition for the gauge

[8]There are extraneous factors of e^K and W in the expression for $\bar{A}^{IJ}_{Z_\alpha, Y_\alpha}$ in (2.36) of I.

loop contribution to the term quadratic in the Yang-Mills field strength, which requires [12] that, averaged over the adjoint PV fields, $\langle \ln f \rangle_\varphi = K/3$. The parameters ν_γ in (3.34) determine the squared PV masses of the PV vector supermultiplets and the corresponding eaten chiral supermultiplets θ_γ; these masses play the role of effective (squared) cut-offs. The PV masses μ_α of the remaining PV chiral multiplets will be introduced through superpotential terms in section 4.2.

4 Anomaly Cancellation

In the context of orbifold compactification of the heterotic string the known mechanisms for cancellation of the effective field theory anomalies are the four-dimensional GS mechanism and string loop threshold corrections. The latter can be parametrized as moduli-dependent PV masses. In this section we outline the needed generalization of the GS mechanism, show that it restricts the form of PV mass terms, and construct an explicit PV superpotential that satisfies these restrictions.

4.1 Strategies for anomaly cancellation

The regularized theory would be invariant under the classical symmetries if it were possible to introduce PV mass terms that respect these symmetries and also cancel all the ultraviolet divergences. That this is not possible can easily be seen by looking at the quadratic divergences. For example, in (2.16)

$$\begin{aligned} \mathcal{D}^\alpha \Gamma_\alpha| &= 2x^{-1}\mathcal{D}_a D_p (T^a z)^p - 2R_{p\bar{m}}\left(e^{-K}\bar{A}^p A^{\bar{m}} + \mathcal{D}_\mu z^p \mathcal{D}^\mu \bar{z}^{\bar{m}}\right), \\ D_p(T^a z)^p &= \mathrm{Tr}T^a + \Gamma^p_{pq}(T^a z)^q. \end{aligned} \tag{4.1}$$

If $U(1)_X$ is anomalous, $\mathrm{Tr}T^X \neq 0$, one cannot regulate the quadratic divergences without introducing at least one mass term for a pair of PV chiral superfields U, U' with $U(1)_X$ charges $q_X + q'_X \neq 0$. Similarly, if the low energy theory possesses a classical invariance under a nonlinear symmetry that effects a Kähler transformation (3.3), a mass term $W_U = \mu U U'$ with constant μ is covariant if $K_{U\bar{U}} K_{U'\bar{U}'} = e^K$. In this case

$$\mathcal{D}^\alpha \Gamma^U_\alpha\big| + \mathcal{D}^\alpha \Gamma^{U'}_\alpha\Big| = 2\,\mathcal{D}^\alpha X_\alpha| + 2x^{-1}\mathcal{D}_X\left(q_X + q'_X\right). \tag{4.2}$$

The first term on the RHS of (4.2) cancels the U, U' counterpart of the term proportional to $N\ \mathcal{D}^\alpha X_\alpha|$ in (2.16). Thus, as discussed in I, quadratic divergences associated with scalar curvature cannot be regulated by PV fields with invariant masses. For the orbifold-derived supergravity models considered here, only a discrete group of modular transformations is expected to survive at the quantum level. As will be seen below it is always possible to construct PV masses that are invariant under this discrete group by including holomorphic functions of the moduli that have well-defined modular weights: $w(T') = e^{\sum_i F^i \omega_i} w(T)$. The appearance of these factors in the effective one-loop Lagrangian would be interpreted as string-loop threshold effects. However it is known from string-loop calculations [8, 9] in orbifold compactifications that these effects do not fully cancel the anomaly. Indeed there are no threshold corrections to the Yang-Mills Lagrangian in Z_3 and Z_7 orbifolds, where the anomaly is completely canceled by the four dimensional analogue [10, 11] of the Green-Schwarz term. Moreover, the $U(1)_X$ anomalies can be canceled only by such a mechanism [32].

The GS mechanism is most easily displayed in the linear multiplet formalism [13] for the dilaton, where the GS term takes the form

$$\mathcal{L}_{GS} = \int d^4\theta E L \left[b g(Z, \bar{Z}) - \frac{1}{2}\delta_X V_X \right] \equiv \int d^4\theta E L V_{GS}, \tag{4.3}$$

where b and δ_X are constants, V_X is the $U(1)_X$ vector superfield, and g is a real superfield which, under a modular transformation (3.3), transforms as

$$g(Z', \bar{Z}') = g(Z, \bar{Z}) + F(Z) + \bar{F}(\bar{Z}). \tag{4.4}$$

The real superfield L satisfies the modified linearity conditions

$$(\bar{\mathcal{D}}^2 - 8R)L = -W_a^\alpha W_\alpha^a, \qquad (\mathcal{D}^2 - 8\bar{R})L = -W_{\dot{\alpha}}^a W_a^{\dot{\alpha}}, \tag{4.5}$$

where W_α^a is the gauge superfield strength. Under (4.4) and a gauge transformation:

$$V_X \to V'_X = V_X + \left(\Lambda + \bar{\Lambda}\right), \tag{4.6}$$

where Λ is a chiral superfield, (4.3) transforms as, using (4.5),

$$\begin{aligned}
\mathcal{L}_{GS} &\to \mathcal{L}'_{GS} = \mathcal{L}_{GS} + \Delta\mathcal{L}_{GS}, \\
\Delta\mathcal{L}_{GS} &= \int d^4\theta E L \left(bF - \frac{\delta_X}{2}\Lambda \right) + \mathrm{h.c} = -\int d^4\theta \frac{E}{8R} \left(\bar{\mathcal{D}}^2 - 8R\right) L \left(bF - \frac{\delta_X}{2}\Lambda \right) + \mathrm{h.c} \\
&= \int d^4\theta \frac{E}{8R} W_a^\alpha W_\alpha^a \left(bF - \frac{\delta_X}{2}\Lambda \right) + \mathrm{h.c},
\end{aligned} \tag{4.7}$$

which has the same form as the quantum anomaly from renormalizable couplings of particles charged under the gauge group $\mathcal{G}_a$ that also carry modular weights and/or $U(1)_X$ charge. The constants b, δ_X can be chosen to cancel the anomalous term for any one $\mathcal{G}_a$. In orbifold compactifications of the heterotic string, the couplings satisfy constraints that allow universal $U(1)_X$ anomaly cancellation. Modular anomaly cancellation is also universal in orbifolds with no $N = 2$ twisted sector. Otherwise there are moduli-dependent threshold corrections that contribute to the anomaly cancellation. As discussed in I, these can be included in the PV regulator masses.

The GS mechanism can be generalized to cancel anomalous coefficients of higher dimension operators by generalizing the modified linear condition:

$$(\bar{\mathcal{D}}^2 - 8R)L = -\Phi, \qquad (\mathcal{D}^2 - 8\bar{R})L = -\bar{\Phi}, \tag{4.8}$$

where Φ is a chiral superfield with chiral Kähler weight $w(\Phi) = 2$, provided higher dimension operators are also included in the tree Lagrangian:

$$\begin{aligned} \mathcal{L}_{tree}(L) &= -\int d^4\theta E\,[3 - 2Ls(L)] \\ &= \int d^4\theta \frac{E}{16R}(\bar{\mathcal{D}}^2 - 8R)\,[3 - 2Ls(L)] + \text{h.c.} \\ &= \int d^4\theta \frac{E}{8R} s(L)\Phi + \text{h.c.} + \cdots, \end{aligned} \tag{4.9}$$

where the ellipsis represents the kinetic terms for the various components of L and the supergravity multiplet. The Kähler potential (2.1) in this formalism is $K = k(L) + G(Z, \bar{Z})$, and the relation

$$k'(L) + 2Ls'(L) = 0 \tag{4.10}$$

between the real functions $k(L)$ and $s(L)$ assures a canonical Einstein term. It follows from (4.8) that Φ is the chiral projection of a real field Ω with Kähler weight $w_K(\Omega) = 0$ and Weyl weight $w_W(\Omega) = w_W(L) = -w_W(E) = 2$, and that it satisfies a generalized Bianchi identity:

$$\left(\bar{\mathcal{D}}^2 - 24R\right)\Phi - \left(\mathcal{D}^2 - 24\bar{R}\right)\bar{\Phi} = \text{total derivative.} \tag{4.11}$$

When $\Phi = W_a^\alpha W_\alpha^a$ as in (4.5), the first term on the RHS of (4.9) reduces to the Yang-Mills kinetic term, and $\Omega = \Omega_{YM}$, the Chern-Simons (CS) superfield. To obtain the Lagrangian for the chiral multiplet formulation we make a superfield duality transformation by writing [13]

$$\mathcal{L} = -\int d^4\theta E\left[3 - 2Ls(L) + (L + \Omega)\left(S + \bar{S}\right)\right], \tag{4.12}$$

where L is unconstrained and S is chiral. Writing S as the chiral projection of an unconstrained complex field Σ: $S = (\bar{\mathcal{D}}^2 - 8R)\Sigma$, the equations of motions for Σ and $\Sigma^\dagger$ give the generalized linearity constraint (4.8). If instead we solve the equations of motion for L, we obtain [13] L as a function of $S+\bar{S}$ such that

$$s(L) = (S+\bar{S})/2, \tag{4.13}$$

and we recover the standard chiral superfield formulation [4] of supergravity, with a canonical Einstein term provided (4.10) holds, except that the standard Yang-Mills term is generalized to

$$\mathcal{L}_{YM} \to \int d^4\theta \frac{E}{8R} S\Phi + \text{h.c.}, \tag{4.14}$$

provided that

$$\frac{\delta}{\delta L}\left(E\Omega\right) = 0. \tag{4.15}$$

When the GS term (4.3) is included, (4.13) is modified to

$$s(L) = (S+\bar{S}-V_{GS})/2. \tag{4.16}$$

Since L is invariant under modular transformations (4.4) and gauge transformations (4.6), the chiral superfield S is not invariant:

$$S \to S' = S + bF(Z) - \delta_X\Lambda/2. \tag{4.17}$$

Then from either (4.3) in the linear formulation or (4.14) in the chiral formulation, we get a variation in the Lagrangian

$$\Delta\mathcal{L} = \int d^4\theta \frac{E}{8R}\Phi\left(bF - \delta_X\Lambda/2\right) + \text{h.c.}. \tag{4.18}$$

In Section 3 we specified the PV Lagrangian in terms of modular and $U(1)_X$ invariant Kähler potential K_{PV} and gauge kinetic functions (3.33). As a consequence the massive Abelian PV vector fields necessarily have modular invariant masses that do not contribute to the anomaly. We further imposed $U(1)_X$ invariance and modular covariance on the part W_2 of the PV superpotential that cancels divergences from light chiral supermultiplet loops. The chiral and conformal anomalies of the low energy quantum field theory arise from linear and logarithmic divergences, respectively. If we assume that they are canceled when the theory is properly regulated, the only noninvariance in the regulated theory appears in the part W_1 of the superpotential that gives a large supersymmetric mass matrix m to PV chiral supermultiplets. In this case the one loop action can be written as

$$\mathcal{L}_1 = \mathcal{L}_{inv} + \mathcal{L}_{ni}, \qquad \mathcal{L}_{ni} = \frac{i}{2}\mathrm{STr}\ln\left[D^2 + H(m)\right]_{ni} + T_{ni}(m), \tag{4.19}$$

where $\mathcal{L}_{inv}$ is modular invariant, T is the helicity-odd fermion contribution, and $\mathcal{L}_{ni}$ arises only from chiral supermultiplet loop contributions. As shown in I, under a transformation on the PV fields that leaves the tree Lagrangian, the PV Kähler potential and the PV gauge couplings invariant, with W_2 covariant:

$$\begin{aligned}\Phi' &= g\Phi, \qquad m'(\Phi) = m(\Phi') \\ \mathcal{L}'_1 &= \mathcal{L}_{inv} + \mathcal{L}_{ni}(\tilde{m}), \qquad \tilde{m} = g^{-1}m'g,\end{aligned} \tag{4.20}$$

because all the operators in the determinants are covariant except the matrix m_{ni} for the chiral multiplets whose PV masses arise from the noncovariant part of W_1. The residual quantum anomaly can be canceled by the GS term provided

$$\mathcal{L}_{ni}(m) = \int d^4\theta E \mathrm{STr}\left[\Omega \ln(m\bar{m})\right], \qquad \tilde{m}_{ni} = e^{\sum_i Q_i F^i + Q_X \Lambda_X} m_{ni}, \tag{4.21}$$

with, for example

$$\mathrm{STr}\left(\Omega_{\mathrm{YM}} Q_i\right) = -b\Omega_{\mathrm{YM}}, \qquad \mathrm{STr}\left(\Omega_{\mathrm{YM}} Q_X\right) = \delta_X \Omega_{\mathrm{YM}}/2, \tag{4.22}$$

where Ω_{YM} is the (matrix valued) Yang-Mills Chern-Simons form:

$$(\bar{\mathcal{D}}^2 - 8R)\Omega_{\mathrm{YM}} = W_a^\alpha W_\alpha^a. \tag{4.23}$$

As mentioned in the introduction, the linear divergences are not completely canceled in the PV regulated theory. Cancellation of *on-shell* quadratic divergences imposed the two constraints given in (2.20) of I, which can be recast in the form

$$N + N' - N_G - N'_G = 3 + 2\alpha, \qquad N + N' - 3N_G - 3N'_G = 7, \qquad \alpha = \sum_C \eta_C \ln f_C/K. \tag{4.24}$$

The first of these assures the cancellation of the linear divergence associated with the spin connection; as a consequence the associated anomaly arises only from PV masses, and, in the absence of threshold corrections, one gets a contribution [see (C.44)]

$$\begin{aligned}\frac{16\pi^2}{\sqrt{g}}\Delta\mathcal{L}^\chi_{\mathrm{spin}} &= -\frac{1}{48} r\cdot\tilde{r}\sum_n\left(N' - N'_G - 2\alpha + 2\sum_p q^p_n\right)\mathrm{Im}F^n \\ &= \frac{1}{48} r\cdot\tilde{r}\sum_n\left(N - N_G - 3 - 2\sum_p q^p_n\right)\mathrm{Im}F^n.\end{aligned} \tag{4.25}$$

However the linear divergence associated with the affine connection in gravitino loops is not canceled, and we get an additional contribution to the chiral anomaly:

$$\frac{16\pi^2}{\sqrt{g}}\Delta\mathcal{L}^\chi_{\rm affine} = \frac{24}{48}r\cdot\tilde{r}{\rm Im}F^n. \tag{4.26}$$

The conformal anomaly counterpart of (4.25), namely

$$\begin{aligned}\frac{16\pi^2}{\sqrt{g}}\Delta\mathcal{L}^c_{\rm spin} &= \sum_n\left(N - N_G - 3 - 2\sum_p q^p_n\right)L_{GB}{\rm Re}F^n, &(4.27)\\ L_{GB} &= \frac{1}{48}\left(r_{\mu\nu\rho\sigma}r^{\mu\nu\rho\sigma} - 4r_{\mu\nu}r^{\mu\nu} + r^2\right), &(4.28)\end{aligned}$$

automatically combines with (4.25) to form the supersymmetric combination contained in the operator Φ_W in (2.8). This is because Pauli-Villars regularization manifestly preserves supersymmetry, in contrast to a straight momentum cut-off procedure, which does not.

The second constraint in (4.24) determines the PV mass-independent coefficient of the Gauss-Bonnet term in the regularized one-loop effective Lagrangian:

$$\frac{16\pi^2}{\sqrt{g}}\mathcal{L}^c_{GB} = L_{GB}(N + N' - 3N_G - 3N'_G + 41)\ln\Lambda = 48L_{GB}\ln\Lambda. \tag{4.29}$$

Since L_{GB} is a total derivative, this drops out of the effective action if Λ is constant. If instead, $\Lambda = \mu_0 e^{\alpha_\Lambda K}$, there is a finite, anomalous contribution to the effective action with

$$\frac{16\pi^2}{\sqrt{g}}\Delta\mathcal{L}^c_{GB} = 96\alpha_\Lambda L_{GB}{\rm Re}F. \tag{4.30}$$

Supersymmetry requires that we take $\alpha_\Lambda = \frac{1}{4}$, in which case the residual contribution to the anomaly in Kähler superspace is given by

$$\Delta\mathcal{L}^{\rm res}_{GB} = \frac{1}{8\pi^2}\int d^4\theta E F^n(T)\Omega_{GB} + {\rm h.c.}, \qquad \Omega_{GB} = -8\Omega_W - \frac{4}{3}\Omega_X - 3\phi_\chi, \tag{4.31}$$

which is of the required form, (4.15), with Ω_W the Chern-Simons form for the curvature superfield strength, and Ω_X the Chern-Simons superfield for the chiral superfield Φ_X introduced in (2.8):

$$(\bar{\mathcal{D}}^2 - 8R)\Omega_W = W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} = \Phi_W, \qquad (\bar{\mathcal{D}}^2 - 8R)\Omega_X = \Phi_X, \tag{4.32}$$

and ϕ_χ is defined in (2.9). Including the PV contribution, the total anomaly involving the curvature strength becomes

$$\Delta\mathcal{L}_{GB} = \frac{1}{8\pi^2}\frac{1}{24}\sum_n\left(N - N_G + 21 - 2\sum_p q^p_n\right)\int d^4\theta E F^n(T)\Omega_{GB} + {\rm h.c.}, \tag{4.33}$$

The off-diagonal gravitino-gaugino connection leads to a contribution to the chiral anomaly (see Appendix B.3)

$$\frac{16\pi^2}{\sqrt{g}}\Delta\mathcal{L}^{\chi}_{\rm off} \quad = \quad -2{\rm Im}F\mathcal{D}^{\mu}\left(xF^{\rho\nu}_{a}\mathcal{D}_{\rho}\tilde{F}^{a}_{\mu\nu}\right), \tag{4.34}$$

which combines with the conformal anomaly associated with total divergences from mixed gauge-gravity loop contributions:

$$\frac{16\pi^2}{\sqrt{g}}\Delta\mathcal{L}^{c}_{\rm mixed} \quad = \quad -8\alpha_{\Lambda}{\rm Re}F\mathcal{D}^{\mu}\left(xF^{\rho\nu}_{a}\mathcal{D}_{\rho}F^{a}_{\mu\nu} - xF^{a}_{\mu\nu}\mathcal{D}_{\rho}F^{\rho\nu}_{a} - 2\mathcal{D}_{a}\mathcal{D}^{\nu}F^{a}_{\nu\mu}\right) + \cdots, \tag{4.35}$$

to give an overall contribution

$$\mathcal{L}_{\rm Mx} = \frac{1}{8\pi^2}\int d^4\theta EF\Omega_{\rm Mx} + {\rm h.c.}, \tag{4.36}$$

where

$$\Omega_{\rm Mx} \quad = \quad -\frac{1}{4}\left[\mathcal{D}^{\alpha},\mathcal{D}^{\dot{\beta}}\right]\left[(S+\bar{S})W_{\alpha}W_{\dot{\beta}}\right] - \frac{1}{16}\left(\mathcal{D}^2\left[(S+\bar{S})W^{\alpha}W_{\alpha}\right] + {\rm h.c.}\right) + \cdots, \tag{4.37}$$

and the ellipses in (4.35) and (4.37) represent terms with derivatives of the dilaton superfield[9] S (as well as fermionic terms). As discribed in Appendix B.3, this contribution to the anomaly is cancelled by mixed PV gauge-chiral matter loops for a specific choice of the V_0-θ_0/φ^a-$\hat{\varphi}^a$ squared PV mass ratio. We will adopt this choice.

In addition there are total derivatives with logarithmically divergent coefficients that generate a D-term conformal anomaly with no chiral counterpart. Specifically there is a contribution from chiral loops and mixed chiral-gravity loops

$$\frac{16\pi^2}{\sqrt{g}}\Delta\mathcal{L}^{c}_{\chi{\rm G}} = -8\alpha_{\Lambda}{\rm Re}FK_{i\bar{m}}\mathcal{D}^{\mu}\left[\mathcal{D}_{\mu}\bar{z}^{\bar{m}}\left(\bar{A}^{i}Ae^{-K} + D_{\nu}\mathcal{D}^{\nu}z^{i}\right) - D_{\nu}\mathcal{D}_{\mu}\bar{z}^{\bar{m}}\mathcal{D}^{\nu}z^{i} + {\rm h.c.}\right]. \tag{4.38}$$

Given that it is not trivial to identify all the total derivatives that were dropped in [16] and [17], we cannot guarantee that there are no other such D-term anomalies, so for present purposes we will subsume these into an operator that we will call Ω'_D.

In order to preserve the correct form of the anomaly, we require that PV chiral supermultiplet mass terms have well-defined modular weights. Those supermultiplets that regulate dilaton loops, and/or

[9] Determination of the explicit expression for these terms requires restoring all the total derivatives dropped in [17]; we have done this only for the term appearing explicitly in (4.35).

that contribute to (3.33), must have invariant masses. Operators that do not satisfy (4.15) include those that contribute to the renormalization of the Kähler potential. These get contributions from all fields with couplings in W_2. We therefore require that these PV fields also have modular and $U(1)_X$ invariant masses; a subset of these also regulate ϕ_3 in (2.9). The requirement that these PV fields have invariant masses (covariant PV superpotential terms) has implications for soft supersymmetry breaking scalar masses.

Other operators Ω_n that satisfy (4.15) include those, like Ω_{YM}, Ω_W and Ω_X, whose chiral projections Φ_n are bilinear in generalized chiral superfield strengths. In the present formalism Ω_X, defined in (4.32) and (2.8), is generalized to the operator:

$$(\bar{\mathcal{D}}^2 - 8R)\Omega_{X^m} = X^\alpha_m X^m_\alpha, \qquad X^m_\alpha = \frac{3}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha \ln \mathcal{M}^2 + X_\alpha, \tag{4.39}$$

where the real superfield $\mathcal{M}^2(Z, \bar{Z}, V_X) = \mathcal{M}\overline{\mathcal{M}}$ is the squared mass of a pair of PV fields. Ω_X and Ω_{X^m} will be explicitly constructed in Appendix E.2 following the construction of Ω_{YM} in [22].

In addition to the above "F-term" operators, which have no bosonic terms in their lowest components, there are "D-term" operators:

$$\Omega_D = \mathcal{M}^2(\mathcal{D}^2 - 8\bar{R})\mathcal{M}^{-2}R^m + \text{h.c.}, \qquad -\Omega_G = G^\mu_m G^m_\mu = \frac{1}{2}G^{\alpha\dot{\beta}}_m G^m_{\alpha\dot{\beta}}, \qquad \Omega_R = R^m \bar{R}^m, \tag{4.40}$$

where

$$R^m = -\frac{1}{8}\mathcal{M}^{-2}(\bar{\mathcal{D}}^2 - 8R)\mathcal{M}^2 = -\frac{1}{8}\mathcal{M}^{-2}\bar{\mathcal{D}}^2\mathcal{M}^2 + R, \qquad G^m_{\alpha\dot{\beta}} = \frac{1}{2}\mathcal{M}[\mathcal{D}_\alpha, \mathcal{D}_{\dot{\beta}}]\mathcal{M}^{-1} + G_{\alpha\dot{\beta}}, \tag{4.41}$$

are generalizations of the supergravity auxiliary fields R, $G_{\alpha\dot{\beta}}$. Although the operators (4.40) satisfy the constraint (4.15), their chiral projections cannot be entirely included in Φ, i.e., the right hand side of the generalized linearity condition (4.8), because they contain terms that are not invariant under the anomalous transformations (or do not transform into a linear multiplet). Therefore full anomaly cancellation appears to require that we add additional counterterms.

To determine which additional terms of the form (4.14) are actually present in the linearity condition (4.8) requires evaluating, in superstring derived supergravity, the higher dimension terms from the underlying string theory. However some insight might be gained by considering the zero-slope limit that gives a supergravity theory in 10 dimensions. The components of the linear multiplet defined in (4.5) include a three-form $h_{\lambda\mu\nu}$ that is a linear combination of the curl of a two-form $b_{\mu\nu}$ and the Yang-Mills CS form $\omega^{\text{YM}}_{\lambda\mu\nu}$. Similarly, the three-form H_{LMN} of 10-d supergravity includes the curl of a two-form B_{MN} and the Yang-Mills CS form ω^{YM}_{LMN}. However in order to cancel all the anomalies of 10-d supergravity, the 3-form H must be modified to include a term proportional to the Lorentz CS form as well. This must also then be the case in the 4-d effective theory, and supersymmetry then implies that in (4.8) and

$$\Phi = c_{\rm YM}\Phi_{\rm YM} + c_{\rm GB}\Phi_{\rm GB} + \cdots, \qquad \Omega = c_{\rm YM}\Omega_{\rm YM} + c_{\rm GB}\Omega_{\rm GB} + \cdots. \tag{4.42}$$

Then (4.9) contains a term proportional to the GB term which must also have a 10-d counterpart. When compactified to four dimensions the 10-d Riemann tensor includes, in addition to the 4-d Riemann tensor, derivatives of the 10-d dilaton Φ and the breathing mode(s) (*e.g.*, $\sigma = \ln\det g_{mn}$, $m, n = 5\ldots 9$). However, in the class of models considered here, with only three diagonal moduli,

$$g^i_{mn} = \delta_{mn}e^{\sigma^i} = \delta_{mn}{\rm Re}(t^i s^{\frac{1}{3}}), \tag{4.43}$$

we find for the Chern-Simons 3-forms

$$(\omega^{\rm YM}_{\mu\nu\rho})_{\rm 10d} = (\omega^{\rm YM}_{\mu\nu\rho})_{\rm 4d}, \qquad (\omega^{\rm Lor}_{\mu\nu\rho})_{\rm 10d} = (\omega^{\rm Lor}_{\mu\nu\rho})_{\rm 4d}, \tag{4.44}$$

and there are no nonvanishing elements of ω_{MNR} involving scalar derivatives. The 10-d expression for the Green-Schwarz counter term is [7]

$$\begin{aligned} L^{(10)}_c &= \int B\wedge X_8 - \left(\frac{2}{3}+\alpha\right)\int\left(\omega^{\rm YM}-\omega^{\rm Lor}\right)\wedge X_7 \\ &= \int\left[H + \left(\frac{1}{3}-\alpha\right)\left(\omega^{\rm YM}-\omega^{\rm Lor}\right)\right]\wedge X_7, \end{aligned} \tag{4.45}$$

where α is an arbitrary parameter, the 8-form curl X_8 of the 7-form X_7, $X_8 = dX_7$, is constructed from the Yang-Mills and curvature field strengths, and the 3-form H is

$$H = dB + \omega^{\rm YM} - \omega^{\rm Lor}. \tag{4.46}$$

In compactifications of the heterotic string the 10-d field strengths

$$\langle R_{mn}\rangle = \langle F_{mn}\rangle \tag{4.47}$$

are nonvanishing in the 4-d vacuum, but

$$\left\langle\left(\omega^{\rm YM} - \omega^{\rm Lor}\right)\right\rangle = 0, \tag{4.48}$$

and no new couplings appear to be generated by direct truncation of (4.45) to four dimensions. However this expression, constructed to cancel the 10-d chiral anomaly, should have a 10-d conformal anomaly counterpart, and it is possible that its 4-d remnant could be at the origin of some of the D-term contributions found here.

4.2 PV masses

Given a set[10] of PV chiral superfields Z^P with Kähler metric $K_{P\bar{Q}}$, a mass term with a well-defined modular weight can be constructed by coupling them to a set Y_P with with inverse Kähler metric multiplied by a light field-dependent function that is modular invariant up to factors $e^{q_i g^i}$:

$$K^Y = f_Y(Z^p, S, V_X) Y_P \bar{Y}_{\bar{Q}} K^{P\bar{Q}}, \tag{4.49}$$

where V_X is the vector supermultiplet[11] of the anomalous $U(1)_X$. The Kähler potentials are modular invariant if under (3.10) for an appropriate choice of the matrix M

$$Z'^P = M^P_Q Z^Q, \qquad Y'_Q = e^{-q_i^Y F^i} N^P_Q Y_P, \qquad N = M^{-1}, \qquad f' = e^{q_i^Y (F^i + \bar{F}^i)} f, \tag{4.50}$$

and the superpotential

$$W_1^Z = \mu_Z(T^j) Z^P Y_P, \quad W_1'^Z = e^{-Q_i^Z F^i} W_1^Z, \qquad Q_i^Z = -\omega_i^Z + q_i^Y, \quad \mu'_Z = \mu_Z e^{\sum_i \omega_i^Z F^i} \tag{4.51}$$

has modular weights Q_i^Z; for example, $\omega_i = n_i$ if $\mu(T^i) = \mu \prod_i [\eta(T^i)]^{2n_i}$. A modular covariant mass term W_1^X has $Q_i^X = 1$. Note also that the squared mass matrix

$$(m_Z^2)^P_Q = K_Z^{P\bar{R}} W^{\bar{S}}_{1\bar{R}} K^{\bar{Y}}_{\bar{S}M} W^M_{1Q} = f^{-1} \delta^P_Q |\mu_Z(T)|^2 \tag{4.52}$$

is diagonal and commutes with the operators that appear in the one-loop effective action (2.2). The moduli-dependence of the PV masses we have introduced in (4.51) can be interpreted as a parameterization of the threshold corrections [9, 33] corrections from higher KK and string mode loops. This would be consistent with an interpretation of the PV masses as fully parameterizing Planck/string scale physics which provides the cancellations that restore the finiteness of the underlying theory, as well as restoring the perturbative symmetries of string theory, up to the terms that require including a field dependence in the (infinite) UV cut-off. The requirement of modular covariant mass terms entails the introduction of factors of Dedekind eta functions $\eta(iT^n)$; these functions, and therefore PV masses [34] with $\omega_i > 0$, are exponentially suppressed in the (strong coupling) limit of large Ret. However we do not expect our perturbative treatment to be applicable in this strong coupling limit.

[10] These may be, e.g., any of $\dot{Z}, \widehat{Z}, \widetilde{Z}$; see Appendix F.

[11] In Yang-Mills superspace [5] only the gauge covariant superfield strengths W^a_α are introduced and the chiral superfields Φ^b are covariantly chiral with a nonholomorphic gauge transformation superfield operator. In the presence of an anomalous gauge symmetry $\mathcal{G}_X$, it is necessary to introduce the corresponding gauge potential superfield V_X. For $U(1)_X$ we use "partial" $U(1)_X$ superspace; see Appendix D.4.

If the fields Z^P have the same Kähler metric as the light fields Z^p: $K_{P\bar{Q}} = K_{p\bar{q}}$, the Z^P and Y_P loop contributions to terms linear in Γ_α in $\Phi_{1,2}$ and $\mathcal{L}_Q$ cancel, while they give a double contribution to terms quadratic in Γ_α in Φ_1. Therefore we need to either 1) introduce other PV fields (with negative signature) with a simpler metric that can mimic the contribution of the light fields or 2) couple one set of (negative signature) fields Z^P to fields Φ_P with the opposite gauge charge but trivial metric, *e.g.* $K^\Phi = \sum_P e^{g^P}|\Phi_P|^2$. Option 1) is possible for sigma models whose Kähler potentials have the special property

$$G(Z^p) = \sum_n g^n(Z^p_n), \qquad g^n_{pq} = c_n g^n_p g^n_q, \tag{4.53}$$

as shown in II for no-scale models that characterize the untwisted sector of orbifold compactifications. This allows full regularization of the theory in a simple way if the twisted sector fields are set to zero in the background. This was justified in II on the grounds that the Kähler potential for orbifolds is not known beyond leading (quadratic) order in the twisted sector superfields Φ^a_T. As a consequence the $O(|\Phi_T|^2)$ loop corrections cannot be determined. However since in realistic models [30] the SM spectrum includes twisted sector fields, one would like to include them in the effective one-loop Lagrangian using a general modular invariant parametrization of the Kähler potential. Therefore we adopt option 2), which entails a squared-mass matrix that is not of the form (4.52) and generally does not commute with other operators in the one-loop effective action. As shown in Appendices B–D, cancellation of on-shell UV divergences can be achieved with a PV sector such that the only masses that are not invariant under the anomalous symmetries are of the form (4.52), and indeed we argued above that the $\dot{Z}$ mass matrix should be modular and $U(1)_X$ invariant. In fact we will adopt the simplest possible approach, with noninvariant masses generally present only for PV fields Φ^C with a Kähler metric of the form $K_{C\bar{D}} = f_C\delta_{CD}$.
To construct invariant masses for $\dot{Z}^\sigma = \dot{Z}^P, \dot{Z}^N$, we introduce fields $\dot{Y}_\sigma = \dot{Y}_P, \dot{Y}_N$, with Kähler potential

$$K^{\dot{Y}} = \sum_A e^{G^A}|\dot{Y}_A|^2 + \sum_N e^{G^N}\left[g^{i\bar{\jmath}}_n\left(\dot{Y}_I - \chi^n_i\dot{Y}_N\right)\left(\dot{\bar{Y}}_{\bar{J}} - \bar{\chi}^{\bar{n}}_{\bar{\jmath}}\dot{\bar{Y}}_{\bar{N}}\right) + |\dot{Y}_N|^2\right], \tag{4.54}$$

where $g = \sum_n g^n$ is either the Kähler potential for just the moduli with ϕ^a all gauge-charged matter, or it is the Kähler potential for the untwisted sector with ϕ^a the twisted sector,[12], and

$$g^{i\bar{\jmath}}_n g^n_{\bar{\jmath}k} = \delta^i_k, \qquad G^{A,N} = \sum_m q^{A,N}_m g^m + \ell^{A,N}, \tag{4.55}$$

[12]In this case we have to slightly modify (4.54) and (4.57); see Appendix D.3.

where $\ell^{A,N}$ is a modular and gauge invariant function of the chiral superfields. The $\dot{Y}_A$ are PV counterparts of the light fields Z^a not included in the g_n; they transform according to the gauge group representation conjugate to that of the Z^a and have modular weights q^A_n. The Kähler potential (4.54) is modular invariant provided under (3.10)

$$\begin{aligned}\chi'^m_i(Z',\bar{Z}') &= n^j_i\left(\chi^n_j(Z,\bar{Z})+\dot{a}F_q\right), \qquad \dot{Y}'_A = e^{-F^A}\dot{Y}_A, \qquad \dot{Y}'_I = e^{-F^N} n^j_i\left(\dot{Y}_J + \dot{a}F^n_j\dot{Y}_N\right),\\ \dot{Y}'_N &= e^{-F^N}\dot{Y}_N, \qquad F^{N,A} = \sum_m q^{N,A}_m F^m. \qquad (4.56)\end{aligned}$$

where n^i_j is the submatrix of N^p_q, defined in (3.12), that acts only on the fields Z^i in $\sum_n g^n$. Then the terms in the superpotential

$$\begin{aligned}W_1^{\dot{Z}} &= \sum_a W_a + \sum_n W_n, \qquad W_n = \dot{\mu}_n(T^i)\left(\sum_{P\in n}\dot{Z}^P\dot{Y}_P + \dot{Z}^N\dot{Y}_N\right), \qquad \dot{\mu}_n(T'^i) = e^{F^N - F}\dot{\mu}_n(T^i),\\ W_a &= \dot{\mu}_a(T^i)\left(\dot{Z}^A\dot{Y}_A - \dot{a}^{-1}q^a_n Z^a\dot{Z}^N\dot{Y}_A\right), \qquad \dot{\mu}_a(T'^i) = e^{F^a + F^A - F}\dot{\mu}_a(T^i), \qquad (4.57)\end{aligned}$$

are modular covariant. The UV-divergent contributions of the $\dot{Y}$ to the loop corrections can be canceled by the fields U and V along with additional fields Φ with very simple transformation properties, Kähler potential and superpotential $W_1(U,V,\Phi)$, that are given explicitly in Appendix D. To give masses to the fields introduced in Section 3.3, we introduce additional fields $\widehat{Z}$ and $\widetilde{Y}$ that transform under the nonanomalous gauge group like Z^p and its conjugate, respectively, as well as gauge singlets $\hat{\phi}_r$, with Kähler potentials

$$\begin{aligned}K^{\widehat{Z}}_\alpha &= f_{\widehat{Z}_\alpha}\left[\sum_{P,Q=p,q}\widehat{Z}^P_\alpha\bar{\widehat{Z}}^{\bar{Q}}_\alpha\left(G_{p\bar{q}} + a^2_\alpha G_p G_{\bar{q}}\right) + a_\alpha\left(\widehat{Z}^P_\alpha\bar{\widehat{Z}}^0_\alpha G_p + \text{h.c.}\right) + |\widehat{Z}^0_\alpha|^2\right],\\ K^{\widetilde{Y}}_\alpha &= f_{\widetilde{Y}_\alpha}\sum_{P,Q=p,q} G^{p\bar{q}}\widetilde{Y}^\alpha_P\bar{\widetilde{Y}}^\alpha_{\bar{Q}},\\ K^{\hat{\phi}}_\alpha &= e^K\left\{e^{\beta^\alpha_0 k}\left[2e^{2k}|\hat{\phi}^S_\alpha|^2 + |\hat{\phi}^0_\alpha|^2 + e^k\left(\bar{\hat{\phi}}^{\bar{S}}_\alpha\hat{\phi}^0_\alpha + \text{h.c.}\right)\right] + e^{\beta^S_\alpha k}|\hat{\phi}^\alpha_S|^2\right\}, \qquad (4.58)\end{aligned}$$

that are modular and $U(1)_X$ invariant with the appropriate transformation properties.[13] We thus modify the signature constraints (3.36) to read

$$\begin{aligned}\eta^{\widehat{Z}}_\alpha &= \eta^{\widehat{Y}}_\alpha = \hat{\eta}_\alpha, \qquad \eta^{\widetilde{Y}}_\alpha = \eta^{\widetilde{Z}}_\alpha = \tilde{\eta}_\alpha, \qquad \sum_\alpha\tilde{\eta}_\alpha + \sum_\alpha\hat{\eta}_\alpha \equiv \tilde{N} + \hat{N} = 0,\\ \eta^{\hat{\phi}^r}_\alpha &= \eta^{\phi^r}_\alpha, \qquad \phi^r = (\phi^S, \phi_S, \phi_0), \qquad (4.59)\end{aligned}$$

[13] The Kähler potential for $\hat{\phi}^{S,0}$ and $\phi_{S,0}$ are the same as for $\widehat{Z}^\sigma$ and $\widehat{Y}_\sigma$, respectively, with the substitution $G(Z^p,\bar{Z}^{\bar{p}}) \to k(S,\bar{S})$ and $a^{\hat{\phi}}_\alpha = -1$, $f_{\hat{\phi}_\alpha} = \exp(K + \beta^\alpha_0 k)$.

and the first two constraints in (3.35) are modified to read

$$\sum_\alpha \eta_\alpha^{\widehat{Y}} \hat{a}_\alpha^2 = -1, \qquad \sum_\alpha \eta_\alpha^{\widehat{Y}} \hat{a}_\alpha^4 = +1. \tag{4.60}$$

because the contribution from $\widehat{Z}_\alpha^I$ to Φ_1^{PV} doubles that from $\widehat{Y}_I^\alpha$.

The remaining PV masses can be generated, for example, by the superpotential

$$\begin{aligned} W_1 &= W^{\dot{Z}} + W^{\widehat{Y}} + W^{\widetilde{Z}} + W^\varphi + W^{\tilde{\varphi}} + W^S + W^\phi + W^V + W^\Phi, \\ W^{\widehat{Y}} &= \sum_\alpha \mu_\alpha^{\widehat{Y}} \sum_{P=A,I,0} \widehat{Z}_\alpha^P \widehat{Y}_P^\alpha, \qquad W^{\widetilde{Z}} = \sum_\alpha \mu_\alpha^{\widetilde{Z}} \sum_{P=A,I} \widetilde{Z}_\alpha^P \widetilde{Y}_P^\alpha, \\ W^\varphi &= \sum_{\alpha,a} \mu_\alpha^\varphi \varphi_\alpha^a \hat{\varphi}_\alpha^a, \qquad W^{\tilde{\varphi}} = \frac{1}{2}\mu_\alpha^{\tilde{\varphi}} \tilde{\varphi}_\alpha^a \tilde{\varphi}_\alpha^a, \qquad W^V = \sum_{A,\gamma}\left(\mu_\gamma^U U_A^\gamma U_\gamma^A + \frac{1}{2}\mu_\gamma^V (V_A^\gamma)^2\right), \\ W^S &= \sum_{\alpha,r} \mu_\alpha^S \phi_\alpha^r \hat{\phi}_r^\alpha, \qquad W^\phi = \frac{1}{2}\sum_{\alpha,\beta} \mu_{\alpha\beta}^\phi \phi^\alpha \phi^\beta, \qquad \eta_\alpha^\phi = \eta_\beta^\phi \text{ if } \mu_{\alpha\beta}^\phi \neq 0, \end{aligned} \tag{4.61}$$

where $W^{\dot{Z}}$ is given by (4.57), and W^Φ is given in Appendix D. The Kähler potential for the fields Φ, whose only role is to cancel unwanted curvature terms generated by the Kähler potential (4.54), is also given in Appendix D. Note that W^φ, W^S and $W^{\dot{Z}}$ (for at least one set $\dot{Z}, \dot{Y}$) are $U(1)_X$ invariant and modular covariant. The mass parameters μ can in general depend on the moduli: $\mu = \mu(T^i)$. When all the above additional PV fields are included, the prefactors in their Kähler potentials and their $U(1)_X$ charges are understood to be included in the sums in (3.37) and (3.38), and in (3.36) N' includes all PV chiral superfields. We further require

$$\sum_\alpha \eta_\alpha^Y \ln f_\alpha^Y - \sum_\alpha \eta_\alpha^Z \ln f_\alpha^Z = 0, \qquad Y = \widehat{Y}, \widetilde{Y}, \qquad Z = \widehat{Z}, \widetilde{Z}. \tag{4.62}$$

The simplest form of the prefactors needed for the cancellation of all UV divergences is

$$f = e^{\alpha K + \beta k + \sum_n q_n g^n + q V_X}. \tag{4.63}$$

Although we are working generally in Yang-Mills superspace [5] in which the gauge potential superfields V_a do not appear explicitly in the superpotential, as explained in Appendix D.4 fields with superpotential mass terms that are not $U(1)_X$ invariant must transform in such a way that the mass term remains holomorphic under a $U(1)_X$ transformation; this requires the presence of qV_X in the exponent of the prefactor for these fields. As shown in Appendix D, all the anomalies arising from gauge-charged matter in the light sector can be absorbed into the mass terms of U, V, Φ. This means in particular that we need to include factors e^{qV_X} only for these fields. These fields also have prefactors of the form $e^{\sum_n q_n g^n}$, similar to $e^{G^{A,N}}$ in (4.54), that accommodate the charged matter and T-moduli contributions to the modular anomaly. For the other PV fields we take the simpler form

$$f_C = e^{\alpha_C K_C + \beta_C k_C}. \tag{4.64}$$

A similar prefactor may be included in the Kähler potential for any chiral superfield $\Phi \neq \theta$ that does not have other couplings except in W_1, provided the moduli-dependence of W_1 assures modular covariance as needed; this includes additional fields with the same Kähler metric, gauge charges and superpotential terms in W_1 as $\dot{Z}, \widehat{Z}, \widetilde{Z}, \dot{Y}, \widehat{Y}, \widetilde{Y}$ but no other couplings. All PV fields with couplings that contribute to the renormalization of the superpotential must have invariant mass terms. Here we will assume only the minimal number of these fields needed to regulate light field contributions to the renormalization of the Kähler potential. Thus we include $\dot{Z}, \dot{Y}$ only with the Kähler potential given in (3.16), (3.18) and (4.54). We take all $\widehat{Y}, \widetilde{Z}$ to have the Kähler potentials given in (3.28), and $\widehat{Z}, \widetilde{Y}$ to have, respectively, the inverse Kähler metrics with prefactors

$$f_{\widehat{Z}_\alpha, \widetilde{Y}_\alpha} = e^{\alpha_\alpha^{\widehat{Z}, \widetilde{Y}} K}, \tag{4.65}$$

such that including the T^i-dependence of $\mu_\alpha^{\widehat{Y}, \widetilde{Z}}$, the masses are modular invariant, and the constraint (4.62), which reduces to

$$\sum_\alpha \hat{\eta}_\alpha \left(\alpha_\alpha^{\widetilde{Y}} - \alpha_\alpha^{\widehat{Z}} \right) = 0, \tag{4.66}$$

is satisfied.

4.3 Quadratic PV mass terms

We have constructed the PV Lagrangian in such a way that PV fields that couple to one another in W_1 have no other common coupling. The the term quadratic in masses takes the form, in a basis where the squared mass matrix is diagonal

$$\begin{aligned} \mathcal{L}_Q^{PV} \quad = \quad & \frac{1}{32\pi^2} \left[\left(3 \sum_\gamma \ell_\gamma^\theta - \sum_C \ell_C \right) \left(\hat{V} + M^2 \right) - \frac{1}{2} \left(\sum_C \ell_C - \sum_\gamma \ell_\gamma^\theta \right) \mathcal{D}^\alpha X_\alpha | \right. \\ & \left. + \sum_\gamma \ell_\gamma^s \, \mathcal{D}^\alpha k_\alpha | + \sum_C \ell_C \, \mathcal{D}^\alpha \Gamma_{C\alpha}^C | \right], \end{aligned} \tag{4.67}$$

where here C refers to all heavy chiral supermultiplets $\Phi^C \neq \theta$, and

$$\ell_X = m_X^2 \ln(\Lambda^2/m_X^2)\eta_X, \qquad X = \Phi^C, \theta. \tag{4.68}$$

Finiteness requires that the coefficient of $\ln \Lambda$ vanish:

$$0 = \sum_\gamma m_{0\gamma}^2 \eta_\gamma^0 = \sum_\gamma m_{s\gamma}^2 \eta_\gamma^s = \sum_C m_C^2 \eta_C = \sum_C m_C^2 \eta_C \, \mathcal{D}^\alpha \Gamma_{C\alpha}^C \big| \,. \tag{4.69}$$

If Φ^C couples to Φ^D in W_1, with

$$m_C^2 = m_D^2 = \mu_C^2 |w_C(t)|^2 e^{K-2(q_C+q_D)V_X} f_C(z,\bar{z}) \equiv \mu_C^2 \Lambda_C^2, \tag{4.70}$$

then $\mathcal{D}_\alpha \left(\Gamma_{C\alpha}^C + \Gamma_{D\alpha}^D\right)$ is completely determined in terms of $(q_C + q_D)$ and scalar derivatives of f_C, so the third equality in (4.69) insures the fourth, since the coefficient of each term of fixed $\Lambda_C^2 \neq$ constant must vanish separately. We have regulated the theory such that all masses are invariant except for some chiral superfields Φ^{C_α} with masses of the form (4.70). These give contributions to the renormalized Kähler potential of the form

$$\begin{aligned} K_Q &\ni \sum_C \frac{\lambda_C \Lambda_C^2}{32\pi^2} - 2\frac{\lambda_0 \Lambda_0^2}{32\pi^2}, \qquad \lambda_X = \sum_{\gamma=1}^{n_X} \eta^{X_\gamma} \mu_{X_\gamma}^2 \ln(\mu_{X_\gamma}^2), \\ m_{X_\gamma}^2 &= \mu_{X_\gamma}^2 \Lambda_X^2, \qquad \sum_{\gamma=1}^{n_X} \eta^{X_\gamma} \mu_{X_\gamma}^2 = 0. \end{aligned} \tag{4.71}$$

We require $\lambda_C = 0$ if Λ_C^2 is not modular and $U(1)_X$ invariant. Note that in order to cancel the chiral anomaly and the logarithmic divergences, we have necessarily

$$\sum_{\alpha=1}^{n_C} \eta^{C_\alpha} \neq 0 \tag{4.72}$$

for at least some sets $\{C, D\}$. For these the constraint $\lambda_C = 0$ requires[14] $n_C \geq 5$.

5 Superfield structure of the anomaly and anomaly cancellation

The bosonic part of the anomaly under infinitesimal modular and $U(1)_X$ transformations that arises from the noninvariance of PV masses is given in (C.70) of Appendix C. Referring to (C.72)

[14]See Appendix C of [35].

and (E.11), when combined with the additional contributions described in Section 4.1–with the cancellation of (4.37) imposed–that expression is the bosonic part of an infinitesimal variation of the superfield operator

$$\begin{aligned} \mathcal{L}_{\rm anom} &= \mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_r=\int d^4\theta E\left(L_0+L_1+L_r\right), & (5.1)\\ L_0 &= \frac{1}{8\pi^2}\left[{\rm Tr}\eta\ln\mathcal{M}^2\Omega_0+K\left(\Omega_{GB}+\Omega'_D\right)\right], \qquad L_r=-\frac{1}{192\pi^2}{\rm Tr}\eta\int d\ln\mathcal{M}\Omega_r. & (5.2)\end{aligned}$$

$\mathcal{L}_1$ is defined by its variation:

$$\Delta L_1=\frac{1}{8\pi^2}\frac{1}{192}{\rm Tr}\eta\Delta\ln\mathcal{M}^2\Omega'_L=\frac{1}{8\pi^2}\frac{1}{192}{\rm Tr}\eta H\Omega'_L+{\rm h.c.}, \tag{5.3}$$

and Ω'_D is the "D-term" anomaly that arises from certain total derivatives with logarithmically divergent coefficients as discussed in Section 4.1. The superfield Ω_r is defined by (E.11) and (E.12), Ω_{GB} is defined in (4.31), the real superfield $\mathcal{M}^2$ is the PV squared mass matrix,

$$\Delta\ln\mathcal{M}^2=2\left(H+\bar{H}\right), \tag{5.4}$$

with H holomorphic, and

$$\begin{aligned} \Omega_0 &= \frac{1}{3}\Omega_W+\Omega^0_{\rm YM}-\frac{1}{36}\Omega_X+\frac{1}{192}\Omega_L-\frac{1}{48}\left(6\phi_\chi+\mathcal{D}^2R+\bar{\mathcal{D}}^2\bar{R}\right)-\frac{1}{192}\Omega'_L\\ &= -\frac{1}{24}\Omega_{GB}+\Omega^0_{\rm YM}-\frac{1}{48}\Omega_\phi,\\ \Omega'_L &= \Omega_L-16\Omega_X, \qquad \Omega_\phi=12\phi_\chi+\mathcal{D}^2R+\bar{\mathcal{D}}^2\bar{R}, \qquad (5.5)\end{aligned}$$

with the Chern-Simons superfields in (5.5) defined in (4.23), (4.32), and (E.7). The above results were also found [29] by performing a superspace calculation of the anomaly in PV regulated superconformal supergravity, and then gauge-fixing to Kähler $U(1)_K$ superspace [5]. Note that the part of Ω'_L that is independent of the $U(1)_X$ charges and modular weights, namely

$${\rm Tr}\eta H\Omega'_L=16\sum_C\tilde{\varphi}^+_C\left[(1-2\alpha^C)^2-1\right]\Omega_X+O(q_X,q_i), \tag{5.6}$$

drops out of (5.3) by virtue of (D.103) and (D.113) [with (D.114)]. In the present approach, the coefficient of the Chern-Simons superfield Ω^X_{YM} comes from a combination of terms in Ω_{X_m} and Ω_G in (C.72). Because the anomalous $U(1)_X$ is not incorporated into the superspace geometry, the superfield strength in the chiral projection of $\Omega^0_{\rm YM}$, defined in (C.72), does not include W^X_α. Instead, the PV masses depend on the the $U(1)_X$ vector field strength V_X, giving a $U(1)_X$-dependent

contribution from Ω_L. We note here that the operator Ω_0 in (5.2) has contributions that contain a dependence on the dilaton field, such as Ω_X and Ω_L; these do not satisfy the condition (4.15) of Section 4.1. We show in Appendix E.2 that the linear/chiral multiplet duality outlined in Section 4.1 still holds in the presence of these operators; although some intermediate equations are modified the action for the dilaton coupling to Ω is the same in both the linear and chiral multiplet formulations, and still takes the form (4.9) or (4.14).
There are additional $U(1)_X$-dependent contributions from the "D-terms"in Ω_r. Integrating (E.11) gives

$$\begin{aligned}\mathcal{L}_r \;=\; -\frac{1}{8\pi^2}\frac{1}{24}\int d^4\theta E \mathrm{Tr}\eta \Big\{ & \ln\mathcal{M}\left(\frac{1}{2}\mathcal{D}^\alpha L_\alpha + 2\mathcal{D}^\alpha X_\alpha\right)(\ln\mathcal{M})^2 + G^{\alpha\dot\beta}\mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\ln\mathcal{M} \\ & -\frac{1}{4}\left(\mathcal{D}^2\ln\mathcal{M}\mathcal{D}_{\dot\beta}\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M} + \mathrm{h.c.}\right) \\ & -\ln\mathcal{M}\left[\frac{1}{8}D^2\bar{\mathcal{D}}^2\ln\mathcal{M} + \mathcal{D}^\alpha(R\mathcal{D}_\alpha\ln\mathcal{M}) + \mathrm{h.c.}\right] \\ & -\frac{1}{2}\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M}\Big\},\end{aligned} \tag{5.7}$$

up to a linear superfield and a total derivative; the variation of L_r under an anomalous transformation is given in (E.16).
In this section we evaluate the anomalies and discuss the counterterms needed to cancel them. Possible connections of these counterterms to the 10d GS term were discussed in Section 4.1. For simplicity we present explicit results only for the case without threshold corrections as in Z_3 and Z_7 orbifold compactifications; the result for any specific compactification with threshold corrections can be reconstructed from the results of Appendix D.4, by matching the parameters ω_n^C to the threshold corrections specific to the model.

5.1 Generalization of the 4d GS mechanism

The traces in $\mathcal{L}_0$ are completely determined, up to possible threshold corrections, in terms of the quantum numbers of the light spectrum. Taking the traces subject to the constraints given in (3.35)–(3.38), Appendix A and Appendix D.4, we obtain a contribution that can be written in the form (up to terms that are invariant under modular[15] and $U(1)_X$ transformations and terms that

[15]In the case that string threshold corrections are present, these include terms of the form $\sum_{n,A} b_n^A \ln\left[(T^n+\bar{T}^n)|\eta(T^n)|^2\right]\Omega_A$.

do not contribute to the variation of the action)

$$L_0 = \left(bg - \frac{1}{2}\delta_X V_X\right)\Omega_0 + \frac{g}{8\pi^2}\left(\Omega'_D + \Omega_\phi\right). \tag{5.8}$$

In contrast, the traces in $\mathcal{L}_1$ depend on the details of the PV regularization procedure. For example if we use either (D.130) or (D.147) we get an expression of the form

$$\begin{aligned}\frac{1}{192}\mathrm{Tr}\eta H\Omega'_L &= \frac{1}{3}\sum_l F^l \Bigg[C_{GS}\Omega^X_{YM} + A_l\Omega_{W_X X} - \sum_{m,n} A_{mnl}\Omega_{nm} \\ &\qquad - \sum_n \left(2B_{nl}\Omega_{W_X n} - A_{nl}\Omega_{nX}\right)\Bigg] \\ &\quad + \frac{1}{3}\Lambda\left[3C'_{GS}\Omega^X_{YM} + a\Omega_{W_X X} - \sum_{m,n} a_{mn}\Omega_{nm} - \sum_n \left(2b_n\Omega_{W_X n} - a_n\Omega_{nX}\right)\right],\end{aligned} \tag{5.9}$$

with

$$\begin{aligned}(\bar{\mathcal{D}}^2 - 8R)\Omega_{W_X X} &= W^\alpha_X X_\alpha, \qquad (\bar{\mathcal{D}}^2 - 8R)\Omega_{nm} = g^\alpha_n g^m_\alpha, \\ (\bar{\mathcal{D}}^2 - 8R)\Omega_{nX} &= g^\alpha_n X_\alpha, \qquad (\bar{\mathcal{D}}^2 - 8R)\Omega_{W_X n} = W^\alpha_X g^n_\alpha,\end{aligned} \tag{5.10}$$

the expressions for the parameters A, B, a, b, c are given in (D.131)–(D.138) for the PV sectors of Appendix D.1 and Appendix D.2 with the choice (D.130), and

$$C_{GS} = 8\pi^2 b, \qquad C'_{GS} = -4\pi^2\delta_X, \tag{5.11}$$

In writing the above we used the known string theory constraints, which for orbifold compactifications with no threshold corrections read

$$\begin{aligned}8\pi^2 b &= \frac{1}{24}\left(2\sum_p q^p_n - N + N_G - 21\right) \quad \forall \quad n \\ &= C_a - C^M_a + 2\sum_b C^b_a q^b_n \quad \forall\ n, a, \quad C_a = \mathrm{Tr}(T_a)^2_{\mathrm{adj}}, \quad C^M_a = \mathrm{Tr}(T_a)^2_{\mathrm{mat}},\end{aligned} \tag{5.12}$$

$$2\pi^2\delta_X = -\frac{1}{24}\mathrm{Tr}T_X = -\frac{1}{3}\mathrm{Tr}T^3_X = -\mathrm{Tr}(T^2_a T_X) \quad \forall \quad a \neq X. \tag{5.13}$$

If the expression (5.9) were to factorize:

$$\frac{1}{192}\mathrm{Tr}\eta H\Omega'_L = \left(C_{GS}F + C'_{GS}\right)\Omega'_0, \tag{5.14}$$

$$\Omega'_0 = \left[\Omega^X_{YM} + A\Omega_{W_X X} - \sum_{m,n} A'_{mn}\Omega_{nm} - \sum_n \left(2B'_n\Omega_{W_X n} - A'_n\Omega_{nX}\right)\right], \tag{5.15}$$

the full F-term contribution to the anomaly could be canceled provided the tree-level Lagrangian includes a contribution [15] (in the chiral formulation for the dilaton)

$$\mathcal{L}_S = -\int d^4\theta(S+\bar{S})\Omega = \frac{1}{8}\int d^4\theta\frac{E}{R}S\Phi + \text{h.c.}, \qquad \Phi = (\bar{\mathcal{D}}^2 - 8R)\Omega, \qquad \Omega = \Omega_0 + \Omega_0', \quad (5.16)$$

with S transforming under modular and $U(1)_X$ transformations as

$$S \to S - \frac{\delta_X}{2}\Lambda + bF, \tag{5.17}$$

so that the dilaton Kähler potential

$$K_S = k(S + \bar{S} + \frac{\delta_X}{2}V_X - bG) \tag{5.18}$$

is invariant. Note that the coefficient of Ω^X_{YM} in Ω is fixed, since the dilaton is known to couple universally to the gauge field strength (for affine level $k=1$). The Lagrangian (5.16) and the Kähler potential (5.18) can be obtained by a duality transformation from the formulation where the dilaton is the lowest component of a modified linear multiplet, as described in Section 4.1. The new couplings in (5.16) of course also contribute ultraviolet divergent contributions in the effective QFT. We expect that these can be regulated by PV fields with modular and $U(1)_X$ invariant masses, as we have shown to be the case for the dilaton coupling to Φ_{YM}, so that they will not induce any further contributions to the anomaly. In the case that string loop corrections of the form

$$\begin{aligned}
\mathcal{L}_{\text{thresh}} &= \frac{1}{4\pi^2}\sum_{n,A}\ln|\eta(T^n)|^2 b_n^A\Omega_A, & (5.19)\\
\Delta\mathcal{L}_{\text{thresh}} &= \frac{1}{8\pi^2}\sum_{n,A}\left[F(T^n) + \bar{F}(\bar{T}^{\bar{n}})\right] b_n^A\Omega_A, & (5.20)
\end{aligned}$$

are present, the constraints (5.12) and (5.13) are modified as in (D.102) and (D.121), and the quantum anomaly arising from the F-term should be canceled by a the combination of (5.17) and (5.20).

If we make a different choice than (D.130) or (D.147) for the invariant functions ℓ in (4.55), there will be additional terms on the right hand side of (5.9). However this model-dependence is removed if we evaluate it with only the Kähler moduli nonvanishing among the background chiral superfields, such that

$$\ell \to 0, \qquad K \to g = \sum_n g^n, \qquad \Omega_{nX} \to \sum_m \Omega_{nm}, \qquad \Omega_{W_X X} \to \sum_n \Omega_{W_X n}, \tag{5.21}$$

so that (5.9) reduces to

$$
\begin{aligned}
\frac{1}{192}\mathrm{Tr}\eta H\Omega'_L &= \frac{1}{3}\sum_l F^l\left[C_{GS}\Omega^X_{YM} + \sum_n\left(A_l - 2B_{nl}\right)\Omega_{W_Xn} - \sum_{m,n}\left(A_{mnl} - A_{nl}\right)\Omega_{nm}\right] \\
&+\frac{1}{3}\Lambda\left[3C'_{GS}\Omega^X_{YM} + \sum_n\left(a - 2b_n\right)\Omega_{W_Xn} - \sum_{m,n}\left(a_{mn} - a_n\right)\Omega_{nm}\right]. \qquad (5.22)
\end{aligned}
$$

One can check that this does not factorize for the models listed in Appendix D.5 if we use the results (D.131)–(D.138) that apply if the prescription of Appendix D.1 or Appendix D.2 is used. With this prescription, the modular weights and $U(1)_X$ charges of the PV fields that enter the sums in (D.131)–(D.138) are exactly those of the light chiral supermultiplets. They arise from the contributions of PV fields Ψ^C with, in the absence of threshold corrections ($\omega^C_n = 0\;\forall\;\Phi^C$) modular weights

$$
q^{\Psi^C}_n = 1 - q^C_n. \qquad (5.23)
$$

In contrast, the more constrained prescription of Appendix D.3 replaces the gauge singlet PV "moduli" Φ^N, Φ^n with

$$
(q^N_m, q^n_m) = (2\delta^N_m, 0) \qquad (5.24)
$$

by the same number of PV "moduli" but with

$$
(q^N_m, q^n_m) = (\delta^N_m, \delta^n_m) \qquad (5.25)
$$

This leaves sums linear in the modular weights unchanged, but slightly modifies the nonlinear sums:

$$
\Delta\sum_C q^{\Psi^C}_{n_1}\ldots q^{\Psi^C}_{n_M} = (2 - 2^M)\prod_{k=1}^{M-1}\delta_{n_k n_{k+1}}. \qquad (5.26)
$$

This does not achieve factorization, but suggests that an even more constrained PV sector, such as "option 1", discussed in Section 4.2 and briefly outlined in the beginning of Appendix D, might allow a significantly different set of modular weights for the PV fields that have noninvariant masses. Note also that in the effective field theory for the $\mathbb{Z}_7$ case of Appendix D.5, there are mixed anomalies associated with $\mathrm{Tr}T_a q_n \neq 0$ and $\mathrm{Tr}T_a q^2_n \neq 0$. No anomalies are present in the corresponding string theory [14] (which has no Wilson lines and therefore no anomalous $U(1)$). Therefore in the appropriately regulated effective field theory these anomalies should disappear. For the case where an anomalous $U(1)_X$ is present, we also need a modification of $\mathrm{Tr}T^2_X q_n$. In addition there may be larger symmetries, or partial symmetries–such as the "Heisenberg symmetry" of the untwisted sector Kähler potential that is used in Appendix D.3–that should be respected in the PV sector.[16] For example, the Kähler metric of the untwisted sector for $\mathbb{Z}_3$ models, including the more realistic FIQS model with Wilson lines and a $U(1)_X$, have a larger symmetry than the $[SL(2,\mathbb{Z})]^3$ group of modular transformations that we have been using. If that symmetry is preserved in the PV sector, this might also give a different result for terms nonlinear in modular and $U(1)_X$ charges.

Cancellation of the remaining contributions to the anomaly, namely the the term proportional to g in (5.8)–that has not been considered previously in this context–and the D-term anomaly $\mathcal{L}_r$ in (5.7) appears to require additional counterterms, although possibly the former, and certainly the latter, will be modified by any modification of the PV sector that can provide factorization in (5.22). Because $\ln\mathcal{M}$ contains a term $(\frac{1}{2}-\alpha)K$, it is not impossible that the the former term can be canceled by a contribution from the latter, in the same way that that the Ω_X term in Ω'_L is canceled in (5.6); this cancellation seems to be independent of the details of the PV sector.

6 Summary of Results

We used on-shell Pauli-Villars regularization of the one-loop ultra-violet divergences of supergravity to determine the anomaly structure of these theories. This regularization procedure requires constraints on the chiral matter representations of the gauge group that are satisfied in any supersymmetric extension of the Standard Model, and by hidden sectors that have been found in orbifold compactifications of the heterotic string.

We showed that the logarithmic and quadratic one-loop divergences in the S-matrix can be canceled by a PV sector that respects the classical symmetries of the theory except for certain superpotential terms that generate large, noninvariant PV masses for a subset of PV chiral superfields that have a very simple Kähler metric. A PV sector with this feature was explicitly constructed for effective supergravity theories with three Kähler moduli T^i obtained from orbifold compactifications of the weakly coupled heterotic string. These theories are classically invariant under a $U(1)_X$ gauge transformation and under the T-duality group of modular transformations that are anomalous at the quantum level.

If all linear divergences were canceled in the regulated theory, it would be anomaly free, with

[16] In writing the transformation properties (3.10) and (3.14) we neglected constant phases [36] and matrices [37] that mix chiral superfields with the same modular weights. These can be trivially incorporated and do not affect the conclusions.

noninvariance of the action arising only from that of the Pauli-Villars masses. Here we calculated their contribution to the bosonic part of the anomaly in the component Lagrangian. Because the regularization procedure is manifestly supersymmetric the full anomaly generated by PV loops necessarily forms a superfield. Since only chiral superfields contribute, and since they have a diagonal Kähler metric, we were able to draw on the recent superfield calculation [29] of the chiral multiplet loop contribution to the UV divergences and anomalies in supergravity. This provided a check of our component results, and greatly simplified the identification of the superfield form of the anomaly.

Pauli-Villars regulator fields allow for the cancellation of all quadratic and logarithmic divergences, but there remain residual linear divergences associated with nonrenormalizable gravitino/gaugino interactions. This result follows directly from the constraints imposed by cancellation of on-shell quadratic divergences, and is therefore independent of the details of the PV sector. These residual divergences result in additional contributions to the chiral anomaly. There is also an additional conformal anomaly that appears if the UV cut-off Λ is field dependent, because there are residual terms in the coefficient of $\ln\Lambda$ that are not canceled by the PV sector. These are total derivatives, so the S-matrix of the regulated theory is finite, but a field-dependent cut-off can induce finite terms that are not invariant under the classical symmetries. These new contributions, which are anomalous only under T-duality invariance, combine to form a supermultiplet provided the ultraviolet cut-off has the field dependence given in (1.1). When this contribution is combined with the PV contribution, we obtain results in agreement with string-loop calculations of the anomalous coefficients of the squared (nonanomalous) Yang-Mills and space-time curvatures.

It is well known that these contributions to the anomaly can be canceled by a four-dimensional version of the Green-Schwarz mechanism. We showed that the remaining contributions depend on the choice of PV sector couplings. Provided that the F-term contributions factorize [14], they can be canceled by a generalization this mechanism, corresponding to a generalization of the modified linearity condition for the linear supermultiplet that is dual to the dilaton chiral supermultiplet. It may be that factorization is possible only with constraints on the superpotential of the type discussed at the beginning of Appendix D. Such constraints would be somewhat analogous to the constraint on gauge charges mentioned above, although the cancellation of UV divergences by itself requires no constraint on the Kähler potential. Such a factorization condition could be used as a tool to probe higher order terms in the twisted sector Kähler potential. This is of importance for phenomenology if some fields require large vacuum values as, for example, in $U(1)_X$ gauge symmetry breaking at the string scale. As an example, we found that if we impose a Kähler potential of the form (D.7) for the twisted sector of the $\mathbb{Z}_{\mathbf{7}}$ model of Appendix D.5 one can find values of the parameters such that terms proprotional to $X^{\alpha} g_{\alpha}^{n}$, with coefficients both linear and quadratic in the modular weights, either factorize or vanish, while the cancellation in (5.6) is unaffected. This approach will be pursued further elsewhere.

There are additional contributions to the conformal anomaly that may require new counter-terms. These include terms that are nonlinear in the parameters of the anomalous transformations. On the other hand it is possible that some or all of these terms can be made to cancel.

We briefly considered the possible connections between the 4-d counterterms that we find and the 10-d GS term; establishing a clear connection requires further work.

Acknowledgments. We are happy to acknowledge the contributions of Andreas Birkedal, Choonseo Park, Matthijs Ransdorp and So-Jong Rey to various early stages of this work. We thank Dan Freedman, Joel Giedt, Brent Nelson, Claudio Scrucca, Marco Serone, Tom Taylor and Lewis Tunstall for discussions and helpful input. MKG would like to thank the Kavli Institute of Physics at U.C. Santa Barbara for hospitality during part of the completion of this work. This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics, of the U.S. Department of Energy under Contract DE-AC02-05CH11231, in part by the National Science Foundation under grants PHY-0457315 and PHY-1002399.

Appendix

A Linear divergences

The one-loop contribution to the bosonic part of the effective action is

$$\begin{aligned} \mathcal{L}_1 &= \frac{i}{2}\mathrm{Tr}\ln(\hat{D}^2_\Phi + H_\Phi) - \frac{i}{2}\mathrm{Tr}\ln(-i\not{D} + M_\Theta) \\ &\quad + i\mathrm{Tr}\ln(D^2_{Gh} + H_{Gh}) - i\mathrm{Tr}\ln(\hat{D}^2_{gh} + H_{gh}) \\ &\equiv \frac{i}{2}\mathrm{STr}\ln(\hat{D}^2 + H) - \frac{i}{2}T_-, \end{aligned} \tag{A.1}$$

where Φ is a $2N + 4N_G + 10$ component boson, Θ is an $N + N_G + 5$ component Majorana fermion, with N is the number of chiral multiplets and N_G is the number of gauge multiplets, and the last two terms are the (4-component fermion) ghostino and ($N_G + 4$ component boson) ghost contributions, respectively. The last equality is obtained after casting the fermionic determinant in the form

$$-\frac{i}{2}\mathrm{Tr}\ln(-i\not{D} + M_\Theta) = -\frac{i}{8}\mathrm{Tr}\ln(\hat{D}^2_\Theta + H_\Theta) - \frac{i}{2}T_-, \tag{A.2}$$

where in (A.2) $\hat{D}_\Theta$ and H_Θ are 8×8 matrices in Dirac space that act on the vector-valued 4-component fermion (Θ_R, Θ_L); for example H_Θ for PV the fields with noninvariant masses is given in (C.9) below. The first term is the helicity averaged part of the fermion contribution, and [17]

$$T_- = \mathrm{Tr}\left\{-\left[\not{D}^2 + i\not{D}M\right]^{-1} i\not{D}\mathcal{N}\right\} \tag{A.3}$$

is the helicity-odd part,[17] where $\mathcal{N}$ is the helicity-odd part of $-i\not{D} + M_\Theta$. T_- is not invariant under chiral transformations because, unlike the the other operators appearing in (A.1), $\mathcal{N}$ cannot be recast in invariant form; it explicitly contains the axial current. If the linear divergences in T_- cancel, the non-invariance can be shifted to the PV masses. This requires

$$\mathrm{Tr}\eta(\mathcal{G}\cdot\tilde{\mathcal{G}}\phi) = 0, \tag{A.4}$$

where the signature $\eta = 1$ for light fields,

$$\mathcal{G}_{\mu\nu} = [D_\mu, \mathcal{D}_\nu] = G_{\mu\nu} + Z_{\mu\nu}, \tag{A.5}$$

[17] As discussed in [1] the separation of the fermion determinant into helicity odd and even parts is ambiguous; there is a unique choice that allows a PV regularization of the quadratic divergences. For the universal axion couplings to gauginos, this choice is also consistent with nonrenormalization [28] of the θ-parameter and with linearity constraints [12] in the dual linear formulation for the dilaton supermultiplet.

is the fermion field strength, the fermion covariant derivative is

$$D_\mu = \partial_\mu + Z_\mu + v_\mu + J_\mu\gamma_5 = d_\mu + Z_\mu, \qquad G_{\mu\nu} = [d_\mu, d_\nu], \tag{A.6}$$

with Z_μ the spin connection:

$$Z_\mu = \frac{1}{4}\gamma_\nu[\nabla_\mu, \gamma^\nu], \qquad Z_{\mu\nu} = \frac{1}{4}r_{\mu\nu\rho\sigma}\gamma^\sigma\gamma^\rho, \tag{A.7}$$

and

$$\delta J_\mu = -i\partial_\mu\phi \tag{A.8}$$

under a chiral transformation. In fact, the trace in (A.4) does not vanish because the gravitino connection contains the affine connection as well as the spin connection, whereas the PV fields are chiral fermions and (Abelian) gauginos, with only the spin connection contributing to space-time curvature dependent terms. In addition there is an off-diagonal gaugino-gravitino connection that is not reproduced in the PV sector. This implies that there is a residual linear divergence. When the PV fields are included all the other terms are reproduced, provided they satisfy condition (A.4). The first equality in (4.24), which follows from the cancellation of on-shell quadratic divergences, ensures the equality

$$\mathrm{Tr}\eta Z \cdot \tilde{Z}\phi = \frac{1}{2}\left(N + N' - N_G - N'_G - 3 - 2\sum_C \eta_C\alpha^C\right) Z\cdot\tilde{Z}\mathrm{Im}F = 0. \tag{A.9}$$

To implement the condition (A.4) for the chiral fermions it is useful to use the relations

$$\begin{aligned}
\left(G^+_{\mu\nu}\right)^{\bar{n}}_{\bar{m}} &= -K_{p\bar{m}}\left(G^-_{\mu\nu}\right)^p_q K^{q\bar{n}} + 2Z_{\mu\nu}\delta^{\bar{n}}_{\bar{m}}, \\
\left(D^+_\mu\right)^{\bar{n}}_{\bar{m}} &= -K_{p\bar{m}}\left(D^-_\mu\right)^p_q K^{q\bar{n}} + K^{q\bar{n}}\partial_\mu K_{q\bar{m}} + 2Z_\mu\delta^{\bar{n}}_{\bar{m}},
\end{aligned} \tag{A.10}$$

Writing

$$D^\pm_\mu = \partial_\mu + Z_\mu + J^\pm_\mu, \qquad (J_\mu)^i_j \equiv (J^-_\mu)^i_j, \qquad (J_\mu)^{\bar{m}}_{\bar{n}} \equiv (J^+_\mu)^{\bar{m}}_{\bar{n}}, \tag{A.11}$$

for a chiral superfield with a general Kähler metric $\mathbf{K}$: $\mathbf{K}_{i\bar{m}} = K_{i\bar{m}}$, the anomalous part of the one loop action under a chiral transformation takes the form [see (B.1)]

$$\begin{aligned}
\frac{1}{2}\left[\mathrm{Tr}\ln(-i\not{D}^+R) - \mathrm{Tr}\ln(-i\not{D}^-L)\right] &= \frac{1}{2}\left[\mathrm{Tr}\ln(-i\not{D}^+R) - \mathrm{Tr}\ln(-i\mathbf{K}\not{D}^-\mathbf{K}^{-1}L)\right] \\
&= \frac{1}{2}\left[\mathrm{Tr}\ln(-i\{\not{\partial} + \not{Z} + \not{J}^+\}R) - \mathrm{Tr}\ln(-i\mathbf{K}\{\not{\partial} + \not{Z} + \not{J}^-\}\mathbf{K}^{-1}L)\right] \\
&= \frac{1}{2}\left[\mathrm{Tr}\ln(-i\{\not{\partial} + \not{Z} + \not{J}^+\}R) - \mathrm{Tr}\ln(-i\{\not{\partial} + \not{Z} - \not{J}^+\}L)\right],
\end{aligned} \tag{A.12}$$

where in the last equality we used (A.10), so we can simply take J^+_μ (or $-J^-_\mu$) to be the axial current: $J_\mu = J^\pm_\mu$. Then

$$G^V_{\mu\nu} = Z_{\mu\nu} + [J^\pm_\mu, J^\pm_\nu], \qquad J_{\mu\nu} = \partial_\mu J^\pm_\nu - \partial_\nu J^\pm_\mu, \tag{A.13}$$

and the condition (A.4) reduces to

$$0 = \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\partial_\mu J^\pm_\nu \partial_\sigma J^\pm_\rho \phi\right). \tag{A.14}$$

This implies, in particular,

$$0 = \mathrm{Tr}T_a^2 T_X, \tag{A.15}$$

and

$$0 = \sum_C \eta_C\left(-8\alpha_C^3 + 12\alpha_C^2 - 6\alpha_C\right) + N + N' - N_G - N'_G - 3 = -8\sum_C \eta_C\alpha_C^3 - 8,$$

$$\sum_C \eta_C\alpha_C^3 = -1. \tag{A.16}$$

where α_C is defined in (4.64) and we used the constraints (3.36)–(3.38). There are also mixed terms in the gauge and Kähler $U(1)$ connections; these require

$$\begin{aligned} 0 &= 2\sum_C \eta_C\alpha_C(T_a)^2_C - \mathrm{Tr}T_a^2 = 2\sum_C \eta_C\alpha_C(T_a)^2_C, && \text{(A.17)}\\ 0 &= 4\sum_C \eta_C\alpha_C^2(T_X)_C - 4\sum_C \eta_C\alpha_C(T_X)_C + \mathrm{Tr}T_X = 4\sum_C \eta_C\alpha_C^2(T_X)_C, && \text{(A.18)}\end{aligned}$$

since cancellation of quadratic divergences (2.16) requires $\mathrm{Tr}T_X = 0$, and cancellation of the logarithmic divergences (2.8) requires $\mathrm{Tr}T_a^2 = \sum_C \eta_C\alpha_C(T_X)_C = 0$ when PV contributions are included in the trace. The contributions to (A.17) and (A.18) from $\widehat{Z}, \widetilde{Y}$ fermion loops are, respectively

$$2\sum_\alpha \hat\eta_\alpha\left(\alpha_\alpha^{\widehat{Z}} + \alpha_\alpha^{\widetilde{Y}}\right)C_a^M, \qquad 4\sum_\alpha \hat\eta_\alpha\left[\left(\alpha_\alpha^{\widehat{Z}}\right)^2 - \left(\alpha_\alpha^{\widetilde{Y}}\right)^2\right](\mathrm{Tr}T_X)_M. \tag{A.19}$$

Since $\sum_\alpha \hat\eta_\alpha = 0$, both terms in (A.19) as well as (4.66) vanish if

$$\alpha_\alpha^{\widehat{Z}} = \alpha^{\widehat{Z}}, \qquad \alpha_\alpha^{\widetilde{Y}} = \alpha^{\widetilde{Y}}, \tag{A.20}$$

independent of α. For example, if there are no threshold corrections, $\mu_\alpha^{\widehat{Z}}$ and $\mu_\alpha^{\widetilde{Y}}$ are constants in (4.61), and modular covariance of $W^{\widehat{Z}}$, $W^{\widetilde{Y}}$ requires $\alpha^{\widehat{Z}} = \alpha^{\widetilde{Y}} = 1$. More generally, (A.20) requires that $\mu_\alpha^{\widehat{Z},\widetilde{Y}} = c_\alpha \mu^{\widehat{Z},\widetilde{Y}}(T^i)$, which simplifies the constraints in Section 4.3. This constraint (A.20) also assures that there are no other contributions to (A.4) from $\widehat{Y}, \widehat{Z}, \widetilde{Y}, \widetilde{Z}$ loops. There is no contribution to (A.17) and (A.18) from the adjoint fermions $\chi_\alpha^a = \mathcal{D}_\alpha \varphi^a|$ which have vector gauge connections and no $U(1)_X$ charges. Then these constraints are trivially satisfied if the superfields ϕ_γ in (3.34) carry no gauge charge, which is natural since they were introduced to regulate gravitational couplings. The constraints involving the parameters β_C are, using (3.37) and (3.38),

$$0 = \sum_C \beta_C^2 \left(2\alpha_C - 1\right) = 2\left(\sum_C \beta_C^2 \alpha_C - 1\right) = \sum_C \beta_C \left(4\alpha_C^2 - 4\alpha_C + 1\right) = 4\sum_C \beta_C \alpha_C^2,$$

$$\sum_C \beta_C^2 \alpha_C = 1, \qquad \sum_C \beta_C \alpha_C^2 = 0, \qquad \sum_C \beta_C^3 = 0. \tag{A.21}$$

Finally, we consider the gravitino and gaugino sector. As discussed in Appendix B.3, cancellation of quadratic divergences arising from the axion connection in the gaugino covariant derivative, which is defined as

$$\begin{aligned} \mathcal{D}_\mu \lambda_L^a &= \left(\partial_\mu + Z_\mu - i\Gamma_\mu + \ell_\mu\right) \lambda_L^a + i(T_c)^a_b A_\mu^c \lambda_L^b, \\ \mathcal{D}_\mu \lambda_R^a &= \left(\partial_\mu + Z_\mu + i\Gamma_\mu + \ell_\mu\right) \lambda_R^a + i(T_c)^a_b A_\mu^c \lambda_R^b, \end{aligned} \tag{A.22}$$

where Γ_μ is the Kähler $U(1)_K$ connection defined in (B.51), and

$$\ell_\lambda = \frac{1}{24}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \frac{\partial_\lambda y}{2x} \tag{A.23}$$

is the "vector" axion connection. Note that although the gauge representation is real, we have, instead of (A.10),

$$\begin{aligned} \left(G^+_{\mu\nu}\right)^a_b &= -K_{cb}\left(G^-_{\mu\nu}\right)^c_d K^{da} + 2\left(Z_{\mu\nu} + \ell_{\mu\nu}\right)\delta^a_b, \\ \left(D^+_\mu\right)^a_b &= -K_{cb}\left(D^-_\mu\right)^c_d K^{da} + K^{da}\partial_\mu K_{db} + 2\left(Z_\mu + \ell_\mu\right)\delta^a_b, \end{aligned} \tag{A.24}$$

where $\ell_{\mu\nu}$ is the field strength associated with the connection ℓ_μ, and $K_{ab} = x\delta_{ab}$, so

$$K_{bc}(T_e)^c_d K^{da} = (T_e)^b_a = (T_e^T)^a_b = -(T_e)^a_b. \tag{A.25}$$

We therefore identify $J^\pm$ in (A.4) as $J^\pm_\mu = \pm i\Gamma_\mu + iT\cdot A_\mu$. The contribution from Γ_μ alone to (A.4) is included in the sum rule (A.16). It remains to cancel the gaugino contribution proportional to

$\Gamma_\mu C_a$, where C_a is the adjoint Casimir. From (3.39) and (B.51) we have the following contribution from $\varphi^a, \hat{\varphi}^a, \tilde{\varphi}^a$:

$$C_a \left[\sum_\gamma \eta_\gamma^\varphi \left(\Gamma_\mu - \partial_\mu y/2x \right) + \sum_\gamma \eta_\gamma^{\hat{\varphi}} \left(\partial_\mu y/2x - \Gamma_\mu \right) - \sum_\gamma \eta_\gamma^{\tilde{\varphi}} \Gamma_\mu \right], \tag{A.26}$$

which cancels the gaugino contribution provided

$$\sum_\gamma \eta_\gamma^\varphi = \sum_\gamma \eta_\gamma^{\hat{\varphi}} = \sum_\gamma \eta_\gamma^{\tilde{\varphi}} = 1, \tag{A.27}$$

in accordance with the overall constraint

$$\sum_\gamma \eta_\gamma^\varphi + \sum_\gamma \eta_\gamma^{\hat{\varphi}} + \sum_\gamma \eta_\gamma^{\tilde{\varphi}} = 3, \tag{A.28}$$

needed for the cancellation of gauge and gaugino one-loop logarthmic divergences that arise from their gauge couplings [1, 3].

Additional constraints on the PV sector required to cancel the remaining contributions to (A.4) from PV chiral supermultiplets that regulate chiral matter loops are discussed in Appendix D.

B Chiral Anomalies

B.1 Chiral anomaly analysis in the regulated theory

The chiral anomaly arises from the helicity odd part of the fermion determinant

$$T_- = \frac{1}{2} \left[\operatorname{Tr} \ln \mathcal{M}(\gamma_5) - \operatorname{Tr} \ln \mathcal{M}(-\gamma_5) \right], \qquad \mathcal{M} = -i \not{D} + M, \tag{B.1}$$

in which the axial connection appears explicitly. In [1]–[3], explicit cancellation of ultraviolet quadratic and logarithmic UV divergences in (A.1) was shown to be possible by the introduction of the PV fields discussed in Section 3. In the previous appendix we found additional constraints that insure cancellation of most linear divergences. We defer to Appendix B.3 the discussion of residual linear divergences that are associated with the gauge-gravity-dilaton sector.

In this section we outline a check that, in the absence of linear divergences, the anomaly appears only through the noncovariance of the PV masses, and show that the "consistent anomaly" of YM theory is recovered under appropriate assumptions. Under an infinitesimal chiral transformation

$$\Theta \to e^{-i\gamma_5\phi}\Theta, \qquad \delta\Theta = -i\gamma_5\phi\Theta, \tag{B.2}$$

where $\Theta = (\psi_\mu, \lambda^a, L\chi^p + R\chi^{\bar{p}}, \alpha)$ is a $2N + 4N_G + 10$ component Majorana spinor[18], we have $\mathcal{M} \to \mathcal{M} + \delta\mathcal{M}$, $\delta \mathrm{Tr} \ln \mathcal{M} = \mathrm{Tr}\mathcal{M}^{-1}\delta\mathcal{M}$, and (B.1) is shifted by

$$\delta T_- = \frac{1}{2}\left[\mathrm{Tr}\mathcal{M}^{-1}(\gamma_5)\delta\mathcal{M}(\gamma_5) - \mathrm{Tr}\mathcal{M}^{-1}(-\gamma_5)\delta\mathcal{M}(-\gamma_5)\right]. \tag{B.3}$$

Using the methods developed in Appendix A of [17] the corresponding shift in the Lagrangian can be cast in the form

$$\begin{aligned} \delta\mathcal{L} &\ni -\frac{i}{2}\delta T_- = -i\int \frac{d^4p}{4(2\pi)^4}\left[T(\gamma_5) - T(-\gamma_5)\right], \\ T(\gamma_5) &\equiv T = \mathrm{Tr}\sum_{\ell=0}^{\infty}(-\mathcal{R})^\ell \delta\mathcal{R}, \end{aligned} \tag{B.4}$$

where now the trace is over only Dirac indices and internal quantum numbers, and Lorentz indices for the gravitino, and

$$\begin{aligned} \mathcal{R} &= \frac{1}{-p^2}\left[p^2 - T^{\mu\nu}\Delta_\mu\Delta_\nu - \frac{i}{2}\hat{\mathcal{G}}\cdot\sigma + X + P_{\mu\nu}\left(p^\nu + \mathcal{G}^\nu\right)\widehat{M}^\mu\right], \\ &\quad \Delta_\mu = p_\mu + \mathcal{G}_\mu + \delta_\mu \\ \delta\mathcal{R} &= \frac{1}{-p^2}P_{\mu\nu}\left(p^\nu + \mathcal{G}^\nu\right)\widehat{N}^\mu = \frac{1}{-p^2}p_\mu N^\mu + O\left(\frac{\partial}{\partial p}\right). \end{aligned} \tag{B.5}$$

The operators in these expression are given in Appendix F.4 as covariant derivative expansions. Specifically, $T^{\mu\nu}, X, P_{\mu\nu}, \delta_\mu$ and $\mathcal{G}_\mu, \hat{\mathcal{G}}_{\mu\nu}$ are expansions in covariant derivatives of, respectively, the space-time curvature tensor[19] and the field strength $\mathcal{G}_{\mu\nu}$ acting on fermions:

$$\mathcal{G}^\pm_{\mu\nu} = [D^\pm_\mu, D^\pm_\nu] = G^\pm_{\mu\nu} + Z_{\mu\nu} + iG^L_{\mu\nu}\gamma_5, \qquad Z_{\mu\nu} = -\frac{1}{4}r_{\rho\sigma\mu\nu}\gamma^\rho\gamma^\sigma, \tag{B.6}$$

where $+(-)$ refers to right(left)-handed fermions, and $G^\pm_{\mu\nu}$, $G^L_{\mu\nu}$ are unit matrices in Dirac space. G^L contributes only through loops in the gaugino-gravitino-dilatino sector. In addition we defined

$$N_\mu = -\begin{pmatrix} R\gamma_\mu i\delta\, \not{D}^+ R & -R\gamma_\mu \delta M L \\ -L\gamma_\mu \delta\bar{M} R & L\gamma_\mu i\delta\, \not{D}^- L \end{pmatrix}, \quad M_\mu = \begin{pmatrix} 0 & R\gamma_\mu M L \\ L\gamma_\mu \bar{M} R & 0 \end{pmatrix}, \tag{B.7}$$

and $T(-\gamma_5)$ is obtained from $T(\gamma_5)$ by the substitutions $(D^+, D^-, M, \bar{M}) \to (D^-, D^+, \bar{M}, M)$, with

$$M = M_0 + M_{\mu\nu}\sigma^{\mu\nu}. \tag{B.8}$$

[18]The chiral fermion α is an auxiliary field used to implement gravitino gauge fixing [16].

[19]There is a sign error in the third and second terms, respectively, of the expressions for $T_{\mu\nu}$ and X in Eq. (A.19) of [17].

The matrix $M_{\mu\nu}$ has nonvanishing elements only between the gauginos λ and either the dilatino χ^S or the auxiliary field α introduced to fix the gravitino gauge [16, 17]. Under the chiral transformation (B.2),

$$\delta D^\pm_\mu = \pm i \left[D^\pm_\mu, \phi\right], \qquad \delta \bar{M} = i\{\bar{M}, \phi\}, \qquad \delta M = -i\{M, \phi\}. \tag{B.9}$$

In order to evaluate the light quark contribution to the chiral anomaly, we must resum the derivative expansion of [17]. This resummation can be expressed in terms of the action of the operators in (B.1) on a function of momentum:

$$\begin{aligned} \hat{F} f(p) &= f(p - iD)F, \qquad \mathcal{G}_\mu f(p) = \int_0^1 d\lambda \lambda \frac{\partial}{\partial p_\rho} f(p - i\lambda D)\mathcal{G}_{\rho\mu}, \\ T_{\mu\nu} f(p) &= g_{\mu\nu} - 2\int_0^1 d\lambda\lambda(1-\lambda)\frac{\partial^2}{\partial p_\rho \partial p_\sigma} f(p - i\lambda D) r_{\mu\rho\sigma\nu} + O(r^2), \end{aligned} \tag{B.10}$$

and, by partial integration

$$\begin{aligned} f(p)\hat{F} &= f(p + iD)F, \qquad f(p)\mathcal{G}_\mu = -\int_0^1 d\lambda \lambda \frac{\partial}{\partial p_\rho} f(p + i\lambda D)\mathcal{G}_{\rho\mu}, \\ T_{\mu\nu} f(p) &= g_{\mu\nu} - 2\int_0^1 d\lambda\lambda(1-\lambda)\frac{\partial^2}{\partial p_\rho \partial p_\sigma} f(p + i\lambda D) r_{\mu\rho\sigma\nu} + O(r^2), \end{aligned} \tag{B.11}$$

where F is any local field operator.

First neglecting G_L and $M_{\mu\nu}$, the anomaly is mass-independent and arises from terms of order p^{-6} in (B.1). Writing

$$T_n = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left\{[\mathcal{R}(\gamma_5)]^n \,\delta\mathcal{R}(\gamma_5) - [\mathcal{R}(-\gamma_5)]^n \,\delta\mathcal{R}(-\gamma_5)\right\} = \int \frac{d^4p}{(2\pi)^4} t_n, \tag{B.12}$$

we have $T_0 = 0$, and we only need to evaluate $t_1 \sim p^{-3}$ and $t_2 \sim p^{-5}$. Since T_2 is finite we can evaluate it directly using (B.10) and (B.11). This is a local operator which is bilinear in the field strength $\mathcal{G}_{\mu\nu}$. Since it is independent of momenta we can set one external momentum to zero. That is, in the product $\mathcal{G}_{\rho\sigma}\mathcal{G}'_{\mu\nu}$ we first set $D_\lambda \mathcal{G}_{\rho\sigma} = 0$ and then $D'_\lambda \mathcal{G}'_{\mu\nu} = 0$, and take the average. This trick considerably simplifies the calculation and, using standard Feynman parameterization techniques and the Bianchi identities, and dropping total derivatives:

$$\mathrm{Tr}(\mathcal{G}\mathcal{G}D\phi) \to -\mathrm{Tr}[\phi D(\mathcal{G}\mathcal{G})], \tag{B.13}$$

we find that the various contributions cancel:

$$T_2 = 0. \tag{B.14}$$

Since T_1 has logarithmically divergent contributions, we must explicitly cancel them against one another before using integration by parts or shifts in the integration variable. To this end we first rewrite

$$\mathrm{Tr}(\mathcal{G}D\phi) \to -\mathrm{Tr}(\phi D\mathcal{G}) \tag{B.15}$$

and

$$[D^\nu, {}^*\hat{G}_{\mu\nu}] = i[\hat{G}^\nu, {}^*\hat{G}_{\mu\nu}] = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}[\hat{G}^\nu, \hat{G}^{\rho\sigma}], \tag{B.16}$$

where

$$\hat{G}_\mu f(p) = \int_0^1 d\lambda \frac{\partial}{\partial p_\rho} f(p - i\lambda D)G_{\rho\mu}, \qquad f(p)\hat{G}_\mu = -\int_0^1 d\lambda \frac{\partial}{\partial p_\rho} f(p + i\lambda D)G_{\rho\mu}. \tag{B.17}$$

Using this result in the expression for T_1 that contains a factor $\sigma \cdot \hat{G}$ one finds the contribution

$$-\frac{i}{2}\delta T_- = +\frac{i}{2}T_1 = \frac{1}{16\pi^2}\left(G \cdot \widetilde{G}\phi\right). \tag{B.18}$$

In the regulated theory, we have to subtract a contribution with $-p^2 \to -(p^2 - m^2)$ in the denominators in (B.5), and drop the terms that vanish in the limit $m^2 \to \infty$. Using

$$G^{\rho\nu}\widetilde{G}'_{\mu\nu} = \frac{1}{2}G \cdot \widetilde{G}' g^\rho_\mu - \widetilde{G}_{\mu\nu}G'^{\rho\nu}, \qquad D_\mu \widetilde{G}^{\mu\nu} = 0, \tag{B.19}$$

and assuming $[m^2, G] = 0$ we get a contribution

$$\begin{aligned}
t_1(m^2) - t_1(0) &= -8\mathrm{Tr}\int_0^1 d\lambda \left[\frac{p_\rho p^\mu}{(p^2 - m^2)^3}G^{\rho\nu}\widetilde{G}_{\mu\nu} + \frac{\widetilde{G}^{\mu\nu}G_{\rho\nu}}{(p^2 - m^2)^2}\left(\frac{p_\mu p^\rho}{p^2 - m^2} - \frac{g^\rho_\mu}{2}\right)\right]\phi \\
&= -4\mathrm{Tr}\int_0^1 d\lambda \left[\left(\frac{p^2}{(p^2 - m^2)^3} - \frac{1}{(p^2 - m^2)^2}\right)G \cdot \widetilde{G}\phi\right] \\
&= -4\mathrm{Tr}\int_0^1 d\lambda \left(\frac{m^2}{(p^2 - m^2)^3}G \cdot \widetilde{G}\phi\right), \\
\int_p \left[t_1(m^2) - t_1(0)\right] &= +\frac{i}{8\pi^2}\mathrm{Tr}\left(G \cdot \widetilde{G}\phi\right), \qquad \int_p \equiv \int \frac{d^4p}{(2\pi)^4}
\end{aligned} \tag{B.20}$$

and we recover (B.18)

$$-\frac{i}{2}\delta T_- = +\frac{i}{2}\int_p \left[t_1(0) - t_1(m^2)\right] = \frac{1}{16\pi^2}\left(G \cdot \widetilde{G}\phi\right) = \frac{i}{2}\int_p t_1(0), \tag{B.21}$$

which is the standard result for the contribution from renormalizable gauge couplings with $G^\pm_{\mu\nu} \to \mp iT \cdot F_{\mu\nu}$. However there are additional contributions from the PV sector when $[m^2, G] \neq 0$, as we will see in Section B.2.

The space-time curvature term in T_1 has two contributions. Those arising from $Z_\mu Z^\mu$ and $(T^{\mu\nu} - g^{\mu\nu})\{p_\mu, Z_\nu\}$ are UV finite and may be straightforwardly evaluated as described above. After dropping a total derivative as in (B.15), the remaining contribution reduces to

$$\frac{1}{p^2}\mathrm{Tr}\left(\{p^\mu, [D^\nu, Z_\mu]\}\frac{\not{p}}{p^2}\gamma_\nu\gamma_5\phi\right). \tag{B.22}$$

To evaluate this we adopt the convention that all p-derivatives act to the left so that

$$\left[\frac{\partial}{\partial p_\alpha}, p^\beta\right] = -g^{\alpha\beta}, \tag{B.23}$$

and we define

$$C_{\mu\nu} = [D_\mu, D_\nu] = [\nabla_\mu, \nabla_\nu] + Z_{\mu\nu}. \tag{B.24}$$

Then writing

$$\{p^\mu, Z_\mu\} = 2p^\mu Z_\mu + [Z_\mu, p^\mu], \tag{B.25}$$

the first term drops out of (B.22) because

$$\frac{p^\mu}{p^2}Z_\mu = \frac{2p^\mu p^\rho - p^2\delta^{\mu\rho}}{p^4}\sum_{m=0}\int_0^1 d\lambda\frac{\lambda}{m!}\left(-i\lambda D\cdot\frac{\partial}{\partial p}\right)^m Z_{\rho\mu} = 0 \tag{B.26}$$

by symmetry, and we obtain

$$[Z_\mu, p^\mu] \;=\; i\int_0^1 d\lambda\lambda D^\mu Z_\mu(\lambda) + \int_0^1 d\lambda\lambda\int_0^\lambda d\eta\eta\,[C^\mu(\eta), Z_\mu(\lambda)], \tag{B.27}$$

where if $G = Z, C$

$$\begin{aligned} G_\mu(\lambda)f(p) &= \frac{\partial}{\partial p_\rho}f(p - i\lambda D)G_{\rho\mu}, \\ f(p)G_\mu(\lambda) &= -\frac{\partial}{\partial p_\rho}f(p - i\lambda D)G_{\rho\mu}. \end{aligned} \tag{B.28}$$

Finally, using (B.17) we obtain

$$\begin{aligned} [D_\nu, [Z_\mu, p^\mu]] \;=\; & -\int_0^1 d\lambda\lambda^2\int_0^\lambda d\eta\,[C_\nu(\eta), D^\mu Z_\mu(\lambda)] + i\int_0^1 d\lambda\lambda D_\nu D^\mu Z_\mu(\lambda) \\ & + \int_0^1 d\lambda\lambda\int_0^\lambda d\eta\eta\,([C^\mu(\eta), D_\nu Z_\mu(\lambda)] + [D_\nu C^\mu(\eta), Z_\mu(\lambda)]). \end{aligned} \tag{B.29}$$

Evaluating this contribution by the above procedure and combining all three contributions gives

$$\begin{aligned}\delta T &= -\int_p t_1 = \frac{i}{192\pi^2} r\cdot\tilde{r}\mathrm{Tr}\phi, \qquad -\frac{i}{2}\delta T = \frac{1}{384\pi^2} r\cdot\tilde{r}\mathrm{Tr}\phi\\ r\cdot\tilde{r} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} r_{\rho\sigma\tau\lambda} r^{\tau\lambda}{}_{\mu\nu}.\end{aligned}\tag{B.30}$$

It is straightforward to check that there is no contribution to this term when $p^2 \to p^2 - m^2$ in the denominators of $\mathcal{R}, \delta\mathcal{R}$ in (B.5).

In the following section we will see that the remaining terms of the "consistent anomaly" [26] will emerge when we include all the contributions from the PV sector.

B.2 Full PV contribution

In this section we consider PV fields with no spin-dependent mass terms. Then the only relevant mass is the PV mass m, and $\mathcal{M}_{PV}$ is given by (A.7) of [17] with $L = 0,\ M = 0,\ m = m_{PV}$. To evaluate (B.1) we follow (A.13) of [17], but take $\mathcal{M}_0 = \mathcal{M}_{PV}(-m, -\vec{\sigma})$. Then we have

$$\begin{aligned}\mathcal{R}_{PV} &= \frac{1}{-p^2 + \mathbf{m}^2}\left[p^2 - \mathbf{m}^2 - T^{\mu\nu}\Delta_\mu\Delta_\nu + \hat{\mathbf{m}}^2 + \hat{h} + X - i\widehat{\not{D}\mathbf{m}}\right],\\ \delta\mathcal{R}_{PV} &= \frac{1}{-p^2 + \mathbf{m}^2}\left(p_\mu N^\mu + N\right) + O\left(\frac{\partial}{\partial p}\right),\end{aligned}\tag{B.31}$$

with $\delta M \to \delta m$ in N_μ, defined in (B.7), and

$$\not{D}\mathbf{m} = \begin{pmatrix} 0 & R[\not{D}^+ m - m\not{D}^-]L \\ L[\not{D}^-\bar{m} - \bar{m}\not{D}^+]R & 0\end{pmatrix} \equiv \begin{pmatrix} 0 & R\not{D}mL \\ L\not{D}\bar{m}R & 0\end{pmatrix}, \tag{B.32}$$

$$N = -\begin{pmatrix} Rm\delta\bar{m}R & -iRm\delta\not{D}^-L \\ -iL\bar{m}\delta\not{D}^+R & L\bar{m}\delta mL\end{pmatrix}, \quad \mathbf{m}^2 = \begin{pmatrix} Rm\bar{m}R & 0 \\ 0 & L\bar{m}mL\end{pmatrix}. \tag{B.33}$$

First consider chiral fermion contributions to the space-time curvature term. The only contribution from the PV sector that is not suppressed by m^{-2} comes from the replacement $\not{p}i\delta\not{D}\phi \to \mathbf{m}\delta\mathbf{m}$ in the term involving $Z_\mu Z^\mu$, giving

$$\begin{aligned}\delta T^{PV} &= -\int_p t_1^{PV} = \frac{r\cdot\tilde{r}}{192\pi^2}\mathrm{Tr}\eta\frac{1}{4}\left(\bar{m}^{-1}\delta\bar{m} - m^{-1}\delta m\right),\\ -\frac{i}{2}\delta T^{PV} &= -\frac{r\cdot\tilde{r}}{384\pi^2}\mathrm{Tr}\eta\frac{i}{4}\left(\bar{m}^{-1}\delta\bar{m} - m^{-1}\delta m\right),\end{aligned}\tag{B.34}$$

and the total contribution from light and heavy modes is

$$-\frac{i}{2}\left(\delta T + \delta T^{PV}\right) = -\frac{r\cdot\tilde{r}}{16\pi^2}\frac{1}{24}\mathrm{Tr}\left[\frac{1}{2}\left(\phi^+ - \phi^-\right) + \frac{i}{4}\eta\left(\bar{m}^{-1}\delta\bar{m} - m^{-1}\delta m\right)\right], \tag{B.35}$$

where for a chiral transformation defined by (B.2) $\phi^+ = -\phi^- = -\phi$. If the PV masses were modular covariant they would satisfy

$$\delta\bar{m} = i(\phi^-\bar{m} - \bar{m}\phi^+) \to i\{\bar{m}, \phi\}, \qquad \delta m = i(\phi^+ m - m\phi^-) \to -i\{m, \phi\}, \tag{B.36}$$

and the regulated theory would be anomaly free, provided (A.9) is satisfied. If we define

$$d\bar{m} = \delta\bar{m} - i\{\bar{m}, \phi\}, \qquad dm = \delta m + i\{m, \phi\}, , \tag{B.37}$$

the total contribution from light and heavy modes reads, setting $\eta = -1$,

$$-\frac{i}{2}\left(\delta T + \delta T^{PV}\right) = -\frac{r\cdot\tilde{r}}{16\pi^2}\frac{1}{24}\mathrm{Tr}\eta\frac{i}{4}\left(\bar{m}^{-1}d\bar{m} - m^{-1}dm\right). \tag{B.38}$$

In other words, we may write

$$\begin{aligned} -\frac{i}{2}\left(\delta T + \delta T^{PV}\right) &= -\frac{i}{2}\left(\delta T + \delta T^{PV}_{m=0}\right) - \frac{i}{2}\left(\delta T^{PV} - \delta T^{PV}_{m=0}\right) \\ &= -\frac{i}{2}\left(\delta T + \delta T^{PV}_{m=0}\right) - \frac{r\cdot\tilde{r}}{16\pi^2}\frac{1}{24}\mathrm{Tr}\eta\frac{i}{4}\left(\bar{m}^{-1}d\bar{m} - m^{-1}dm\right). \end{aligned} \tag{B.39}$$

The first term on the right hand side of (B.39) vanishes by virtue of (A.9), which is consistent with the absence of linear divergences associated with the spin connection.

To evaluate the remaining contributions we have to take into account that m is *a priori* matrix-valued and field dependent. Then (B.20) is replaced by

$$\begin{aligned} \tau_1 &= -2\mathrm{Tr}\eta\left(\frac{m^2}{(p^2-m^2)^2}G^{\mu\nu}\widetilde{G}_{\mu\nu}\frac{1}{p^2-m^2} + \frac{1}{p^2-m^2}\widetilde{G}_{\mu\nu}G^{\mu\nu}\frac{m^2}{(p^2-m^2)^2}\right)\phi, \\ \int_p \tau_1(p^2) &= -\frac{i}{16\pi^2}\int_0^\infty dp^2 p^2\tau_1(-p^2) \\ &= \frac{i}{8\pi^2}\mathrm{Tr}\eta\int_0^\infty dp^2\left(\left\{G\cdot\widetilde{G}, \frac{m^2}{(p^2+m^2)^2}\right\} - \frac{m^2}{p^2+m^2}\left\{G\cdot\widetilde{G}, \frac{1}{p^2+m^2}\right\}\frac{m^2}{p^2+m^2}\right)\phi \\ &= \frac{i}{8\pi^2}\mathrm{Tr}\eta\left[2G\cdot\widetilde{G} + \int_0^\infty dp^2\frac{\partial}{\partial p^2}\left(\frac{m^2}{p^2+m^2}G\cdot\widetilde{G}\frac{m^2}{p^2+m^2}\right)\right]\phi \\ &= \frac{i}{8\pi^2}\mathrm{Tr}\eta\left(G\cdot\widetilde{G}\phi\right), \end{aligned} \tag{B.40}$$

where here and below we use a shorthand notation with $\phi = -\phi^+ = \phi^-$, and, for example,

$$\mathrm{Tr}\eta\left(G\cdot\widetilde{G}\phi\right) \equiv \frac{1}{2}\mathrm{Tr}\eta\left[\left(G^+\cdot\widetilde{G}^+ + G^-\cdot\widetilde{G}^-\right)\phi\right]. \tag{B.41}$$

That is, we average over helicities with the convention that

$$O\phi \equiv -\frac{1}{2}\left(O^+\phi^+ - O^-\phi^-\right) = \frac{1}{2}\left(O^+ + O^-\right)\phi. \tag{B.42}$$

If $\delta m \neq 0$, there is another contribution with $\not{\phi} i\delta \not{D}\phi \to m\delta m$:

$$\begin{aligned}
\sigma_2' &= \frac{1}{4}\mathrm{Tr}\eta\left(\frac{\sigma^{\mu\nu}}{p^2-m^2}G_{\mu\nu}\right)^2\frac{1}{p^2-m^2}m\delta m\gamma_5 \\
&= 2i\mathrm{Tr}\eta\frac{1}{p^2-m^2}\widetilde{G}_{\mu\nu}\frac{1}{p^2-m^2}G^{\mu\nu}\frac{1}{p^2-m^2}mdm \\
&\quad -2\mathrm{Tr}\eta\left\{m,\frac{1}{p^2-m^2}\widetilde{G}_{\mu\nu}\frac{1}{p^2-m^2}G^{\mu\nu}\frac{1}{p^2-m^2}\right\}m\phi.
\end{aligned} \tag{B.43}$$

These are the only contributions if $[D_\mu, m] = 0$; in this case $[G, m] = 0$, and if also $dm = 0$ this reduces to

$$T_2^{PV} \quad \to \quad \int_p \sigma_2' \to \int_p \tau_1 = T_1^{PV}, \qquad T_1 - T_2 \to 0, \tag{B.44}$$

as expected. The remaining contributions involve covariant derivatives on the mass matrix. Writing

$$T_n^{PV} = -\frac{1}{2}\int_p \mathrm{Tr}\eta\left\{[\mathcal{R}(\gamma_5)]^n\,\delta\mathcal{R}(\gamma_5) - [\mathcal{R}(-\gamma_5)]^n\,\delta\mathcal{R}(-\gamma_5)\right\}_{PV} = \int_p (\tau_n + \sigma_n), \tag{B.45}$$

where τ_n and σ_n are the contributions without and with, respectively, a factor δm. They take the general form

$$\begin{aligned}
\tau_n &= \mathrm{Tr}\eta\left(O_n^\tau \not{D}\phi\right) \to -\mathrm{Tr}\eta\left([\not{D}, O_n^\tau]\,\phi\right), \\
\sigma_n &= \mathrm{Tr}\eta\left(O_n^\sigma \delta m\right) = \mathrm{Tr}\eta\left(O_n^\sigma dm\right) + i\mathrm{Tr}\eta\left(\{m, O_n^\sigma\}\phi\right).
\end{aligned} \tag{B.46}$$

Using relations such as

$$\begin{aligned}
\not{D}^2 m &= D^2 m - \frac{i}{2}[\sigma\cdot G, m], \\
[\not{D}, m^2] &= \{m, \not{D}m\}, \qquad [\not{D}, P] = \{m, P[\not{D}, m]P\}, \\
P\frac{\partial}{\partial p}\cdot Dm^2 P &= \frac{\partial}{\partial p}\cdot[D,P] + 2P[p\cdot D, P] = [D,P]\cdot\frac{\partial}{\partial p} - 2[p\cdot D, P]P, \\
\{m, P[G,m]P\} &= [G,P], \qquad P = \frac{1}{p^2-m^2},
\end{aligned} \tag{B.47}$$

it is possible to show that

$$\sum_n \tau_n + \sum_n \sigma_n = \sum_n \mathrm{Tr}\eta\left(O_n^\sigma dm\right). \tag{B.48}$$

As a check of our anomaly calculation, consider the case with constant PV masses such that $[D, m^2] = [G, m^2] = \delta m = 0$. Then $\sigma_i = 0$ and

$$
\begin{aligned}
\tau_2 &= -\mathrm{Tr}\eta \left[Pi\widehat{\not{D}m} \right]^2 P \not{p}[\not{D}\phi]\gamma_5 + \frac{1}{2}\mathrm{Tr}\eta \left\{ P[\not{D}, m], P\sigma \cdot G \right\} Pm[\not{D}, \phi]\gamma_5 \\
&= -2\mathrm{Tr}\eta P^3 m^2 P[\widetilde{G}^{\mu\nu}, m][G_{\mu\nu}, m]\phi - 2\mathrm{Tr}\eta P^3 \left\{ [\widetilde{G}^{\mu\nu}, m], G_{\mu\nu} \right\} m\phi \\
&\quad -2\mathrm{Tr}\eta p^2 P^4 [\widetilde{G}_{\mu\nu}, [D^\mu, m]][D^\nu, m]\phi + 4\mathrm{Tr}\eta P^2 \left\{ [D^\mu, m], \widetilde{G}_{\mu\nu} \right\} [D^\nu, m]\phi \\
&\quad +2\mathrm{Tr}\eta P^3 [D^\mu, m][G^{\mu\nu}, [D_\nu, m]] \left(p^2 P - 2 \right) \phi, \\
\tau_3 &= \mathrm{Tr}\eta \left[Pi\,\not{D}m \right]^3 Pm[\not{D}, \phi]\gamma_5 \\
&= -4\mathrm{Tr}\eta P^4 \{ [\widetilde{G}^{\mu\nu}, m], D_\mu m D_\nu m \} m\phi + 4\mathrm{Tr}\eta P^4 D_\mu m [\widetilde{G}^{\mu\nu}, m] D_\nu m m\phi \\
&\quad +4\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\eta P^4 D_\mu m D_\nu m D_\rho D_\sigma m\phi.
\end{aligned} \tag{B.49}
$$

The covariant derivative on chiral fermions is given in (A.6) with

$$
J^\pm_\mu = \Gamma^\pm_\mu \mp i\Gamma_\mu \mp iT^\pm \cdot A_\mu, \qquad T^+ = (T^-)^T = (T^-)^*, \tag{B.50}
$$

where A_μ is a gauge field and

$$
\Gamma_\mu = \frac{i}{4}\left(K_i \mathcal{D}_\mu z^i - K_{\bar{m}} \mathcal{D}_\mu \bar{z}^{\bar{m}} \right), \qquad \left(\Gamma^-_\mu \right)^p_q = \Gamma^p_{qk} \mathcal{D}_\mu z^k = \left[\left(\Gamma^+_\mu \right)^{\bar{p}}_{\bar{q}} \right]^\dagger. \tag{B.51}
$$

are the Kähler $U(1)$ and reparameterization connections, respectively. It follows from gauge invariance of the Kähler potential that

$$
K_{i\bar{m}} (D^+_\mu)^{\bar{m}}_{\bar{n}} K^{\bar{n}j} = (\partial_\mu + Z_\mu)\delta^j_i - (J^-_\mu)^j_i. \tag{B.52}
$$

For chiral fermions the field strength $G_{\mu\nu}$ can be expressed in terms of the general two-forms $T_{\mu\nu}$ defined[20] in (E.2); explicitly:

$$
\begin{aligned}
\left(G^-_{\mu\nu} \right)^p_q &= \frac{1}{2} X_{\mu\nu}\delta^p_q + iF^a_{\mu\nu}(T_a)^p_q - \Gamma^p_{q\mu\nu}, \\
\left(G^+_{\mu\nu} \right)^{\bar{n}}_{\bar{m}} &= -K_{p\bar{m}} K^{q\bar{n}} \left(G^-_{\mu\nu} \right)^p_q, \quad X_{\mu\nu} = K_{\mu\nu}.
\end{aligned} \tag{B.53}
$$

For constant masses we have

$$
[D_\mu, m] = [J_\mu, m] = [a_\mu, m], \qquad J_\mu = j_\mu + a_\mu = \begin{pmatrix} J^+_\mu & 0 \\ 0 & J^-_\mu \end{pmatrix}, \tag{B.54}
$$

[20]The two-forms $T_{\mu\nu}$ used here and below should not confused with the tensor $T^{\mu\nu}$ introduced in the derivative expansion in (B.5) and defined in (F.27).

where a_μ is the connection associated with the anomalous symmetry:

$$[j_\mu, m] = \{a_\mu, m\} = 0, \qquad G_{\mu\nu} = g_{\mu\nu} + a_{\mu\nu}, \qquad [g_{\mu\nu}, m] = \{a_{\mu\nu}, m\} = 0. \tag{B.55}$$

For the helicity components of these matrices, this gives

$$m a^-_{\mu\nu} = -a^+_{\mu\nu} m, \qquad a^-_{\mu\nu}\bar{m} = -\bar{m} a^+_{\mu\nu}, \qquad m g^-_{\mu\nu} = g^+_{\mu\nu} m, \qquad g^-_{\mu\nu}\bar{m} = \bar{m} g^+_{\mu\nu}, \tag{B.56}$$

and we obtain in this case

$$\begin{aligned}
T_2 &= \frac{i}{12\pi^2}\mathrm{Tr}\eta\left(\tilde{a}^{\mu\nu}a_{\mu\nu}\phi\right) \\
&\quad + \frac{i}{3\pi^2}\mathrm{Tr}\eta\left[\left(\{\tilde{g}^{\mu\nu}, a_\mu a_\nu\} + a_\mu \tilde{g}^{\mu\nu} a_\nu + \{\tilde{a}^{\mu\nu}, a_\mu a_\nu\} - a_\mu \tilde{a}^{\mu\nu} a_\nu\right)\phi\right] \\
T_3 &= \frac{i}{3\pi^2}\mathrm{Tr}\eta\left[\left(\{\tilde{a}^{\mu\nu}, a_\mu a_\nu\} - a_\mu \tilde{a}^{\mu\nu} a_\nu\right)\phi\right] \\
&\quad + \frac{2i}{3\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\eta\left(a_\mu a_\nu a_\rho a_\sigma \phi\right).
\end{aligned} \tag{B.57}$$

Terms linear and cubic in a drop out of $\delta T = \sum_{n=1}^{3}(-1)^n T_n$, and, provided there is a PV particle with negative signature, $\eta = -1$, for every light particle, we recover the standard [26] result:

$$\begin{aligned}
-\frac{i}{2}\delta T &= \frac{1}{16\pi^2}\mathrm{Tr}\phi\left(g\cdot\tilde{g} + \frac{1}{3}a\cdot\tilde{a} + \frac{r\cdot\tilde{r}}{24}\right. \\
&\quad \left. -\frac{8}{3}\left[\{\tilde{g}_{\mu\nu}, a^\mu a^\nu\} + a^\mu \tilde{g}_{\mu\nu} a^\nu\right] + \frac{16}{3}\epsilon_{\mu\nu\rho\sigma}a^\mu a^\nu a^\rho a^\sigma\right).
\end{aligned} \tag{B.58}$$

B.3 Gauge, gravity and dilaton sector: nonrenormalizable operators

In the preceeding subsections the cancellation of linear divergences was implicitly assumed, that is, we assumed

$$\delta T = -\delta T^{PV}_{m=0}, \qquad \mathrm{Tr}(\phi G\cdot\widetilde{G})_{\mathrm{light}} = -\mathrm{Tr}(\eta\phi G\cdot\widetilde{G})_{\mathrm{PV}}, \tag{B.59}$$

However contributions from nonrenormalizable interactions to the gaugino and gravitino connections, including an additional curvature term in the gravitino connection and [17] an off-diagonal gravitino-gaugino connection, do not have counterparts in the PV sector. These have to be treated separately. The contribution from the gravitino field strength:

$$\mathrm{Tr}\left(g\cdot\tilde{g}\right)_\psi \ni -r\tilde{r} \tag{B.60}$$

leads to the contribution (4.26) to the chiral anomaly. There is a an additional contribution from the off-diagonal gaugino-gravitino connection; the corresponding field strength $g_{\mu\nu}$ has terms linear and quadratic in the Yang-Mills field $F_{\mu\nu}$. The terms quartic in $F_{\mu\nu}$ cancel between gaugino and gravitino loops:

$$g\cdot\tilde{g}\ni\pm\frac{x^2}{2}\epsilon_{\mu\nu\rho\sigma}F_a^{\alpha\rho}F^a_\beta{}^{\ \nu}F_{b\alpha}{}^{\sigma}F^{b\beta\mu}\quad\text{for}\quad\begin{cases}\lambda\\ \psi\end{cases}. \tag{B.61}$$

The remaining contribution takes the form

$$\begin{aligned}\mathrm{Tr}\,(g\cdot\tilde{g})_{\lambda+\psi}\ &\ni\ 4\left[\mathcal{D}^\mu\left(\sqrt{x}F_a^{\rho\nu}\right)\mathcal{D}_\rho\left(\sqrt{x}\tilde{F}^a_{\mu\nu}\right)-\mathcal{D}_\rho\left(\sqrt{x}F_a^{\rho\nu}\right)\mathcal{D}^\mu\left(\sqrt{x}\tilde{F}^a_{\mu\nu}\right)\right]\\ &\quad+2x\tilde{F}_a^{\mu\nu}F^a_{\rho\sigma}r_{\mu\nu}{}^{\rho\sigma}-xrF^a_{\mu\nu}\tilde{F}_a^{\mu\nu}-4xc_{abc}\tilde{F}^c_{\mu\nu}F^{a\rho\mu}F^b_\rho{}^{\ \nu}\\ &=\ 4\mathcal{D}^\mu\left(xF^{\rho\nu}\mathcal{D}_\rho\tilde{F}_{\mu\nu}\right).\end{aligned} \tag{B.62}$$

The second expression on the right hand side of (B.62) is obtained from the first by using the identities (B.19) and [17]

$$\tilde{F}^a_{\mu\nu}[\mathcal{D}^\mu,\mathcal{D}_\rho]F_a^{\rho\nu}=-F^a_{\mu\nu}[\mathcal{D}^\mu,\mathcal{D}_\rho]\tilde{F}_a^{\rho\nu}=c_{abc}\tilde{F}^a_{\mu\nu}F^{b\mu\rho}F^{c\nu}{}_\rho+r^\mu_\nu\tilde{F}^a_{\mu\rho}F_a^{\nu\rho}-\frac{1}{2}r_{\mu\nu}{}^{\rho\sigma}\tilde{F}_a^{\mu\nu}F^a_{\rho\sigma}, \tag{B.63}$$

The contribution from (B.62) to the chiral anomaly is given in (4.34). As described in Section 4.1, these anomalies arise from uncanceled linear divergences that have counterparts in uncanceled logarithmic divergences which are total divergences, and so do not affect the finiteness of the S-matrix, but the resulting anomalies form superfields provided the the cut-off has the form (1.1). The field strength G^L in (B.6) arises from a term in the gaugino connection, $i\gamma_5L_\mu=-i\gamma_5\partial_\mu y/2x$, that, as was shown in [1], must be defined as a “vector” (rather than an “axial vector”) connection, through the use of the identity

$$\gamma_5=(i/24)\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma, \tag{B.64}$$

in order to allow Pauli-Villars regularization of the quadratic divergences. This choice further insures that the nonrenormalization [28] of the topological charge, $\theta=8\pi^2y$, is consistent with linear-chiral multiplet duality [38] for the dilaton supermultiplet, and preserves [12] the modified linearity condition in the linear multiplet formulation. Since BRST invariance requires (and supersymmetry allows) the regulation of nonabelian multiplet gauge loops by PV chiral multiplets, it is clear that only the chiral multiplet axial connections can appear in the anomalies associated with the regulated pure Yang-Mills sector. As a result, contributions from δL_μ are absent[21] from $\delta\mathcal{R}$. The operator T_- is defined in such a way that the contribution to the anomaly involving only G^L,

[21] In Feynman diagram language the gaugino-loop contribution to the L_μ anomaly arising from a shift in the axion y is canceled by a gauge vector loop contribution [39].

and G is proportional to

$$\begin{aligned}
& \frac{1}{2}\mathrm{Tr}\left\{\left[\left([G^+ + i\gamma^5 G^L]\cdot\sigma\right)^2 + \left([G^- + i\gamma^5 G^L]\cdot\sigma\right)^2\right]\gamma_5\phi\right\} \\
= \quad & \mathrm{Tr}\phi\left\{\left[(g\cdot\sigma)^2 + (a\cdot\sigma)^2 - (G^L\cdot\sigma)^2\right]\gamma_5 + i\{(g\cdot\sigma),(G^L\cdot\sigma)\}\right\}. \qquad \text{(B.65)}
\end{aligned}$$

From the expressions for the field strengths given in Eq. (C.18) of [17], it is easy to see that the traces involving G^L vanish identically in the above expression. Those involving only G have already been taken into account. In particular, the dilatino χ^S, gauginos λ and gravitino ψ transform only under modular transformations:

$$\chi^S \to e^{\frac{i}{2}\mathrm{Im}F}\chi^S, \quad \lambda \to e^{-\frac{i}{2}\mathrm{Im}F}\lambda, \quad \psi \to e^{-\frac{i}{2}\mathrm{Im}F}\psi. \qquad \text{(B.66)}$$

To fix the gravitino gauge, we follow [16, 17] and introduce an auxiliary field α that transforms like a chiral fermion: $\alpha \to e^{\frac{i}{2}\mathrm{Im}F}\alpha$. With this procedure, the ghostino makes no contribution to the chiral anomaly, but we must include the contribution of the auxiliary field.

The fields in this sector are $U(1)_X$-neutral, and λ transforms under modular transformations with the opposite phase from the phase of χ^S, α. Therefore the masses (B.8) are modular invariant, and terms quadratic or quartic in $M\cdot\sigma$ and linear in ϕ cannot contribute to (B.1); the only nonvanishing contributions involve the additional, modular invariant, connections $i\gamma_5 L_\mu$ and $(\Gamma^\pm)^S_S$. The λ-α,λ-χ_S matrix elements (B.8) satisfy [17]

$$\begin{aligned}
M_0 \;&=\; M_0^T, \qquad M_{\mu\nu} = -M^T_{\mu\nu}, \qquad M^0_{\lambda\alpha} = -\bar{M}^0_{\lambda\alpha}, \qquad M^{\mu\nu}_{\lambda\alpha} = \bar{M}^{\mu\nu}_{\lambda\alpha} \\
0 \;&=\; M^{\mu\nu}_{\lambda\chi}\bar{M}^{\lambda\chi}_{\mu\nu}, \qquad \widetilde{M}^{\mu\nu}_{\lambda\chi} = iM^{\mu\nu}_{\lambda\chi}, \qquad \tilde{\bar{M}}^{\mu\nu}_{\lambda\chi} = -i\bar{M}^{\mu\nu}_{\lambda\chi}, \qquad M^0_{\lambda\chi} = -\bar{M}^0_{\lambda\chi}. \qquad \text{(B.67)}
\end{aligned}$$

The contributions to the anomaly that may arise from these masses can be determined by identifying local operators that could be obtained from the terms of order p^{-6} in the expansion (B.1). The masses in (B.67) are gauge invariant and modular covariant:

$$\delta\bar{M} = i\left(\bar{M}\phi + \phi\bar{M}\right) = 0 \qquad \text{(B.68)}$$

because $\phi_\chi = \phi_\alpha = -\phi_\lambda = \frac{1}{2}\mathrm{Im}F$. Then since only even powers of M can occur, the *a priori* possibilities are $T_{2,3,4}$ with

$$\begin{aligned}
T_i \;&=\; \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left(t_i + \mathrm{h.c.}\right), \\
t_2 \;&=\; -\frac{1}{p^2}\left(\not{p} + \not{\mathcal{G}}\right)\widehat{M}\frac{1}{p^2}\left(\not{p} + \not{\mathcal{G}}\right)\widehat{\bar{M}}\frac{1}{\not{p}}\not{D}\phi\gamma_5 \equiv t_\infty + t_2', \qquad t_\infty = -\frac{1}{p^2}\not{p}\widehat{M}\frac{1}{p^2}\not{p}\widehat{\bar{M}}\frac{1}{\not{p}}\not{D}\phi\gamma_5,
\end{aligned}$$

$$
\begin{aligned}
t_3 &= -\left[\frac{1}{p^2}\left(\{\mathcal{G}_\mu, p^\mu\} + \frac{i}{2}\hat{\mathcal{G}}\cdot\sigma\right)\frac{1}{\not{p}}\widehat{M}\frac{1}{\not{p}}\widehat{\bar{M}} + \text{permutations}\right]\frac{1}{\not{p}}\,\not{D}\phi\gamma_5, \\
t_4 &= -\left(\frac{1}{\not{p}}\widehat{M}\frac{1}{\not{p}}\widehat{\bar{M}}\right)^2\frac{1}{\not{p}}\,\not{D}\phi\gamma_5.
\end{aligned}
\tag{B.69}
$$

Because $M^{\mu\nu}_{a\alpha(\chi)} = c_{\alpha(\chi)}F^{\mu\nu}_a$, no Lorentz and gauge invariant operator can be constructed from $M^2_{\mu\nu}G_{\rho\sigma}\phi$ or from three factors of $M_{\mu\nu}$ in the trace. The only invariant involving the space-time Riemann tensor has two factors of $M^{\mu\nu}$ and vanishes due to (B.67). Then from power counting in $p_\mu \sim D_\mu$, we can get the following local operators:

$$
\begin{aligned}
T_4 &\propto \operatorname{Tr}\left(M\bar{M}M\bar{M} + \text{h.c.}\right)\phi\gamma_5 = 0, \\
T_3 &\sim t'_2 \sim \operatorname{Tr}\left(\mathcal{G}^+_{\mu\nu}M^{\mu\nu}\bar{M}_0 + \mathcal{G}^-_{\mu\nu}\bar{M}^{\mu\nu}M_0 + \text{permutations} + \text{h.c.}\right)\phi.
\end{aligned}
\tag{B.70}
$$

Terms even $M_{\mu\nu}$ cancel between λ and α, χ_S. For the odd terms, from (B.67) we have

$$
M^{\mu\nu}_{\lambda\alpha}\bar{M}^{\alpha\lambda}_0 + \bar{M}^{\mu\nu}_{\lambda\alpha}M^{\alpha\lambda}_0 = 0,
\tag{B.71}
$$

and

$$
\left(M^{\mu\nu}\bar{M}_0 + M_0\bar{M}^{\mu\nu}\right)_{\lambda\lambda} + \text{h.c.} = \left(M^{\mu\nu}\bar{M}_0 + M_0\bar{M}^{\mu\nu}\right)_{\chi\chi} + \text{h.c.} = 0.
\tag{B.72}
$$

It follows that T_4 vanishes, as does any term with $\mathcal{G}_{\mu\nu}$ real. A part that is not real is $X_{\mu\nu}$ which cancels between λ and α, χ_S. There are also terms with $G^\pm_\lambda \to i\gamma_5 G^L$, $G^\pm_\chi \to \mp i G^L$, where

$$
G^L_{\mu\nu} = \frac{1}{2x}\left(\partial_\mu x\partial_\nu y - \partial_\nu x\partial_\mu y\right).
\tag{B.73}
$$

Since for these contributions

$$
G^+_\chi R = -G^+_\lambda R, \quad G^-_\chi L = -G^-_\lambda L,
\tag{B.74}
$$

if we keep explicit the helicity projection operators in the expressions (B.5), it becomes clear that these also cancel. The divergent piece t_∞ in (B.69) is proportional to a total derivative, as we will explicitly display below.[22]

Next consider the PV sector. The λ-θ PV mass that is generated by the supersymmetric Higgs mechanism, satisfies

$$
\delta m = 0, \qquad \phi_\lambda = -\phi_\chi = -\frac{1}{2}\operatorname{Im}F, \qquad dm = -i\{\phi, m\} = 0,
\tag{B.75}
$$

[22] See (B.79) for $m^2 = 0$; (B.81) vanishes identically in this case.

and there are no linear divergences or residual anomalies generated by λ,θ loops. The divergent contribution t_∞ is regulated by the coupling of λ_0 to $\hat{\varphi}$ defined by (3.33), giving rise to masses $M_{\lambda^0\hat{\varphi}^a_-}$ of the form (B.8) which have the same properties (B.67) as $M_{\lambda\chi}$. N_μ is still defined as in (B.31) since $\delta M = 0$, but R_{PV} contains an additional term

$$\{\hat{\mathbf{m}} + (p_\nu + \mathcal{G}_\nu)\, P^{\mu\nu}\gamma_\mu\}\, \widehat{M} \tag{B.76}$$

inside the square brackets. Under a chiral modular transformation the phases satisfy (B.75) provided the PV masses m are constant. Since $M\mathbf{m}M = \delta m = 0$, for constant masses the only potentially nonvanishing contribution analogous to (B.69) is

$$t_2 \;=\; -\frac{1}{p^2-m^2}\left(\not{p}\widehat{\bar{M}}\frac{\not{p}}{p^2-m^2}\widehat{M}\;\not{p} + \bar{m}\widehat{M}\frac{\not{p}}{p^2-m^2}\widehat{\bar{M}}m\right)\frac{1}{p^2-m^2}\;\not{D}\phi\gamma_5. \tag{B.77}$$

Terms in (B.77) with no factor of $\sigma\cdot M$ vanish identically; evaluation of the terms with one of $\sigma\cdot M$ gives

$$\begin{aligned} T_2 \;&=\; 4\int\frac{d^4p}{(2\pi)^4}\frac{p^2}{(p^2-m^2)}\left[i\mathrm{Tr}\eta\left(D_\mu\widetilde{M}^{\mu\nu}\frac{1}{p^2-m^2}\bar{M}_0 + \widetilde{M}^{\mu\nu}\frac{1}{p^2-m^2}D_\mu\bar{M}_0\right.\right.\\ &\qquad\left.\left. -D_\mu M_0\frac{1}{p^2-m^2}\widetilde{\bar{M}}^{\mu\nu} - M_0\frac{1}{p^2-m^2}D_\mu\widetilde{\bar{M}}^{\mu\nu}\right) + \mathrm{h.c.}\right]\mathcal{D}_\nu\mathrm{Im}F, \end{aligned} \tag{B.78}$$

$$\begin{aligned} T_2 \;&\ni\; 4\int\frac{d^4p}{(2\pi)^4}\frac{p^2}{p^2-m^2}D_\mu\left[i\mathrm{Tr}\eta\left(\widetilde{M}^{\mu\nu}\frac{1}{p^2-m^2}\bar{M}_0 - M_0\frac{1}{p^2-m^2}\widetilde{\bar{M}}^{\mu\nu}\right) + \mathrm{h.c.}\right]\mathcal{D}_\nu\mathrm{Im}F\\ &=\; \text{total derivative}, \end{aligned} \tag{B.79}$$

since

$$2D_\nu D_\mu\left(M^{\mu\nu}\bar{M}_0\right) = \left[G^+_{\nu\mu}, M^{\mu\nu}\bar{M}_0\right] = 0. \tag{B.80}$$

Note that the coefficient of the divergent integral in (B.78) is just the variation of the first line of the expression for $\mathrm{Tr}\mathcal{R}^2\mathcal{R}_5$ in (B.29) of [17]. The condition on $\sum\eta e^2$ in (3.35) assures that the divergence is canceled. Finally, taking into account the symmetry properties (B.67), terms containing two factors of $\sigma\cdot M$ vanish in the limit of equal PV masses: $m^2_{\lambda_0\theta_0} = m^2_{\chi\hat{\chi}}$, and we get a contribution

$$-\frac{i}{2}T_2 = \frac{\mathrm{Im}F}{16\pi^2}f(r)\mathcal{D}^\mu\left(xF^{\rho\nu}\mathcal{D}_\rho\tilde{F}_{\mu\nu}\right), \qquad r = m^2_{\lambda_0\theta_0}/m^2_{\chi\hat{\chi}}, \qquad f(1) = 0. \tag{B.81}$$

The function $f(r)$ takes all possible values $-\infty \le f(r) \le \infty$ over the range $0 \le r \le \infty$; therefore the PV mass ratio can be chosen so that (B.81) exactly cancels the contribution (B.62) from the off-diagonal gaugino-gravitino connection. Since the PV regularization procedure respects supersymmetry, the conformal anomaly necessarily includes the supersymmetric completion, proportional to (4.35), of (B.81), and the contribution (4.36) can thus be cancelled. Therefore in the following section we include only PV fields that have noninvariant PV masses through superpotential couplings to one another.

C The full anomaly

In this appendix we calculate the full contribution to the anomaly that is generated by superpotential couplings of PV chiral superfields resulting in noninvariant masses.
The one-loop effective action from chiral multiplet loops is given by

$$S_1 = \frac{i}{2}\mathrm{Tr}\eta \ln\left(\hat{D}^2_\Phi + H_\Phi(\mathbf{m})\right) - \frac{i}{2}\mathrm{Tr}\eta\ln\left[-i\,\not{D}_\Theta + M_\Theta(\mathbf{m})\right], \tag{C.1}$$

where the subscripts Φ and Θ denote scalar and fermion loops, respectively. We require that under an infinitesimal transformation on superfields $Z(\theta) \to Z'(\theta') = g(\theta)Z(\theta)$, such that $\delta\Phi = \phi_\Phi\Phi$, $\delta\Theta = \phi_\Theta\Theta$,

$$\delta S_1 = S_1(\mathbf{m}+d\mathbf{m}) - S_1(\mathbf{m}) = \frac{i}{2}\mathrm{STr}\eta[1+(1+\mathcal{R})^{-1}d\mathcal{R}] = \frac{i}{2}\mathrm{STr}\eta\sum_{n=0}^{4}(-\mathcal{R})^n d\mathcal{R}, \tag{C.2}$$

where

$$\begin{aligned}
d\mathbf{m}_\Theta &= e^{-\phi_\Theta}\mathbf{m}(z',V'_X)e^{\phi_\Theta} - \mathbf{m}(z,V_X) = \delta\mathbf{m} - [\phi_\Theta, m] = \tilde{\mathbf{m}} - \mathbf{m} \\
&= \begin{pmatrix} 0 & (\delta m - \phi^+_\Theta m + m\phi^-_\Theta)R \\ (\delta\bar{m} - \phi^-_\Theta\bar{m} + \bar{m}\phi^+_\Theta)L & 0 \end{pmatrix} = \begin{pmatrix} 0 & dmL \\ d\bar{m}R & 0 \end{pmatrix} \qquad \text{(C.3)} \\
d\mathbf{m}_\Phi &= e^{-\phi_\Phi}\mathbf{m}(z',V'_X)e^{\phi_\Phi} - \mathbf{m}(z,V_X) = \delta\mathbf{m} - [\phi_\Phi, m] \\
&= \begin{pmatrix} 0 & (\delta m - \phi^+_\Phi m + m\phi^-_\Phi) \\ (\delta\bar{m} - \phi^-_\Phi\bar{m} + \bar{m}\phi^+_\Phi) & 0 \end{pmatrix}. \qquad \text{(C.4)}
\end{aligned}$$

Here we define, using the notation introduced in (B.5) and (B.31), with now $\delta\,\not{D} \to 0$, $\delta m \to dm$,

$$\begin{aligned}
\mathrm{STr}Fd\mathcal{R} &= \mathrm{Tr}(Fd\mathcal{R})_\Phi - \mathrm{Tr}(Fd\mathcal{R})_\Theta, \qquad \text{(C.5)} \\
\mathcal{R}_{\Phi,\Theta} &= \frac{1}{p^2-\mathbf{m}^2}\left[T^{\mu\nu}\Delta_\mu\Delta_\nu - \hat{H} - X - p^2 + \mathbf{m}^2\right]_{\Phi,\Theta}, \qquad \text{(C.6)}
\end{aligned}$$

$$
\begin{aligned}
d\mathcal{R}_\Theta &= -\frac{1}{p^2-\mathbf{m}^2}\left[-\not p + M_\Theta(\mathbf{m})\right] d\mathbf{m}, && \text{(C.7)}\\
d\mathcal{R}_\Phi &= -\frac{1}{p^2-\mathbf{m}^2}\left[e^{-\phi_\Phi} H_\Phi(z', V'_X) e^{\phi_\Phi} - H_\Phi(z, V_X)\right]. && \text{(C.8)}
\end{aligned}
$$

We argued in Section 4.1 that fields Φ^P with couplings in W_2, or with Kähler curvature terms $R_{P\bar{m}Q\bar{n}} \neq 0$, must have modular and $U(1)_X$ invariant masses. Then $M_\Theta(\mathbf{m}) = \mathbf{m}$, and $\hat{H}, \mathcal{G}_\mu$ are given by the derivative expansions of Appendix A of [17] in terms of the following operators:[23]

$$
\begin{aligned}
H_\Phi &= \begin{pmatrix} \bar{h} + \Delta\bar{H} & h' \\ \bar{h}' & h + \Delta H \end{pmatrix}, \qquad H_\Theta = \begin{pmatrix} \bar{h}R & 0 \\ 0 & hL \end{pmatrix} - \frac{i}{2}\sigma\cdot\mathcal{G}_\Theta - i[\not{D}, \mathbf{m}], && \text{(C.9)}\\
(\Delta H)^Q_P &= \delta^Q_P\left(\hat{V} + M^2\right) + R^Q_{P\bar{m}k}\left(e^{-K}\bar{A}^k A^{\bar{m}} + \mathcal{D}_\mu z^k \mathcal{D}^\mu \bar{z}^{\bar{m}}\right) + \frac{1}{x}\mathcal{D}_a D_P (T^a z)^Q, && \text{(C.10)}\\
h &= \bar{m}m, \qquad m_{PQ'} = \delta_{PQ} e^{K/2}\mu_P, && \text{(C.11)}\\
h'_{PQ} &= e^{-K}\left(\bar{A}^k D_k - \bar{A}\right) e^{K/2} m_{PQ} - (q^P_X + q^Q_X) F_X m_{PQ}, \qquad F_X = -\frac{1}{4}\mathcal{D}^2 V_X\Big|, && \text{(C.12)}\\
D_\mu m_{PP'} &= \mathcal{D}_\mu z^q\left[m_{PP'}\left(K_q + \partial_q \ln\mu_P\right) - m_{QP'}\Gamma^Q_{Pq} - m_{PQ'}\Gamma^{Q'}_{P'q}\right]\\
&\quad + iA^X_\mu\left(q^A_X + q^{P'}_X\right) m_{PP'}, && \text{(C.13)}\\
[\not{D}, \mathbf{m}] &= \begin{pmatrix} 0 & \not{D}mL \\ \not{D}\bar{m}R & 0 \end{pmatrix}, \qquad \mathcal{G}^\Theta_{\mu\nu} = G^\Phi_{\mu\nu} + Z_{\mu\nu} - \gamma_5\Gamma_{\mu\nu}, && \text{(C.14)}
\end{aligned}
$$

where indices are raised and lowered with the Kähler metric, the operator A and its covariant derivatives are defined in Section 2 [see (2.3)], $Z_{\mu\nu}$ is defined in (B.6), and

$$
\begin{aligned}
\Gamma_{\mu\nu} &= \frac{1}{2}(\mathcal{D}_\nu \bar{z}^{\bar{m}}\mathcal{D}_\mu z^i - \mathcal{D}_\mu \bar{z}^{\bar{m}}\mathcal{D}_\nu z^i) K_{i\bar{m}} = \frac{1}{2}X_{\mu\nu}, && \text{(C.15)}\\
\left(G^{\Phi-}_{\mu\nu}\right)^P_Q &= iF^a_{\mu\nu}(T_a)^P_Q - \Gamma^P_{Q\mu\nu} = -K^{P\bar{M}}\left(G^{\Phi+}_{\mu\nu}\right)^{\bar{N}}_{\bar{M}} K_{\bar{N}Q}, \qquad a \neq X, && \text{(C.16)}
\end{aligned}
$$

are the field strengths associated with the Kähler and reparameterization + gauge connections, respectively. The expression for $d\mathcal{R}_\Theta$ is obtained using the methods of Appendix A of [17], with $\mathcal{R}_5 \to d\mathcal{R}$, $\mathcal{M}_0 \to \mathcal{M}_4(-M, -\vec{\sigma})$. Since $d\mathcal{R}$ appears only on the far right in (C.2), we can drop all momentum derivatives in the resulting operator.

[23] In the interest of simplification we have evaluated the component Lagrangian in WZ gauge for V_X: $V_X| = F_X = 0$, since the regulated theory, including counterterms, must be $U(1)_X$ gauge invariant. However we have left explicit the F_X term in (C.12) because it gives a contribution to $d\mathbf{m}_\Phi$ through $F_X \to F_\Lambda = -\frac{1}{4}\mathcal{D}^2\Lambda$ if $V_X \to \Lambda + \bar{\Lambda}$. This term arises from mixed $F^P F_X$ terms in the Lagrangian when Kähler potential terms take the form $e^{q^P_X V_X} f(Z, \bar{Z})|\Phi^P|^2$.

For a gauge transformation $\phi^{\pm}_{\Phi} = \phi^{\pm}_{\Theta}$. Now consider a Kähler transformation that is induced by a chiral field redefinition:

$$Z'^p(\theta) = f^p[Z(\theta')], \qquad W' = e^{-F}W, \qquad K' = K + F + \bar{F}, \qquad \theta' = e^{\frac{i}{2}\mathrm{Im}F}\theta. \tag{C.17}$$

For an infinitesimal transformation

$$dZ'^p = \frac{\partial f^p}{\partial Z^q} dZ^q \approx dZ^p + (\phi^-)^p_q dZ^q. \tag{C.18}$$

Then for example if $K(Z^p)$ is the light field Kähler potential, the corresponding light loop divergences can be canceled by PV fields Z^P with Kähler potential

$$K_{PV} = Z^P \bar{Z}^{\bar{Q}} K_{p\bar{q}}, \tag{C.19}$$

which is modular invariant provided

$$\delta Z^P = (\phi^-)^p_q Z^Q \tag{C.20}$$

under C.17. This gives

$$\delta z^P = (\phi^-)^p_q z^Q, \qquad \delta\chi^P = (\phi^-)^p_q \chi^Q + \frac{i}{2}\mathrm{Im}F\chi^P + (\partial_k\phi^-)^p_q z^Q \chi^k. \tag{C.21}$$

Since the last term does not contribute to the effective bosonic Lagrangian that we are evaluating, from now on we will set

$$\phi_\Theta = \phi_\Phi - \frac{i}{2}\gamma_5 \mathrm{Im}F = \phi, \qquad \left[\phi^+_\Phi, m\bar{m}\right] = \left[\phi^+_\Theta, m\bar{m}\right]. \tag{C.22}$$

Then, since $m\bar{m} = \bar{m}m \equiv m^2$ for the chiral PV fields that contribute to the anomaly, we obtain

$$\begin{aligned}
d\mathcal{R}_\Phi &= -\frac{1}{p^2 - m^2}\begin{pmatrix} d\bar{h}_\Phi & dh' \\ d\bar{h}' & dh_\Phi \end{pmatrix}, \qquad d\mathcal{R}_\Theta = -\frac{1}{p^2 - m^2}\begin{pmatrix} d\bar{h}_\Theta R & -\not{p}dmL \\ -\not{p}d\bar{m}R & dh_\Theta L \end{pmatrix}, && \text{(C.23)}\\
dh_\Theta &= \bar{m}dm, \qquad dh_\Phi = d\bar{m}m + \bar{m}dm, && \text{(C.24)}\\
dh'^{\bar{P}}_Q &= \left[\left\{e^{-K}\left(\bar{A}^k D_k - \bar{A}\right)e^{K/2} - (q^P_X + q^Q_X)F_X\right\}dm\right]^{\bar{P}}_Q - (q^P_X + q^Q_X)F_\Lambda m^{\bar{P}}_Q \\
&= \tilde{\varphi}^+ h'^{\bar{P}}_Q - \left[F^k D_k \tilde{\varphi}^+ - (q^P_X + q^Q_X)F_\Lambda\right] m^{\bar{P}}_Q, \qquad F_\Lambda = -\left.\frac{1}{4}\mathcal{D}^2\Lambda\right|, && \text{(C.25)}
\end{aligned}$$

where $F^k = e^{-K/2}\bar{A}^k = -\frac{1}{4}\mathcal{D}^2 Z^k$ at lowest order in the loop expansion.

A priori the expansion (C.2) contains ultraviolet divergences which must vanish if the regulated theory is truly finite. The quadratic divergences are contained in

$$\mathrm{STr}d\mathcal{R} = -\mathrm{Tr}\eta\frac{1}{p^2 - m^2}\left(dh_\Phi - 2dh_\Theta\right) + \mathrm{h.c.} = 0, \tag{C.26}$$

where the trace is over internal indices only. The logarithmic divergences occur in

$$\begin{aligned}\mathrm{STr}\mathcal{R}d\mathcal{R} &= \mathrm{Tr}\eta\frac{1}{p^2-m^2}\left[-i\widehat{\not{D}\bar{m}}\frac{1}{p^2-m^2}\not{p}dm+\left(T^{\mu\nu}\mathcal{G}^-_{\Theta\mu}\mathcal{G}^-_{\Theta\nu}-\frac{\hat{r}}{4}\right)\frac{1}{p^2-m^2}dh_\Theta\right]L\\ &+\mathrm{Tr}\eta\frac{1}{p^2-m^2}\left[\left(\widehat{\Delta H}-T^{\mu\nu}\mathcal{G}^-_{\Phi\mu}\mathcal{G}^-_{\bar{\Phi}\nu}\right)\frac{1}{p^2-m^2}dh_\Phi+\hat{\bar{h}}'\frac{1}{p^2-m^2}dh'_\Phi\right]\\ &+\mathrm{Tr}\eta\frac{1}{p^2-m^2}\left(\hat{h}-m^2\right)\frac{1}{p^2-m^2}\left(dh_\Phi-dh_\Theta L\right)+\mathrm{h.c.},\end{aligned}\tag{C.27}$$

where the traces are over both Dirac and internal indices. Since PV fields with $dm\neq 0$ have no superpotential couplings and have $K^{PV}_{PQ}=0$, there are no terms odd in m, and the logarithmically divergent part of this expression is proportional to

$$\mathrm{Tr}\eta\left[\frac{r}{2}\bar{m}dm-\left(d\bar{m}m+\bar{m}dm\right)\Delta H-\bar{h}'dh'-\frac{1}{2}D^2\bar{m}dm\right]+\mathrm{h.c.}\tag{C.28}$$

where here the traces are over internal indices only. The expression (C.28) is just the variation of the logarithmically divergent m-dependent part of the one loop action,[24] which, under the above assumptions, is proportional to [2]

$$\mathrm{Tr}\eta\left[\{\bar{m}m,\Delta H\}-\frac{r}{2}\bar{m}m-\frac{1}{2}D_\mu mD^\mu\bar{m}+h'\bar{h}'\right].\tag{C.29}$$

The ultraviolet divergent terms that are independent of m have been constructed to cancel the light loop divergences. We also require that the logarithmically divergent terms proportional to m^2 vanish, which means that for a given functional form of $m^2_C(z,\bar{z},c_X)$, where $c_X=V_X|$,[25] we have to introduce a set of PV fields with masses

$$m^2_{C_\gamma}=\rho^C_\gamma m^2_C(z,\bar{z},c_X),\qquad \sum_\gamma\eta^C_\gamma\rho^C_\gamma=0.\tag{C.30}$$

This condition assures the vanishing of (C.29) and (C.28) as well as the finite terms proportional to m^2 that arise from $\frac{i}{2}\mathrm{Tr}\mathcal{R}^2d\mathcal{R}$. For the masses that are not modular and $U(1)_X$ invariant, we need also to eliminate the residual finite terms proportional to $m^2\ln m^2$, which requires the additional constraint

$$\sum_\gamma\eta^C_\gamma\rho^C_\gamma\ln(\rho^C_\gamma)=0.\tag{C.31}$$

[24] There is no quadratically divergent m-dependent contribution to the one loop action S_1.

[25] Here and throughout this appendix $\Phi|$ is the $\theta=\bar{\theta}=0$ component of the superfield Φ with all fermion fields also set to zero.

for these masses. Both the PV masses and the functions H are controlled by the choice of Kähler metric for the PV fields, which is constrained by the requirement of cancellation of quadratic divergences. Assuming that $D^{2n}m^2$ commutes with other operators if $dm \neq 0$, the last line drops out in (C.27). Dropping terms that vanish as $m^2 \to \infty$, the finite part of (C.27) is

$$\begin{aligned}
-\frac{i}{2}\,\mathrm{STr}\eta\mathcal{R}d\mathcal{R}|_{\rm finite} \;=\; & -\mathrm{Tr}\eta\frac{1}{192\pi^2m^2}\Bigg\{\frac{1}{2}\left(G^-_\Phi\cdot G^-_\Phi\right)(d\bar{m}m-\bar{m}dm)-G^-_{\mu\nu}X^{\mu\nu}\bar{m}dm \\
& -\left[D^4\bar{m}-\frac{1}{2}G_{\mu\nu}G^{\mu\nu}\bar{m}-\frac{2}{3}\left([D_\mu,G^{\mu\nu}]+D_\mu r^{\mu\nu}\right)D_\nu\bar{m}\right]dm \\
& +\frac{1}{8}\left(r_{\mu\nu\rho\sigma}r^{\mu\nu\rho\sigma}-ir\cdot\tilde{r}+2X_{\mu\nu}X^{\mu\nu}-4\nabla^2r\right)\bar{m}dm \\
& +\left(D^2\Delta H\right)(d\bar{m}m+\bar{m}dm)+\left(D^2\bar{h}'\right)dh'\Bigg\}+\mathrm{h.c.},
\end{aligned}\tag{C.32}$$

where we define

$$D_\mu\bar{m}=D^-_\mu\bar{m}-\bar{m}D^+_\mu,\qquad G_{\mu\nu}\bar{m}=[D_\mu,D_\nu]\bar{m}=G^-_{\mu\nu}\bar{m}-\bar{m}G^+_{\mu\nu},\qquad etc.,\tag{C.33}$$

and $G^\pm_{\mu\nu}=(G^\pm_\Phi)_{\mu\nu}\mp\frac{1}{2}X_{\mu\nu}$ is defined as in (B.6). If $\mathcal{L}_m$ is the part of the Lagrangian that contains the mass matrix m for PV chiral fermions $f^P=f^P_L$:

$$m^{\bar{Q}}_{P'}=-K^{\bar{Q}R}(\mathcal{L}_m)_{RP'},\qquad(\mathcal{L}_m)_{RP'}=\frac{\partial^2}{\partial f^R\partial f^{P'}}\mathcal{L}_m,\tag{C.34}$$

under a transformation $Z^p\to Z'^p$, $\phi^P\to g\phi^P$, $f^P\to e^{i\alpha}gf^P$, we have

$$m'^{\bar{Q}}_{P'}=-e^{-i(\alpha+\alpha')}g^{\bar{Q}}_{\bar{M}}K^{\bar{M}R}(\mathcal{L}'_m)_{RN'}(g^{-1})^{N'}_{P'},\tag{C.35}$$

since $K'_{PV}=K_{PV}$ by construction. For superpotential fermion mass terms $\mathcal{L}_m=-e^{K/2}\mu_{PQ'}\chi^P\chi^{Q'}$, $\alpha=\frac{1}{2}\mathrm{Im}F$, and if under (C.17) $\mu_{PQ}(Z')=e^{\omega^{PQ}_iF^i}\mu_{PQ}(Z)$, we have

$$\begin{aligned}
\mathcal{L}'_m &= -e^{K'/2+i\mathrm{Im}F}g^P_Rg^{M'}_{Q'}\mu'_{PM'}\chi^R\chi^{Q'}=-e^{K/2+F+\omega^{PM'}_iF^i}g^P_Rg^{M'}_{Q'}\mu_{PM'}\chi^R\chi^{Q'}, \\
m'^{\bar{Q}}_{P'} &= -e^{-i\mathrm{Im}F}g^{\bar{Q}}_{\bar{N}}K^{\bar{N}R}(\mathcal{L}'_m)_{RM'}(g^{-1})^{M'}_{P'}=e^{\mathrm{Re}F+\omega^{PM'}_iF^i}g^{\bar{Q}}_{\bar{N}}K^{\bar{N}R}g^M_RK_{M\bar{S}}m^{\bar{S}}_{P'}, \\
\tilde{m}^{\bar{Q}}_{P'} &= e^{i\mathrm{Im}F}(g^{-1})^{\bar{Q}}_{\bar{N}}m'^{\bar{N}}_{M'}g^{M'}_{P'}=e^{F+\omega^{PM'}_iF^i}K^{\bar{Q}R}g^M_RK_{M\bar{S}}m^{\bar{S}}_{N'}g^{N'}_{P'},
\end{aligned}\tag{C.36}$$

For the chiral PV fields $\Psi^P_\gamma,\Psi'^Q_\gamma$, $\Psi=U,V,\Phi$, with couplings defined by (D.19)–(D.21) in Appendix D, we have

$$\mu^\gamma_{PQ'}\;=\;\delta_{PQ}\mu^P_\gamma(T),\qquad m^{\bar{P}}_{Q'}=e^{K/2}K^{\bar{P}Q}\mu^P_\gamma(T),$$

$$
\begin{aligned}
K_{P\bar{Q}} &= \delta_{PQ} f^P_\gamma, \qquad K_{P'\bar{Q}'} = \delta_{PQ} f^{P'}_\gamma, \\
(g_\gamma)^R_Q &= \delta^R_Q e^{\phi^-_{R\gamma}}, \qquad (g_\gamma)^{R'}_{Q'} = \delta^R_Q e^{\phi^-_{R'\gamma}}.
\end{aligned} \tag{C.37}
$$

Explicitly

$$
\begin{aligned}
\ln f^P_\gamma &= \sum_n q^{P_\gamma}_n g^n + q^{P_\gamma}_X c_X, \qquad \phi^-_{P_\gamma} = -\sum_i q^{P_\gamma}_n F^n - q^{P_\gamma}_X \lambda, \\
c_X &= V_X|, \qquad \lambda = \Lambda|.
\end{aligned} \tag{C.38}
$$

where Λ is the $U(1)_X$ gauge transformation superfield introduced in (4.6) and (D.91), and q_n and q_X are modular weights and $U(1)_X$ charges, respectively. Then we obtain

$$
\sum_R g^R_Q g^{R'}_{P'} = e^{\varphi^-_{P_\gamma} + \varphi^-_{P'_\gamma}} \delta_{PQ}, \qquad \tilde{m}^P_\gamma = e^{\sum_i F^i (1+\omega^{P_\gamma}_i) + \varphi^-_{P_\gamma} + \varphi^-_{P'_\gamma}} m_\gamma \equiv e^{\tilde{\varphi}^+_{P_\gamma}} m^P_\gamma, \tag{C.39}
$$

To evaluate (C.2) we need only consider fields with noninvariant masses: $\tilde{m} \neq m$. In addition to the PV superfields U, V, Φ, considered above, these include the gauge singlets ϕ_γ for which we take

$$
\ln f^\phi_\gamma = \alpha^\phi_\gamma K, \qquad \varphi^-_{\phi_\gamma} = -\alpha^\phi_\gamma F, \qquad \tilde{\varphi}^+_{\phi_\gamma} = \sum_i \left(1 - 2\alpha^\phi_\gamma + \omega^\gamma_i\right) F^i, \qquad \mu^\phi_{\gamma\gamma'} \neq 0, \tag{C.40}
$$

and the adjoint chiral multiplet $\tilde{\varphi}^a_\alpha$ with

$$
f^{\tilde{\varphi}}_\alpha = 1, \qquad \varphi^-_\gamma = 0, \qquad \tilde{\varphi}^+_\gamma = \sum_i \left(1 + \omega^\gamma_i\right) F^i. \tag{C.41}
$$

Let us first consider the coefficient of $r \cdot \tilde{r}$ in (C.32). To evaluate this we use the conditions (3.36), (3.37), (D.97) and (D.98). These include contributions from chiral PV fields that have covariant masses: $\tilde{m} = m$, $\tilde{\varphi}^\pm = 0$; we can also include them in the sums here, since their net contribution vanishes. However we must exclude the fields θ^γ; their masses arise from D-terms rather than F-terms. Each pair $\Phi^P, \Phi'^{P'}$ gives gives an identical contribution, and we remove from $\mathrm{Tr}\eta = N'$ the contribution $N'_G = \sum_\gamma \eta^\theta_\gamma$. On the other hand, the moduli-dependent prefactors in (C.38), with, referring to (D.19),

$$
q^{U^A_\gamma}_n = \alpha^A_\gamma q^A_n, \qquad etc., \tag{C.42}
$$

that have been chosen to cancel loop contributions from the PV fields $\dot{Y}$ are not included; to include these we use (D.22)–(D.28) and the identifications (C.42), giving the result in (D.99).

However we also have to exclude the net contribution of all the fields[26] $\dot{Z}, \dot{Y}, \Psi$ from N' where $N_\Psi = -N_{\dot{Y}} = -N_{\dot{Z}} = N+2$. That is, we have to add a factor $N+2$ to N', giving

$$\begin{aligned}
\frac{16\pi^2}{\sqrt{g}}\delta S_r &= -\frac{i}{96} r\cdot\tilde{r}\mathrm{Tr}\eta(\tilde{\varphi}^- - \tilde{\varphi}^+) = -\frac{1}{24} r\cdot\tilde{r}\mathrm{Tr}\eta\phi_{PV} = \frac{1}{24} r\cdot\tilde{r}\mathrm{Tr}\phi \\
&= -\frac{1}{48} r\cdot\tilde{r}\left[\left(N' - N'_G - 2\alpha_1\right)\mathrm{Im}F + 2\mathrm{Tr}T_X a_X + \sum_n\left(\omega_n + 2\sum_p q^p_n\right)\mathrm{Im}F^n\right] \\
&= \frac{1}{48} r\cdot\tilde{r}\left[\sum_n\left(N - N_G - 3 - \omega_n - 2\sum_p q^p_n\right)\mathrm{Im}F^n - 2\mathrm{Tr}T_X a_X\right], \qquad \text{(C.43)}
\end{aligned}$$

where [see (3.37)],

$$\alpha_1 = \sum_C \eta_C\alpha^C = -10, \qquad \omega_n = \sum_C \eta_C\omega^C_n, \qquad a_X = \mathrm{Im}\lambda. \tag{C.44}$$

Apart from the "threshold corrections" $\omega_n(T^i)$, this is precisely the result in (B.30) (with the identification $\phi = -\eta\phi_{PV} = \eta\phi^+_{PV} = -\frac{i}{2}\eta\tilde{\varphi}^-$) for N chiral fermions with $\phi^p = \sum_n(\frac{1}{2} - q^p_n)\mathrm{Im}F^n - q^p_X a_X$, the auxiliary fermion needed for gravitino gauge fixing [16] with $\phi^\alpha = \frac{1}{2}\sum_n \mathrm{Im}F^n$, and $N_G + 4$ gaugino and gravitino degrees of freedom with $\phi = -\frac{1}{2}\sum_n \mathrm{Im}F^n$.

To evaluate the full anomaly, we can simplify further by setting

$$\mu^\phi_{\alpha\beta} = \mu^\phi_\alpha\delta_{\alpha\beta} \tag{C.45}$$

in (4.61), and imposing (D.38) and (D.118) with $q^{U^A}_X = q^{U_A}_X$.

Then for the fields with noninvariant PV masses introduced in Appendix D.4 we have

$$\begin{aligned}
\hat{\phi}^0, \hat{\phi}^\pm: &\qquad \ln f_{\hat{\phi}_\gamma} = \hat{\alpha}^{\hat{\phi}}_\gamma K, \qquad \hat{\alpha}^0_\gamma = \hat{\alpha}^+_\gamma = 0, \qquad \hat{\alpha}^-_\gamma = 1, &\text{(C.46)}\\
\Psi^P: &\qquad \ln f_{\Psi^P} = \sum_n q^P_n g^n + q^P_X V_X, \qquad q^P_n = 1 - q^p_n + \dot{\omega}_n, \qquad q^N_X = q^I_X = 0, &\text{(C.47)}\\
\tilde{\varphi}^a: &\qquad f^{\tilde{\varphi}^a} = 1, &\text{(C.48)}
\end{aligned}$$

and the q^A_X are subject to the constraints given in (D.40), (D.41) and (D.43). Here $P = N, I, A$, and we identify [see (D.19)]

$$\Psi^N = \Phi^n, \qquad \Psi^I = \delta^I_N\Phi^N, \qquad \Psi^A = U^A, U_A, V^A, \qquad q^n_m = 0, \qquad q^i_n = 2\delta^i_n. \tag{C.49}$$

[26]Equivalently, we remove N_Ψ which is already included in (D.99), and include the reparameterization + threshold contributions from $\dot{Z}, \dot{Y}$ which exactly cancels their Kähler connection contributions $N_{\dot{Z}} + N_{\dot{Y}}$.

The PV superfields $\Psi^P, \tilde{\varphi}^a$ and $\hat{\phi}_\gamma,\ \gamma = 1,\ldots,5$, have $\eta = +1$ and $\hat{\phi}^\pm_\gamma,\ \gamma = 1,\ldots 2N-4$ has $\eta = -1$. Using (C.45), and denoting the fields $U, V, \Phi, \hat{\phi}, \tilde{\varphi}$ collectively by Φ^C, the covariant derivative in (C.14) reduces to

$$\begin{aligned} D_\mu m_{CC'} &= \left\{\mathcal{D}_\mu z^i \left[K_i + \partial_i \ln(\mu_C - 2\ln f_C)\right] - 2q_X(iA^X_\mu + \partial_\mu c_X)\right\} m_{CC'} \\ &\equiv \frac{1}{2}\left(V^C_\mu + iA^C_\mu\right) m_{CC'} \equiv V^+_\mu m_{CC'} = (V^-_\mu)^\dagger m_{CC'}. \end{aligned} \tag{C.50}$$

Since the Kähler metric is covariantly constant: $D_q K^{C\bar{C}} = 0$, we also have

$$D_\mu m^{\bar{C}}_{C'} = K^{\bar{C}C}\mathcal{D}_\mu m_{CC'}, \qquad \partial_i \ln \mu_C = \delta_{it^n}\omega^C_n \zeta(t^n), \qquad \zeta(t) = \partial_t \eta(t)/\eta(t). \tag{C.51}$$

The vectors V_μ, A_μ satisfy

$$\begin{aligned} iA^C_{\mu\nu} &= i\left(\mathcal{D}_\mu A^C_\nu - \mathcal{D}_\nu A^C_\mu\right) = 2f^C_{\mu\nu} - X_{\mu\nu}, \qquad \mathcal{D}_\mu V^C_\nu - \mathcal{D}_\nu V^C_\mu = 0, \\ f^C_{\mu\nu} &= \partial_i\partial_{\bar{m}} \ln f_C\left(\mathcal{D}_\mu z^i \mathcal{D}_\nu \bar{z}^{\bar{m}} - \mathcal{D}_\nu z^i \mathcal{D}_\mu \bar{z}^{\bar{m}}\right) - iF^a_{\mu\nu}(T_a z)^i D_i \ln f_C - 2iq^C_X F^X_{\mu\nu}, \end{aligned} \tag{C.52}$$

and from (C.14)–(C.16) and (C.33) we have

$$\left(G^\pm_\Phi\right)^C_{\mu\nu} = \pm\left(f^C_{\mu\nu} - iT^C \cdot F_{\mu\nu}\right), \qquad F_{\mu\nu} \neq F^X_{\mu\nu}, \tag{C.53}$$

$$\left(G^\pm_\Theta\right)^C_{\mu\nu} \equiv (G^\pm_\Phi)^C_{\mu\nu} \mp \frac{1}{2}X_{\mu\nu} = \pm i\left(\frac{1}{2}A^C_{\mu\nu} - T^C \cdot F_{\mu\nu}\right), \qquad F_{\mu\nu} \neq F^X_{\mu\nu} \tag{C.54}$$

$$G_{\mu\nu} m_C = iA^C_{\mu\nu} m_C, \qquad G_{\mu\nu}\bar{m}_C = -iA^C_{\mu\nu}\bar{m}_C, \qquad G_{\mu\nu}\bar{m}m = 0 \tag{C.55}$$

We are interested only in the variation of the on shell action. Therefore we may drop terms proportional to

$$g^{-\frac{1}{2}} g_{\mu\nu}\frac{\partial \mathcal{L}_{tree}}{\partial g_{\mu\nu}} = \frac{r}{2} - 2V + \mathcal{D}_\mu \bar{z}^m \mathcal{D}^\mu z^i K_{i\bar{m}} = \frac{r}{2} - \frac{1}{2}\mathcal{D}^\alpha X_\alpha\Big| - 3\left(\hat{V} + M^2\right). \tag{C.56}$$

Then, defining

$$\begin{aligned} f_\alpha &= \mathcal{P}\mathcal{D}_\alpha \ln f = -\frac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha \ln f, \\ f^P_\alpha &= \sum_n q^P_n g^n_\alpha + q^P_X W^X_\alpha, \qquad f^{\hat{\phi}_\gamma}_\alpha = \alpha^{\hat{\phi}}_\gamma K_\alpha, \qquad f^{\tilde{\varphi}^a}_\alpha = 0, \end{aligned} \tag{C.57}$$

we have

$$\Delta H^C_D = \delta^C_D\left(\hat{V} + M^2 + \frac{1}{2}\,\mathcal{D}^\alpha f^C_\alpha\big|\right) + \frac{1}{x}\mathcal{D}_a (T^a)^C_D$$

$$
\begin{aligned}
&= \delta^C_D\left[\frac{1}{6}\left(r-\mathcal{D}^\alpha X_\alpha|\right)+\frac{1}{2}\,\mathcal{D}^\alpha f^C_\alpha\big|\right]+\frac{1}{x}\mathcal{D}_a(T^a)^C_D, \\
a &\neq X, \qquad (\mathrm{Tr}T_a)_C = 0, \qquad\qquad \text{(C.58)} \\
\left(m^{-1}h'\right)^{\bar{C}'}_C &= e^{-K/2}\left[\bar{A}^k\left(K_k-2\partial_k\ln f_C+\partial_k\ln\mu_C\right)-\bar{A}\right]-2q^C F_X \\
&= -F^i\mathcal{D}_i\ln m^2+2q_X F_X-\bar{M}=\frac{1}{4}\left(\mathcal{D}^2\ln\mathcal{M}^2-8\bar{R}\right)\big|\equiv\bar{y}, \qquad \text{(C.59)} \\
(m^{-1}dh')^{\bar{C}'}_C &= \bar{y}\tilde{\varphi}^+ +\sum_n\left(1-2q^C_n+\omega^C_n\right)e^{-K/2}\bar{A}^kF^n_k-2q^CF_\Lambda \\
&= \bar{y}\tilde{\varphi}^+ + F_{\tilde{\varphi}^+_C} \qquad\qquad F_{\tilde{\varphi}^+_C}=-\frac{1}{4}\,\mathcal{D}^2\tilde{\varphi}^+_C\big|\,, \qquad \text{(C.60)}
\end{aligned}
$$

where

$$
\mathcal{M}^2 = e^{K-2f}|\mu|^2, \qquad \mathcal{M}^2\big| = m^2, \tag{C.61}
$$

is a real superfield, and the identifications on the right in (C.59) and (C.60) hold to the order we are working in because to that order the auxiliary fields can be interchanged with their tree-level values[27]

$$
F^k = -e^{-K/2}\bar{A}^k, \qquad M = e^{K/2}W = e^{-K/2}A = 2\,R|\,. \tag{C.62}
$$

Since the relevant PV masses have a diagonal Kähler metric, it follows from (B.53) that $G^+ = -G^-$, so the first term in (C.32) drops out, and, from, (C.33), $G\bar{m} = \{G^-,\bar{m}\}$, etc. Then writing

$$
T = \sum_k(-)^kT_k, \qquad T_k = \frac{i}{2}\,\mathrm{STr}\eta\mathcal{R}^kd\mathcal{R}\Big|_{\mathrm{finite}}, \tag{C.63}
$$

we obtain, imposing (C.30),

$$
\begin{aligned}
T_1 \;=\; &+\frac{\sqrt{g}}{192\pi^2}\mathrm{Tr}\eta\Bigg[\tilde{\varphi}^+\Bigg\{\frac{1}{8}\left(r_{\mu\nu\rho\sigma}r^{\mu\nu\rho\sigma}-ir\cdot\tilde{r}+2X_{\mu\nu}X^{\mu\nu}+4iA_{\mu\nu}X^{\mu\nu}\right)-\left(\nabla^\mu V^-_\mu+V^\mu_-V^-_\mu\right)^2 \\
&\quad +\Box\left(\mathcal{D}^\alpha f_\alpha|-\frac{1}{3}\,\mathcal{D}^\alpha X_\alpha|-\nabla^\mu V^-_\mu-V^\mu_-V^-_\mu-\frac{1}{6}r\right) \\
&\quad -2V^-_\nu\nabla^\nu\left[\left(\nabla_\mu+V^-_\mu\right)V^\mu_-\right]-\frac{1}{2}A_{\mu\nu}A^{\mu\nu}-\frac{2i}{3}V^-_\nu\mathcal{D}_\mu A^{\mu\nu} \\
&\quad +\frac{2}{3}\nabla_\mu\left(r^{\mu\nu}V^-_\nu\right)+\frac{2}{3}r^{\mu\nu}V^-_\mu V^-_\nu\Bigg\} \\
&+\left(\bar{y}\tilde{\varphi}^+ + F_{\tilde{\phi}^+}\right)\left(\Box y+2V^-_\mu\partial^\mu y+y\nabla^\mu V^-_\mu+yV^-_\mu V^\mu_-\right)\Bigg]+\mathrm{h.c.}
\end{aligned} \tag{C.64}
$$

[27] Our normalization of the auxiliary field M differs by a factor -3 from that of Binétruy, Girardi and Grimm [5]: $M = -\frac{1}{3}M_{\mathrm{BGG}}$.

$$
\begin{aligned}
T_4 &= -\frac{\sqrt{g}}{32\pi^2}\mathrm{Tr}\eta\tilde{\varphi}^+\left[\frac{1}{6}(y\bar{y})^2 - \frac{1}{2}y\bar{y}V_\mu V^\mu + \frac{1}{3}\left(V_\mu V_-^\mu\right)^2 - \frac{1}{6}\left(V_\mu^- V_-^\mu\right)^2 + \frac{1}{3}V_\mu^- V_-^\mu V_\nu V^\nu\right] \\
&\quad + \text{h.c.},
\end{aligned} \tag{C.65}
$$

The expressions for T_2 and T_3 are more complicated; they combine to give

$$
\begin{aligned}
T_2 - T_3 &= -\frac{\sqrt{g}}{32\pi^2}\mathrm{Tr}\eta\left[\tilde{\varphi}^+\left\{\left(\frac{1}{2}\,\mathcal{D}^\alpha f_\alpha| - \frac{1}{6}\,\mathcal{D}^\alpha X_\alpha| + \frac{1}{6}r\right)^2 - \frac{1}{3}V_\mu\partial^\mu\left(\frac{1}{2}\,\mathcal{D}^\alpha f_\alpha| - \frac{1}{6}\,\mathcal{D}^\alpha X_\alpha| + \frac{1}{6}r\right)\right.\right. \\
&\qquad - \frac{1}{3}\left(\nabla_\mu V^\mu + 3y\bar{y} + r\right)\left(\frac{1}{2}\,\mathcal{D}^\alpha f_\alpha| - \frac{1}{6}\,\mathcal{D}^\alpha X_\alpha| + \frac{1}{6}r\right) \\
&\qquad + \frac{1}{12}\nabla^\mu\left(V_\mu r\right) + \frac{r^2}{48} + \frac{4i}{9}V_-^\nu\mathcal{D}^\mu A_{\mu\nu} \\
&\qquad - \frac{1}{6}\left[y\Box\bar{y} + \bar{y}\Box y + \partial_\mu y\partial^\mu\bar{y} + \bar{y}(V_\mu^- - V_\mu)\partial^\mu y + y(V_\mu^+ - V_\mu)\partial^\mu\bar{y}\right. \\
&\qquad\qquad \left. - y\bar{y}\left(\nabla_\mu V^\mu + 3V_\mu V^\mu + V_\mu^+ V_-^\mu\right)\right] \\
&\qquad + \left[\frac{1}{x^2}D^a D^b - \frac{1}{2}\left(F^a\cdot F^b - i\tilde{F}^a\cdot F^b\right)\right]T_a T_b + \frac{1}{24}\left(A\cdot A + i\tilde{A}\cdot A\right) \\
&\qquad + \frac{1}{6}\left[\Box\left(V_-^\mu V_\mu^-\right) + V^\mu\nabla_\mu\nabla^\nu V_\nu^- + \left(\nabla^\mu V_\mu\right)\left(\nabla^\nu V_\nu^-\right) - \left(\nabla^\mu V_-^\nu\right)\nabla_\mu V_\nu^-\right] \\
&\qquad + \frac{1}{6}\nabla^\mu\left(V_\mu V_\nu^- V_-^\nu + V_\mu^- V_-^\nu V_\nu^-\right) \\
&\qquad - \frac{1}{3}\left[V_\mu V^\mu V_-^\nu V_\nu^- + (V_\mu^- V^\mu)^2 - (V_\mu^- V_-^\mu)^2\right] \\
&\qquad \left. - \frac{1}{36}\left[10V_\mu^- V_\nu^- r^{\mu\nu} + \nabla^\mu\left(3rV_\mu^- - 2r_{\mu\nu}V_-^\nu\right) + 3rV_\mu^- V_-^\mu\right] + \frac{1}{6}ry\bar{y}\right\} \\
&\qquad \left. - \frac{1}{6}\left(y^2\bar{y} + V_\mu\partial^\mu y + V_\mu V_-^\mu y + y\nabla^\mu V_\mu + y\,\mathcal{D}^\alpha X_\alpha| - 3y\,\mathcal{D}^\alpha f_\alpha|\right)\left(F_{\tilde{\phi}^+} + \bar{y}\tilde{\varphi}^+\right)\right] \\
&\quad + \text{h.c.},
\end{aligned} \tag{C.66}
$$

In writing (C.66) we used (C.52), the Bianchi identities:

$$
0 = D^\mu\widetilde{G}_{\mu\nu} = 2\nabla^\mu r_{\mu\nu} - \nabla_\nu r = \epsilon^{\lambda\mu\nu\rho}r_{\mu\nu\rho\sigma}, \qquad V_\mu^- V_\nu\nabla^\mu V_-^\nu = V_\mu V_\nu^-\nabla^\mu V_-^\nu + iA^{\mu\nu}V_\mu^+ V_\nu^-, \tag{C.67}
$$

the identities

$$
0 = [D_\mu, D_\nu]V_\rho + r^\sigma_{\rho\nu\mu}V_\sigma = \left([D_\mu, D_\nu]V_\rho^+ + r^\sigma_{\rho\nu\mu}V_\sigma^+ - iA_{\mu\nu}V_\rho^+\right)m = \left([D_\mu, D_\nu] - iA_{\mu\nu}\right)m, \tag{C.68}
$$

and the conjugate relations. Combining these contributions using, from (C.52)

$$
A_{\mu\nu}A^{\mu\nu} = 2i\mathcal{D}_\mu(V_\nu^- A^{\mu\nu}) - 2iV_\nu^-\mathcal{D}_\mu A^{\mu\nu},
$$

$$V_-^\mu \Box V_\mu^- \;=\; \frac{1}{2}V_-^\mu\{\nabla_\mu, \nabla^\nu\}V_\nu^- - \frac{1}{2}V_\nu^- V_\mu^- r^{\mu\nu} - iV_\nu^- \mathcal{D}_\mu A^{\mu\nu}. \tag{C.69}$$

gives

$$\begin{aligned}
\frac{i}{2}T \;=\; & \frac{\sqrt{g}}{64\pi^2}\mathrm{Tr}\eta\Bigg[\tilde{\varphi}^+\Bigg\{\left(F^a\cdot F^b - i\tilde{F}^a\cdot F^b - \frac{2}{x^2}D^aD^b\right)T_aT_b + \frac{1}{12}\left(3A\cdot A - i\tilde{A}\cdot A\right)\\
& -\frac{1}{24}\left(r_{\mu\nu\rho\sigma}r^{\mu\nu\rho\sigma} - ir\cdot\tilde{r} + 2X_{\mu\nu}X^{\mu\nu}\right) + \frac{r^2}{72} - \frac{i}{6}A_{\mu\nu}X^{\mu\nu}\\
& +\frac{1}{3}\nabla^\mu\left(\nabla_\mu - V_\mu\right)\left(\frac{1}{3}\,\mathcal{D}^\alpha X_\alpha| - \mathcal{D}^\alpha f_\alpha| + \frac{1}{6}r + \nabla^\nu V_\nu^- + V_-^\nu V_\nu^-\right)\\
& -\frac{1}{2}\left(\mathcal{D}^\alpha f_\alpha| - \frac{1}{3}\,\mathcal{D}^\alpha X_\alpha|\right)^2 + \frac{1}{3}\nabla^\mu\left[y\left(\partial_\mu - V_\mu^-\right)\bar{y} - \nabla^\nu\left(V_\mu^- V_\nu^-\right)\right]\\
& +\frac{2}{3}\nabla^\mu\left[V_\mu^-\nabla^\nu V_\nu^- + \frac{1}{2}V_\mu^- V_-^\nu V_\nu^-\right] + \frac{1}{6}\nabla^\mu\left(rV_\mu^- - 2r_{\mu\nu}V_-^\nu\right)\Bigg\}\\
& -\frac{1}{3}F_{\tilde{\varphi}^+}\left\{\Box y - \mathcal{D}_\mu yV^\mu + y\left(\mathcal{D}^\mu V_\mu^- + V_-^\mu V_\mu^- - \mathcal{D}_\mu V^\mu - V^\mu V_\mu^-\right)\right.\\
& \left. +2V_\mu^-\mathcal{D}^\mu y - y^2\bar{y} + y\left(3\mathcal{D}^\alpha f_\alpha - \mathcal{D}^\alpha X_\alpha\right)\right\}\Bigg] + \mathrm{h.c.}
\end{aligned} \tag{C.70}$$

This expression is the bosonic part of the superfield expression

$$\delta\mathcal{L}_1 \;=\; \frac{1}{8\pi^2}\int d^4\theta\,\mathrm{Tr}\left[\eta\widetilde{\Phi}^+(T^n, \Lambda_X)\Omega_1\right] + \mathrm{h.c.} \tag{C.71}$$

$$\Omega_1 \;=\; -\frac{1}{48}\Omega_m + \frac{1}{3}\Omega_W + \Omega^0_{\mathrm{YM}} - \frac{1}{36}\Omega_{X^m}, \qquad \Omega_m = \Omega_D - 4\Omega_G + 8\Omega_R, \tag{C.72}$$

where the operators in (C.72) are defined in (4.23) and (4.39)–(4.41), Ω^0_{YM} is the Chern-Simons superfield for the nonanomalous gauge group, and $\widetilde{\Phi}^+\Big| = \tilde{\varphi}^+$. The expression (C.71) is the result of an infinitesimal transformation, and must be integrated to give the expression for a finite transformation; in particular the modular transformations are discrete and therefore finite. This is possible if the coefficient of $\widetilde{\Phi}^+$ contains only 1) operators such as Ω_W, Ω_{YM} and Ω_{X^m} whose chiral projections are invariant under $U(1)_X$ and modular transformations, 2) linear multiplets that drop out of the superspace integral and 3) derivatives of $\ln\mathcal{M}$. That this is indeed the case is shown in Section E.1.

D Orbifold compactification: PV sector for matter

We argued in Section 4.1 that PV mass terms must have well-defined modular weights, with invariant masses for those fields that contribute to the renormalization of the Kähler potential. One way to achieve this would be to couple in the PV mass superpotential all PV fields Z^σ with metric $K^Z_{\sigma\bar\rho}$ to fields Y_σ with a metric proportional to its inverse, as in (4.52). However each Z, Y pair gives no contribution linear in the scalar curvature, and doubles the contribution quadratic in scalar curvature. This requires an even number of pairs with signatures that sum to zero, and the introduction of other fields that reproduce the curvature terms from the light fields. This was done in [3] for the untwisted sector with Kähler potential:

$$\begin{aligned} G_u &= \sum_n G^n(Z^i_n), \quad G^n = -\delta^n_i \ln\left(T^i + \bar T^i - \sum_{a=1}^{N_n-1}|\Phi^{ai}|^2\right) = \delta^n_i\left[g^i - \ln(1 - e^{g^i}\sum_{a=1}^{N_n-1}|\Phi^{ai}|^2)\right], \\ g^i &= -\ln(T^i + \bar T^i), \end{aligned} \tag{D.1}$$

where the special property $G^n_{ij} = G^n_i G^n_j$ was exploited to mimic their curvature terms by other fields. The twisted sector Kähler potential is not known beyond leading order:

$$G = G_u + f(Z, \bar Z), \quad f = X^a + O(\Phi^3), \quad X^a = e^{g^a}|\Phi^a|^2, \quad g^a = \sum_i q^a_i g^i. \tag{D.2}$$

Under a nonlinear transformation $Z^i_n \to Z'^i_n(Z^j_n)$ such that

$$G_u(Z') = G_u(Z) + F + \bar F, \quad G^n(Z'_n) = G^n(Z_n) + F^n + \bar F^n, \quad F = \sum_n F^n, \tag{D.3}$$

X^a is invariant provided

$$\Phi^a \to \Phi'^a = e^{-F^a}\Phi^a, \quad F^a = \sum_n q^a_n F^n. \tag{D.4}$$

To use the same trick including the twisted sector fields requires a constraint on the overall Kähler potential analogous to the constraint on gauge charges discussed in Section 3.2. For example this is possible with a Kähler potential of the form

$$G = g + f(X^a) + H(Z) + \bar H(\bar Z), \qquad g = \sum_n g^n(Z^i_n), \quad g^a = \sum_n q^a_n g^n, \tag{D.5}$$

where g^n transforms like G^n under a modular transformation, for example $g^n = \delta^n_i g^i$ or $g^n = G^n$. The modular invariant holomorphic function H is constructed from operators of the form

$$\prod_a[\Phi^a \prod_{i=1}^3 \eta^{2q^a_{n_i}}(iT^i)], \tag{D.6}$$

and the fields Φ^a are separated into groups $\Phi_m \ni \Phi^a_m,\ a = 1, \ldots N_m$ with the function $f(X^a)$ in (D.5) restricted to the form

$$f = \sum_m k^m, \quad k^m = -\lambda_m^{-1} \ln \left(1 - \lambda_m \sum_{a=1}^{N_m} X^a_m \right). \tag{D.7}$$

This is option 1) of Section 4.2; its implementation requires the introduction of a number of additional PV fields. Here we will focus on option 2) which entails no restriction on the twisted sector Kähler potential. In Section D.3 we will consider a hybrid case where option 1) is implemented for the untwisted sector in order to include the possibility of maximal Heisenberg invariance.

In order to make the PV Kähler potential and superpotential fully modular invariant we introduced a set of PV fields $\dot{Z}^\sigma = \dot{Z}^N, \dot{Z}^P$, with negative signature, that regulate the UV divergent contributions to the renormalization of the light field Kähler potential and to the operator ϕ_3 in (2.9). Their Kähler potential is given by (3.1) and (3.18) with $\zeta_{m\bar{n}} = \delta_{m\bar{n}}$:

$$\begin{aligned} K(\dot{Z}^\sigma) &= K^{\dot{Z}} + K(\dot{Z}^N) = K^{\dot{Z}}_{\rho\bar{\sigma}} \dot{Z}^\rho \bar{\dot{Z}}^{\bar{\sigma}} + \frac{1}{2}\left(K_{\sigma\rho} \dot{Z}^\sigma \dot{Z}^\rho + \text{h.c.} \right), \\ \sigma &= P, N, \qquad P = I, A, \end{aligned} \tag{D.8}$$

which is modular invariant provided $\dot{Z}^\sigma$ transforms as in (3.15) under (3.3). These fields couple in the PV superpotential $W_1^{\dot{Z}}$ given in (4.57) to superfields $\dot{Y}_\sigma = \dot{Y}_N, \dot{Y}_P$, with the Kähler potential given in (4.54). To calculate the contributions of these fields to $\mathcal{L}_Q, \Phi_1, \Phi_2$, we need the affine connection derived from the PV Kähler metric. For $\dot{Z}^\alpha$ we have

$$\begin{aligned} \dot{\Gamma}^P_{Nr} &= K^{p\bar{q}} \partial_r \bar{\chi}^{\bar{n}}_{\bar{q}}, \quad \dot{\Gamma}^P_{Qr} = \Gamma^p_{qr} + \chi^n_q \dot{\Gamma}^P_{Nr}, \quad \dot{\Gamma}^N_{N'r} = -\chi^n_q \dot{\Gamma}^Q_{N'r}, \\ \dot{\Gamma}^N_{Pr} &= +D_r \chi^n_p - \chi^n_q \chi^{n'}_p \dot{\Gamma}^Q_{N'r}, \end{aligned} \tag{D.9}$$

If $\chi^n_p = \dot{a}\partial_p h^n(Z)$, with $h^n(Z)$ holomorphic, $\partial_r \bar{h}(\bar{z})^{\bar{n}}_{\bar{q}} = \dot{\Gamma}^P_{Nr} = 0$, χ^n_p drops out of the traces of products of $\dot{\Gamma}$ and its derivatives. In addition, since $(T_a)^q_p \chi^n_q = 0$, it also drops out of the traces of $\dot{\Gamma}$ with gauge generators. Alternatively, since χ^n_p is proportional to the parameter $\dot{a}$ by virtue of the condition (3.18), we can cancel the UV divergent terms involving χ^n_p by including several copies of the $\dot{Z}^\sigma \to \dot{Z}^\sigma_\lambda$, $\lambda = 1, \ldots, 2n_{\dot{Z}} + 1$ with signatures η^Z_λ and parameters $\dot{a}^Z_\lambda$ such that

$$\dot{n} = \sum_\lambda \eta^{\dot{Z}}_\lambda = -1, \qquad \sum_\lambda \eta^{\dot{Z}}_\lambda \dot{a}^2_\lambda = \sum_\lambda \eta^{\dot{Z}}_\lambda \dot{a}^4_\lambda = 0. \tag{D.10}$$

We also need to impose the condition (A.4). This is again automatically satisfied if χ^n is holomorphic because it also drops out of the trace of products of $J^\pm_\mu$ with ϕ because $\phi^\rho_N = 0$, where

$$(J^-_\mu)^\rho_\sigma = \dot{\Gamma}^\rho_{\sigma r} \mathcal{D}_\mu z^r + i T^\rho_\sigma \cdot A_\mu + i\delta^\rho_\sigma \Gamma_\mu = \left[(J^+_\mu)^\rho_\sigma \right]^\dagger, \qquad \Gamma_\mu = \frac{i}{4}\left(K_i \mathcal{D}_\mu z^i - K_{\bar{m}} \mathcal{D}_\mu \bar{z}^{\bar{m}} \right). \tag{D.11}$$

For the more general case, (A.4) requires the conditions on $\dot{a}^2$ in (D.10) and the additional constraint

$$\sum_\lambda \eta_\lambda^{\dot{Z}} \dot{a}_\lambda^6 = 0. \tag{D.12}$$

Only one set of the $\dot{Z}_\lambda$, say $\dot{Z}_0$, with negative signature, $\eta_0^{\dot{Z}} = -1$, need have couplings in the superpotential (3.16) and in the additional terms $K_{\sigma\rho}$ in the Kähler potential (D.8). Gauge and modular invariant mass terms for all the above fields may be constructed as in (4.57) by introducing a superfield $\dot{Y}_\sigma^\lambda$, with the Kähler potential (4.54), for each $\dot{Z}_\lambda^\sigma$.
Rather than use the most general parameterization, we will simply set

$$\chi_p^n = \partial_p \chi^n, \tag{D.13}$$

and consider three choices for the pair of functions g^n in (4.54) and χ^n that we take to be the same in (D.8) and in (4.54). In our second and third scenarios we make some assumptions on the form of the Kähler potential for the light fields.

D.1 χ^n holomorphic

We set

$$\begin{aligned} g^n &= \delta_i^n g^n(T^i + \bar{T}^{\bar{\imath}}) = -\delta_i^n \ln(T^i + \bar{T}^{\bar{\imath}}), \qquad i, n = 1,2,3, \\ \chi^n &= 2\dot{a}\delta_i^n \ln \eta(T^i), \qquad \chi_i^n = 2\dot{a}\frac{\partial \eta(T^i)}{\partial T^i}\delta_i^n \equiv 2\dot{a}\zeta(T^i)\delta_i^n, \end{aligned} \tag{D.14}$$

in (3.18) and (4.57). Then for $\dot{Z}^\sigma$:

$$\dot{\Gamma}_{Qr}^P = \dot{\Gamma}_{qr}^p, \qquad \dot{\Gamma}_{Qr}^N = D_r \chi_q^n, \qquad \dot{\Gamma}_{Nr}^P = \dot{\Gamma}_{Nr}^N = 0, \tag{D.15}$$

and the UV divergent contributions from $\dot{Z}$ simply cancel the UV divergences from the light fields Z. The affine connections for $\dot{Y}_{I,N}$ are

$$\dot{\Gamma}_{Ij}^I = G_j^N - 2g_j^n, \qquad \dot{\Gamma}_{Ij}^N = -\left(\partial_j - 2g_i^n\right)\chi_i^n, \qquad \dot{\Gamma}_{Nj}^I = 0, \qquad \dot{\Gamma}_{Nj}^N = G_j^N, \tag{D.16}$$

and the UV divergent contributions from $\dot{Y}_{I,N}$, with $\sum_\lambda \dot{\eta}_\lambda = -1$, involve the operators

$$\sum_\lambda \dot{\eta}_\lambda (\Gamma_{\dot{Y}}^n)_{\sigma\alpha}^\sigma = -2\left(G_\alpha^N - g_\alpha^n\right), \tag{D.17}$$

$$\sum_\lambda \dot{\eta}_\lambda (\Gamma_{\dot{Y}}^n)_{\rho\alpha}^\sigma (\Gamma_{\dot{Y}}^n)_{\sigma\alpha}^\rho = -\left(G_\alpha^N - 2g_\alpha^n\right)\left(G_\beta^N - 2g_\beta^n\right) - G_\alpha^N G_\beta^N. \tag{D.18}$$

All the UV divergent contributions from $\dot{Y}$ can be canceled by additional PV fields $\Psi = U, V, \Phi$ with Kähler and superpotential

$$
\begin{aligned}
K(\Psi) &= \sum_\gamma \left[\sum_A \left(e^{\alpha_\gamma^A G^A} |U_\gamma^A|^2 + e^{\alpha_\gamma^A G^A} |U_A^\gamma|^2 + e^{\gamma_\gamma^A G^A} |V_\gamma^A|^2 \right) \right. \\
&\qquad \left. + \sum_{N=n} \left(e^{\delta_\gamma^N (G^N - 2g^n)} |\Phi_\gamma^N|^2 + e^{\epsilon_\gamma^n G^N} |\Phi_\gamma^n|^2 \right) \right], \qquad \text{(D.19)} \\
W(\Psi) &= \sum_\gamma \left[\sum_A \mu_\gamma^U U_\gamma^A U_A^\gamma + \frac{1}{2} \left(\sum_A \mu_\gamma^V (V_\gamma^A)^2 + \sum_N \mu_\gamma^N (\Phi_\gamma^N)^2 + \sum_n \mu_\gamma^n (\Phi_\gamma^n)^2 \right) \right], \qquad \text{(D.20)} \\
T_a(U_\gamma^A) &= -T_a^T(U_A^\gamma), \quad a \neq X; \qquad T_a(V) = -T_a^T(V), \quad T_a(\Phi) = 0 \quad \forall \ a, \qquad \text{(D.21)}
\end{aligned}
$$

if we require

$$
\begin{aligned}
\sum_{\Psi,\gamma} \eta_\gamma^\Psi &= 6 + d_M = N + 2, \qquad \text{(D.22)} \\
C_M^a &= \sum_\gamma \left\{ \eta_\gamma^U \left[(\mathrm{Tr} T_a^2)_{U_\gamma} + (\mathrm{Tr} T_a^2)_{U^\gamma} \right] + \eta_\gamma^V (\mathrm{Tr} T_a^2)_{V_\gamma} \right\}, \qquad \text{(D.23)} \\
1 &= \sum_\gamma \eta_\gamma^N \delta_\gamma^N = \sum_\gamma \eta_\gamma^n \epsilon_\gamma^n = \sum_\gamma \eta_\gamma^N (\delta_\gamma^N)^2 = \sum_\gamma \eta_\gamma^n (\epsilon_\gamma^n)^2, \qquad \text{(D.24)} \\
(\mathrm{Tr} T_X)_M &= -\sum_\gamma \eta_\gamma^U \left[(\mathrm{Tr} T_X)_{U_\gamma} + (\mathrm{Tr} T_X)_{U^\gamma} \right], \qquad \text{(D.25)} \\
(\mathrm{Tr} T_b)_{M^a} &= -\sum_\gamma \eta_\gamma^{UA} \alpha_\gamma^A \left[(\mathrm{Tr} T_b)_{U_\gamma^A} + (\mathrm{Tr} T_b)_{U_A^\gamma} \right], \qquad \text{(D.26)} \\
d_{M^a} &= \sum_\gamma \left(2\eta_\gamma^{UA} \alpha_\gamma^A d_{U_\gamma^A} + \eta^{V_\gamma^A} \delta_\gamma^A d_{V^A\gamma} \right) \qquad \text{(D.27)} \\
&= \sum_\gamma \left[2\eta_\gamma^{UA} (\alpha_\gamma^A)^2 d_{U_\gamma^A} + \eta_\gamma^{VA} (\delta_\gamma^A)^2 d_{V_\gamma^A} \right], \qquad \text{(D.28)}
\end{aligned}
$$

where the subscript M stands for the gauge-charged light sector, and d_{M^a} is the dimension of the (generally reducible) gauge group representation in the matter sector with modular weights q_n^a. In addition to (D.10) and (D.12), the constraint (A.4) imposes further conditions on the parameters in $K(\Psi)$. The $\dot{Y}$ contributions to the first condition in (A.4), namely

$$
\sum_{\lambda,A} \left(\sum_i \partial_\mu t^i G_i^A + i T_A \cdot A_\mu + i\Gamma_\mu \right) \left(\sum_j \partial_\nu t^j G_j^A + i T_A \cdot A_\nu + i\Gamma_\nu \right) q_X^A \mathrm{Im}\lambda, \qquad \text{(D.29)}
$$

and

$$
\mathrm{Im}F^m(t^m)\left[\sum_{\lambda,A}\left(\sum_i \partial_\mu t^i G_i^A + iT_A\cdot A_\mu + i\Gamma_\mu\right)\left(\sum_j \partial_\nu t^j G_j^A + iT_A\cdot A_\nu + i\Gamma_\nu\right) q_m^A\right.
$$

$$
+\sum_{\lambda,N}\left(\sum_j \partial_\mu t^j G_j^N - 2\partial_\mu t^n g_n^n + i\Gamma_\mu\right)\cdot\left(\sum_k \partial_\mu t^k G_k^N - 2\partial_\mu t^n g_n^n + i\Gamma_\mu\right)
$$

$$
\left.+\sum_{\lambda,N}\left(\sum_j \partial_\mu t^j G_j^N + i\Gamma_\mu\right)\left(\sum_k \partial_\nu t^k G_k^N q_m^N + i\Gamma_\mu\right)\right], \tag{D.30}
$$

require

$$
(\mathrm{Tr}T_a^2 T_X)_M = -\sum_\gamma \eta_\gamma^U\left[(\mathrm{Tr}T_a^2 T_X)_{U_\gamma} + (\mathrm{Tr}T_a^2 T_X)_{U^\gamma}\right] \tag{D.31}
$$

$$
1 = \sum_\gamma \eta_\gamma^N(\delta_\gamma^N)^3 = \sum_\gamma \eta_\gamma^n(\epsilon_\gamma^n)^3, \tag{D.32}
$$

$$
(\mathrm{Tr}T_b T_c)_{M^a} = \sum_\gamma\left\{\eta^{U_\gamma^A}\alpha_\gamma^A\left[(\mathrm{Tr}T_b T_c)_{U_\gamma^A} + (\mathrm{Tr}T_b T_c)_{U_A^\gamma}\right] + \eta^{V_\gamma^A}(\mathrm{Tr}T_b T_c)_{V_\gamma^A}\delta_\gamma^A\right\}, \tag{D.33}
$$

$$
(\mathrm{Tr}T_b)_{M^a} = -\sum_\gamma \eta_\gamma^{UA}(\alpha_\gamma^A)^2\left[(\mathrm{Tr}T_b)_{U_\gamma^A} + (\mathrm{Tr}T_b)_{U_A^\gamma}\right], \tag{D.34}
$$

$$
d_{M^a} = \sum_\gamma\left[2\eta_\gamma^{UA}(\alpha_\gamma^A)^3 d_{U_\gamma^A} + \eta_\gamma^{VA}(\delta_\gamma^A)^3 d_{V_\gamma^A}\right\}. \tag{D.35}
$$

Note that, for example, since the right hand side of (D.18) is equal to $-2\left(G_\alpha^N - g_\alpha^n\right)\left(G_\beta^N - g_\beta^n\right) - 2g_\alpha^n g_\beta^n$, we can cancel (D.17) and (D.18) if we replace the last term in (D.19) and the condition (D.24) by, respectively

$$
\sum_{N=n}\left(e^{\delta_\gamma^N(G^N - g^n)}|\Phi_\gamma^N|^2 + e^{\epsilon_\gamma^n g^n}|\Phi_\gamma^n|^2\right) \tag{D.36}
$$

and

$$
2 = \sum_\gamma \eta_\gamma^N\delta_\gamma^N = \sum_\gamma \eta_\gamma^N(\delta_\gamma^N)^2 = \sum_\gamma \eta_\gamma^n(\epsilon_\gamma^n)^2, \qquad 0 = \sum_\gamma \eta_\gamma^n\epsilon_\gamma^n. \tag{D.37}
$$

Imposing the additional constraints (A.4) removes this ambiguity. However we will see in Appendix D.3 that there is at least one other choice of PV sector with this (and the following) choice of $\dot{Z}, \dot{Y}$ Kähler potentials.

A simple solution to the constraints (D.22)–(D.28) and (D.31)–(D.35) is[28]

$$\delta^N = \epsilon^n = \alpha^A = \delta^A = 1, \tag{D.38}$$

in which case they reduce to

$$\begin{aligned}
1 &= \sum_\gamma \eta^N_\gamma = \sum_\gamma \eta^n_\gamma, && \text{(D.39)}\\
(\mathrm{Tr}T_b)_{M^a} &= -\sum_\gamma \eta^{U^A_\gamma}\left[(\mathrm{Tr}T_b)_{U^A_\gamma} + (\mathrm{Tr}T_b)_{U^\gamma_A}\right], && \text{(D.40)}\\
(\mathrm{Tr}T_bT_c)_{M^a} &= \sum_\gamma \left\{\eta^{U^A_\gamma}\left[(\mathrm{Tr}T_bT_c)_{U^A_\gamma} + (\mathrm{Tr}T_bT_c)_{U^\gamma_A}\right] + \eta^{V^A_\gamma}(\mathrm{Tr}T_bT_c)_{V^A_\gamma}\right\}, && \text{(D.41)}\\
d_{M^a} &= \sum_\gamma \left[2\eta^{U^A_\gamma} d_{U^A_\gamma} + \eta^{V^A_\gamma} d_{V^A{}_\gamma}\right], && \text{(D.42)}\\
(\mathrm{Tr}T_a^2T_X)_M &= -\sum_{A,\gamma} \eta^{U^A{}_\gamma}\left[(\mathrm{Tr}T_a^2T_X)_{U^A_\gamma} + (\mathrm{Tr}T_a^2T_X)_{U^\gamma_A}\right]. && \text{(D.43)}
\end{aligned}$$

Note that (D.39) and (D.42) are equivalent to (D.22).

D.2 Preserving shift symmetry

The shift symmetries $\mathrm{Im}t^I \to \mathrm{Im}t^I + \alpha^I$, $\mathrm{Im}s \to \mathrm{Im}s + \beta$, $\alpha^I, \beta \in \mathcal{R}$, of the classical Kähler potential are preserved if $K = K[(T^I + \bar{T}^I), (S + \bar{S}), \Phi^a]$. Then shift symmetry and modular covariance of the light particle Kähler potential is preserved if it takes the form

$$K = k(S+\bar{S}) + g + f(X^a), \quad X^a = e^{g^a}|\Phi^a|^2, \quad g^a = \sum_n q^a_n g^n, \quad f(X) = \sum_a X^a + O(X^2). \tag{D.44}$$

The metric is

$$\begin{aligned}
K_{a\bar{b}} &= e^{g^a}\delta_{ab} f_a + \phi^b\bar{\phi}^{\bar{a}} f_{ab} e^{g^a+g^b}, \quad K_{a\bar{\imath}} = \sum_b g^b_{\bar{\imath}}\bar{\phi}^{\bar{b}} K_{a\bar{b}},\\
K_{j\bar{\imath}} &= g_{j\bar{\imath}} f_{n_i} + \sum_{a,b} g^a_j g^b_{\bar{\imath}} \phi^a \bar{\phi}^{\bar{b}} K_{a\bar{b}}, \quad f_{n_i} = \left(1 + \sum_a q^a_{n_i} X^a f_a\right),
\end{aligned} \tag{D.45}$$

where

$$f_a = \frac{\partial f}{\partial X^a}, \qquad f_{ab} = \frac{\partial f}{\partial X^a \partial X^b}. \tag{D.46}$$

[28] The constraints (D.26), and (D.34) are nontrivial only if T_b is a $U(1)$ generator; V^A is a $U(1)$ singlet.

$$\chi^n = \dot{a} g^n = \dot{a}\delta^n_i g(T^i + \bar{T}^i). \tag{D.47}$$

If we define

$$k_{a\bar{b}} = \partial_a \partial_{\bar{b}} f(X^a) = K_{a\bar{b}}, \quad k_{a\bar{b}} k^{\bar{b}c} = \delta^a_c, \quad \tilde{g}_{i\bar{\jmath}} = f_{n_i} g_{i\bar{\jmath}} = f_{n_i} g_i^2 \delta_{ij}, \quad \tilde{g}^{k\bar{\jmath}} \tilde{g}_{i\bar{\jmath}} = \delta^k_i, \tag{D.48}$$

the inverse metric is:

$$K^{a\bar{b}} = k^{a\bar{b}} + \phi^a \bar{\phi}^{\bar{b}} \sum_{\bar{\imath} j} g^a_j g^b_{\bar{\imath}} \tilde{g}^{\bar{\imath} j}, \quad K^{j\bar{\imath}} = \tilde{g}^{\bar{\imath} j} = f^{-1}_{n_i} g_i^{-2} \delta^{ij}, \quad K^{a\bar{\imath}} = -\phi^a \sum_j g^a_j \tilde{g}^{j\bar{\imath}}. \tag{D.49}$$

Using the properties $K^{\bar{\imath} j} = \delta^{ij}$, $g_i = g_{\bar{\imath}} = e^{g_{n_i}}$ and $(T_a)^i_j g_i = 0$, one can see that only (D.10) is required for (A.4) to be satisfied in this case; however a minimum of 3 sets of the $\dot{Z}, \dot{Y}$ is still required.

To maintain invariance of the PV Kähler potential we take

$$\chi^n = \dot{a} g^n = \dot{a}\delta^n_i g(T^i + \bar{T}^i). \tag{D.50}$$

For $\dot{Y}$ we now have

$$\Gamma^I_{Nj} = -\dot{a}\delta^i_j \delta^i_n, \qquad \Gamma^I_{N\alpha} = \frac{\dot{a}}{8}\delta^i_n (\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha T^i = 0, \tag{D.51}$$

because T^i is gauge invariant, and once (D.10) is imposed the contributions from $\dot{Y}$ involve the same operators as in (D.18), and are canceled by contributions from the fields $\Psi = U, V, \Phi$ introduced in (D.19)–(D.21), subject to the conditions (D.22)–(D.28) and (D.31)–(D.35).

D.3 Heisenberg invariant untwisted sector Kähler potential

Here we set $\chi^n = \dot{a} g^n = \dot{a} G^n$, where G^n is defined in (D.1). Once the conditions (D.10) and (D.12) are imposed, the $\dot{Z}$ cancel the UV divergences from the light fields Z^p. The contributions from $\dot{Y}_I$ decouple from those from $\dot{Y}_N$ in the relevant sums. Those from $\dot{Y}_I$ reduce to those from fields with metric equal to the inverse of the untwisted sector metric derived from (D.1), with an extra overall factor e^{G^N}, and those from $\dot{Y}_N$ are the same as the contributions from a field with metric e^{G^N}. We may cancel their contributions to the UV divergences with the method used in [3] to cancel the UV divergences from the light untwisted sector. That is, the $\dot{Y}$ contributions to the UV divergences can be canceled with $\Psi = U, V, \Phi^N, \Phi^n_I, \Phi^I_n$ where U, V, have the same Kähler potential as in (D.19). The Φ^N, which are no longer all gauge singlets, have Kähler potential

$$K(\Phi^N) = \sum_\gamma \sum_{N=n} e^{\delta^N_\gamma (G^N - g^n)} |\Phi^N_\gamma|^2, \tag{D.52}$$

and the constraints (D.22)–(D.28) are appropriately modified. The fields Φ^n_I, $I=0,\ldots,N_n$, have a Kähler potential of the same form as $\dot{Y}_{I,N}$, but without the prefactor e^{G^N} in (4.54), and the fields Φ^I have the inverse Kähler metric:

$$K^{\Phi^I}_{I\bar{J}}K^{\bar{J}K}_{\Phi_I}=\delta^K_J,\qquad I,J,K=0,\ldots,N_n. \tag{D.53}$$

This set cancels the remaining $\dot{Y}_{I,N}$ UV contributions if we impose

$$\eta^{\Phi^I}_\gamma=\eta^{\Phi_I}_\gamma\equiv\eta^\Phi_\gamma,\qquad\sum_\gamma\eta^\Phi_\gamma=0,\qquad\sum_\gamma\eta^\Phi_\gamma(a^\Phi_\gamma)^2=-\sum_\gamma\eta^\Phi_\gamma(a^\Phi_\gamma)^4=\frac{1}{2}. \tag{D.54}$$

However, it is not possible to cancel all the contributions from $\dot{Y}$ to (A.4) because terms odd in $(\Gamma^{\Phi^n_\gamma}_\mu)^I_J$, $(T^{\Phi^n_\gamma}_a)^I_J$ and $(\phi^{\Phi^n_\gamma})^I_J$ cancel between Φ^I and Φ_I. Terms proportional to $\mathrm{Tr}\left(\dot{\Gamma}^Y_\mu\dot{\phi}^Y\right)$ are indeed canceled by virtue of (D.54), but there is no contribution from the Φ sector to cancel terms like $G_i\mathcal{D}_\mu z^j(\dot{T}^Y_a\dot{\phi}^Y)^i_j$, for example. These terms could be made to vanish with $\sum_\lambda\eta^{\dot{Z}}_\lambda\dot{a}^2_\lambda=-1$, but this contradicts the condition in (D.10). So in this case we need to modify (4.54)–(4.57) as follows:

$$K^{\dot{Y}}=\sum_A e^{G^A}|\dot{Y}_A|^2+\sum_N e^{G^N}\left[g^{i\bar{j}}_n\left(\dot{Y}_I-bG^n_i\dot{Y}_N\right)\left(\dot{\bar{Y}}_{\bar{J}}-bG^n_{\bar{j}}\dot{\bar{Y}}_{\bar{N}}\right)+|\dot{Y}_N|^2\right], \tag{D.55}$$

which is modular invariant provided under (3.10)

$$\dot{Y}'_A=e^{-F^A}\dot{Y}_A,\qquad\dot{Y}'_I=e^{-F^N}m^j_i\left(\dot{Y}_J+bF^n_j\dot{Y}_N\right),\quad\dot{Y}'_N=e^{-F^N}\dot{Y}_N,\quad F^{N,A}=\sum_m q^{N,A}_mF^m, \tag{D.56}$$

and the superpotential is given by (4.57), except that

$$W_n=\dot{\mu}_n(T^i)\left(\sum_{P\in n}\dot{Z}^P\dot{Y}_P+\dot{a}^{-1}b\dot{Z}^N\dot{Y}_N\right). \tag{D.57}$$

Now we have (see Appendix D of [3])

$$\begin{aligned}\sum_\lambda\dot{\eta}_\lambda(\Gamma^n_{\dot{Y}})^\sigma_{\sigma\alpha}&=-(N_n+1)\left(G^N_\alpha-G^n_\alpha\right), &\text{(D.58)}\\ \sum_\lambda\dot{\eta}_\lambda(\Gamma^n_{\dot{Y}})^\sigma_{\rho\beta}(\Gamma^n_{\dot{Y}})^\rho_{\sigma\alpha}&=-(N_n+1)\left(G^N_\alpha-G^n_\alpha\right)\left(G^N_\beta-G^n_\beta\right)+2(b_2+b_4)G^{n\alpha}_i\bar{Z}^i_\alpha\\ &\quad-(1+2b_2+b_4)\left[G^n_\alpha G^n_\beta+(\hat{G}^n)^i_{j\alpha}(\hat{G}^n)^j_{i\beta}\right], &\text{(D.59)}\\ \sum_\lambda\dot{\eta}_\lambda(\Gamma^n_{\dot{Y}})^J_{I\alpha}(T^a)^I_J&=(\mathrm{Tr}T^a)_{Z_n}\left(G^N_\alpha-G^n_\alpha\right)-(1+b_2)(T^a)^j_i(\hat{G}^n)^i_{j\alpha}, &\text{(D.60)}\\ (\hat{G}^n)^i_{j\alpha}&=\mathcal{P}\left(G^n_j\mathcal{D}_\alpha Z^i_n\right),\qquad(\hat{G}^n)_{i\alpha}=\mathcal{P}\left(G^n_iG^n_j\mathcal{D}_\alpha Z^j_n\right), &\text{(D.61)}\\ Z^i_\alpha&=\mathcal{P}\mathcal{D}_\alpha Z^i,\qquad\mathcal{P}=-\frac{1}{8}(\bar{\mathcal{D}}^2-8R),\qquad b_p=-\sum_\lambda\dot{\eta}_\lambda b^p_\lambda &\text{(D.62)}\end{aligned}$$

If $b_p = 0$, we recover the inverse of the untwisted sector Kähler potential. However if

$$b_4 = -b_2 = 1, \tag{D.63}$$

we eliminate both the terms arising from this contribution that cannot be eliminated by U, V, Φ^I_N, with

$$K(\Phi^I_N) = \sum_{\gamma,N} \sum_{I=0}^{N_n} e^{\delta^N_\gamma} (G^N - G^n) |\Phi^I_N|^2, \tag{D.64}$$

as well as the new term involving the last two operators in (D.61). A simple solution to (D.63) with three sets $\dot{Y}_\gamma$ is, for example

$$(\eta, b) = (1, 3), (-1, 2), (-1, 2). \tag{D.65}$$

The constraints (D.22)–(D.28) now read

$$\begin{aligned}
\sum_{\Psi,\gamma} \eta^\Psi_\gamma &= 3 + \sum_n N_n + d_{M^T} = N + 2, &\text{(D.66)}\\
C^a_M &= \sum_\gamma \left\{ \eta^U_\gamma \left[(\mathrm{Tr} T^2_a)_{U_\gamma} + (\mathrm{Tr} T^2_a)_{U^\gamma} \right] + \eta^V_\gamma (\mathrm{Tr} T^2_a)_{V_\gamma} + \sum_N \eta^N_\gamma (\mathrm{Tr} T^2_a)_{\Phi^N_\gamma} \right\}, &\text{(D.67)}\\
1 &= \sum_\gamma \eta^N_\gamma \delta^N_\gamma = \sum_\gamma \eta^N_\gamma (\delta^N_\gamma)^2, \qquad (\mathrm{Tr} T_X)_{Z_n} = -\sum_\gamma \eta^N_\gamma \delta^N_\gamma T^{N_\gamma}_X, &\text{(D.68)}\\
(\mathrm{Tr} T_X)_M &= -\sum_\gamma \left(\eta^U_\gamma \left[(\mathrm{Tr} T_X)_{U_\gamma} + (\mathrm{Tr} T_X)_{U^\gamma} \right] + \eta^N_\gamma (T_X)_{\Phi^N_\gamma} \right), &\text{(D.69)}\\
(\mathrm{Tr} T_b)_{M^a} &= -\sum_\gamma \eta^{UA}_\gamma \alpha^A_\gamma \left[(\mathrm{Tr} T_b)_{U^A_\gamma} + (\mathrm{Tr} T_b)_{U^\gamma_A} \right], &\text{(D.70)}\\
d_{M^a} &= \sum_\gamma \left(2\eta^{UA}_\gamma \alpha^A_\gamma d_{U^A_\gamma} + \eta^{VA}_\gamma \delta^A_\gamma d_{V^A_\gamma} \right)\\
&= \sum_\gamma \left[2\eta^{UA}_\gamma (\alpha^A_\gamma)^2 d_{U^A_\gamma} + \eta^{VA}_\gamma (\delta^A_\gamma)^2 d_{V^A_\gamma} \right], &\text{(D.71)}
\end{aligned}$$

where M^a now includes only twisted sector matter.
For the relevant $\dot{Y}$ contributions to (A.4) we have

$$\begin{aligned}
\sum_\lambda \dot{\eta}_\lambda \mathrm{Tr} \left[(\Gamma^n_{\dot{Y}})_{\mu\nu} \Phi_{\dot{Y}} \right] &= -\left(G^N_{\mu\nu} - G^n_{\mu\nu} \right) \left[(N_n + 1)(F - F_N) + \sum_i q^i_X \lambda_X \right]\\
&\quad + (1 + b_2) Y_{\mu\nu}, &\text{(D.72)}
\end{aligned}$$

$$\sum_\lambda \dot\eta_\lambda \mathrm{Tr}\left(\tilde\Gamma^n_{\dot Y}\cdot\Gamma^n_{\dot Y}\Phi_{\dot Y}\right) = -\left(\widetilde{G}^N-\widetilde{G}^n\right)\cdot\left(G^N-G^n\right)\left[(N_n+1)(F-F^N)+\sum_i q^i_X\lambda_X\right] + (1+b_2)Y+(b_2+b_4)Z, \tag{D.73}$$

$$\sum_\lambda \dot\eta_\lambda \mathrm{Tr}\left[(\Gamma^n_{\dot Y})_{\mu\nu}(T^a)_{\dot Y}\Phi_{\dot Y}\right] = \left(G^N_{\mu\nu}-G^n_{\mu\nu}\right)\left[(\mathrm{Tr}T^a)_{Z_n}(F-F^N)+(\mathrm{Tr}T^aT_X)_{Z_n}\lambda_X\right] + (1+b_2)Y^a_{\mu\nu}, \tag{D.74}$$

$$\sum_\lambda \dot\eta_\lambda \mathrm{Tr}\left[(T^a)_{\dot Y}(\Gamma^n_{\dot Y})_{\mu\nu}\Phi_{\dot Y}\right] = \left(G^N_{\mu\nu}-G^n_{\mu\nu}\right)\left[(\mathrm{Tr}T^a)_{Z_n}(F-F^N)+(\mathrm{Tr}T^aT_X)_{Z_n}\lambda_X\right] + (1+b_2)Z^a_{\mu\nu}. \tag{D.75}$$

The precise form of the various operators Y, Z in the above is unimportant, since they drop out when (D.63) is imposed, and the remaining contribution is canceled by U, V, Φ^N, with the additional conditions

$$(\mathrm{Tr}T_a^2T_X)_M = -\sum_\gamma \eta^U_\gamma\left[(\mathrm{Tr}T_a^2T_X)_{U_\gamma}+(\mathrm{Tr}T_a^2T_X)_{U^\gamma}\right]-\sum_{N,\gamma}\eta^N_\gamma(\mathrm{Tr}T_a^2T_X)_{\Phi^N_\gamma} \tag{D.76}$$

$$(\mathrm{Tr}T_X)_{Z_n} = \sum_\gamma \eta^N_\gamma(\mathrm{Tr}T_X)_{\Phi^N_\gamma}(\delta^N_\gamma)^2, \qquad (\mathrm{Tr}T_bT_c)_{Z_n}=\sum_\gamma \eta^N_\gamma(\mathrm{Tr}T_bT_c)_{\Phi^N_\gamma}\delta^N_\gamma, \tag{D.77}$$

$$(\mathrm{Tr}T_bT_c)_{M^a} = \sum_\gamma\left\{\eta^{U^A_\gamma}\alpha^A_\gamma\left[(\mathrm{Tr}T_bT_c)_{U^A_\gamma}+(\mathrm{Tr}T_bT_c)_{U^\gamma_A}\right]+\eta^{V^A_\gamma}(\mathrm{Tr}T_bT_c)_{V^A_\gamma}\delta^A_\gamma\right\}, \tag{D.78}$$

$$(\mathrm{Tr}T_b)_{M^a} = -\sum_\gamma \eta^{UA}_\gamma(\alpha^A_\gamma)^2\left[(\mathrm{Tr}T_b)_{U^A_\gamma}+(\mathrm{Tr}T_b)_{U^\gamma_A}\right], \tag{D.79}$$

$$1 = \sum_\gamma \eta^N_\gamma(\delta^N_\gamma)^3, \tag{D.80}$$

$$d_{M^a} = \sum_\gamma\left[2\eta^{UA}_\gamma(\alpha^A_\gamma)^3+\eta^{VA}_\gamma(\delta^A_\gamma)^3 d_{V^A_\gamma}\right]. \tag{D.81}$$

As before these constraints have a simple straightforward solution:

$$1 = \alpha^A=\gamma^N=\delta^N, \qquad 1=\sum_\gamma \eta^N_\gamma, \tag{D.82}$$

$$C^a_{M_b} = \sum_\gamma\left\{\eta^{UB}_\gamma\left[(\mathrm{Tr}T_a^2)_{U^B_\gamma}+(\mathrm{Tr}T_a^2)_{U^\gamma_B}\right]+\eta^{VB}_\gamma(\mathrm{Tr}T_a^2)_{V^B_\gamma}\right\}, \tag{D.83}$$

$$(\mathrm{Tr}T_b)_{M_a} = -\sum_\gamma \eta^{UA}_\gamma\left[(\mathrm{Tr}T_b)_{U^A_\gamma}+(\mathrm{Tr}T_b)_{U^\gamma_A}\right] \qquad (\mathrm{Tr}T_b)_{Z_n}=-\sum_\gamma \eta^N_\gamma T_b^{N_\gamma}, \tag{D.84}$$

$$d_{M^a} = \sum_\gamma\left(2\eta^{UA}_\gamma d_{U^A_\gamma}+\eta^{VA}_\gamma d_{V^A_\gamma}\right), \qquad C^a_{Z_n}=\sum_\gamma \eta^N_\gamma(\mathrm{Tr}T_a^2)_{N_\gamma}. \tag{D.85}$$

Note that the Kähler potential g^n used in the previous two subsections is just the limiting case $\Phi_n^A \to 0$ of G^n, so this gives an alternative PV sector for those cases. They differ in the resulting sum rules that are quadratic and cubic in the modular weights.

D.4 Anomalous masses

The conditions (D.22)–(D.28) and (D.31)–(D.35) imply that the the Ψ masses are not modular invariant (at least in the absence of moduli-dependent mass factors) and are not in general $U(1)_X$ invariant. Their Kähler potential is invariant under Kähler transformations arising from nonlinear transformations on the light fields:

$$Z^p \to Z'^p(Z), \quad dZ'^p = M_q^p dZ^p, \quad K(Z') = K(Z) + F(Z), \quad W(Z') = e^{-F(Z)}W(Z), \tag{D.86}$$

provided

$$U'^A_\gamma = e^{-\alpha_\gamma^A F^A} U_\gamma^A, \qquad U'^\gamma_A = e^{-\alpha_\gamma^A F^A} U_A^\gamma, \qquad V'^A_\gamma = e^{-\gamma_\gamma^A F^A} V_\gamma^A, \tag{D.87}$$

$$\Phi'^N_\gamma = e^{-\delta_\gamma^N (F^N - F^n)} \Phi_\gamma^N, \qquad \Phi'^n_\gamma = e^{-\epsilon_\gamma^n F^n} \Phi_\gamma^n. \tag{D.88}$$

The Kähler potential is also gauge invariant; in Yang-Mills superspace [5] the superfields Z^p are defined to be covariantly chiral. In particular, under $U(1)_X$

$$Z^a \to g^{q_a} Z^a, \qquad Z^{\bar{a}} \to g^{-q_a} Z^{\bar{a}}, \qquad \mathcal{A}_M \to \mathcal{A}_M + g^{-1}\mathcal{D}_M g \tag{D.89}$$

with the gauge covariant superspace derivative $\mathcal{D}_M$ given (neglecting other connections) by

$$\mathcal{D}_M Z^p = (D_M + q^p \mathcal{A}_M) Z^p. \tag{D.90}$$

For the PV fields $\Psi^A = U^A, U_A$ we take instead, with (4.6),

$$\Psi^A \to e^{-q^A \Lambda} \Psi^A, \qquad \bar{\Psi}^{\bar{A}} \to e^{-q^A \bar{\Lambda}} \bar{\Psi}^{\bar{A}}. \tag{D.91}$$

It is necessary to introduce the vector superfield V_X in order that the $U(1)_X$-violating terms in the PV superpotential remain holomorphic under a $U(1)_X$ transformation. In addition, in the regulated theory noninvariance under $U(1)_X$ arises from the PV masses; this must cancel the noninvariance of the GS term (4.3) that explicitly involves V_X. Working in "partial" $U(1)_X$ superspace allows us to meet these conditions while keeping the notation from becoming too cumbersome. With this modification the Kähler potential in (D.21) for U is replaced by

$$K(U) = \sum_{\gamma,A} \left(e^{q_\gamma^A V_X + \alpha_\gamma^A G^A} |U_\gamma^A|^2 + e^{p_\gamma^A V_X + \alpha_\gamma^A G^A} |U_A^\gamma|^2 \right). \tag{D.92}$$

In this way we have put all the anomaly associated with the T-moduli and gauge charged chiral fields in the superpotential $W(\Psi)$ in (D.20). We define

$$\dot\omega^n_m = q^N_m - 1, \quad \dot\omega^a_m = q^A_m + q^a_m - 1, \tag{D.93}$$

as the (negative of the) modular weights of $\dot\mu_{n,a}$ in the superpotential (4.57). If, for illustration, we use the PV sector of Appendix D.1 or Appendix D.2, the Ψ have modular weights

$$\begin{aligned} q_m^{\Phi^N_\gamma} &= q^N_m - 2\delta^n_m = \dot\omega^n_m - 2\delta^n_m + 1, \qquad q_m^{\Phi^n_\gamma} = q^N_m = \dot\omega^n_m + 1, && \text{(D.94)}\\ q_m^{U^A_\gamma} &= q^A_m = \dot\omega^a_m - q^a_m + 1, \qquad q_m^{U^\gamma_A} = q^A_m = \dot\omega^a_m - q^a_m + 1, && \text{(D.95)}\\ q_m^{V^A_\gamma} &= q^A_m = \dot\omega^a_m - q^a_m + 1, && \text{(D.96)}\end{aligned}$$

where we have assumed[29] (D.38). Then, if we further assume that $\omega_n^{\Psi_\gamma}$ is independent of γ, using the sum rules (D.22)–(D.28) we have

$$\begin{aligned}\sum_{\Psi,\gamma}\eta^\Psi_\gamma q_m^{\Psi_\gamma} &= \sum_a(\dot\omega^a_m - q^a_m + 1) + 2\sum_n(\dot\omega^n_m - \delta^n_m + 1)\\ &= N + 2 - \sum_\lambda \dot\eta_\lambda\left(2\sum_n \dot\omega^n_m + \sum_a \dot\omega^a_m\right) - \sum_p q^p_m, \end{aligned}\tag{D.97}$$

where p refers to t-moduli and all gauge-charged matter, with, for the untwisted sector charged matter $q^{ai}_m = \delta^i_m$, and for t-moduli $q^i_m = 2\delta^i_m$. In writing the last line of (D.97) we used $\sum_\lambda \dot\eta_\lambda = -1$ and, including the dilaton,

$$N = d_M + 4 = \sum_{\Psi,\gamma}\eta^\Psi_\gamma - 2. \tag{D.98}$$

Then $\tilde m^\Psi_\gamma$ is given by (C.39) with, using also the constraint (D.25) on $(\mathrm{Tr}T_X)_\Psi$,

$$\begin{aligned}\sum_{\Psi,\gamma}\eta^\Psi_\gamma\tilde\varphi^+_{\Psi_\gamma} &= \sum_{\Psi,\gamma}\eta^\Psi_\gamma\left[\sum_m\left(1 - 2q_m^{\Psi_\gamma} + \omega_m^{\Psi_\gamma}\right)F^m - 2q_X^{\Psi_\gamma}\lambda\right]\\ &= \sum_m\left(2\sum_p q^p_m - N - 2 + \sum_{P=\Psi,\dot Y,\dot Z}\eta^P\omega^P_m\right)F^m + 2\mathrm{Tr}T_X\lambda. \end{aligned}\tag{D.99}$$

When combined with other PV loop contributions, we get the general result (C.43). Similarly, there is a contribution proportional to (for $a \neq X$)

$$\sum_C \eta^C\tilde\varphi^+_C(T^a)^2_C + \text{h.c.} = \sum_{\Psi,\gamma}\eta^\Psi_\gamma\tilde\varphi^+_{\Psi_\gamma}C^a_{\Psi_\gamma} + \sum_{\varphi_\gamma}\eta^{\varphi_\gamma}\tilde\varphi^+_{\varphi_\gamma}C_a + \text{h.c.}$$

[29] The assumption does not affect the results below, except for the case with $p = 4$ in (D.145).

$$
\begin{aligned}
= & \sum_n \left[C_a \left(1 + \sum_\gamma \tilde{\omega}_n^{\tilde{\varphi}_\gamma^a} \right) - \sum_b C_{M^b}^a \left(1 - 2q_n^b - \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^{P^B} \omega_n^{P^B} \right) \right] F^n \\
& + 2\mathrm{Tr}(T^a)^2 T_X \lambda + \mathrm{h.c.}, \qquad \text{(D.100)}
\end{aligned}
$$

where we used (D.31), (D.33) and (A.27). This result is again completely general, and agrees with (B.58) when multiplied by $1/64\pi^2$ with $\phi^b = i\left(q_n^b \mathrm{Im} F^n - \frac{1}{2}\mathrm{Im} F + q_X^b \mathrm{Im}\lambda\right)$. The conditions that the anomalies be canceled places constraints on the threshold factors as functions of charges; the coefficients of F^n must be independent of n in (C.43), and independent of n and a in (D.100). To insure this, we set

$$
\tilde{\omega}_n^{\tilde{\varphi}^a} \equiv \sum_\gamma \tilde{\omega}_n^{\tilde{\varphi}_\gamma^a} = \frac{1}{C_a} \left[C_{GS} + \sum_b C_{M^b}^a \left(1 - 2q_n^b - \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^{P^B} \omega_n^{P^B} \right) \right] - 1, \qquad \text{(D.101)}
$$

where $C_{GS} = 8\pi^2 b$ is the Green-Schwarz coefficient, and and we identify the coefficients b_n^a of the threshold corrections as

$$
b_n^a = C_a \tilde{\omega}_n^{\tilde{\varphi}^a} + \sum_b C_{M^b}^a \sum_P \eta^{P^B} \omega_n^{P^B} = C_{GS} - C_a + \sum_b C_{M^b}^a \left(1 - 2q_n^b \right). \qquad \text{(D.102)}
$$

For the anomalous coefficient of $r \cdot \tilde{r}$, the expression in square brackets in (C.43) is also equal to the explicit contribution from just the fields with noninvariant masses:

$$
\begin{aligned}
& -\sum_m \left[2\sum_p q_m^p - N - 2 + N_G + \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^P \omega_m^P + \sum_a \tilde{\omega}_m^{\tilde{\varphi}^a} + \sum_\gamma \hat{\eta}_\gamma \left(1 - 2\hat{\alpha}_\gamma + \hat{\omega}_m^{\hat{\phi}_\gamma} \right) \right] F^m \\
& \quad -2\mathrm{Tr} T_X \lambda = -\sum_C \eta_C \tilde{\varphi}_C^+ = -24 \left(C_{GS} F + C'_{GS} \lambda \right) - 24F, \qquad \text{(D.103)}
\end{aligned}
$$

where $\hat{\phi}_\gamma$ is the subset of neutral chiral multiplets ϕ_γ that has noninvariant masses, and the coefficient C'_{GS} of the $U(1)_X$ GS term is defined in (5.11) and (5.13). In writing the last equality in (D.103) we used string theory results for the anomalous coefficient of Ω_W in the full result (4.33), including the contributions (4.26) and (4.30) that are not included in the PV contribution. String results require that the coefficient of F^m in (D.103) be independent of m.

If we calculate the coefficient of $X_{\mu\nu}\tilde{X}^{\mu\nu}$ following the arguments leading to (C.43), using (3.38) and (A.16) as well as (3.36) and (3.37), we obtain -2 times the RHS of (C.43) with ω_n in the expression in brackets replaced by[30]

$$\sum_C \omega_n^C \eta^C (1-\alpha_C)^2 = \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^P \omega_n^P + \sum_a \tilde{\omega}_n^{\tilde{\varphi}^a} + \sum_\gamma \eta_\gamma^\phi \omega_n^{\phi_\gamma} (1-\alpha_\gamma)^2. \tag{D.104}$$

Alternatively, if we calculate directly using only the noninvariant PV fields, we obtain (-2 times) (D.103) with the last term replaced by

$$\sum_{\gamma,m} \hat{\eta}_\gamma \left[(1-2\hat{\alpha}_\gamma)^3 + (1-2\hat{\alpha}_\gamma)^2 \hat{\omega}_m^{\hat{\phi}_\gamma} \right] F^m. \tag{D.105}$$

Comparing (D.103) with(C.43) and (D.105) with (D.104), consistency requires[31]

$$\begin{aligned} \sum_\gamma \hat{\eta}_\gamma \left(1-2\hat{\alpha}_\gamma + \hat{\omega}_m^{\hat{\phi}_\gamma}\right) &= 5 + \sum_\gamma \eta_\gamma^\phi \omega_m^{\phi_\gamma}, & \text{(D.106)} \\ \sum_\gamma \hat{\eta}_\gamma \left[(1-2\hat{\alpha}_\gamma)^3 + (1-2\hat{\alpha}_\gamma)^2 \hat{\omega}_m^{\hat{\phi}_\gamma} \right] &= 5 + \sum_\gamma \eta_\gamma^\phi (1-2\alpha_\gamma)^2 \omega_m^{\phi_\gamma}, & \text{(D.107)} \end{aligned}$$

in conformity with the sum rules

$$\begin{aligned} \sum \eta_\gamma^\phi &= N' - N'_G + N + 2 - 3N_G = -15 - 2N_G, & \text{(D.108)} \\ \sum \eta_\gamma^\phi \alpha_\gamma &= -10 - N_G, & \text{(D.109)} \\ \sum \eta_\gamma^\phi \alpha_\gamma^2 &= -4 - N_G, & \text{(D.110)} \\ \sum \eta_\gamma^\phi \alpha_\gamma^3 &= -1 - N_G, & \text{(D.111)} \end{aligned}$$

which are obtained by subtracting the contributions of $\theta_\gamma, \dot{Z}, \varphi^a, \hat{\varphi}^a, \tilde{\varphi}^a$ from N' and subtracting the contribution of φ^a, with $\alpha^{\varphi^a} = 1$ from the sum rules for α^n. The constraints (D.108)–(D.111) imply in particular

$$\sum_\gamma \eta_\gamma^\phi (1-2\alpha_\gamma) = \sum_\gamma \eta_\gamma^\phi (1-2\alpha_\gamma)^3 = 5, \tag{D.112}$$

and we obtain for the coefficient of $X^\alpha X_\alpha$ in Φ_L [see (E.9)]

$$\sum_m \left[2\sum_p q_m^p - N + 3 + N_G + \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^P \omega_m^P + \sum_a \tilde{\omega}_m^{\tilde{\varphi}^a} + \sum_\gamma \hat{\eta}_\gamma (1-2\hat{\alpha}_\gamma)^2 \hat{\omega}_m^{\hat{\phi}_\gamma} \right] F^m + 2\mathrm{Tr} T_X \lambda$$

[30]The superpotential mass term μ^φ is constant with $\omega^\varphi = 0$ for $\varphi^a, \hat{\varphi}^a$ as required by modular covariance for this term.

[31]The constraints (3.36) and (A.20) assure that $\omega^{\tilde{Z},\hat{Y}}$ drops out of these equations.

$$
\begin{aligned}
&= \sum_C \eta_C \tilde{\varphi}_C^+ (1-2\alpha_C)^2 \\
&= 24\left(C_{GS}F + C'_{GS}\lambda\right) + 24F + \sum_{\gamma,m} \hat{\eta}_\gamma \left[(1-2\hat{\alpha}_\gamma)^2 - 1\right] \hat{\omega}_m^{\hat{\phi}_\gamma} F^m. \qquad \text{(D.113)}
\end{aligned}
$$

The last term in(D.113) vanishes in the absence of threshold corrections, or more generally if we impose

$$
\sum_\gamma \hat{\eta}_\gamma \left(1-2\hat{\alpha}_\gamma\right)^2 \hat{\omega}_m^{\hat{\phi}_\gamma} = \sum_\gamma \hat{\eta}_\gamma \hat{\omega}_m^{\hat{\phi}_\gamma}. \qquad \text{(D.114)}
$$

If we define $\phi_\gamma = \hat{\phi}_\gamma, \tilde{\phi}_\gamma$, where $\tilde{\phi}_\gamma$ has a modular invariant mass term: $1 - 2\tilde{\alpha}_\gamma + \tilde{\omega}_m^{\tilde{\phi}_\gamma} = 0$, so that

$$
\tilde{\omega}_m^{\tilde{\phi}_\gamma} \equiv \tilde{\omega}_\gamma^{\tilde{\phi}} \qquad \text{(D.115)}
$$

is independent of m, it is straightforward to show that (D.112) is equivalent to (D.106)–(D.107). A third constraint reads

$$
\sum_\gamma \eta_\gamma^\phi \left(1-2\alpha_\gamma\right)^2 = \sum_\gamma \hat{\eta}_\gamma \left(1-2\hat{\alpha}_\gamma\right)^2 + \sum_\gamma \tilde{\eta}_\gamma \left(\tilde{\omega}_\gamma^{\tilde{\phi}}\right)^2 = -2N_G + 9, \qquad \text{(D.116)}
$$

A simple solution to these constraints, that satisfies (D.114) is to set $\tilde{\omega}_\gamma^{\tilde{\phi}} = (1-2\tilde{\alpha}_\gamma) = 0$ with

$$
\tilde{\eta}_\gamma = -1, \qquad \sum_\gamma \tilde{\eta}_\gamma = -24, \qquad \tilde{\alpha}_\gamma = \frac{1}{2}, \qquad \text{(D.117)}
$$

and to take $\hat{\phi}_\gamma = (\hat{\phi}^0, \hat{\phi}^\pm)$ with

$$
\hat{\alpha}_\gamma^0 = 0, \qquad \sum_\gamma \hat{\eta}_\gamma^0 = 5, \qquad 1 - 2\hat{\alpha}_\gamma^\pm = \pm 1, \qquad \sum_\gamma \hat{\eta}_\gamma^\pm = -N_G + 2. \qquad \text{(D.118)}
$$

In this case the coefficient of F^m in (D.103) is independent of m provided $\hat{\omega}_n^{\hat{\phi}_\gamma}$ satisfies

$$
\hat{\omega}_n \equiv \sum_\gamma \hat{\omega}_n^{\hat{\phi}_\gamma} \;=\; 24C_{GS} - \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^P \omega_n^P - \sum_a \tilde{\omega}_n^{\tilde{\varphi}^a} + N - N_G + 21 - 2\sum_p q_n^p, \qquad \text{(D.119)}
$$

where

$$
\omega_n^p = \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^{Pp} \omega_n^{Pp} = \sum_\lambda \eta_\lambda^{\Psi^p} \omega_n^{\Psi^p} - 2\dot{\omega}_n^p, \qquad \omega_n = \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^P \omega_n^P = \sum_\lambda \eta_\lambda^\Psi \omega_n^\Psi - 2\sum_p \dot{\omega}_n^p. \qquad \text{(D.120)}
$$

We identify the corresponding threshold corrections as

$$b_n^r = \sum_P \omega_n^P + \sum_a \tilde{\omega}_n^{\tilde{\varphi}^a} + \hat{\omega}_n = 24C_{GS} - N_G + N + 21 - 2\sum_p q_n^p. \tag{D.121}$$

The "F-term operator Ω_{X^m} and the "D-term" operator Ω_m also contain terms linear in the PV sector parameters; the linear terms in (E.5)–(E.14) have a single factor

$$\ln \mathcal{M} \quad = \quad \frac{1}{2}\left(K - 2\ln f + \ln|\mu^2|\right). \tag{D.122}$$

The sum rule (D.116) assures consistency between the coefficient of derivatives of K as calculated by summing over all PV states as in (C.43) and (D.103), and by summing over only states with noninvariant masses as in (D.104) and (D.105). However in the former case, when using the sum rules (3.37) and (3.38), we have to subtract a spurious contribution[32]

$$\sum_a \left[(1-2\alpha^{\varphi^a})^2 + (1-2\alpha^{\hat{\varphi}^a})^2\right] = 2N_G, \qquad \alpha^{\varphi^a} = 1, \qquad \alpha^{\hat{\varphi}^a} = 0, \tag{D.123}$$

giving

$$\begin{aligned}
&\sum_n \left(N + N_G - 7 - \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^P \omega_n^P - \sum_a \tilde{\omega}_n^{\tilde{\varphi}^a} - \sum_\gamma \eta_\gamma^\phi \omega_n^{\phi_\gamma}(1-2\alpha_\gamma) - 2\sum_p q_n^p\right) F^n - 2\mathrm{Tr}T_X\lambda \\
= \quad & -\sum_m \left[2\sum_p q_m^p - N - 2 + N_G + \sum_{P=\Psi,\dot{Y},\dot{Z}} \eta^P \omega_m^P + \sum_a \tilde{\omega}_m^{\tilde{\varphi}^a}\right. \\
&\left. + \sum_\gamma \hat{\eta}_\gamma (1-2\hat{\alpha}_\gamma)\left(1 - 2\hat{\alpha}_\gamma + \hat{\omega}_m^{\hat{\phi}_\gamma}\right)\right] F^m - 2\mathrm{Tr}T_X\lambda \\
= \quad & -\sum_C \eta_C \tilde{\varphi}_C^+ (1-2\alpha_C),
\end{aligned} \tag{D.124}$$

which is consistent with (D.116).

The chiral projection Φ_L of Ω_L is given by

$$\Phi_L = (\bar{\mathcal{D}}^2 - 8R)\Omega_L = 16\,(X^\alpha - 2f^\alpha)(X_\alpha - 2f_\alpha)\,. \tag{D.125}$$

Using the above results we obtain

$$\frac{1}{192}\mathrm{Tr}\left(\eta\tilde{\Phi}^+\Phi_L\right) = \frac{1}{12}\sum_C \eta_C \Phi_C^+ (1-2\alpha_C)^2 X^\alpha X_\alpha + \frac{1}{3}\sum_{\Psi,\gamma} \eta_\gamma^\Psi \Phi_{\Psi_\gamma}^+ f_{\Psi_\gamma}^\alpha \left(f_\alpha^{\Psi_\gamma} - X_\alpha\right) \tag{D.126}$$

[32] The actual contribution from these fields is $2\sum_a(1-\alpha^{\varphi^a} - \alpha^{\hat{\varphi}^a})^2 = 0$.

where Φ^+_C is a chiral superfield with $\Phi^+_C\big| = \tilde{\varphi}^+_C$,

$$f^n_\alpha = \sum_m (1+\dot{\omega}^n_m) g^m_\alpha - \ell^n_\alpha, \qquad f^P_\alpha = \sum_n (1+\dot{\omega}^p_n - q^p_n) g^n_\alpha - q^p_X W^X_\alpha - \ell^P_\alpha \tag{D.127}$$

$$\Phi^+_n = -\sum_m (1-\omega^n_m) F^m, \qquad \Phi^+_p = -\sum_n (1-\omega^p_n - 2q^p_n) F^n + 2q^p_X \Lambda_X. \tag{D.128}$$

The term in brackets can be evaluated using the conditions (D.93)–(D.96), together with the relation

$$C_{GS} = \sum_p (\omega^p_l + 2q^p_l - 1)(q^p_X)^2 \qquad \forall \quad l, \tag{D.129}$$

which, like (D.102), is known to be satisfied in orbifold compactifications of the heterotic string. If we take, again for illustration,

$$\ell^C(Z,\bar{Z}) = 0, \qquad C = n, P, \tag{D.130}$$

when (D.126) is combined with the other terms in Ω'_L, defined in (5.5), we obtain the result given in (5.9) with

$$\begin{aligned} A_{mnl} &= \sum_k (\omega^k_l - 1)(1+\dot{\omega}^k_n)(1+\dot{\omega}^k_m) \\ &\quad + \sum_p (\omega^p_l + 2q^p_l - 1)(1+\dot{\omega}^p_n - q^p_n)(1+\dot{\omega}^p_m - q^p_m) \end{aligned} \tag{D.131}$$

$$A_{nl} = \sum_k (\omega^k_l - 1)(1+\dot{\omega}^k_n) + \sum_p (\omega^p_l + 2q^p_l - 1)(1+\dot{\omega}^p_n - q^p_n) \tag{D.132}$$

$$A_l = \sum_p (1-\omega^p_l - 2q^p_l) q^p_X \tag{D.133}$$

$$B_{nl} = \sum_p (1-\omega^p_l - 2q^p_l)(1+\dot{\omega}^p_n - q^p_n) q^p_X \tag{D.134}$$

$$a_{mn} = 2\sum_p q^p_X (1+\dot{\omega}^p_n - q^p_n)(1+\dot{\omega}^p_m - q^p_m) \tag{D.135}$$

$$a_n = 2\sum_p q^p_X (1+\dot{\omega}^p_n - q^p_n) \tag{D.136}$$

$$a = -2\sum_p (q^p_X)^2 = -2C^M_X, \tag{D.137}$$

$$b_n = -2\sum_p (q^p_X)^2 (1+\dot{\omega}^p_n - q^p_n). \tag{D.138}$$

The anomalous part of the Lagrangian also contains terms quartic in the PV parameters; these are not constrained by the requirement of finiteness alone. As an example we can assume the simple solutions defined by (D.38), (D.39), and (D.117)–(D.118). If we further assume $\hat{\omega}_n^{\hat{\phi}_\gamma^{0,\pm}} = \hat{\omega}_n^{0,\pm}$ is independent of γ, The real superfields $\mathcal{M}_{\hat{\phi}_C}$ take the form

$$\mathcal{M}_\gamma^{0,\pm} = e^{K/2}\mu_\gamma^{0,\pm}\prod_n |\eta(T^n)|^{2\hat{\omega}_n^{0,\pm}}. \tag{D.139}$$

The constant parameters $\ln\mu$ drop out in the variation of the action and in derivatives of $\ln\mathcal{M}$, and we obtain

$$\begin{aligned}\sum_\gamma \hat{\eta}_\gamma \left(\mathcal{O}\ln\mathcal{M}_{\hat{\phi}_\gamma}\right)^p &\to \frac{5}{2^p}\left(\mathcal{O}K + 4\sum_n \hat{\omega}_n^0 \mathcal{O}|\eta(T^n)|\right)^p \\ &+ \frac{2-N_G}{2^p}\left[\left(\mathcal{O}K + 4\sum_n \hat{\omega}_n^+ \mathcal{O}|\eta(T^n)|\right)^p + \left(4\sum_n \hat{\omega}_n^- \mathcal{O}|\eta(T^n)| - \mathcal{O}K\right)^p\right], \end{aligned} \tag{D.140}$$

$$\sum_a \eta_{\tilde{\varphi}^a}(\mathcal{O}\ln\mathcal{M}_{\tilde{\varphi}^a})^p \to \frac{N_G}{2^p}\left(\mathcal{O}K + 4\sum_n \tilde{\omega}_n^{\tilde{\varphi}^a}\mathcal{O}|\eta(T^n)|\right)^p \tag{D.141}$$

where $\mathcal{O}$ stands for any operator, and

$$\hat{\omega}_n = 5\hat{\omega}^0 + (2-N_G)\left(\hat{\omega}^+ + \hat{\omega}^-\right), \qquad \hat{b}_n^\phi = (N_G - 2)\hat{\omega}^-, \tag{D.142}$$

with $\hat{\omega}^n$ given by (D.119).

If in addition to (D.39)–(D.43) we assume

$$\begin{aligned}(\mathrm{Tr}T_bT_cT_d)_{M^a} &= -\sum_\gamma \eta^{U_\gamma^A}\left[(\mathrm{Tr}T_bT_cT_d)_{U_\gamma^A} + (\mathrm{Tr}T_bT_cT_d)_{U_A^\gamma}\right] \\ &\quad + \eta^{V_\gamma^A}(\mathrm{Tr}T_bT_cT_d)_{V_\gamma^A}, \end{aligned} \tag{D.143}$$

$$(\mathrm{Tr}T_a^2T_X^2)_M = \sum_{A,\gamma}\eta^{U^A\gamma}\left[(\mathrm{Tr}T_a^2T_X^2)_{U_\gamma^A} + (\mathrm{Tr}T_a^2T_X^2)_{U_A^\gamma}\right], \tag{D.144}$$

we obtain

$$\sum_P \eta_P^\Psi(\mathcal{O}\ln\mathcal{M}_{\Psi^P})^p \to \sum_{n=1}^3(\mathcal{O}\ln\mathcal{M}_n)^p + \sum_{q=1}^{N-1}(\mathcal{O}\ln\mathcal{M}_q)^p, \tag{D.145}$$

with

$$\ln\mathcal{M}_n = K/2 - g - \sum_m \dot{\omega}_m^n g^m, \qquad \ln\mathcal{M}_p = K/2 - g + \sum_n (q_n^p - \dot{\omega}_n^p)g^n + q_X^p V_X, \tag{D.146}$$

where $q = m = 1,2,3; a$ with $q_n^m = 2\delta_n^m$, $q_X^m = 0$ for the moduli, and a denotes the gauge-charged matter fields.

As mentioned above, the terms quadratic and higher in weights are modified if use instead the PV sector of Appendix D.3. They are also modified if $\ell^p \neq 0$; a natural choice might be

$$\ell^P = K - g. \tag{D.147}$$

D.5 Orbifold models

Here we list the quantum numbers needed to evaluate the anomaly coefficients for three orbifold compactification models for which all the modular weights and gauge charges are known. These are Z_7 and Z_3 models with no string threshold corrections: $b_n^{a,r} = \omega_n^C = 0$, so the conditions (D.102) and (D.121) reduce to (5.12). One can check that these are satisfied[33] from the tables given below, where we group the chiral multiplets according to their representation (rep) under the semi-simple part of the gauge group, and their modular weights q_n, and give the multiplicity (n_s) of each set with fixed quantum numbers, as well as the total multiplicity n for each set: $N = \sum n$. For the two Z_3 models the superfields $T^{i\bar{j}}$ are the Kähler moduli of which only the diagonal elements $T^i = T^{i\bar{\imath}}$ are assumed to have nonvanishing vacuum values. In all the tables S is the dilaton superfield, and the indices $i, j = 1, 2, 3$. The first two models were studied at the string level in [14]. They have no Wilson lines and no anomalous $U(1)$ so they have

$$8\pi^2 b = C_{GS} = 30, \qquad \delta_X = C'_{GS} = 0, \tag{D.148}$$

and all chiral multiplets are E_8 singlets. For the third (FIQS) model,[34] which is a Z_3 orbifold model with Wilson lines, we also list the $U(1)_X$ charges q_X.
$\mathbb{Z}_\mathbf{3}$: The gauge group is $E_8 \otimes E_6 \otimes SU(3)$ with $N_G = 334$.

name	rep	q_n	n_s	$n/3$
U^i	(27,3)	δ_n^i	1	81
T	(27,1)	$\frac{2}{3}$	27	243
Y^i	(1,$\bar{3}$)	$\frac{2}{3} + \delta_n^i$	27	81
$T^{i\bar{j}}$	(1,1)	$\delta_n^i + \delta_n^j$	1	3
S	(1,1)	0	1	$\frac{1}{3}$

[33] They are also satisfied for $T_a \in U(1)_a$ for the nonanomalous $U(1)$ factors whose charges are not listed. The fields in a given set are not degenerate under under these $U(1)$'s.

[34] See the last reference in [30].

From the table we have

$$N = 1225, \qquad \sum_p q_n^p = 816, \tag{D.149}$$

and

$$\sum_p q_n^p q_m^p = 542 + 168\delta_{mn}, \tag{D.150}$$

$$\sum_p q_n^p q_m^p q_l^p = 396 + 56\left(\delta_{mn} + \delta_{nl} + \delta_{lm}\right) + 168\delta_{mn}\delta_{nl}. \tag{D.151}$$

$\mathbb{Z}_7$: The gauge group is $E_8 \otimes E_6 \otimes U(1)^2$ with $N_G = 328$.

name	rep	q_n	n_s	$n/3$
U_1^i	27	δ_n^i	1	27
U_2^i	1	δ_n^i	1	1
t^i	27	q_n^i	7	189
Y_1^i	1	$q_n^i + \delta_n^i$	7	7
Y_2^i	1	$q_n^i + 2\delta_n^i$	7	7
Y_3^i	1	$q_n^i + 4\delta_n^i$	7	7
Y_4^i	1	$q_n^i + \delta_n^{M^i}$	7	7
Y_5^i	1	$q_n^i + \delta_n^{m^i}$	7	7
Y_6^i	1	$q_n^i + 2\delta_n^{M^i}$	7	7
Y_7^i	1	$q_n^i + 2\delta_n^i + \delta_n^{M^i}$	7	7
Y_8^i	1	$q_n^i + \delta_n^i - \delta_n^{m^i}$	7	7
T^i	1	$2\delta_n^i$	1	1
S	1	0	1	$\frac{1}{3}$

where[35]

$$q_n^1 = \left(\frac{6}{7}, \frac{5}{7}, \frac{3}{7}\right), \qquad q_n^2 = \left(\frac{3}{7}, \frac{6}{7}, \frac{5}{7}\right), \qquad q_n^3 = \left(\frac{5}{7}, \frac{3}{7}, \frac{6}{7}\right), \tag{D.152}$$

and

$$M^i = i + 1 \mod 3, \qquad m^i = i - 1 \mod 3. \tag{D.153}$$

We have

$$N = 823, \qquad \sum_p q_n^p = 618, \tag{D.154}$$

[35] These are respectively ρ^1, ρ^4, ρ^2 of [14].

and

$$\sum_p q_n^p q_m^p = 438 + 342\delta_{mn}, \tag{D.155}$$

$$\sum_p q_n^p q_m^p q_l^p = 291 + 1425\delta_{mn}\delta_{nl} + 279\Delta^+_{nml} + 191\Delta^-_{nml}, \tag{D.156}$$

where

$$\begin{aligned} \Delta^+_{ijk} &= \delta_{ij}\left(\delta_i^1\delta_k^2 + \delta_i^2\delta_k^3 + \delta_i^3\delta_k^1\right) + \text{cyclic}(ijk), \\ \Delta^-_{ijk} &= \delta_{ij}\left(\delta_i^2\delta_k^1 + \delta_i^3\delta_k^2 + \delta_i^1\delta_k^3\right) + \text{cyclic}(ijk). \end{aligned} \tag{D.157}$$

The FIQS model: The gauge group is $SO(10)\otimes SU(3)\otimes SU(2)\otimes[U(1)]^7\otimes U(1)_X$ and $N_G = 64$.

name	rep	$\sqrt{6}q_X$	q_n	n_s	$n/3$
Q^i	(1,3,2)	0	δ_n^i	1	6
u^i	(1,$\bar{3}$,1)	0	δ_n^i	1	3
L^i	(1,1,2)	0	δ_n^i	1	2
Φ	(16,1,1)	$\frac{3}{2}$	δ_n^i	1	16
D	(1,3,1)	$\frac{2}{3}$	$\frac{2}{3}$	12	12
d	(1,$\bar{3}$,1)	$\frac{2}{3}$	$\frac{2}{3}$	15	15
L_1	(1,1,2)	$\frac{2}{3}$	$\frac{2}{3}$	33	22
L_2	(1,1,2)	$-\frac{4}{3}$	$\frac{2}{3}$	3	2
T_1	(1,1,1)	$\frac{2}{3}$	$\frac{2}{3}$	114	38
T_2	(1,1,1)	$-\frac{4}{3}$	$\frac{2}{3}$	30	10
Y^i	(1,1,1)	$\frac{2}{3}$	$\frac{2}{3}+\delta_n^i$	9	9
$T^{i\bar{j}}$	(1,1,1)	0	$\delta_n^i+\delta_n^j$	1	3
S	(1,1,1)	0	0	1	$\frac{1}{3}$

We have

$$N = 415, \qquad \sum_p q_n^p = 258, \tag{D.158}$$

and

$$\begin{aligned} \sum_p q_n^p q_m^p &= 158 + 42\delta_{mn}, \qquad \sum_p (q_X^p)^2 = 50, \qquad \sum_p q_n^p q_X^p = 21\sqrt{6}, \\ \sum_p q_n^p q_m^p q_X^p &= \sqrt{6}\left(12 + 5\delta_{mn}\right), \qquad \sum_p (q_X^p)^2 q_n^p = 28, \end{aligned} \tag{D.159}$$

$$\sum_p q_n^p q_m^p q_l^p \quad = \quad 108 + 8\left(\delta_{mn} + \delta_{nl} + \delta_{lm}\right) + 42\delta_{mn}\delta_{nl}. \tag{D.160}$$

E Construction of the GS Term

In this appendix we detail the steps in the construction of the GS term. As in Appendix C, the notation $\Phi|$ will denote the bosonic part of the lowest component ϕ of the superfield Φ.

E.1 Anomaly superfields

Part of the anomaly can be expressed in term of supersymmetric field operators of the form[36]

$$\begin{aligned} L(T,T',H) &= \frac{1}{2}\int d^4\theta \frac{E}{R} T^\alpha T'_\alpha H(Z) + \text{h.c.} \\ &= \frac{\sqrt{g}}{2} H(z)\, \mathcal{D}_\alpha T'_\beta \mathcal{D}^\alpha T^\beta \Big| + \text{h.c.} + \text{fermions} \\ &= \sqrt{g}\left(2\text{Re}HT_0T'_0 + \text{Re}HT'_{\mu\nu}T^{\mu\nu} + \text{Im}H\tilde{T}'_{\mu\nu}T^{\mu\nu}\right) + \text{fermions}, \end{aligned} \tag{E.1}$$

where $H(Z)$ is a holomorphic function of the chiral fields, with T_α defined by (2.12) and [3]:

$$\begin{aligned} T_0 &= \frac{1}{2}\,\mathcal{D}^\alpha T_\alpha| = -D_{\bar{m}}T_i\left(e^{-K}\bar{A}^i A^{\bar{m}} + \mathcal{D}_\mu z^i \mathcal{D}^\mu \bar{z}^{\bar{m}}\right) + x^{-1}\mathcal{D}_a T_i (T^a z)^i, \\ T_{\mu\nu} &= \left[\left(\mathcal{D}_\mu z^i \mathcal{D}_\nu \bar{z}^{\bar{m}} - \mathcal{D}_\nu z^i \mathcal{D}_\mu \bar{z}^{\bar{m}}\right) D_{\bar{m}} - iF^a_{\mu\nu}(T_a z)^i\right] T_i. \end{aligned} \tag{E.2}$$

In particular we have contributions with $T_\alpha = X_\alpha$ and X^m_α; X^m_α is defined in (4.39). In addition we have the supersymmetric operators

$$\begin{aligned} L(\text{YM},H) &= \frac{1}{2}\int d^4\theta \frac{E}{R} H(Z) W^\alpha_a W^a_\alpha + \text{h.c.} \\ &= \frac{\sqrt{g}}{2}\, H(Z)\mathcal{D}_\alpha W^a_\beta \mathcal{D}^\alpha W^\beta_a \Big| + \text{h.c.} + \text{fermions} \\ &= -\sqrt{g}\left[\text{Re}H\left(F^{\mu\nu}_a F^a_{\mu\nu} - \frac{2}{x^2}\mathcal{D}_a\mathcal{D}^a\right) + \text{Im}HF_a\cdot\tilde{F}^a\right] + \text{fermions}. \end{aligned} \tag{E.3}$$

and $(X_{\mu\nu} = K_{\mu\nu})$

$$L(W,H) \quad = \quad \frac{1}{2}\int d^4\theta \frac{E}{R} H(Z) W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \text{h.c.}$$

[36]The sign on the RHS of (A12) in I is incorrect. As a consequece the signs of the L_α terms should be flipped in (A7) and in the expressions for $\tilde{L}_0$ and $\tilde{L}_G$ in (2.11). The coefficient of $\mathcal{L}_\alpha$ in (2.11) should be $-1/18$, and the coefficient of $X^\alpha X_\alpha$ in (A8) should be $+1/6$.

$$
\begin{aligned}
&= \frac{\sqrt{g}}{2}\, H(Z)\mathcal{D}_\alpha W_{\beta\gamma\delta}\mathcal{D}^\alpha W^{\beta\gamma\delta}\Big| + \text{h.c.} + \text{fermions} \\
&= \frac{\sqrt{g}}{8}\left[\mathrm{Re}H\left(r^{\mu\nu\rho\sigma}r_{\mu\nu\rho\sigma} - 2r_{\mu\nu}r^{\mu\nu} + \frac{1}{3}r^2\right) + \mathrm{Im}H r\cdot\tilde{r}\right] \\
&\quad + \frac{\sqrt{g}}{12}\left(\mathrm{Re}H X_{\mu\nu}X^{\mu\nu} + \mathrm{Im}H \widetilde{X}_{\mu\nu}X^{\mu\nu}\right) + \text{fermions}. \qquad \text{(E.4)}
\end{aligned}
$$

For the full cancellation of the anomalies we will also need the operators introduced in Section 4.1, equations (4.40) and (4.41). The "D-term" combination Ω_m defined in (C.72) can be written

$$
\begin{aligned}
\Omega_m &= -\left(\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}^2\ln\mathcal{M} + [\mathcal{D}^2,\mathcal{D}_{\dot\beta}]\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M} - \bar{\mathcal{D}}^2\ln\mathcal{M}\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}_\alpha\ln\mathcal{M} + \text{h.c.}\right) \\
&\quad + \frac{3}{2}\mathcal{D}^2\ln\mathcal{M}\bar{\mathcal{D}}^2\ln\mathcal{M} - \frac{1}{2}\{\mathcal{D}_\alpha,\mathcal{D}_{\dot\beta}\}\ln\mathcal{M}\{\mathcal{D}^\alpha,\mathcal{D}^{\dot\beta}\}\ln\mathcal{M} + 2\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M}[\mathcal{D}_\alpha,\mathcal{D}_{\dot\beta}]\ln\mathcal{M} \\
&\quad + \left(4R\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}_\alpha\ln\mathcal{M} - 2R\mathcal{D}^2\ln\mathcal{M} - 4\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}_\alpha R + \mathcal{D}^2 R + \text{h.c.}\right) \\
&\quad - 8R\bar{R} + 2G_{\alpha\dot\beta}\left(2\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M} - [\mathcal{D}^\alpha,\mathcal{D}^{\dot\beta}]\ln\mathcal{M} + G^{\alpha\dot\beta}\right) \\
&= -\Omega_L - 2\Omega_{LX} - 8R\bar{R} + \mathcal{D}^2R + \bar{\mathcal{D}}^2\bar{R} + 2G_{\alpha\dot\beta}G^{\alpha\dot\beta} + \ln\mathcal{M}\left(\frac{1}{2}\mathcal{D}^\alpha L_\alpha + 2\ln\mathcal{M}\mathcal{D}^\alpha X_\alpha\right) \\
&\quad - \frac{\partial}{\partial\ln\mathcal{M}}\left[\frac{1}{4}\left(\mathcal{D}^2\ln\mathcal{M}\mathcal{D}_{\dot\beta}\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M} + \text{h.c.}\right) - 2G_{\alpha\dot\beta}\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M}\right. \\
&\qquad + \left(\ln\mathcal{M}\left\{\frac{1}{8}\bar{\mathcal{D}}^2\mathcal{D}^2\ln\mathcal{M} + \mathcal{D}^\alpha(R\mathcal{D}_\alpha\ln\mathcal{M})\right\} + \text{h.c.}\right) \\
&\qquad \left. + \frac{1}{2}\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M}\right] \\
&\equiv -\Omega_L - 2\Omega_{LX} - 8R\bar{R} + \mathcal{D}^2R + \bar{\mathcal{D}}^2\bar{R} + 2G_{\alpha\dot\beta}G^{\alpha\dot\beta} - \frac{\partial}{\partial\ln\mathcal{M}}f(\ln\mathcal{M}). \qquad \text{(E.5)}
\end{aligned}
$$

The result (E.5) is determined only up to a linear supermultiplet L_0

$$
(\bar{\mathcal{D}}^2 - 8R)L_0 = (\mathcal{D}^2 - 8\bar{R})L_0 = 0. \qquad \text{(E.6)}
$$

which does not contribute to the variation of the action, and the argument of the $\ln\mathcal{M}$ derivative is determined up to a total spinorial derivative. The real superfields

$$
\begin{aligned}
\Omega_L &= L^\alpha\mathcal{D}_\alpha\ln\mathcal{M} + \mathcal{D}_{\dot\beta}(\ln\mathcal{M}L^{\dot\beta}) = L_{\dot\beta}\mathcal{D}^{\dot\beta}\ln\mathcal{M} + \mathcal{D}^\alpha(\ln\mathcal{M}L_\alpha), \qquad \text{(E.7)} \\
\Omega_{LX} &= X^\alpha\mathcal{D}_\alpha\ln\mathcal{M} + \mathcal{D}_{\dot\beta}(\ln\mathcal{M}X^{\dot\beta}) = X_{\dot\beta}\mathcal{D}^{\dot\beta}\ln\mathcal{M} + \mathcal{D}^\alpha(\ln\mathcal{M}X_\alpha) \qquad \text{(E.8)}
\end{aligned}
$$

are the Chern-Simons superfields [40], respectively, for the chiral superfields

$$
\Phi_L = L^\alpha L_\alpha, \qquad \Phi_{LX} = X^\alpha L_\alpha, \qquad L_\alpha = (\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha\ln\mathcal{M}, \qquad \text{(E.9)}
$$

as discussed in the next subsection. If we define Ω'_m by

$$\frac{1}{48}\Omega_m + \frac{1}{36}\Omega_{X_m} = \frac{1}{48}\Omega'_m + \frac{1}{36}\Omega_X, \qquad \Omega'_m = \Omega_m + \frac{3}{4}\Omega_L + 2\Omega_{LX}, \tag{E.10}$$

the mixed term Ω_{LX} drops out and we obtain

$$\begin{aligned}\Omega'_m \;=\;& -\frac{1}{4}\Omega_L - 8R\bar{R} + 2G_{\alpha\dot\beta}G^{\alpha\dot\beta} + \mathcal{D}^2 R + \bar{\mathcal{D}}^2\bar{R} \\ & - \frac{\partial}{\partial \ln\mathcal{M}}\left[\frac{1}{4}\left(\mathcal{D}^2\ln\mathcal{M}\mathcal{D}_{\dot\beta}\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M} + \text{h.c.}\right) - 2G_{\alpha\dot\beta}\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M}\right. \\ & \quad + \left(\ln\mathcal{M}\left\{\frac{1}{8}\bar{\mathcal{D}}^2\mathcal{D}^2\ln\mathcal{M} + \mathcal{D}^\alpha(R\mathcal{D}_\alpha\ln\mathcal{M})\right\} + \text{h.c.}\right) \\ & \quad + \frac{1}{2}\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M} \\ & \quad \left. - (\ln\mathcal{M})^2\left(\frac{1}{4}\mathcal{D}^\alpha L_\alpha + \ln\mathcal{M}\mathcal{D}^\alpha X_\alpha\right)\right].\end{aligned} \tag{E.11}$$

The first five terms on the right hand side of (E.11) are invariant under modular and $U(1)_X$ transformations and have the correct chiral and Weyl weights for some part of them be included in the chiral projection of the modified linear superfield L; for example the second and third terms complete the PV contribution to the anomalous coefficent of Ω_{GB}. The remainder may require additional counterterms for anomaly cancellation. Defining Ω_r by

$$\Omega'_m = -\frac{1}{4}\Omega_L - 8R\bar{R} + 2G_{\alpha\dot\beta}G^{\alpha\dot\beta} + \Omega_r + \mathcal{D}^2R + \bar{\mathcal{D}}^2\bar{R}, \tag{E.12}$$

it is the coefficient of

$$\delta\ln\mathcal{M}^2 = 2\delta\ln\mathcal{M} = 2\left(\delta H + \delta\bar{H}\right), \qquad \mathcal{D}_{\dot\beta}H = \mathcal{D}_\alpha\bar{H} = 0, \tag{E.13}$$

under an infinitesimal modular and/or $U(1)_X$ transformation on the corresponding part

$$\mathcal{L}_r = -\frac{1}{8\pi^2}\frac{1}{48}\ell_r = -\frac{1}{8\pi^2}\frac{1}{48}\int d^4\theta E\,\text{Tr}\eta\int 2d\ln\mathcal{M}\Omega_r(\ln\mathcal{M}) \tag{E.14}$$

of the Lagrangian. Under a finite transformation

$$\Delta\ln\mathcal{M} = H + \bar{H}, \tag{E.15}$$

we have[37]

$$\Delta\ell_r \;=\; 2\int d^4\theta E\,\text{Tr}\eta\left\{\left(\frac{1}{2}\mathcal{D}^\alpha L_\alpha + 2\mathcal{D}^\alpha X_\alpha\right)\left[H\bar{H} + 2\left(H + \bar{H}\right)\ln\mathcal{M}\right]\right.$$

[37] The integrated anomaly for pure supergravity was given in ref. [41].

$$
\begin{aligned}
&-\frac{1}{4}\left(\mathcal{D}^2 H\mathcal{D}_{\dot\beta}\bar H\mathcal{D}^{\dot\beta}\bar H + 2\mathcal{D}^2 H\mathcal{D}_{\dot\beta}\bar H\mathcal{D}^{\dot\beta}\ln\mathcal{M} + \mathcal{D}^2\ln\mathcal{M}\mathcal{D}_{\dot\beta}\ln\bar H\mathcal{D}^{\dot\beta}\bar H + \text{h.c.}\right)\\
&-\frac{1}{4}\left(\mathcal{D}^2 H\mathcal{D}_{\dot\beta}\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M} + 2\mathcal{D}^2\ln\mathcal{M}\mathcal{D}_{\dot\beta}\bar H\mathcal{D}^{\dot\beta}\ln\mathcal{M} + \text{h.c.}\right)\\
&+G^{\alpha\dot\beta}\left(\mathcal{D}_\alpha H\mathcal{D}_{\dot\beta}\bar H + \mathcal{D}_\alpha H\mathcal{D}_{\dot\beta}\ln\mathcal{M} + \mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\bar H\right)\\
&-\left(\{H+\ln\mathcal{M}\}\left\{\frac{1}{8}D^2\bar{\mathcal{D}}^2\bar H + \mathcal{D}^\alpha(R\mathcal{D}_\alpha H)\right\} + \text{h.c.}\right)\\
&-\left(H\left\{\frac{1}{8}\mathcal{D}^2\bar{\mathcal{D}}^2\ln\mathcal{M} + \mathcal{D}^\alpha(R\mathcal{D}_\alpha\ln\mathcal{M})\right\} + \text{h.c.}\right)\\
&-\frac{1}{2}\mathcal{D}^\alpha H\mathcal{D}_\alpha H\mathcal{D}_{\dot\beta}\bar H\mathcal{D}^{\dot\beta}\bar H - \mathcal{D}^\alpha H\mathcal{D}_\alpha H\mathcal{D}_{\dot\beta}\bar H\mathcal{D}^{\dot\beta}\ln\mathcal{M}\\
&-\mathcal{D}^\alpha H\mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\bar H\mathcal{D}^{\dot\beta}\bar H - 2\mathcal{D}^\alpha H\mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\bar H\mathcal{D}^{\dot\beta}\ln\mathcal{M}\\
&-\frac{1}{2}\mathcal{D}^\alpha H\mathcal{D}_\alpha H\mathcal{D}_{\dot\beta}\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M} - \frac{1}{2}\mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\bar H\mathcal{D}^{\dot\beta}\bar H\\
&-\mathcal{D}^\alpha H\mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\ln\mathcal{M}\mathcal{D}^{\dot\beta}\ln\mathcal{M} - \mathcal{D}^\alpha\ln\mathcal{M}\mathcal{D}_\alpha\ln\mathcal{M}\mathcal{D}_{\dot\beta}\bar H\mathcal{D}^{\dot\beta}\ln\mathcal{M}\Big\}. \qquad \text{(E.16)}
\end{aligned}
$$

Note that since $\mathcal{D}^\alpha L_\alpha, \mathcal{D}^\alpha X_\alpha$ are linear supermultiplets subject to the condition (E.6), it is obvious that the constant $\ln\mu$ in the expression (D.122) for $\ln\mathcal{M}$ drops out of (E.16).

E.2 Chern-Simons superfields

We wish to impose the modified linearity condition

$$(\bar{\mathcal{D}}^2 - 8R)(L+\Omega) = (\mathcal{D}^2 - 8\bar R)(L+\Omega) = 0, \qquad \text{(E.17)}$$

where Ω is a superfield with $U(1)_K$ chiral weight $w_\chi(\Omega) = 0$ and Weyl weight $w_W(\Omega) = 2$ such that

$$\mathcal{L}(\Omega) = \frac{\sqrt{g}}{32\pi^2}\int d^4\theta E\Omega, \qquad \Omega = e^{K/3}\Omega_0, \qquad \text{(E.18)}$$

is invariant under superweyl transformations. For example, the standard case has $\Omega = \Omega_{YM}$. In this case the Lagrangian (E.18) reads in component form

$$\mathcal{L}(\Omega_{YM}) = -\frac{\sqrt{g}}{32\pi^2}F^{\mu\nu}_a F^a_{\mu\nu} + \cdots = -\frac{\sqrt{g_0}}{32\pi^2}F^{\mu\nu}_a F^a_{\mu\nu} + \cdots, \qquad \text{(E.19)}$$

where we dropped fermions and auxiliary fields; these have nontrivial transformations under the superweyl transformation [5]. Girardi and Grimm [22] constructed an explicit expression for the

Yang-Mills CS superfield in terms of prepotential superfields $\varphi_\alpha, \bar{\varphi}^{\dot{\beta}}$, defined as the (ordinary) spinorial derivatives of a hermetian superfield U (called Υ in the notation of [22]) such that the (covariant) chiral(anti-chiral) projection of $\varphi(\bar{\varphi})$ is the chiral(anti-chiral) Yang-Mills superfield strength:

$$\varphi_\alpha = -U^{-1}D_\alpha U, \quad \bar{\varphi}^{\dot{\beta}} = \mathcal{D}^{\dot{\beta}}UU^{-1}, \quad (\bar{\mathcal{D}}^2 - 8R)\varphi_\alpha = -8W_\alpha, \quad (\mathcal{D}^2 - 8\bar{R})\bar{\varphi}^{\dot{\beta}} = 8W^{\dot{\beta}}. \tag{E.20}$$

The CS superfield is constructed from these prepotentials and their covariant spinorial derivatives:

$$\begin{aligned}
\Omega_{YM} &= -\frac{1}{8}\mathrm{Tr}\left(\varphi^\alpha W_\alpha\right) - \frac{i}{32}\mathcal{D}^{\dot{\beta}}\left(\chi_{\dot{\beta}} + \sigma_{\dot{\beta}}\right), \\
\chi_{\dot{\beta}} &= \frac{i}{2}\mathrm{Tr}\left[Y_{\dot{\beta}}\mathcal{D}^\alpha Y_\alpha + 2Y^\alpha \mathcal{D}_\alpha \bar{\varphi}_{\dot{\beta}}\right], \\
\sigma_{\dot{\beta}} &= \mathrm{Tr}\left(\int_0^1 dtY_t\left[Y^\alpha, Y_{\alpha\dot{\beta}}\right] + \frac{i}{2}\int_0^1 dtY_t\left[\mathcal{D}^\alpha Y_\alpha, Y_{\dot{\beta}}\right]\right), \\
Y_A &= \mathcal{D}_A UU^{-1}, \quad Y_t = \partial_t U(t)U^{-1}(t),
\end{aligned} \tag{E.21}$$

and the interpolating superfield $U(t)$ is defined to satisfy:

$$U(0) = 1, \quad U(1) = U. \tag{E.22}$$

The above construction is quite general. In particular we can take U to be a real function $U(Z, \bar{Z}, V_X)$ of the chiral superfields and the $U(1)_X$ vector superfield. The above construction with

$$U \to U_K = e^{-K/8} \tag{E.23}$$

is the CS form Ω_X for $X^\alpha X_\alpha$. Similarly,

$$U \to U_m = \mathcal{M} \tag{E.24}$$

gives the CS form Ω_L for $L^\alpha L_\alpha$. For these Abelian cases, the integration is trivial, giving, up to a linear superfield, the expression (E.7) for Ω_L, and an analogous expression with $\ln\mathcal{M} \to -\frac{1}{8}K$ for Ω_X.

The CS forms Ω_X and Ω_L are explicitly dependent on the dilaton through the presence of the Kähler potential K and therefore do not satisfy the condition (4.15). In the remainder of this section we show that the linear/chiral multiplet duality outlined in (4.8)–(4.18) of Section 4.1 still holds.

The action is

$$\mathcal{S} = -\int E\left[3 - 2Ls(L) + (S + \bar{S})(L + \Omega)\right], \qquad K = k(L) + G(T + \bar{T}, \Phi), \tag{E.25}$$

with

$$\Omega = e^{K/3}\Omega_0, \qquad S = (\bar{\mathcal{D}}^2 - 8R)\Sigma, \tag{E.26}$$

and L, Σ unconstrained. To assure a canonical Einstein term in the chiral formulation we require

$$2Ls(L) - L(S + \bar{S}) = 0. \tag{E.27}$$

First consider the standard case:

$$\frac{\partial}{\partial L}\Omega_0 = \frac{\partial}{\partial S}\Omega_0 = 0. \tag{E.28}$$

The equation of motion for Σ gives

$$\frac{1}{E}\frac{\delta \mathcal{S}}{\delta \Sigma} = -(\bar{\mathcal{D}}^2 - 8R)(L + \Omega) = 0, \tag{E.29}$$

which is just the modified linearity condition for L and which implies

$$\int ES(L + \Omega) + \text{h.c.} = 0, \tag{E.30}$$

so the action (E.25) reduces to

$$\mathcal{S} = -\int E\left[3 - 2Ls(L)\right]. \tag{E.31}$$

On the other hand the equation of motion for L is

$$\begin{aligned}\frac{1}{E}\frac{\delta \mathcal{S}}{\delta L} &= 2s(L) + 2Ls'(L) - S - \bar{S} + \frac{\partial K}{\partial L} \\ &\quad -\frac{1}{3}\frac{\partial K}{\partial L}\left[2Ls(L) - (S + \bar{S})L\right] \\ &= 2Ls'(L) + K'(L) = 0,\end{aligned} \tag{E.32}$$

where the last line, which assures a canonical Einstein term in the linear formulation, was obtained using (E.27). Integrating this using (E.27) gives

$$S + \bar{S} = 2s(L) = -\int \frac{dL}{L}\frac{\partial K}{\partial L}. \tag{E.33}$$

So for example if $K = \ln L$, $S + \bar{S} = 1/L$. Using (E.27), the action expressed in terms of S reduces to

$$\mathcal{S} = -\int E\left[3 + (S + \bar{S})\Omega\right]. \tag{E.34}$$

To cancel the anomaly we can add a constant of integration $V = V(Z),\ Z \neq S, L$ to the right hand side of (E.33):

$$S + \bar{S} = 2s(L) = -\int \frac{dL}{L}\frac{\partial K}{\partial L} + V. \tag{E.35}$$

Or, equivalently, we can add a term

$$-\int ELV \tag{E.36}$$

to the action (E.25).

Now consider instead the case $\Omega_0 = \Omega_0(\sigma),\ \sigma = S + \bar{S}$. Although the superfield σ is no longer simply a Lagrange multiplier, it is still a nonpropagating field that can be removed by its equation of motion. In this case Ω still drops out of the equation of motion for L, leaving (E.27) and (E.32)–(E.34) unchanged. However the equation of motion for Σ now reads

$$\frac{1}{E}\frac{\delta \mathcal{S}}{\delta \Sigma} = -(\bar{\mathcal{D}}^2 - 8R)\left[L + \Omega + (S + \bar{S})\frac{\partial}{\partial S}\Omega\right] = 0, \tag{E.37}$$

which gives a different modified linearity condition for L and the action (E.25) gets an extra term

$$\mathcal{S} = -\int E\left[3 - 2Ls(L) + (S + \bar{S})\left(S\frac{\partial}{\partial S}\Omega + \text{h.c.}\right)\right]. \tag{E.38}$$

When we calculate the component Lagrangian, after inserting the operator $(\bar{\mathcal{D}}^2 - 8R)$ and integrating by parts and neglecting the Einstein term $-3\int E$, we obtain

$$\begin{aligned}\mathcal{S} \ &\ni\ -\frac{1}{8}\int \frac{E}{R}\left[s(L)(\bar{\mathcal{D}}^2 - 8R)L - S(S + \bar{S})(\bar{\mathcal{D}}^2 - 8R)\frac{\partial}{\partial S}\Omega\right] + \text{h.c.} \\ &=\ \frac{1}{8}\int \frac{E}{R}\left[s(L)(\bar{\mathcal{D}}^2 - 8R)\Omega + \{s(L) - S\}(S + \bar{S})(\bar{\mathcal{D}}^2 - 8R)\frac{\partial}{\partial \sigma}\Omega\right] + \text{h.c.}.\end{aligned} \tag{E.39}$$

Using (E.33) the last term cancels out and the first term is just the counterpart of the second term in (E.34) in the linear formulation.

Finally if we take $\Omega_0 = \Omega_0(L)$, the equation of motion for Σ just gives (E.29)–(E.31). but using (E.27) the equation of motion for L gives

$$\frac{1}{E}\frac{\delta \mathcal{S}}{\delta L} = 2Ls'(L) + k'(L) - 2s(L)e^{K/3}\frac{\partial \Omega_0}{\partial L} = 0, \tag{E.40}$$

so now we get a different differential equation for $s(L)$ but we still get the same action.

F Notations and conventions

In this Appendix we summarize our notation and conventions.

F.1 Sign conventions

Our Dirac matrices and space-time metric signature $(+---)$ are those of Bjorken and Drell or Itzykson and Zuber. We use upper case notation (R, Γ) for derivatives of the Käher metric $K_{i\bar{m}}$, and lower case (r, γ) for derivatives of the space-time metric $g_{\mu\nu}$. Our sign conventions for, respectively, the Riemann tensor, Ricci tensor, and curvature scalar are

$$r^{\mu}_{\nu\rho\sigma} = g^{\mu\lambda} r_{\lambda\nu\rho\sigma} = \partial_\sigma \Gamma^{\mu}_{\nu\rho} - \partial_\rho \Gamma^{\mu}_{\nu\sigma} + \Gamma^{\mu}_{\sigma\tau}\Gamma^{\tau}_{\nu\rho} - \Gamma^{\mu}_{\rho\tau}\Gamma^{\tau}_{\nu\sigma}, \qquad r_{\mu\nu} = r^{\rho}_{\mu\rho\nu}, \qquad r = g^{\mu\nu} r_{\mu\nu}. \tag{F.1}$$

The general coordinate covariant derivative is defined by

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \gamma^{\rho}_{\mu\nu} A_\rho, \qquad \text{etc.} \tag{F.2}$$

We use identical definitions for the Kähler curvature and scalar field reparameterization derivatives with

$$g_{\mu\nu} \to K_{i\bar{m}} = K_{\bar{m}i}, \qquad \gamma \to \Gamma, \qquad r \to R, \qquad \nabla_\mu \to D_i, D_{\bar{m}}. \tag{F.3}$$

We use the Yang-Mills sign conventions of [4] and [42]:

$$\mathcal{D}_\mu \phi^b = \nabla_\mu \phi^b + i A^a_\mu (T_a)^b_c \phi^c, \tag{F.4}$$

$$\mathcal{D}_\mu \bar{\phi}^{\bar{b}} = \nabla_\mu \bar{\phi}^{\bar{b}} - i A^a_\mu (T_a)^{\bar{b}}_{\bar{c}} \bar{\phi}^{\bar{c}}, \qquad (T_a)^{\bar{b}}_{\bar{c}} = (T^*_a)^b_c = (T_a)^c_b. \tag{F.5}$$

Note that $\mathcal{D}_\mu$ is used for general coordinate and Yang-Mills covariant derivatives, while D_μ is fully covariant. Thus for a fermion ψ, $D_\mu\psi$ includes the spin, Yang-Mills, Kähler and reparameterization (as well as the affine connection for the graviton ψ_μ), and for a function of scalar fields $z^i, \bar{z}^{\bar{m}}$,

$$D_\mu f(z, \bar{z}) = \mathcal{D}_\mu z^i D_i f(z, \bar{z}) + \mathcal{D}_\mu z^{\bar{m}} D_{\bar{m}} f(z, \bar{z}), \tag{F.6}$$

with repeated indices summed.

F.2 Supergravity conventions

We work in Kähler $U(1)$ superspace [5]. The tree level Lagrangian we are working with is defined by the Kähler potential K, superpotential W and gauge kinetic function f_{ab} given in (2.1). A generic chiral superfield is denoted by Z^i:

$$Z^i = S, T, \Phi = (z^i, \chi^i, F^i), \tag{F.7}$$

with components

$$z^i = Z^i\big|\,, \qquad \chi^i_\alpha = \frac{1}{\sqrt{2}}\,\mathcal{D}_\alpha Z^i\big|\,, \qquad F^i = -\frac{1}{4}\,\mathcal{D}^\alpha\mathcal{D}_\alpha Z^i\big| \equiv -\frac{1}{4}\,\mathcal{D}^2 Z^i\big|\,, \tag{F.8}$$

where $Z|$ denotes the lowest component, and the antichiral superfields are

$$\bar{Z}^{\bar{m}} = (Z^m)^\dagger = (\bar{z}^{\bar{m}}, \bar{\chi}^{\bar{m}}, \bar{F}^{\bar{m}}). \tag{F.9}$$

We also use the notation

$$Z^p = T^i, \Phi^a = (z^p, \chi^p, F^p), \qquad i = 1,2,3, \tag{F.10}$$

for the moduli

$$T^i = (t^i, \chi^{t^i}, F^{t^i}) \tag{F.11}$$

and the gauge charged chiral multiplets

$$\Phi^a = (\phi^a, \chi^a, F^a). \tag{F.12}$$

The dilaton chiral supermultiplet decomposes as

$$S = (s, \chi^s, F^s), \qquad s = x + iy. \tag{F.13}$$

The Yang-Mills superfield strengths are

$$W^a_\alpha = (\lambda^a_\alpha, F^a_{\mu\nu}, D^a), \tag{F.14}$$

and the supergravity supermultiplet includes the vierbein e^α_μ, the gravitino ψ_μ and the auxiliary fields $G_{\alpha\dot{\beta}}, R, \bar{R}$. Solution of the tree level equations of motion determine the auxiliary fields as

$$\begin{aligned} F^i\big| &= -e^{-K/2}A^i = -e^{K/2}K^{i\bar{m}}\left(\bar{W}_{\bar{m}} + K_{\bar{m}}\bar{W}\right), \qquad D^a| = \frac{1}{x}\mathcal{D}^a = \frac{1}{x}K_i(T^a z)^i, && \text{(F.15)}\\ R| &= \frac{1}{2}M = e^{-K/2}A = e^{K/2}W(z) = \left(\bar{R}|\right)^\dagger, \qquad G_{\alpha\dot{\beta}}\Big| = 0, && \text{(F.16)} \end{aligned}$$

where here (and throughout the text) the notation $\mathcal{O}(\Psi)|$ denotes the bosonic component of the functional $\mathcal{O}$ of the superfields Ψ, that is, the $\theta = \bar{\theta} = 0$ component with all fermions set to zero. Our definition of M is such that the vacuum value $\langle M\rangle$ is the gravitino mass; it differs by a factor $-1/3$ from that of [5]. The covariant derivatives

$$A_{i_1\cdots i_n} = D_{i_1}\cdots D_{i_n}A \tag{F.17}$$

of the modular covariant operator

$$A = e^K W, \qquad A \to e^{\bar{F}} A, \qquad A_i \to e^{\bar{F}} A_i, \qquad \text{etc.} \tag{F.18}$$

are defined in (2.5). Note that for the Kähler potential and superpotential W we use the conventional notation for ordinary derivatives

$$K_{i_1 \cdots i_n} = \partial_{i_1} \cdots \partial_{i_n} K, \qquad W_{i_1 \cdots i_n} = \partial_{i_1} \cdots \partial_{i_n} W. \tag{F.19}$$

F.3 PV superfields

To regulate the superpotential couplings we introduce modular covariant PV fields

$$\dot{Z}^P = \dot{T}^I, \dot{\Phi}^A \tag{F.20}$$

that transform like dZ^p under gauge and modular transformations. To make the Kähler potential for these fields modular invariant and their superpotential modular covariant, we need to introduce three additional fields $\dot{Z}^N,\ N = 1,2,3$, and we also use the notation

$$\dot{Z}^\rho = \dot{Z}^N, \dot{Z}^P. \tag{F.21}$$

These fields acquire mass through invariant couplings to the PV superfields

$$\dot{Y}_\rho = \dot{Y}_N, \dot{Y}_P, \tag{F.22}$$

where $\dot{Y}_P$ transforms like $K_{p\bar{q}} d\bar{Z}^{\bar{q}}$ under YM transformations. These have no other superpotential couplings; their divergent contributions to field strength terms are canceled by the PV fields, introduced in Appendix D.1,

$$\Psi^C = U^A, U_A, V^A, \Phi^N, \Phi^n, \qquad N, n = 1,2,3, \tag{F.23}$$

where $\Phi^{N,n}$ are gauge singlets, and U^A, U_A, V^A form a real (reducible) representation of the gauge group such that (3.25) is satisfied. Their masses not invariant under $U(1)_X$ and modular transformations; they reflect chiral matter contributions to the anomaly.

To regulate all the nonrenormalizable terms in the superpotential and YM couplings, we also need fields $\widetilde{Z}^P$; they acquire invariant masses through coupling to $\widetilde{Y}_P$. These transform, respectively, like dZ^p and $K_{p\bar{q}} d\bar{Z}^{\bar{q}}$ under YM and modular transformations.

To regulate the renormalizable Yang-Mills couplings we introduce chiral PV fields $\widehat{Y}_P$ that transform like $K_{p\bar{q}}d\bar{Z}^{\bar{q}}$ under YM and modular transformations, and acquire invariant masses through coupling to chiral superfields $\widehat{Z}^P$ that transform like $\dot{Z}^P$. We also need chiral superfields $\varphi^a, \hat{\varphi}^a, \tilde{\varphi}^a$ in the adjoint representation of the YM gauge group. The fields $\varphi^a, \hat{\varphi}^a$, regulate gauge couplings to matter and to the gravity sector, respectively, and couple to one another in an invariant mass term; $\tilde{\varphi}^a$ has no couplings to light matter and its noninvariant mass term reflects the gaugino/gauge contribution to the modular anomaly.
To regulate nonrenormalizable dilaton/YM couplings we introduce chiral superfields ϕ^S, that acquire invariant masses through superpotential couplings to chiral fields ϕ_S, and chiral superfields θ_s and Abelian vector superfields V_s that acquire invariant masses through a superhiggs mechanism. Finally, to regulated additional nonrenormalizable gravity couplings we need chiral superfields ϕ^C, with noninvariant masses that reflect the gravity sector contribution to the modular anomaly, and chiral superfields θ_0 and Abelian vector superfields V_0 that acquire invariant masses through a superhiggs mechanism.
Unless otherwise specified, Φ^C denotes any chiral PV superfield with PV signature

$$\eta^C = \pm 1. \tag{F.24}$$

F.4 The covariant derivative expansion

Here we collect the operators [17] that appear in the covariant derivative expansion used in the evaluation of the variation of the action. If $F(x)$ is a scalar field operator, we define

$$\hat{F} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(D \cdot \frac{\partial}{\partial p} \right)^n F(x), \qquad D \cdot \frac{\partial}{\partial p} F(x) \equiv [D_\mu, F(x)] \frac{\partial}{\partial p_\mu}. \tag{F.25}$$

In addition we define

$$\mathcal{G}_\mu = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!} \left(-iD \cdot \frac{\partial}{\partial p} \right)^n \mathcal{G}_{\nu\mu} \frac{\partial}{\partial p_\nu}, \qquad \mathcal{G}_{\nu\mu} = [D_\mu, D_\nu]. \tag{F.26}$$

The full formal expansions of the following space-time curvature-dependent operators

$$T^{\mu\nu} = g^{\mu\nu} - \frac{1}{3} r^{\mu\rho\sigma\nu} \frac{\partial^2}{\partial p^\rho \partial p^\sigma} + \frac{i}{6} \nabla^\tau r^{\mu\rho\sigma\nu} \frac{\partial^3}{\partial p^\rho \partial p^\sigma \partial p^\tau} + O\left(\frac{\partial^4}{\partial p^4} \right), \tag{F.27}$$

$$P^{\mu\nu}\gamma_\nu = P^\mu = \gamma^\mu - \frac{1}{6} r^{\mu\rho\sigma\nu} \gamma_\nu \frac{\partial^2}{\partial p^\rho \partial p^\sigma} + O\left(\frac{\partial^3}{\partial p^3} \right), \tag{F.28}$$

$$\delta_\mu = \frac{i}{9}\left(\nabla_\mu r_{\rho\nu} - \nabla_\nu r_{\rho\mu}\right)\frac{\partial^2}{\partial p_\rho \partial p_\nu} + O\left(\frac{\partial^3}{\partial p^3}\right), \tag{F.29}$$

$$X = -\frac{r}{3} + \frac{i}{3}\nabla_\mu r \frac{\partial}{\partial p_\mu} + O\left(\frac{\partial^2}{\partial p^2}\right), \tag{F.30}$$

are given[38] in Appendix A of [35].

References

[1] M.K. Gaillard, Phys.Lett. B **342**, 125 (1995) and B **347**, 284 (1995).

[2] M.K. Gaillard, Phys. Rev. D **58**, 105027 (1998).

[3] M.K. Gaillard, Phys. Rev. D **61**, 084028 (2000).

[4] E. Cremmer, S. Ferrara, L. Girardello, and A. Van Proeyen, Phys. Lett. B **116**, 231 (1982) and Nucl. Phys. B **212**, 413 (1983); J. Bagger Nucl. Phys. **B212**, 211 (1983).

[5] P. Binétruy, G. Girardi, R. Grimm and M. Muller, Phys. Lett. B **189**, 83 (1987); P. Binétruy, G. Girardi and R. Grimm, Physics Reports, **343**, 255 (2001).

[6] A. Giveon, N. Malkin and E. Rabinovici, Phys. Lett. B **220**, 551 (1989); E. Alvarez and M. Osorio, Phys. Rev. D **40**, 1150 (1989).

[7] M. B. Green and J. H. Schwarz, Phys. Lett. B **149**, 117 (1984).

[8] I. Antoniadis, K.S. Narain and T.R. Taylor, Phys Lett. B **267**, 37 (1991).

[9] L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B **355**, 649 (1991).

[10] G.L. Cardoso and B.A. Ovrut, Nucl. Phys. B **369**, 351 (1992).

[11] J.-P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, Phys. Lett. B **271**, 307 (1991).

[12] M.K. Gaillard and T.R. Taylor, Nucl. Phys. B **381**, 577 (1992).

[13] P. Binétruy, G. Girardi, R. Grimm and M. Müller, Phys. Lett. B **265** 111 (1991); P. Adamietz, P. Binétruy, G. Girardi and R. Grimm, Nucl. Phys. B **401**, 257 (1993).

[38] There is a sign error in the third and second terms, respectively, of the expressions for $T_{\mu\nu}$ and X in Eq. (A.19) of [17].

[14] C. A. Scrucca and M. Serone, JHEP **0102**, 019 (2001).

[15] D. Butter and M. K. Gaillard, Phys. Lett. B **679**, 519 (2009).

[16] M.K. Gaillard and V. Jain, Phys. Rev. D **49**, 1951 (1994).

[17] M.K. Gaillard, V. Jain and K. Saririan, Phys. Rev. D **55**, 833 (1997).

[18] M.K. Gaillard, V. Jain and K. Saririan, Phys. Lett. B **387**, 520 (1996).

[19] D.Z. Freedman and B. Kors, JHEP **0611**, 067 (2006).

[20] I. Antoniadis, E. Gava and K. S. Narain, Phys. Lett. B **283**, 209 (1992) and Nucl. Phys. B **383**, 93 (1992).

[21] G.L. Cardoso and B.A. Ovrut, Nucl. Phys. **392**, 315 (1993), ibid. B **418**, 535 (1993).

[22] G. Girardi and R. Grimm, Annals Phys. **272**, 49 (1999).

[23] P. Binetruy, G. Dvali, R. Kallosh and A. Van Proeyen, Class. Quant. Grav. **21** 3137 (2004).

[24] H. Elvang, D. Z. Freedman and B. Kors, JHEP **0611**, 068 (2006).

[25] W. A. Bardeen, Phys. Rev. **184**, 1848 (1969).

[26] J. Wess and B. Zumino Phys. Lett. B **37**, 95 (1971).

[27] W. A. Bardeen and B. Zumino, Nucl. Phys. B **244**, 421 (1984).

[28] M.A. Shifman, Nucl. Phys. B **352**, 87 (1991); M.A. Shifman, and A.I. Vainshtein Nucl. Phys. B **365**, 312 (1991); A.A. Iogansen, Sov. J. Nucl. Phys. **54** (1991).

[29] D. Butter, arXiv:0911.5426 [hep-th].

[30] L.E. Ibàñez, H.-P. Nilles and F. Quevedo, *Phys. Lett.* **187**, (1987) 25; A. Font, L. Ibàñez, D. Lust and F. Quevedo, *Phys. Lett.* B **245**, (1990) 401; A. Font, L. E. Ibàñez, F. Quevedo and A. Sierra, Nucl. Phys. B **331**, (1990) 421.

[31] Joel Giedt, Annals Phys. **289**, 251 (2001) and **297**, 67 (2002).

[32] M. Dine, N. Seiberg, E. Witten, Nucl. Phys. **B289**, 585 (1987); J. J. Atick, L. Dixon, A. Sen, Nucl. Phys. B **292**, 109 (1987); M. Dine, I. Ichinose, N. Seiberg, Nucl. Phys. B **293**, 253 (1987).

[33] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, Nucl.Phys. B **483**, 141 (1997); H.P. Nilles, S. Stieberger, Nucl. Phys. B **499**, 3 (1997); S. Stieberger, Nucl. Phys. B **541**, 109 (1999).

[34] We thank Thomasz Taylor for this observation.

[35] P. Binétruy and M.K. Gaillard, Nucl. Phys. **312**, 341 (1989).

[36] S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B **225**, 363 (1989).

[37] E. J. Chun, J. Mas, J. Lauer and H. P. Nilles, Phys. Lett. B **233**, 141 (1989); J. Lauer, J. Mas and H. P. Nilles, Nucl. Phys. B **351**, 353 (1991).

[38] E.S. Fradkin and A.A. Tseytlin, Ann. Phys. **162**, 31 (1985). This paper showed equivalence up to finite topological anomalies; subsequently full equivalence has been shown; S.J. Rey, private communication.

[39] M.A. Shifman and A.I. Vainshtein, Nucl. Phys. B **277**, 456 (1986); V.S. Kaplunovsky and J. Louis, Nucl. Phys. B **444**, 191 (1995).

[40] S. Cecotti, S. Ferrara and M. Villasante, Int. J. Mod. Phys. A **2**, 1839 (1987).

[41] I. L. Buchbinder and S. M. Kuzenko, Phys. Lett. B **202**, 233 (1988).

[42] S. Ferrara and B. Zumino, Nucl. Phys. B **79**, 413 (1974).