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In this paper, we study the quantization of the (1+1)-dimensional projectable Hořava-Lifshitz (HL) gravity, and find that, when only gravity is present, the system can be quantized by following the canonical Dirac quantization, and the corresponding wavefunction is normalizable for some orderings of the operators. The corresponding Hamilton can also be written in terms of a simple harmonic oscillator, whereby the quantization can be carried out quantum mechanically in the standard way. When the HL gravity minimally couples to a scalar field, the momentum constraint is solved explicitly in the case where the fundamental variables are functions of time only. In this case, the coupled system can also be quantized by following the Dirac process, and the corresponding wavefunction is also normalizable for some particular orderings of the operators. The Hamilton can be also written in terms of two interacting harmonic oscillators. But, when the interaction is turned off, one of the harmonic oscillators has positive energy, while the other has negative energy. A remarkable feature is that orderings of the operators from a classical Hamilton to a quantum mechanical one play a fundamental role in order for the Wheeler-DeWitt equation to have nontrivial solutions. In addition, the space-time is well quantized, even when it is classically singular.

I. INTRODUCTION

Quantum field theory (QFT) provides a general framework for all interactions of the nature, but with one exception, gravitation. However, the universal coupling of gravity to all forms of energy makes it plausible that gravity should be also implemented in such a framework. In addition, around the singularities of the Big Bang and black holes, space-time curvatures become so high and it is generally expected that general relativity (GR), as a classical theory, is no longer valid, and Planck physics must be taken into account, whereby these singularities will be smoothed out and a physically meaningful description near these singular points is achieved. Moreover, one it is confirmed [1], the recent detection of the primordial gravitational waves by BICEP2 [2] indicates that in high energy gravity is quantized [3].

Motivated by the above considerations, quantization of gravity has been one of the main driving forces in physics in the past decades [4], and various approaches have been pursued, including string/M-Theory [5] and loop quantum gravity [6]. However, it is fair to say that so far a well-established quantum theory of gravity is still absent, and many questions remain open.

Recently, Hořava [7] proposed a theory of quantum gravity in the framework of QFTs, with the perspective that Lorentz symmetry (LS) appears only as an emergent symmetry at low energies, but can be fundamentally absent at high energies [1]. In Hořava's theory, the LS is bro-

ken via the anisotropic scaling between time and space in the ultraviolet (UV),

$$t \to b^{-z}t, \quad x^i \to b^{-1}x^i, \ (i = 1, 2, ..., d), \quad (1.1)$$

where z denotes the dynamical critical exponent. This is a reminiscent of Lifshitz scalars in condensed matter physics [11], hence the theory is often referred to as the Hořava-Lifshitz (HL) gravity. For the theory to be power-counting renormalizable, the critical exponent z must be $z \geq d$ [7, 12], while the relativistic scaling corresponds to $z = 1$.

With Eq.(1.1) as the guideline, Hořava assumed that the breaking of the LS and thus the 4-dimensional diffeomorphism invariance is only down to the so-called foliation-preserving diffeomorphism,

$$t \to t'(t), \quad x^i \to x'^i \left(t, x^k\right), \quad (1.2)$$

often denoted by Diff(M, $\mathcal{F}$). This gauge symmetry provides a crucial ingredient to the construction of the HL gravity: its action includes only higher-dimensional spatial (but not time) derivative operators, so that the UV behavior is dramatically improved, and in particular can be made power-counting renormalizable. The exclusion of high-dimensional time derivative operators, on the other hand, prevents the ghost instability, so that the long-standing problem of unitarity is resolved [13]. In the infrared (IR) the lower dimensional operators take over, and presumably provide a healthy IR limit.

Applying the HL theory to cosmology, various remarkable features were found [14]. In particular, the higher-order spatial curvature terms can give rise to a bouncing

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[1] Note that the breaking of LS in the matter sector is highly restricted by experiments, but in the gravitational sector the re-

strictions are much weaker [8–10].

universe [15], and may ameliorate the flatness problem [16]. The anisotropic scaling provides a solution to the horizon problem and generation of scale-invariant perturbations either with [17] or without [18] inflation. The scalar perturbations become adiabatic, not because of the conservation of energy as in GR [19], but because of the slow-roll condition [17]. Similar results were also obtained for tensor perturbations of primordial gravitational waves [20], while the vector perturbations are still trivial. The dark sector can have its purely geometric origins [21, 22], and so on.

Despite all these remarkable features, it was soon found that the original version of the HL gravity is plagued with several undesirable issues, including the IR instability [7, 23] and strong coupling [24]. To address these problems, various models were proposed [14]. So far, several models are free of these problems, and are consistent with the solar system tests [25, 26] and cosmological observations [27, 28]: One is the healthy extension of the HL gravity [29], and another is the nonprojectable general covariant theory [30]. The latter has been recently embedded into string theory [31].

In this paper, we study another important issue of the HL gravity - the quantization. It is well-known that normally this becomes extremely complicated and very mathematically involved in (3+1)-dimensions [4]. To bypass these technical issues, in this paper we shall study the quantization of the HL gravity in (1+1)-dimensional (2d) spacetimes, so the problem becomes tractable, and may still be able to shed lights on some basic nature of the quantization of the theory, as various important examples of the (3+1)-dimensional (4d) gravity belong to this class, including spherically symmetric black holes and the FRW universe, not to mention the string inspired models [32], although it is also well-known that GR in 4-dimensions is quite different from that in lower dimensions [33].

Specifically, the rest of the paper is organized as follows: In Sec. II we shall provide a brief review on the 2d HL gravity, from which it can be seen that, unlike the 2d GR, the 2d HL gravity is non-trivial even without coupling to matter. This can be further seen from the non-trivial (classical) vacuum solutions of the theory with the projectability condition, presented in Sec. III, in which the local and global properties of the solutions are also studied. In Sec. IV, the quantization of the 2d HL gravity is carried out explicitly by the canonical Dirac quantization. In addition, we find that the problem can also be reduced to the quantization of a simple harmonic oscillator [34]. In Sec. V, we generalize these studies to the case where the HL gravity is minimally coupled to a scalar field, which shares the same gauge symmetry as the 2d gravitational sector. Unlike the vacuum case, we find that now the momentum constraint cannot be solved explicitly in general. Then, we restrict ourselves only to the case in which the fundamental variables depend only on time. Similar to the vacuum case, now the system can also be quantized by the standard Dirac quantization.

The corresponding Hamilton can be also written in two interacting harmonic oscillators. When the interaction vanishes, one of the two oscillators has positive energy and the other has negative energy. The paper is ended in Sec. VI, in which we derive our main conclusions. A remarkable feature is that orderings of the operators from a classical Hamilton to a quantum mechanical one play a fundamental role in order for the Wheeler-DeWitt equation $\hat{H}|\Psi\rangle = 0$ to have nontrivial solutions. In addition, space-times can be still well quantized, although they are classically singular [cf. Fig.1]. This is true not only for the vacuum case, but also for the case coupled with the scalar field [cf. Fig.4].

Note that the quantization of the 2d HL gravity was studied recently in [34], and showed that it is equivalent to the 2d causal dynamical triangulations (CDT) when the projectability condition is imposed. In addition, the 3d projectable HL gravity was also studied numerically in terms of CDT [35], and found evidence for the consistency of the quantum phases of solutions to the equations of motion of classical HL gravity. Benedetti and Guarnieri, on the other hand, studied one-loop renormalization in a toy model of the HL gravity, that is, the conformal reduction of the $z = 2$ projectable HL theory [36]. They found that the would-be asymptotic freedom associated to the running Newton's constant is exactly balanced by the strong coupling of the scalar mode as the Weyl invariant limit is approached. Then, they concluded that in such model the UV limit is singular at one loop order, and argued that a similar phenomenon should be expected in the full theory, even in higher dimensions. Loop corrections and renormalization group flows were also studied in some particular models of the HL gravity [10, 37], and different conclusions were obtained for different models.

II. HORAVA-LIFSHITZ THEORY OF GRAVITY IN (1+1)-DIMENSIONS

The Einstein's theory of gravity in (1+1)-dimensional spacetimes is trivial, as the Riemann and Ricci tensors $\mathcal{R}_{\mu\nu\beta\gamma}$ and $\mathcal{R}_{\mu\nu}$ are uniquely determined by the Ricci scalar $\mathcal{R}$ via the relations [33],

$$\begin{aligned}
\mathcal{R}_{\mu\nu\beta\gamma} &= \frac{1}{2}\left(g_{\mu\beta}g_{\nu\gamma} - g_{\mu\gamma}g_{\nu\beta}\right)\mathcal{R}, \\
\mathcal{R}_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}\mathcal{R},
\end{aligned} \tag{2.1}$$

where the Greek letters run from 0 to 1. Then, the Einstein tensor $E_{\mu\nu}[=\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}]$ always vanishes, and

the Einstein-Hilbert action [2]

$$S_{EH} = \zeta^2 \int d^2x \sqrt{^{(2)}g}\left(\mathcal{R} - 2\Lambda + \zeta^{-2}\mathcal{L}_M\right), \quad (2.2)$$

leads to a set of non-dynamical field equations, in which the metric $g_{\mu\nu}$ is directly related to the energy-momentum tensor $T_{\mu\nu}$ via the relation,

$$\Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (2.3)$$

where $\zeta^2 = 1/(16\pi G)$ [3]. Therefore, in order to have a non-trivial theory of gravity in 2-dimensions (2d), extra degrees are often introduced, such as a dilaton [32] or a Liouville field [38].

However, this is not the case for the HL gravity [7], as the latter has a different symmetry, the foliation-preserving diffeomorphisms (1.2). Then, the general gravitational action takes the form,

$$S_{HL} = \zeta^2 \int dt dx N \sqrt{g}\left(\mathcal{L}_K - \mathcal{L}_V\right), \quad (2.4)$$

where N denotes the lapse function in the Arnowitt-Deser-Misner (ADM) decompositions [39], and $g \equiv \det(g_{ij})$, here g_{ij} is the spatial metric defined on the leaves $t = $ Constant. $\mathcal{L}_K$ is the kinetic part of the action, given by

$$\mathcal{L}_K = K_{ij}K^{ij} - \lambda K^2, \quad (2.5)$$

where λ is a dimensionless constant, and K_{ij} denotes the extrinsic curvature tensor of the leaves $t = $ Constant, given by

$$K_{ij} = \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right), \quad (2.6)$$

and $K \equiv g^{ij}K_{ij}$. Here $\dot{g}_{ij} \equiv \partial g_{ij}/\partial t$, ∇_i denotes the covariant derivative of the metric g_{ij}, and N^i the shift vector. In the (1+1)-dimensional case, since there is only one spatial dimension, we have $i,j = 1$, and

$$K = g^{11}K_{11} = -\frac{1}{N}\left(\frac{\dot{\gamma}}{\gamma} - \frac{N_1'}{\gamma^2} + \frac{N_1\gamma'}{\gamma^3}\right), \quad (2.7)$$

where $\gamma \equiv \sqrt{g_{11}}$, $\gamma' \equiv \partial\gamma/\partial x$, etc.

On the other hand, $\mathcal{L}_V$ denotes the potential part of the action, and is made of R, ∇_i and a_i, that is,

$$\mathcal{L}_V = \mathcal{L}_V\left(R, \nabla_i, a_i\right), \quad (2.8)$$

where $a_i \equiv N_{,i}/N$ and R denotes the Ricci scalar of the leaves $t = $ Constant, which identically vanishes in one-dimension, i.e., $R = 0$. Power-counting renormalizibility

condition requires that $\mathcal{L}_V$ should contain spatial operators with the highest dimensions that are not less than $2z$, where $z \geq d$ [7, 12], and d denotes the number of the spatial dimensions. Taking the minimal requirement, that is, $z = d$, we find that in the current case ($d = 1$) we have

$$\mathcal{L}_V = 2\Lambda - \beta a_i a^i, \quad (2.9)$$

where Λ denotes the cosmological constant, and β is another dimensionless coupling constant. Collecting all the above together, we find that the gravitational action of the HL gravity in $(1+1)$-dimensional spacetimes can be cast in the form,

$$S_{HL} = \zeta^2 \int dt dx N \sqrt{g}\left[(1-\lambda)K^2 - 2\Lambda + \beta a_i a^i\right]. \quad (2.10)$$

III. CLASSICAL SOLUTIONS OF THE 2D HL GRAVITY WITH THE PROJECTABLE CONDITION

Assuming the projectabilility condition, we have [7]

$$N = N(t), \quad (3.1)$$

from which we immediately find $a_i = 0$. In the rest of this section, we shall assume this condition. Then, the variations of the action S_{HL} with respect to N and N_1 yield the Hamiltonian and momentum constraints, and are given, respectively, by

$$\int dx\gamma(K^2 + 4\tilde{\Lambda}) = 0, \quad (3.2)$$

$$K' = 0, \quad (3.3)$$

where $\tilde{\Lambda} \equiv \Lambda/[2(1-\lambda)]$. The variation of the action S_{HL} with respect to γ, on the other hand, yields the dynamical equation,

$$\dot{K} + \frac{1}{2}N(K^2 - 4\tilde{\Lambda}) + \frac{K\dot{\gamma}}{\gamma} - \frac{2KN_1'}{\gamma^2}$$
$$+ \left(\frac{N_1 K}{\gamma^2}\right)' + \frac{3KN_1\gamma'}{\gamma^3} = 0. \quad (3.4)$$

Using the gauge freedom of Eq.(1.2), without loss of the generality, we can always set

$$N = 1, \quad N_1 = 0, \quad (3.5)$$

so that the 2d metric takes the form,

$$ds^2 = -dt^2 + \gamma^2(t,x)dx^2. \quad (3.6)$$

It should be noted that Eq.(3.5) uniquely fixes the gauge only up to

$$t' = t + t_0, \quad x' = \zeta(x), \quad (3.7)$$

where t_0 is a constant, and $\zeta(x)$ is an arbitrary function of x only.

With the above gauge choice, Eq.(3.4) reduces to

$$K^2 - 2\dot{K} + 4\tilde{\Lambda} = 0. \tag{3.8}$$

On the other hand, from the momentum constraint (3.3) we can see that K is independent of x, so the Hamiltonian constraint Eq.(3.2) reduces to,

$$(K^2 + 4\tilde{\Lambda}) \int dx \gamma(t,x) = 0. \tag{3.9}$$

Therefore, there exist two possibilities,

$$i)\ K^2 + 4\tilde{\Lambda} = 0, \quad ii)\ \int dx \gamma(t,x) = 0. \tag{3.10}$$

In the following, we consider them separately.

A. $K^2 + 4\tilde{\Lambda} = 0$

In this case, the extrinsic curvature K is just a constant given by

$$K = \pm 2\sqrt{-\tilde{\Lambda}}, \tag{3.11}$$

which makes sense only when $\tilde{\Lambda} < 0$. From Eq.(3.15), we can find

$$\gamma = e^{\pm 2\sqrt{-\tilde{\Lambda}}t + F(x)}, \tag{3.12}$$

here $F(x)$ is an arbitrary function of x only. Using the gauge residual (3.7), we can always set $F(x) = 0$, so the metric reduces to,

$$ds^2 = -dt^2 + e^{4\sqrt{-\tilde{\Lambda}}t} dx^2. \tag{3.13}$$

This is nothing but the de Sitter spacetime.

B. $\int dx \gamma(t,x) = 0$

In this case, we can see that $\gamma(t,x)$ has to be an odd function of x, i.e., $\gamma(t,x) = -\gamma(t,-x)$. Then, from Eq.(3.8) we find that

$$\frac{dK}{K^2 + 4\tilde{\Lambda}} = \frac{1}{2}dt. \tag{3.14}$$

Since K is independent of x, we find

$$\frac{\dot{\gamma}}{\gamma} = -K(t). \tag{3.15}$$

To solve the above equations under the constraint $\int dx \gamma(t,x) = 0$, it is found convenient to consider the cases $\tilde{\Lambda} > 0$, $\tilde{\Lambda} < 0$, and $\tilde{\Lambda} = 0$, separately.

1. $\tilde{\Lambda} > 0$

Straightforward integration of Eq. (3.14) gives us

$$K = \beta \tan\left[\frac{\beta}{2}(t - t_0)\right], \tag{3.16}$$

where $\beta \equiv \sqrt{4|\tilde{\Lambda}|}$. Then, from Eq.(3.15) we find,

$$\gamma = \cos^2\left(\frac{\beta(t - t_0)}{2}\right) \hat{\gamma}(x). \tag{3.17}$$

To satisfy the Hamiltonian constraint, $\hat{\gamma}(x)$ must be an odd function of x, so that

$$\int_{-L_\infty}^{L_\infty} \hat{\gamma}(x)dx = 0, \tag{3.18}$$

where $x = \pm L_\infty$ denote the boundaries of the spacetime in the spatial direction, which can be taken to infinity. With this in mind, we can introduce a new coordinate x' by $dx' = \hat{\gamma}(x)dx$, so the metric takes the form,

$$ds^2 = -dt^2 + \cos^4\left(\frac{\beta t}{2}\right) dx'^2. \tag{3.19}$$

Note that in writing the above expression, we had set $t_0 = 0$ by using another gauge freedom given in Eq.(3.7). Setting

$$T = \frac{2}{\beta} \tan\left(\frac{\beta t}{2}\right), \tag{3.20}$$

the above metric can be cast in the conformally-flat form,

$$ds^2 = \left(1 + \frac{\beta^2}{4}T^2\right)^{-2} \left(-dT^2 + dx'^2\right), \tag{3.21}$$

for which we have

$$K = \frac{\beta^2}{2}T. \tag{3.22}$$

That is, the space-time is singular at $T = \pm\infty$. This is a real space-time singularity in the HL gravity [40], since it is a scalar one and cannot be removed by any coordinate transformations allowed by the symmetry of the theory. The corresponding Penrose diagram is given by Fig. 1.

2. $\tilde{\Lambda} < 0$

In this case, Eq.(3.14) has the solution

$$K = \begin{cases} -\beta \tanh\left[\frac{\beta}{2}(t - t_0)\right], & |K| < \beta, \\ -\beta \coth\left[\frac{\beta}{2}(t - t_0)\right], & |K| > \beta. \end{cases} \tag{3.23}$$

In the following, let us consider the two cases separately, as they will have different properties.
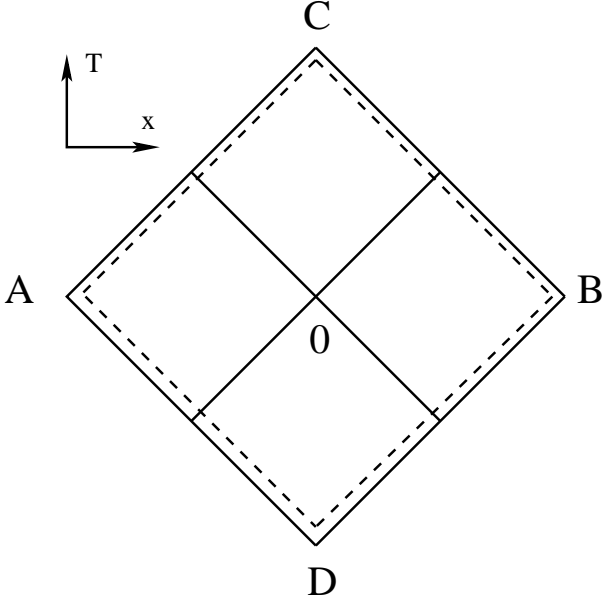
FIG. 1: The Penrose diagram for the solution (3.21), in which the space-time is singular at both past and further null infinities ($T = \pm\infty$), denoted by the lines $\overline{AC}$, $\overline{AD}$, $\overline{BC}$ and $\overline{BD}$.

Case a) $|K| < \beta$: Then, from Eq.(3.15) we find that

$$\gamma = \cosh^2 \left[\frac{\beta}{2}(t - t_0) \right] \hat{\gamma}(x). \qquad (3.24)$$

Again, using the gauge residual (3.7), without loss of the generality, we can always set $t_0 = 0$ and $dx' = \hat{\gamma}(x)dx$, so the metric finally takes the form,

$$ds^2 = -dt^2 + \cosh^4 \left(\frac{\beta t}{2} \right) dx^2. \qquad (3.25)$$

Note that we dropped the prime from x in writing down the above expression. Then, we can see that the metric is singular at $t = \pm\infty$. However, Eq.(3.23) shows that K is finite at these two limit. In addition, the corresponding 2d Ricci scalar $\mathcal{R}$ is given by

$$\mathcal{R} = \beta^2 \frac{\cosh(\beta t)}{\cosh^2 \left(\frac{\beta t}{2} \right)}, \qquad (3.26)$$

which is also finite as $t \to \pm\infty$. To further study the properties of these singularities, let us consider the tidal forces experienced by a free-falling observer, whose trajectory is given by the timelike geodesics, satisfying the Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}_p}{\partial x^\mu} - \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}_p}{\partial \dot{x}^\mu} \right) = 0, \qquad (3.27)$$

where τ denotes the affine parameter along the geodesics, and

$$\mathcal{L}_p \equiv \left(\frac{ds}{d\tau} \right)^2 = -\dot{t}^2 + \cosh^4 \left(\frac{\beta t}{2} \right) \dot{x}^2, \qquad (3.28)$$

but now with $\dot{t} \equiv dt/d\tau$, etc. For timelike geodesics we have $\mathcal{L}_p = -1$. Since the metric (3.25) does not depend on x explicitly, Eq.(3.27) yields the conservation law of momentum,

$$2 \cosh^4 \left(\frac{\beta t}{2} \right) \dot{x} = p, \qquad (3.29)$$

where p denotes the momentum of the observer. Inserting the above expression into Eq.(3.28), we find that

$$\dot{t} = \pm \frac{\sqrt{4 \cosh^4 \left(\frac{\beta t}{2} \right) + p^2}}{2 \cosh^2 \left(\frac{\beta t}{2} \right)}, \qquad (3.30)$$

where "+" ("-") corresponds to the observer moving along the positive (negative) direction of the x-axis. Setting $e_{(0)}^\mu = dx^\mu/d\tau$, we can construct another space-like unit vector, $e_{(1)}^\mu$ as

$$e_{(1)}^\mu = \left(\pm \frac{p}{2 \cosh^2 \left(\frac{\beta t}{2} \right)}, \frac{\sqrt{4 \cosh^4 \left(\frac{\beta t}{2} \right) + p^2}}{2 \cosh^4 \left(\frac{\beta t}{2} \right)} \right), \qquad (3.31)$$

which is orthogonal to $e_{(0)}^\mu$, and parallelly transported along the time-like geodesics,

$$g_{\mu\nu} e_{(a)}^\mu e_{(b)}^\nu = \eta_{ab}, \quad e_{(1);\nu}^\mu e_{(0)}^\nu = 0, \qquad (3.32)$$

where $\eta_{ab} = \text{diag.}(-1, 1)$, and a semicolon ";" denotes the covariant derivative with respect to the 2d metric $g_{\mu\nu}$. Projecting the 2d Ricci tensor onto the above orthogonal frame, we find that

$$R_{(0)(0)} = -R_{(1)(1)} = -\frac{\beta^2 \cosh(\beta t)}{2 \cosh^2 \left(\frac{\beta t}{2} \right)},$$
$$R_{(0)(1)} = 0, \qquad (3.33)$$

which are all finite as $t \to \pm\infty$. Therefore, the singularities at $t = \pm\infty$ must be coordinate ones. In fact, they represent the boundaries of the space-time. To see this, let us consider the proper time that the observer needs to travel from a given time t_0 to $t = \infty$, which is given by

$$\Delta\tau = \int_{t_0}^{\infty} \frac{2 \cosh^2 \left(\frac{\beta t}{2} \right)}{\sqrt{4 \cosh^4 \left(\frac{\beta t}{2} \right) + p^2}} = \infty, \qquad (3.34)$$

for any finite t_0. That is, starting at any given finite moment, t_0, the observer always needs to spend infinite proper time to reach at the time $t = \infty$. In other words, $t = \infty$ indeed represents the future timelike infinity of the space-time. Similarly, one can see that $t = -\infty$ represents the past timelike infinity.

To study its global structure, let us first introduce the new timelike coordinate T via the relation,

$$T = \frac{2}{\beta}\tanh\left(\frac{\beta t}{2}\right), \qquad (3.35)$$

we find that the metric takes the form,

$$ds^2 = \left(1 - \frac{\beta^2}{4}T^2\right)^{-2}(-dT^2 + dx^2), \ (|T| \le 2/\beta). \qquad (3.36)$$

It is interesting to note that the above metric is singular at $T = \pm 2/\beta$. But, as shown above, this corresponds to coordinate singularities. In fact, they are the space-time boundaries, and any observer will need infinite proper time to reach them starting from any finite time. The corresponding Penrose diagram is given by Fig. 2.

Finally, we note that the similarity of the metric (3.25) with the dS_2 metric,

$$ds_{dS_2}^2 = -dt^2 + \cosh^2(\beta t)d\chi^2, \qquad (3.37)$$

where $0 \le \chi \le \pi$ with the hypersurfaces $\chi = 0$ and $\chi = \pi$ identified, so the whole space-time has a $R^1 \times S^1$ topology. The space-time is complete in these coordinates. This can be seen clearly by embedding Eq.(3.37) into a 3-dimensional Minkowski space-time $ds_3^2 = -dv^2 + dw^2 + dX^2$ with [41],

$$v = \frac{1}{\beta}\sinh(\beta t), \quad w = \frac{1}{\beta}\cosh(\beta t)\cos\left(\frac{\chi}{\beta}\right),$$

$$X = \frac{1}{\beta}\cosh(\beta t)\sin\left(\frac{\chi}{\beta}\right), \qquad (3.38)$$

which is a hyperboloid,

$$-v^2 + w^2 + X^2 = \beta^{-2}, \qquad (3.39)$$

in the 3-dimensional Minkowski space-time. The two metrics (3.25) and (3.37) becomes asymptotically identified when $|t| \gg \beta^{-1}$, provided that the coordinate χ is unrolled to $-\infty < \chi < \infty$.

Case b) $|K| > \beta$: In this case, following what was done in the last case, it can be shown that

$$K = -\beta\coth\left(\frac{\beta t}{2}\right), \quad \gamma = \sinh^2\left(\frac{\beta t}{2}\right)\hat{\gamma}(x), \ (3.40)$$

and the corresponding line element takes the form,

$$ds^2 = -dt^2 + \sinh^4\left(\frac{\beta}{2}t\right)dx'^2. \qquad (3.41)$$

Similar to the last case, the metric is singular at $t = \pm\infty$. However, these are coordinate ones, as in the last case. In fact, following what we did there, we find that the following forms a freely-falling frame,

$$e^\mu_{(0)} = \left(\pm\sqrt{1 + \frac{p^2}{4\sinh^4\left(\frac{\beta t}{2}\right)}}, \frac{p}{2\sinh^4\left(\frac{\beta t}{2}\right)}\right),$$
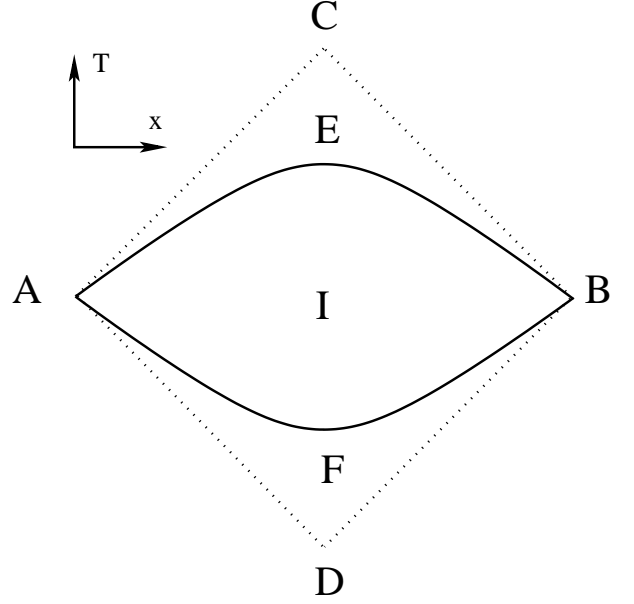

FIG. 2: The Penrose diagram for the solution (3.25), in which the singularities at $t = \pm\infty$, denoted by the curves $\widehat{AEB}$ and $\widehat{AFB}$, are coordinate ones, and represent the physical boundaries of the space-time.

$$e^\mu_{(1)} = \left(\pm\frac{p}{2\sinh^2\left(\frac{\beta t}{2}\right)}, \frac{\sqrt{1 + \frac{p^2}{4\sinh^4\left(\frac{\beta t}{2}\right)}}}{\sinh^2\left(\frac{\beta t}{2}\right)}\right), \ (3.42)$$

for which we have

$$R_{(0)(0)} = -R_{(1)(1)} = -\frac{1}{2}\beta^2\cosh(\beta t)\cosh^{-2}\left(\frac{\beta t}{2}\right),$$

$$R_{(1)(0)} = 0. \qquad (3.43)$$

It is clear that all of these components, representing the tidal forces exerted on the observer, are finite. From Eq.(3.42) one can also show that

$$\Delta\tau = \int_{t_0}^\infty \frac{2\sinh^2\left(\frac{\beta t}{2}\right)}{\sqrt{4\sinh^4\left(\frac{\beta t}{2}\right) + p^2}} = \infty, \qquad (3.44)$$

for any finite t_0. That is, starting at any given finite moment, t_0, the observer will reach $t = \infty$ after spending infinite proper time, i.e., $t = \infty$ represents the space-time boundary. Similarly, one can show that $t = -\infty$ represents the past timelike infinity.

However, in contrast to the last case, the space-time now becomes singular at $t = 0$. This singularity is a scalar singularity, as one can see from Eq.(3.40) and the expression for the 2-dimensional Ricci scalar,

$$\mathcal{R} = \beta^2\left[1 + \coth^2\left(\frac{\beta}{2}t\right)\right]. \qquad (3.45)$$
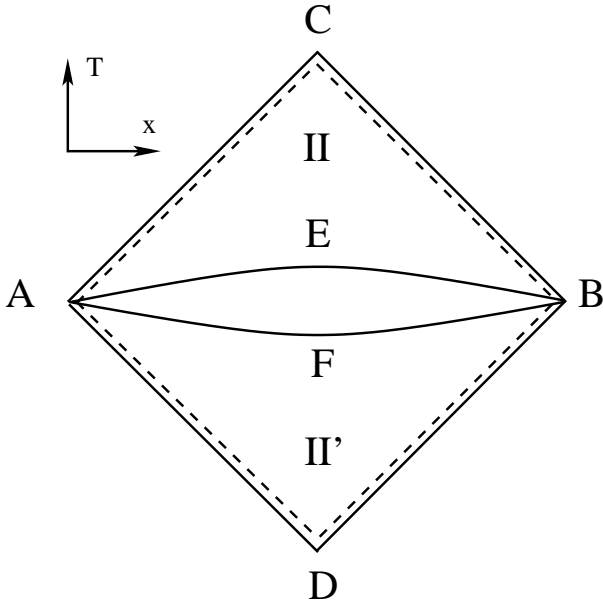
FIG. 3: The Penrose diagram for the solution (3.41), in which the space-time is singular at both past and further null infinities ($T = \pm\infty$ or $t = 0$), denoted by the lines $\overline{AC}$, $\overline{AD}$, $\overline{BC}$ and $\overline{BD}$. The curved lines, $\widehat{AEB}$ and $\widehat{AFB}$, are free of spacetime singularities, and represent the physical boundaries of the space-time.

To study its global properties, we first introduce the new coordinate T via the relation

$$T = -\frac{2}{\beta} \coth\left(\frac{\beta}{2}t\right), \tag{3.46}$$

which maps $t \in (-\infty, 0)$ into the region $T \in (2/\beta, \infty)$, and $t \in (0, \infty)$ into the region $T \in (-\infty, -2/\beta)$. In particular, the time $t = 0^{\pm}$ are mapped to $T = \mp\infty$, and $t = \pm\infty$ to $T = \mp 2/\beta$. In terms of T, we find that

$$ds^2 = \left[1 - \frac{\beta^2}{4}T^2\right]^{-2} \left(-dT^2 + dx'^2\right), \ (|T| \geq 2/\beta). \tag{3.47}$$

The corresponding Penrose diagram is given by Fig. 3, from which we can see that the nature of the singularity at $t = 0$ is null.

It is remarkable to note that the metrics (3.36) and (3.47) take the same form, but with different covering ranges. In Eq.(3.36) we have $|T| \in (0, 2/\beta)$, while in Eq.(3.47) we have $|T| \in (2/\beta, \infty)$. The metrics are singular at $|T| = 2/\beta$, which represent the boundaries of the spacetimes, represented, respectively, by Eqs.(3.36) and (3.47).

Following what we have done in the above, it can be shown that

$$K = -\frac{2}{t}, \qquad \gamma = t^2 \hat{\gamma}(x), \tag{3.48}$$

and the line element takes the form

$$ds^2 = -dt^2 + t^4 dx'^2. \tag{3.49}$$

Setting

$$T = 1/t, \tag{3.50}$$

then in the new coordinates we find that the metric takes the form

$$ds^2 = \frac{1}{T^4}\left(-dT^2 + dx'^2\right), \tag{3.51}$$

for which we have

$$K = -2T. \tag{3.52}$$

That is, the space-time is singular at $T = \pm\infty$, and the corresponding Penrose diagram is similar to that given in Fig.1.

IV. QUANTIZATION OF 2D HL GRAVITY

In the projectable HL gravity, the action (2.10) reduces to

$$S_{HL} = \zeta^2 \int dt dx N\gamma \left[(1-\lambda)K^2 - 2\Lambda\right], \tag{4.1}$$

where K is given by Eq.(2.7). In the following, we'll quantize the field by following Dirac's approach.

A. Hamiltonian Formulation and Dirac Quantization

Starting from the action Eq.(4.1), if we treat γ as a dynamical variable, its canonical momentum is found to be

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = 2\zeta^2(\lambda - 1)K. \tag{4.2}$$

After the Legendre transformation, the corresponding canonical Hamilton is given by

$$H_c(t) = \int dx \left(N\mathcal{H}(x) + N_1(x)\mathcal{H}_1(x)\right), \tag{4.3}$$

here the time variable is suppressed. With the projectability condition, the momentum constraint is local

while the Hamiltonian constraint is global, that is,

$$\mathcal{H}_1 = -\frac{\pi'}{\gamma} \approx 0, \tag{4.4}$$

$$\int dx \mathcal{H}(x) = \int dx \left(\frac{\pi^2 \gamma}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2\gamma \right)$$
$$\approx 0. \tag{4.5}$$

Straightforward calculations give us their Poisson brackets,

$$\begin{aligned}
\{\mathcal{H}(x), \mathcal{H}(x')\} &= 0, \\
\{\mathcal{H}(x), \mathcal{H}_1(x')\} &= \frac{\mathcal{H}(x')}{\gamma^2(x')}\delta_{x'}(x-x') \\
&\quad + \frac{\pi\mathcal{H}_1\delta(x-x')}{\zeta^2(1-\lambda)} \approx 0, \\
\{\mathcal{H}_1(x), \mathcal{H}_1(x')\} &= \frac{2\mathcal{H}_1(x')\delta_{x'}(x-x')}{\gamma^2(x')} \\
&\quad - \frac{2\gamma'\mathcal{H}_1}{\gamma^3}\delta(x-x') + \frac{\mathcal{H}_1'}{\gamma^2}\delta(x-x') \\
&\approx 0. \tag{4.6}
\end{aligned}$$

Therefore, we've got all the constraints and the physical degrees of freedom of the theory per space-time point $(\mathcal{N})$ is given by

$$\begin{aligned}
\mathcal{N} &= \frac{1}{2}\big(\dim\mathcal{P} - 2\mathcal{N}_1 - \mathcal{N}_2\big), \\
&= \frac{1}{2}\big(4 - 2*2 - 0\big) = 0. \tag{4.7}
\end{aligned}$$

Here $\dim\mathcal{P}$ means the dimension of the phase space, $\mathcal{N}_1(\mathcal{N}_2)$ denotes the number of first-class (second-class) constraints. Meanwhile, the local momentum constraint indicates that π is a function of time only, i.e.,

$$\pi(x,t) = \pi(t). \tag{4.8}$$

Note also that the canonical momentum $\pi(t)$ is invariant under the gauge transformation, as can been seen from the expression,

$$\left\{\pi(x), \int dx'\xi(x')\mathcal{H}_1(x')\right\} = \frac{\xi(x)\mathcal{H}_1(x)}{\gamma(x)}, \tag{4.9}$$

which vanishes on the constraint surface. For completeness, we also give the variation of γ under the spatial diffeomorphism,

$$\left\{\gamma(x), \int dx'\xi(x')\mathcal{H}_1(x')\right\} = \left(\frac{\xi}{\gamma}\right)'. \tag{4.10}$$

Since the momentum π is only a function of time, we can obtain an equivalent constraint by integrating Eq.(4.5) directly, and then we have

$$H(\pi, L) = \frac{\pi^2 L}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2 L \approx 0, \tag{4.11}$$

with

$$L(t) = \int dx\gamma(t,x), \tag{4.12}$$

which is gauge-invariant owing to Eq.(4.10). It's worth noting that $\pi(t)$ can be regarded as conjugate momentum to the invariant length $L(t)$. Starting from the basic relation

$$\{\gamma(x), \pi(y)\} = \delta(x-y), \tag{4.13}$$

then integrating both sides with respect to x and y, since π is independent of spatial coordinate y, we directly get

$$\{L(t), \pi(t)\} = 1. \tag{4.14}$$

Now following Dirac's approach, by promoting Eq.(4.14) to the commutation relation $[\hat{L}, \hat{\pi}] = i$, we get the Wheeler-DeWitt equation in the coordinate representation,

$$\hat{H}\Psi = 0. \tag{4.15}$$

However, there is ordering ambiguity arising from the term $L\pi^2$ in Eq.(4.11) [34]. In the following we consider each of the possible orderings, separately.

1. $\quad : \pi^2 L := \hat{L}\hat{\pi}^2$

In this case, the Hamiltonian constraint reads

$$L\left(\frac{\partial^2}{\partial L^2} - \epsilon_{\tilde{\Lambda}}\mu^2\right)\Psi = 0, \tag{4.16}$$

where $\mu \equiv 4\zeta^2|1-\lambda|\sqrt{|\tilde{\Lambda}|}$, and $\epsilon_{\tilde{\Lambda}}$ is a sign function which is one for $\tilde{\Lambda} > 0$, zero for $\tilde{\Lambda} = 0$ and negative one for $\tilde{\Lambda} < 0$. For $\tilde{\Lambda} > 0$, the general solution is

$$\Psi(L,t) = C_1 e^{\mu L} + C_2 e^{-\mu L}. \tag{4.17}$$

It can be shown that this solution is not normalizable even with $C_1 = 0$ with respect to the measure $L^{-1}dL$ in the interval $(0, +\infty)$. For $\tilde{\Lambda} = 0$, we have

$$\Psi(L,t) = A_1 L + A_2, \tag{4.18}$$

while when $\tilde{\Lambda} < 0$, we find

$$\Psi(L,t) = B_1 \sin(\mu L + B_2), \tag{4.19}$$

here A_1, A_2, B_1 and B_2 are some parameters independent of L. Again none of these wavefunctions are normalizable with respect to the measure $L^{-1}dL$.

2. $\quad : \pi^2 L : = \hat\pi \hat L \hat\pi$

In this case, we have

$$\frac{\partial}{\partial L}\left(L\frac{\partial\Psi}{\partial L}\right) - \epsilon_{\tilde\Lambda}\mu^2 L\Psi = 0. \qquad (4.20)$$

When $\tilde\Lambda > 0$, its general solution is given by the linear combination of modified Bessel functions of the first and second kind, denoted, respectively, by I and K, that is,

$$\Psi(L,t) = C_3 I_0(\mu L) + C_4 K_0(\mu L). \qquad (4.21)$$

However, the normalizable condition with the flat measure dL in the interval $(0, +\infty)$ leads to

$$C_3 = 0, \quad C_4 = \frac{2}{\pi}\sqrt{\mu}. \qquad (4.22)$$

For $\tilde\Lambda = 0$, we obtain

$$\Psi(L,t) = A_3 \ln L + A_4, \qquad (4.23)$$

which cannot be normalized in the interval $(0, +\infty)$. When $\tilde\Lambda < 0$, the general solution is given by

$$\Psi(L,t) = B_3 J_0(\mu L) + B_4 Y_0(\mu L), \qquad (4.24)$$

which is a linear combination of Bessel functions of the first and second kind. This wave function can't be normalized either.

3. $\quad : \pi^2 L : = \hat\pi^2 \hat L$

In this case, we have

$$\frac{\partial^2}{\partial L^2}\left(L\Psi\right) - \epsilon_{\tilde\Lambda}\mu^2 L\Psi = 0. \qquad (4.25)$$

When $\tilde\Lambda > 0$, the general solution of the above equation is given by,

$$\Psi(L,t) = \frac{1}{L}\left(C_5 e^{-\mu L} + C_6 e^{\mu L}\right), \qquad (4.26)$$

where C_5 and C_6 are the integration constants. Similar to the first case, the wavefunction now is also not normalizable for any given C_5 and C_6 with respect to the measure $L dL$ in the interval $(0, +\infty)$. When $\tilde\Lambda = 0$, the solution is

$$\Psi(L,t) = A_5 + \frac{A_6}{L}, \qquad (4.27)$$

For $\tilde\Lambda < 0$, we find

$$\Psi(L,t) = \frac{1}{L}\left[B_5 \sin\left(\mu L + B_6\right)\right]. \qquad (4.28)$$

None of these wavefunctions are normalizable with respect to the measure $L dL$ in the interval $(0, +\infty)$.

B. Simple Harmonic Oscillator

In this subsection, we shall show that under canonical transformation the above system can be reduced to that of a simple harmonic oscillator. By using the gauge freedom, we can always set

$$N(t) = 1, \qquad N_1 = 0. \qquad (4.29)$$

Then, applying the momentum constraint (4.8), the canonical Hamilton (4.3) reduces to

$$H(L,\pi) = L\left[\frac{\pi^2}{4\zeta^2(1-\lambda)} + 2\zeta^2\Lambda\right], \qquad (4.30)$$

with L given by Eq.(4.12). After the canonical transformation,

$$L = x^2, \quad \pi = \frac{p}{2x}, \qquad (4.31)$$

we find that Eq.(4.14) yields $\{x,p\} = 1$, and Eq.(4.30) takes the form,

$$H'(x,p) = \frac{p^2}{16\zeta^2(1-\lambda)} + 2\Lambda\zeta^2 x^2. \qquad (4.32)$$

However, this new Hamilton constraint (4.32) can only be equivalent to the original one (4.30) on the classical level. One can immediately understand this point when trying to find the solution to the corresponding Wheeler-DeWitt equation,

$$H'(\hat x,\hat p)\Psi = 0, \qquad (4.33)$$

which yields no physical states due to non-vanishing of the energy of the ground state of the quantized oscillator. Hence, we employ the following ansatz for the quantum canonical transformation,

$$\hat L = \hat x^2, \quad \hat\pi = \frac{1}{4}\left(\frac{1}{\hat x}\hat p + \hat p\frac{1}{\hat x}\right). \qquad (4.34)$$

Correspondingly, some terms can be transformed into the forms,

$$\hat L\hat\pi^2 \implies \frac{\hat p^2}{4} + \frac{i}{2}\frac{1}{\hat x}\hat p - \frac{5}{16\hat x^2}, \qquad (4.35)$$

$$\hat\pi\hat L\hat\pi \implies \frac{\hat p^2}{4} - \frac{1}{16\hat x^2}, \qquad (4.36)$$

$$\hat\pi^2\hat L \implies \frac{\hat p^2}{4} - \frac{i}{2}\frac{1}{\hat x}\hat p + \frac{3}{16\hat x^2}. \qquad (4.37)$$

Now setting,

$$L\pi^2 \mapsto \frac{1}{3}\left(\hat L\hat\pi^2 + \hat\pi\hat L\hat\pi + \hat\pi^2\hat L\right), \qquad (4.38)$$

we find that the new Hamilton under the canonical transformation (4.34) is given by,

$$\tilde H = \frac{\hat p^2}{16\zeta^2(1-\lambda)} + 2\Lambda\zeta^2\hat x^2 - \frac{1}{64\zeta^2(1-\lambda)\hat x^2}. \qquad (4.39)$$

Then, we can introduce the creation and annihilation operators,

$$
\begin{aligned}
a &= c_0 \left(x + \frac{ip}{8\zeta^2(1-\lambda)\sqrt{\tilde{\Lambda}}} \right), \\
a^\dagger &= c_0 \left(x - \frac{ip}{8\zeta^2(1-\lambda)\sqrt{\tilde{\Lambda}}} \right),
\end{aligned}
\tag{4.40}
$$

with $c_0 \equiv 2\zeta\sqrt{1-\lambda}\tilde{\Lambda}^{1/4}$, and

$$
[a, a^\dagger] = 1. \tag{4.41}
$$

In terms of a, $a^\dagger$ and $\hat{x}$, we find

$$
\tilde{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) - \frac{1}{64\zeta^2(1-\lambda)\hat{x}^2}, \tag{4.42}
$$

where $\hbar\omega \equiv \sqrt{\tilde{\Lambda}}$. Clearly, to have a well defined vacuum, we must require $\tilde{\Lambda} > 0$, that is

$$
\frac{\Lambda}{1-\lambda} > 0. \tag{4.43}
$$

Then, the Wheeler-DeWitt equation reads,

$$
\tilde{H}(\hat{x}, \hat{p})|\Psi\rangle = 0. \tag{4.44}
$$

Expanding $|\Psi\rangle$ in terms of the complete set $\{|n\rangle\}$,

$$
|\Psi\rangle = \sum_{n=0}^{\infty} a_n |n\rangle, \tag{4.45}
$$

we find that

$$
a_0 + 10\sqrt{2}a_2 = 0, \tag{4.46}
$$
$$
17a_1 + 14\sqrt{6}a_3 = 0, \tag{4.47}
$$

and for $n \geq 2$,

$$
\begin{aligned}
(4n-6)\sqrt{n(n-1)}a_{n-2} + (8n^2 + 8n + 1)a_n \\
+ (4n+10)\sqrt{(n+1)(n+2)}a_{n+2} = 0. \tag{4.48}
\end{aligned}
$$

Therefore, the wavefunction is given by

$$
\Psi(x) = \langle x|\Psi\rangle = \sum_{n=0}^{\infty} a_n\psi_n(x), \tag{4.49}
$$

where $x = \sqrt{L}$, and

$$
\begin{aligned}
\psi_n(x) &\equiv \langle x|n\rangle = \frac{(2\mu)^{2n+1}}{\pi^{1/4}\sqrt{2^n n!}} \\
&\times \left(x - \frac{1}{2\mu}\frac{d}{dx} \right)^n e^{-\mu x^2}. \tag{4.50}
\end{aligned}
$$

Thus, we find that $\Psi(L) \propto e^{-\mu L}$, which is similar to the ones obtained by the Dirac quantization, although

they are not precisely equal, as we used two quite different approaches to obtain the corresponding Hamiltons of quantum mechanics, as one can see from Eqs.(4.11) and (4.42).

From Eq.(4.40), on the other hand, we find

$$
\begin{aligned}
\langle m|L|n\rangle = \ell_{HL}\Big[(2n+1)\delta_{m,n} + \sqrt{n(n-1)}\delta_{m,n-2} \\
+ \sqrt{(n+1)(n+2)}\delta_{m,n+2}\Big], \tag{4.51}
\end{aligned}
$$

where $\ell_{HL} \equiv 1/(4c_0^2)$ denotes the meanvalue of the gauge-invariant length operator L [cf. Eq.(4.12)] in the ground-state $|0\rangle$, i.e.,

$$
\langle 0|L|0\rangle = \ell_{HL}. \tag{4.52}
$$

0

C. Quantization of Spacetimes with $L(t) = 0$

It should be noted that, in the above studies, either in terms of the Dirac quantization or in terms of the harmonic oscillator, we implicitly assumed $L(t) \neq 0$. Classically, this corresponds to the case studied in Sec. III.A, in which solutions exist only when $\tilde{\Lambda} > 0$, and the corresponding space-time is de Sitter. But, quantum mechanically the quantization can be carried out for any given $\tilde{\Lambda} \in (-\infty, \infty)$.

In addition, classical solutions exist even when $L(t) = \int_{-L_\infty}^{L_\infty} \gamma(t,x)dx = 0$. In this case, the classical solutions are given by

$$
\gamma(t,x) = \gamma_0(x)\hat{\gamma}(t), \tag{4.53}
$$

where

$$
\int_{-L_\infty}^{L_\infty} \gamma_0(x)dx = 0, \tag{4.54}
$$

that is, $\gamma_0(x)$ is an odd function of x. The function $\hat{\gamma}(t)$ satisfies the equation of motion, Eq.(3.8). Of course, in this case the Hamiltonian constraint (3.2) is satisfied identically, while the momentum constraint is satisfied, provided that

$$
\pi(t,x) = \pi(t). \tag{4.55}
$$

Then, Eq.(3.8) reads,

$$
2\ddot{\hat{\gamma}}\hat{\gamma} - \dot{\hat{\gamma}}^2 + 4\tilde{\Lambda}\hat{\gamma}^2 = 0, \tag{4.56}
$$

which can be obtained from the effective action,

$$
S_{\hat{\gamma}} = \int \mathcal{L}_{\hat{\gamma}} dt, \tag{4.57}
$$

where

$$
\mathcal{L}_{\hat{\gamma}} = \frac{\dot{\hat{\gamma}}^2}{\hat{\gamma}} - 4\tilde{\Lambda}\hat{\gamma}. \tag{4.58}
$$

After the Legendre transformation, the corresponding Hamilton turns out to be

$$H_{\hat{\gamma}} = \frac{1}{4}\hat{\gamma}\pi_{\hat{\gamma}}^2 + 4\tilde{\Lambda}\hat{\gamma}, \qquad (4.59)$$

where $\pi_{\hat{\gamma}}$ is momentum conjugate of $\hat{\gamma}$. Remarkably, this Hamilton is nothing but the one precisely given by Eq.(4.11). Therefore, its quantization can be carried out in the same ways as we just did above: either by the standard Dirac quantization or by the harmonic oscillator quantization. Thus, in the following we shall not repeat the above processes.

V. COUPLING WITH A SCALAR FIELD

When the 2d HL gravity couples to a scalar field ϕ, the total action becomes

$$S = S_{HL} + S_{\phi}, \qquad (5.1)$$

where S_{ϕ} denotes the action of the scalar field. To be power-counting renormalizable, the marginal terms of S_{ϕ} must be at least of dimension $2z$ with $z \geq d$. Since ϕ is dimensionless, one can see that the marginal terms are $\nabla_i \phi \nabla^i \phi$ and $a_i \nabla^i \phi$. Then, S_{ϕ} must take the form,

$$S_{\phi} = \int dt dx N \sqrt{g}\Big[\frac{1}{2}(\partial_{\perp}\phi)^2 - \alpha_0(\nabla_i\phi)^2 - V(\phi)$$
$$-\alpha_1\phi\nabla^i a_i - \alpha_2\phi a^i\nabla_i\phi\Big]. \qquad (5.2)$$

Here $\partial_{\perp} \equiv N^{-1}(\partial_t - N^i\nabla_i)$, $V(\phi)$ denotes the potential of the scalar field, and α_n are dimensionless coupling constants [4]. In the relativistic limit, we have $(\alpha_0, \alpha_1, \alpha_2)_{GR} = (1, 0, 0)$.

A. Classical Field Equations

In the projectable case, we have $a_i = 0$ and the last two terms in Eq.(5.2) vanish. Then, the variations of the total action with respect to N, γ, N_1 and ϕ, yield, respectively,

$$\int dx\Bigg\{\Big[\frac{\dot{\gamma}^2}{\kappa\gamma} + 8\zeta^2\Lambda\gamma\Big] + \frac{2c_{\phi}^2}{\gamma}\phi'^2$$
$$+ \Big[2\gamma\dot{\phi}^2 + 4\gamma V(\phi)\Big]\Bigg\} = 0, \qquad (5.3)$$

$$\Big(\frac{\dot{\gamma}}{\gamma}\Big)^{\cdot} + \frac{1}{2}\Big(\frac{\dot{\gamma}}{\gamma}\Big)^2 + 2\tilde{\Lambda} = \kappa\Big(\dot{\phi}^2 + \frac{c_{\phi}^2}{\gamma^2}\phi'^2 - 2V(\phi)\Big),$$

---

[4] Since the scalar field ϕ is dimensionless, these coefficients in principle can be arbitrary functions of ϕ. In this paper, we consider only the case where they are constants.

$$\qquad (5.4)$$

$$\Big(\frac{\dot{\gamma}}{\gamma}\Big)' = 2\kappa\dot{\phi}\phi', \qquad (5.5)$$

$$\big(\gamma\dot{\phi}\big)^{\cdot} - c_{\phi}^2\Big(\frac{\phi'}{\gamma}\Big)' + \gamma\frac{dV(\phi)}{d\phi} = 0, \qquad (5.6)$$

where $c_{\phi}^2 \equiv 2\alpha_0$ must be non-negative in order for the scalar field to be stable, and

$$\kappa = \frac{1}{4\zeta^2(1 - \lambda)}. \qquad (5.7)$$

Note that in the vacuum case γ is a function of t only, as shown previously. However, because of the presence of the scalar field, now it in general is a function of both t and x. To compare it with the vacuum case, in the following let us consider the case $\gamma = \gamma_0(x)\gamma(t)$ only. In fact, as to be shown below, this is also the case where the corresponding Hamiltonian constraint becomes local, while the momentum constraint can be solved explicitly.

Setting $\gamma = \gamma_0(x)\gamma(t)$, from Eq.(5.5) we can choose that $\phi = \phi(t)$. Then, Eqs.(5.3), (5.4) and (5.6) reduce, respectively, to

$$\int dx\Big\{\frac{\dot{\gamma}^2}{\kappa\gamma} + 8\zeta^2\Lambda\gamma + 2\gamma\dot{\phi}^2 + 4\gamma V(\phi)\Big\} = 0, (5.8)$$

$$\Big(\frac{\dot{\gamma}}{\gamma}\Big)^{\cdot} + \frac{1}{2}\Big(\frac{\dot{\gamma}}{\gamma}\Big)^2 + 2\tilde{\Lambda} = \kappa\big(\dot{\phi}^2 - 2V(\phi)\big), \quad (5.9)$$

$$\big(\gamma\dot{\phi}\big)^{\cdot} + \gamma\frac{dV(\phi)}{d\phi} = 0. \qquad (5.10)$$

To solve the above equations, we further assume that $V(\phi) = \tilde{\Lambda} = 0$. Then from Eq.(5.10), we know

$$\dot{\phi} = \frac{\phi_0}{\gamma(t)}, \qquad (5.11)$$

here ϕ_0 is a constant. Combining with Eq.(5.9), we derive an equation for $\gamma(t)$,

$$\ddot{\gamma}(t)\gamma(t) - \frac{1}{2}\dot{\gamma}(t)^2 = \kappa\phi_0^2. \qquad (5.12)$$

One of the solutions can be easily obtained, and is given by

$$\gamma(t) = (c_0 + c_1 t)^2 + \frac{\kappa\phi_0^2}{2c_1^2}, \qquad (5.13)$$

$$\phi(t) = \sqrt{\frac{2}{\kappa}}\arctan\Big(\sqrt{\frac{2}{\kappa}}\frac{c_1(c_0 + c_1 t)}{\phi_0}\Big) + \phi_1, \quad (5.14)$$

where c_0, c_1 and ϕ_1 are constants. In order to make our solution consistent with the integral constraint (5.8), we require $\gamma(t, x)$ to be an odd function of x, so that Eq.(3.18) also holds here. Keeping this in mind and then using the residual gauge freedom, we find the metric takes the form,

$$ds^2 = -dt^2 + \big(t^2 + \epsilon_{\kappa}t_s^2\big)^2 dx^2, \qquad (5.15)$$
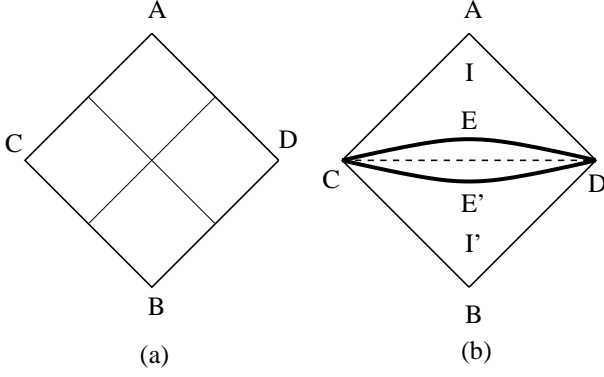
FIG. 4: (a) The Penrose diagram for the solution (5.15) with $\lambda \leq 1$ (or $\kappa \geq 0$), in which the whole space-time is free of space-time singularities. (b) The Penrose diagram for the solution (5.15) with $\lambda > 1$ (or $\kappa < 0$), in which the space-time is singular on $t = \pm t_s$, denoted by the thick solid curves $\widehat{CED}$ and $\widehat{CE'D}$. Thus, in this case the two regions I and I' are causally disconnected.

here $\epsilon_\kappa \equiv \text{sign}(\kappa)$, and

$$t_s^2 \equiv \frac{|\kappa|\phi_0^2}{2c_1^4}. \qquad (5.16)$$

Following what we did in Section III, we can derive the extrinsic curvature K, Ricci scalar R, and the components of the tidal forces, given, respectively by,

$$K = -\frac{t}{2}R = -\frac{2c_1^2 t}{t^2 + \epsilon_\kappa t_s^2}, \qquad (5.17)$$

$$R_{(1)(1)} = -R_{(0)(0)} = \frac{2c_1^2}{t^2 + \epsilon_\kappa t_s^2}. \qquad (5.18)$$

Therefore, the singularities of the spacetime are determined directly by the signs of κ. In particular, if $\lambda \leq 1$, the spacetime is free of space-time singularities. For $\lambda > 1$, on the other hand, there is a curvature singularity located at

$$t = \pm t_s. \qquad (5.19)$$

The corresponding Penrose diagrams are given in Fig.4.

It should be noted that, instead of imposing the condition (3.18), we can set the integrand of the Hamiltonian constraint to zero. But, this will require $C_1 = 0$, and the corresponding space-time is flat.

B. Hamiltonian Structure and Canonical Quantization

When coupling with the scalar field, the Hamiltonian and momentum constraints become,

$$\int dx \mathcal{H}(x) = \int dx \left[\frac{\pi^2 \gamma}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2\gamma + \frac{\pi_\phi^2}{2\gamma} \right.$$

$$\left. + \frac{\alpha_0 \phi'^2}{\gamma} + \gamma V(\phi) \right], \qquad (5.20)$$

$$\mathcal{H}_1 = -\frac{\pi'}{\gamma} + \frac{\pi_\phi \phi'}{\gamma^2}, \qquad (5.21)$$

here π_ϕ denotes the canonical moment conjugate to the scalar field ϕ. Similarly, the Poisson brackets of the two constraints are given by,

$$\{\mathcal{H}(x), \mathcal{H}_1(x')\} = \frac{\mathcal{H}(x')\delta_x(x-x')}{\gamma^2(x')} + \frac{\pi\mathcal{H}_1\delta(x-x')}{\zeta^2(1-\lambda)},$$

$$\{\mathcal{H}_1(x), \mathcal{H}_1(x')\} = \frac{2\mathcal{H}_1(x)\delta_{x'}(x-x')}{\gamma^2(x)}$$

$$+ \frac{2\gamma'\mathcal{H}_1}{\gamma^3}\delta(x-x') - \frac{\mathcal{H}_1'}{\gamma^2}\delta(x-x'). \qquad (5.22)$$

For the non-local Hamiltonian constraint we also find

$$\left\{ \int dx \mathcal{H}(x), \int dx' \mathcal{H}(x') \right\} = 0, \qquad (5.23)$$

as long as $\pi_\phi \phi'/\gamma^2$ vanishes on boundaries.

In the rest of this paper, we only consider the quantization of the system for the case,

$$\phi' = 0 = \pi', \qquad (5.24)$$

in order to compare with what we obtained in the pure gravity case. As a matter of fact, this also makes the problem considerably simp lied and becomes tractable. Under the above assumption, the Hamiltonian constraint reads,

$$H(t) = \frac{\pi^2 L}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2 L + \frac{L\dot{\phi}^2}{2} + LV(\phi) \simeq 0. \quad (5.25)$$

It must be noted that in writing down the above expression, we performed the spatial integration and used the fact that

$$\pi_\phi = \gamma\dot{\phi}, \qquad (5.26)$$

with the gauge choice $N = 1$ and $N_1 = 0$. On the other hand, from the canonical relation,

$$\{\phi(x), \pi_\phi(y)\} = \delta(x-y), \qquad (5.27)$$

we can integrating both sides with respect to the spatial coordinates x and y, and then use Eq.(5.26) and the constraint $\phi = \phi(t)$, to obtain

$$\{\phi(t), L(t)\dot{\phi}(t)\} = 1, \qquad (5.28)$$

which enables us to identify π_ϕ as $\pi_\phi = L\dot{\phi}$. Now making this substitution in the Hamiltonian constraint (5.25), we find the Hamilton with two discrete physical degrees of freedom, L and ϕ, takes the form,

$$H(t) = \frac{\pi^2 L}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2 L + \frac{\pi_\phi^2}{2L} + LV(\phi). \quad (5.29)$$

Thus, the Wheeler-Dewitt equation now reads,

$$\hat{H}(t)\Psi(L,\phi;t) = 0. \quad (5.30)$$

If we further assume that the potential of the scalar field can be ignored, $V(\phi) \simeq 0$, we are able to find solutions to Eq.(5.30) by separation of variables. In this case, assuming

$$\Psi(L,\phi) = X(L)Y(\phi), \quad (5.31)$$

we obtain two independent equations,

$$Y''(\phi) + mY(\phi) = 0, \quad (5.32)$$

$$\left[L\pi^2\right]X(L) + \left(\mu^2 L + \epsilon_\lambda \frac{m\mu}{2\sqrt{\tilde{\Lambda}L}}\right)X(L) = 0. \quad (5.33)$$

Here $[L\pi^2]$ means some specific ordering of L and π, m is an undetermined parameter, μ is given as in the pure gravity case, and ϵ_λ is one for $\lambda < 1$ and negative one for $\lambda > 1$. Eq.(5.32) has the general solution,

$$Y(\phi) = D_1 \sin\left(\sqrt{m}\phi + D_2\right), \quad (5.34)$$

where $D_{1,2}$ are tow integration and possibly complex constants. To solve Eq.(5.33), just as in the pure gravity case, there are three different orderings, which will be considered below, separately.

1. $:\pi^2 L := \hat{L}\hat{\pi}^2$

In this case, the Hamiltonian constraint reads

$$L^2 X'' - \left(\epsilon_{\tilde{\Lambda}}\mu^2 L^2 + k^2\right)X = 0, \quad (5.35)$$

where $k^2 \equiv 2\epsilon_\lambda m\zeta^2|1-\lambda|$ and $\epsilon_{\tilde{\Lambda}}$ is defined in Sec. IV. For $\tilde{\Lambda} > 0$, the general solution is given by the linear combination of modified Bessel functions of the first and second kind, denoted by I_ν and K_ν, respectively, that is,

$$X = \sqrt{L}\left\{C_1 I_\nu(L\mu) + C_2 K_\nu(L\mu)\right\}, \quad (5.36)$$

Here $\nu \equiv \sqrt{1+4k^2}/2$. Generally, this wave-function is not normalizable with respect to the measure dL/L in the interval $(0,+\infty)$. However, if $|\Re(\nu)| < 1/2$, we have the normalized function, given by

$$X_{\text{norm}} = \frac{1}{\pi}\sqrt{\frac{4\mu}{\sec(\pi\nu)}}K_\nu(L\mu). \quad (5.37)$$

In this particular case for $-1/4 \le k^2 < 0$, depending on the value of λ, the parameter m can be either positive or negative. When it is positive, in order to have a normalizable wave function $\Psi(L,\phi)$ of Eq.(5.31), we need to restrict the domain of ϕ to some finite region, for example $(0, 2\pi)$, then it would be straightforward to normalize $Y(\phi)$ from Eq.(5.34) in that finite region. When

m is negative, $Y(\phi)$ can be normalizable either in the region $\phi \in (-\infty, 0)$ or $\phi \in (0, \infty)$, so does the wavefunction $\Psi(L,\phi)$.

For $\tilde{\Lambda} = 0$, the solution is given by

$$X = \sqrt{L}\left(A_1 L^{+\nu} + A_2 L^{-\nu}\right), \quad (5.38)$$

while for $\tilde{\Lambda} < 0$, we find

$$X = \sqrt{L}\left(B_1 J_\nu(\mu L) + B_2 Y_\nu(\mu L)\right). \quad (5.39)$$

Here ν is defined as in the case $\tilde{\Lambda} > 0$. None of these two wave functions are normalizable with respect to the measure $L^{-1}dL$ in the interval $(0,+\infty)$.

2. $:\pi^2 L := \hat{\pi}\hat{L}\hat{\pi}$

In this case, we have

$$L^2 X'' + LX' - \left(\epsilon_{\tilde{\Lambda}}\mu^2 L^2 + k^2\right)X = 0. \quad (5.40)$$

Thus, for $\tilde{\Lambda} > 0$, the general solution is given by,

$$X = C_1 I_k(L\mu) + C_2 K_k(L\mu). \quad (5.41)$$

Again, for $0 \le k^2 < 1/4$, we have the normalized function $X(L)$ given by

$$X_{\text{norm}} = \frac{1}{\pi}\sqrt{\frac{4\mu}{\sec(\pi k)}}K_k(L\mu). \quad (5.42)$$

Similar to the last case, $Y(\phi)$ is normalized only in some restricted domains, depending on the signs of m.

When $\tilde{\Lambda} = 0$, its general solution is

$$X = A_1 L^k + A_2 L^{-k}, \quad (5.43)$$

while for $\tilde{\Lambda} < 0$, it is given by

$$X = B_1 J_k(\mu L) + B_2 Y_k(\mu L). \quad (5.44)$$

It can be shown that none of these two wavefunctions are normalizable in the interval $(0,+\infty)$.

3. $:\pi^2 L := \hat{\pi}^2\hat{L}$

In this case, we have

$$L^2 X'' + 2LX' - \left(\epsilon_{\tilde{\Lambda}}\mu^2 L^2 + k^2\right)X = 0. \quad (5.45)$$

Then, for $\tilde{\Lambda} > 0$, we find

$$X = C_1 j_{-\nu-1/2}(-iL\mu) + C_2 y_{-\nu-1/2}(-iL\mu), \quad (5.46)$$

here j_ν, y_ν denote spherical Bessel functions of the first and second kind. When $\tilde{\Lambda} = 0$, we find that

$$X = L^{-1/2}\left(A_1 L^\nu + A_2 L^{-\nu}\right), \quad (5.47)$$

while for $\tilde{\Lambda} < 0$, we have

$$X = C_1 j_{\nu-1/2}(\mu L) + C_2 y_{\nu-1/2}(\mu L). \quad (5.48)$$

It can be shown that in this case none of these wave functions are normalizable with respect to the measure LdL in the interval $(0,+\infty)$.

C. Two Interacting Simple Harmonic Oscillators

Similar to what we have done in the pure gravity case, we can also treat the Hamilton given by Eq.(5.29) as consisting of harmonic oscillators. To this goal, let us first make the transformations

$$
\begin{aligned}
L(t) &= y_1^2(t) - y_2^2(t), \\
\phi(t) &= \sqrt{2\zeta^2(\lambda-1)}\ln\left(\frac{y_1(t)+y_2(t)}{y_1(t)-y_2(t)}\right), \quad (5.49)
\end{aligned}
$$

for which we are able to convert Eq.(5.26) into the form,

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}m\Big[\left(\dot{y}_1^2 - \omega^2 y_1^2\right) - \left(\dot{y}_2^2 - \omega^2 y_2^2\right)\Big] \\
&\quad - V_e(y_1, y_2), \quad (5.50)
\end{aligned}
$$

but now with

$$
\begin{aligned}
m &\equiv 8(1-\lambda)\zeta^2, \quad \omega^2 \equiv \frac{\Lambda}{2(1-\lambda)}, \\
V_e(y_1,y_2) &\equiv \left(y_1^2 - y_2^2\right) V\Big(\phi(y_1,y_2)\Big). \quad (5.51)
\end{aligned}
$$

Clearly, Eq.(5.50) describes the interaction between two simple harmonic oscillators, one with positive energy and the other with negative energy. Thus, in order for the system to have a total positive energy, the interaction between them is important.

To process further, we need to consider particular potential $V(\phi)$, which will be model-dependent. So, in the following we shall not pursue the quantization alone this direction further.

D. Quantization of Spacetimes with $L(t) = 0$

Just like in the pure gravity part, when $L(t) = 0$, we again have $\gamma(t,x) = \gamma_0(x)\hat{\gamma}(t)$, where $\gamma_0(x)$ is an odd function of x, so Eq.(4.54) is satisfied, which in turn guarantees that the Hamiltonian constraint (5.20) is automatically satisfied, while the momentum constraint (5.21) will be also satisfied when $\pi = \pi(t)$ and $\phi = \phi(t)$. Then, the equations of motion (5.4) and (5.6) can be obtained from the effective Lagrange,

$$
\mathcal{L}_{(\gamma,\phi)} \equiv \frac{\dot{\gamma}^2}{\gamma} + 2\kappa\gamma\dot{\phi}^2 - 4\tilde{\Lambda}\gamma - 4\kappa\gamma V(\phi). \quad (5.52)
$$

Note that in writing the above equation, we had dropped the hat from γ. Then, the corresponding Hamilton is given by

$$
H_{(\gamma,\phi)} = \frac{\pi_\gamma^2}{4}\gamma + \frac{\pi_\phi^2}{8\kappa\gamma} + 4\tilde{\Lambda}\gamma + 4\kappa\gamma V(\phi), \quad (5.53)
$$

which has the same form as the Hamilton given by Eq.(5.29). Therefore, its quantization can be followed precisely what we did in the above, which will not be repeated here.

VI. CONCLUSIONS

In this paper, we have studied the quantization of the (1+1)-dimensional projectable HL gravity. In particular, after giving a brief review of the theory with or without the projectability condition in Sec. II, we have devoted Sec. III to study vacuum solution of the classical HL gravity, and found all the solutions in the projectable case. These solutions can be divided into several classes, and each of them have different local and global properties. Their corresponding Penrose diagrams are given, respectively, by Figs. 1, 2 and 3.

In Sec. IV, after working out the Hamiltonian structure and solving the momentum constraint explicitly for the projectable vacuum HL gravity, we have showed that the resulting Hamilton can be quantized by following the standard Dirac quantization. When moving from the classical Hamilton to the quantum mechanical one, ordering ambiguity always appears. We have found that for some orderings the corresponding wavefunctions are normalizable. In addition, the Hamilton can also be written in the form of a simple harmonic oscillator, whereby its quantization can be carried out in the standard way. Again, the orderings of relevant operators play an essential role, so that the Weeler-DeWitt equation $\hat{H}|\Psi\rangle = 0$ has non-trivial solutions.

In Sec. V, we have extended the studies carried out in Sec. IV to couple minimally with a scalar field, and solved the momentum constraint in the case where the fundamental variables are functions of time only. In this particular case, the quantization of the coupled system can also be carried out by the standard Dirac process. After writing the corresponding Hamilton in terms of two interacting harmonic oscillators, we have found that one of them has positive energy, and the other has negative energy, once the interaction is turned off.

A remarkable feature is that the space-time can be quantized, even it classically has various singularities [cf. Figs.1 and 4]. In this sense, the classical singularities are indeed smoothed out by the quantum effects.

It should be noted that in this paper we have mainly studied the case with the projectability condition. It would be very interesting to see what will happen if such condition is relaxed. We wish to come back to this case soon.

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