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Quasi-Normal Modes for Subtracted Rotating and Magnetised Geometries

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ABSTRACT

We obtain explicit separable solutions of the wave equation of massless minimally coupled scalar fields in the subtracted geometry of four-dimensional rotating and Melvin (magnetised) four-charge black holes of the STU model, a consistent truncation of maximally supersymmetric supergravity with four types of electromagnetic fields. These backgrounds possess a hidden SL(2,R)×SL(2,R)×SO(3) symmetry and faithfully model the near horizon geometry of these black holes, but locate them in a confining asymptotically conical box. For each subtracted geometry we obtain two branches of quasi-normal modes, given in terms of hypergeometric functions and spherical harmonics. One branch is over-damped and the other under-damped and they exhibit rotational splitting. No black hole bomb is possible because the Killing field which co-rotates with the horizon is everywhere timelike outside the black hole. A five-dimensional lift of these geometries is given locally by the product of a BTZ black hole with a two-sphere. This allows an explicit analysis of the minimally coupled massive five-dimensional scalar field. Again, there are two branches, both damped, however now their oscillatory parts are shifted by the quantised wave number k along the fifth circle direction.

1 Introduction

The wave equation in the black hole backgrounds provides very useful insights into its internal structure and the relationship with a conformal symmetry [1] [2]. The wave equation of a massless scalar field in the background of a general multi-charged rotating black hole turns out to be separable and acquires an SL(2,R)×SL(2,R)×SO(3) symmetry, when certain terms are omitted. In [3] it was suggested that this symmetry is a ”hidden conformal symmetry” of the black hole that is spontaneously broken.

In [4, 5] an explicit example of the general multi-charged rotating black hole geometry, which exhibits the SL(2,R)×SL(2,R)×SO(3) conformal symmetry of the wave equation, was constructed. It has been dubbed ”subtracted geometry” because it is constructed by subtracting certain terms from the warp factor of the metric. The subtracted geometry preserves the internal structure of the black hole because it has the same horizon area and periodicity of the angular and time coordinates in the near horizon regions as the original black hole geometry it was constructed from. The new geometry is asymptotically conical and may be interpreted physically as a black hole in an asymptotically confining box.

This paper is concerned with subtracted geometries that arise in four-dimensional N=2 STU supergravity. This is a consistent truncation of maximally supersymmetric ungauged supergravity theories, which arise as an effective theory of toroidally compactified Type IIA (N=8) or Heteortic (N=4) string theory. The original four-charge rotating solution [6][1], along with the explicit expressions for all four gauge potentials was given in [8] as a solution of the bosonic sector of the $\mathcal{N} = 2$ supergravity coupled to three vector supermultiplets. In [9], it was shown that the corresponding subtracted geometry may be obtained by taking a particular scaling limit of the four-charge rotating black hole solution. Furthermore, it was shown [9] that the subtracted geometry for the Schwarzchild black hole can be obtained by applying Harrison transformations of the STU model, which comprise a part of the larger set of symmetries of the black holes when the four-dimensional black hole Lagrangian is reduced on time to three dimensions. In [10, 11, 12], this was generalized to the case of four-charge rotating black holes of the STU model and the interpolating solutions between the rotating black holes and their subtracting geometry were obtained [11, 13] by continuously varying the boost parameters of the Harrison transformations from zero to an infinite boost. (For related works on extremal subtracted geometries, see [14, 15].)

This paper will also be dealing with the subtracted geometry of Melvin STU black

[1]The full rotating charged black hole seed solution, parameterised by an additional charge parameter was recently obtained in [7].

holes which arise as a scaling limit of magnetised four-charge black holes. The magnetised four charge black holes of the STU model were constructed in [16] by solution generating techniques. Special cases include Schwarzschild and Reissner-Nordström black holes in the magnetic field of Maxwell-Einstein gravity. The subtracted geometries of Melvin STU black holes were also constructed there. These geometries faithfully model the near-horizon region of multi-charged black holes in magnetic field backgrounds.

The physical properties of the black holes in the magnetic backgrounds can typically be studied only numerically. We shall see that these magnetised backgrounds can be analysed analytically.

The main aim of this paper is to analyse the quasi-normal solutions of the scalar wave equation in the background of the above mentioned subtracted rotating geometry and the subtracted magnetised geometry, by employing their hidden SL(2,R)×SO(2,R)×SO(3) symmetry. We do so by first explicitly solving the wave equation for a massless scalar field in four dimensions, which due to the very special structure of the metric is separable and solvable in terms of hypergeometric functions and spherical harmonics both for subtracted rotating and subtracted magnetised geometries. In each case we obtain two branches of quasi-normal modes, with remarkably simple values of complex eigenfrequencies, one over-damped and one under-damped. Specifically, in the case of magnetised geometries the effect of the magnetic field turns out to be an additive shift of the real part of the eigenfrequency of the quasi-normal modes. The regularity of these solutions near the outer horizon is analysed in terms of Kruskal-Szekeres coordinates. These results are presented for subtracted rotating geometries in Section 2 and for subtracted magnetised geometries in Section 3.

The analysis is further extended by studying the wave equation for a minimally coupled massive scalar field in the five-dimensional lift of these subtracted geometries. For both rotating and magnetised cases, the lift on a circle S^1 results in a geometry that is locally BTZ $\times$ S^2, a product of the BTZ black hole and a two-sphere. As a consequence, the wave equation for a massive minimally coupled scalar field is separable and may be solved again in terms of the hypergeometric functions, spherical harmonics and a plane wave along the S^1 circle direction. Remarkably simple, explicit expressions for the frequencies of the two branches of the quasi-normal modes are obtained , where the quantised wave number along the S^1 circle shifts the real part of he eigenfrequencies. For the special case of the zero wave number and zero five-dimensional mass, one reproduces the results of Sections 2 and 3 as expected. Solutions for the non-zero wave numbers can be interpreted as quasi-normal modes for the massive four-dimensional Kaluza-Klein modes whose electric charge is proportional to the wave number. The regularity of these modes near the outer horizon is manifest after performing a Kaluza-Klein U(1) gauge transformation on the wave function. All of these results are presented in Section 4.

The paper also contains a number of Appendices (Section 5) collecting together results needed for the calculations described above in a uniform notation. Section 5.1 provides explicit formulae for the subtracted rotating geometry and all the fields of the STU model, which were worked out in [9]. Section 5.2 does the same for the subtracted magnetised geometry in the STU model by elaborating on results given in [16]. Section 5.3 gives detailed expressions for the lift of these geometries on a circle to five dimensions, leading to the BTZ $\times S^2$ geometry. Earlier partial results for the rotating geometry were given in [5, 9], and for the magnetised one in [16]. Here particular care is taken of the dimensions and of the periodicities of metric coordinates. In Section 5.4 a map is provided taking the BTZ coordinates to the local AdS_3 metric from [18, 19]. Section 5.5 contains the formulae for the Kalulza-Klein reduction of the scalar wave equation on a circle.

2 Subtracted Rotating Geometry

The metric for the four-charge rotating black hole solution of the STU model can be written in the form [6, 8, 5]:

$$ds_4^2 = -\Delta_0^{-\frac{1}{2}} G(dt + \mathcal{A})^2 + \Delta_0^{\frac{1}{2}} (\frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2\theta d\phi^2), \tag{2.1}$$

with

$$\begin{aligned} X &= r^2 - 2mr + a^2 \ , \\ G &= r^2 - 2mr + a^2\cos^2\theta \ , \\ \mathcal{A} &\equiv \frac{a\sin^2\theta A_{red}}{G} = \frac{2ma\sin^2\theta}{G} \left[(\Pi_c - \Pi_s)r + 2m\Pi_s\right] d\phi \ , \end{aligned} \tag{2.2}$$

and the warp factor Δ_0 given by

$$\begin{aligned} \Delta_0 = \ & \prod_{i=1}^{4}(r + 2m\sinh^2\delta_i) + 2a^2\cos^2\theta[r^2 + mr\sum_{i=1}^{4}\sinh^2\delta_i + \ 4m^2(\Pi_c - \Pi_s)\Pi_s \\ & -2m^2\sum_{i<j<k}\sinh^2\delta_i\sinh^2\delta_j\sinh^2\delta_k] + a^4\cos^4\theta \, . \end{aligned} \tag{2.3}$$

The mass, four charges and the angular momentum are parameterised as

$$
\begin{aligned}
G_4 M &= \frac{1}{4} m \sum_{i=1}^{4} \cosh 2\delta_i \,, \\
G_4 Q_i &= \frac{1}{4} m \sinh 2\delta_i \,, \quad i = 1, 2, 3, 4 \,, \\
G_4 J &= ma(\Pi_c - \Pi_s) \,,
\end{aligned}
\tag{2.4}
$$

with G_4 the four-dimensional Netwon's constant and we employ the abbreviations

$$
\Pi_c \equiv \prod_{i=1}^{4} \cosh \delta_i \ , \quad \Pi_s \equiv \prod_{i=1}^{4} \sinh \delta_i \ . \tag{2.5}
$$

The two horizons, given by $X = 0$, are at

$$
r_\pm = m \pm \sqrt{m^2 - a^2} \,. \tag{2.6}
$$

It was shown in [5] that the replacement

$$
\Delta_0 \to \Delta = (2m)^3 r(\Pi_c^2 - \Pi_s^2) + (2m)^4 \Pi_s^2 - (2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos^2\theta \,, \tag{2.7}
$$

in the metric (2.1) reduces the highest power of r in Δ_0 and renders in the radial part of the massless scalar wave equation the irregular singular point at infinity regular, allowing for solutions in terms of hypergeometric functions. Moreover, the massless scalar wave equation is separable in terms of ordinary spherical harmonics, rather than the complicated spheroidal functions needed for the full four-charge black hole solution. This new metric has been dubbed a "subtracted geometry" and the massless scalar wave equation in this background exhibits a hidden SL(2,R)×SL(2,R)×SO(3) symmetry. Furthermore, at the outer and inner horizons the entropies

$$
S_\pm = \frac{2\pi m}{G_4} \left[(\Pi_c + \Pi_s) m \pm (\Pi_c - \Pi_s) \sqrt{m^2 - a^2} \right] \,, \tag{2.8}
$$

the inverse surface gravities

$$
\frac{1}{\kappa_\pm} = 2m \left[\frac{m}{\sqrt{m^2 - a^2}} (\Pi_c + \Pi_s) \pm (\Pi_c - \Pi_s) \right] \,, \tag{2.9}
$$

and the angular velocities

$$
\Omega_\pm = \kappa_\pm \frac{a}{\sqrt{m^2 - a^2}} \,, \tag{2.10}
$$

remain unchanged by this replacement, thus preserving the local geometry and thermodynamic properties of the metric. The expressions simplify significantly in the static case when $a = 0$.

It is straightforward to see that these black hole solutions and their subtracted geometry encompasses the following special cases:

$$\begin{aligned} \text{Kerr-Newman:} \quad & \delta_1 = \delta_2 = \delta_3 = \delta_4 \,, \\ \text{Kerr:} \quad & \delta_i = 0 \,, \quad i = 1,2,3,4 \,, \\ \text{Reissner-Nordström:} \quad & \delta_1 = \delta_2 = \delta_3 = \delta_4 \,, \quad a = 0 \,, \\ \text{Schwarzschild:} \quad & \delta_i = 0 \,, \quad a = 0 \,, \quad i = 1,2,3,4 \,. \end{aligned} \tag{2.11}$$

2.1 Kruskal-Szekeres Coordinates for Subtracted Rotating Geometry

In the following we construct Kruskal-Szekeres type coordinates to cover the outer horizon which allow us to identify suitable boundary conditions there[2]. At infinity the appropriate boundary condition is boundedness of the solution. The construction of Kruskal-Szekeres coordinates is in fact considerably simpler than that used for the Kerr solution [20, 21].

The subtracted metric (2.1), (2.2) with (2.7) can be cast in the following remarkably simple form[3]:

$$ds^2 = \sqrt{\Delta}\frac{X}{F^2}\left(-dt^2 + \frac{F^2 dr^2}{X^2}\right) + \sqrt{\Delta}d\theta^2 + \frac{F^2 \sin^2\theta}{\sqrt{\Delta}}(d\phi + W dt)^2 \,, \tag{2.12}$$

with

$$W = -\frac{aA_{red}}{F^2} \,, \quad F^2 = (2m)^2 \left[2m(\Pi_c^2 - \Pi_s^2)r + (2m)^2\Pi_s^2 - a^2(\Pi_c - \Pi_s)^2\right] . \tag{2.13}$$

X and A_{red} are defined in (2.2) and we display them again

$$X = r^2 - 2mr + a^2 \,, \quad A_{red} = 2m(\Pi_c - \Pi_s)r + (2m)^2\Pi_s \,. \tag{2.14}$$

Importantly, X, F and W are only functions of r. We also note that the factor Δ (2.7) can be written in terms of F^2 as

$$\Delta = F^2 + (2m)^2 a^2 (\Pi_c - \Pi_s)^2 \sin^2\theta \,. \tag{2.15}$$

It is straightforward to show that

$$\frac{1}{\kappa_\pm} = \frac{2F(r_\pm)}{r_+ - r_-} \,, \tag{2.16}$$

and

$$\Omega_\pm = -W(r_\pm) \,. \tag{2.17}$$

[2] One can analogously construct Kruskal-Szekeres type coordinates to cover the inner horizon region.

[3] This structure was also anticipated in [5] by evaluating the Laplacian of the subtracted rotating geometry.

This special property of the angular velocities and surface gravities leads to an asymmetry of two branches of the quasi-normal modes as analysed later in this Section.

We now construct Kruskal-Szekeres type coordinates to cover the horizon which allow us to identify suitable boundary conditions there. Due to the structure of the metric (2.12) the construction of Kruskal-Szekeres coordinates is straightforward.

The metric (2.12) allows for the introduction of retarded and advanced co-rotating Eddington-Finkelstein coordinates:

$$u = t - r* \,, \qquad v = t + r* \,, \qquad \phi_+ = \phi + W(r_+)t \,, \tag{2.18}$$

which satisfy

$$g^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0 = g^{\alpha\beta}\partial_\alpha v \partial_\beta v \,. \tag{2.19}$$

The Hamilton-Jacobi equation is separable, yielding a solution

$$r* = \int^r \frac{F dr}{X} \,, \tag{2.20}$$

which is manifest for the metric (2.12).

The co-rotating Kiling vector

$$l^+ = \frac{\partial}{\partial t} - W(r_+)\frac{\partial}{\partial \phi} \,, \tag{2.21}$$

coincides with the null generator of the horizon. The angle ϕ_+ is constant along the orbits of the co-rotating Killing vector l^+:

$$l^+\phi_+ = (\partial_t - W(r_+)\partial_\phi)\phi_+ = 0 \,. \tag{2.22}$$

We introduce Kruskal-Szekeres coordinates:

$$U = -e^{-\kappa_+ u} \,, \qquad V = e^{\kappa_+ v} \,, \tag{2.23}$$

and thus

$$\begin{aligned} \frac{dV}{V} + \frac{dU}{U} &= \frac{2\kappa_+ F dr}{X} \,, \\ \frac{dV}{V} - \frac{dU}{U} &= 2\kappa_+ dt \,. \end{aligned} \tag{2.24}$$

In terms of Kruskal-Szekeres coordinates the metric (2.12) takes the following form:

$$\begin{aligned} ds^2 &= \sqrt{\Delta}\,\frac{X}{F^2}\frac{dUdV}{\kappa_+^2 UV} + \sqrt{\Delta}d\theta^2 \\ &+ \frac{F^2\sin^2\theta}{\sqrt{\Delta}}\left[d\phi_+ + \frac{1}{2\kappa_+}(W(r) - W(r_+))(\frac{dV}{V} - \frac{dU}{U})\right]^2 \,. \end{aligned} \tag{2.25}$$

In the vicinity of the outer horizon $r \sim r_+$ one has

$$r* = \int^r \frac{F(r)dr}{X} \sim \frac{F(r_+)}{r_+ - r_-} \ln(r - r_+) = \frac{1}{2\kappa_+} \ln(r - r_+)\,, \tag{2.26}$$

where we used (2.16) at the last step. This ensures

$$-UV = e^{2\kappa_+ r*} \sim (r - r_+)\,, \tag{2.27}$$

and the metric (2.25) is regular and analytic.

An argument given by Hawking and Reall [22] in the asymptotically AdS case may be adapted to show that if the co-rotating Killing vector l^+ (2.21) is timelike outside the horizon then there can be no super-radiance instability or a black hole bomb [23, 24].

The length squared of the co-rotating Killing vector l^+ (2.21) is

$$g^{\alpha\beta} l^+_\alpha l^+_\beta = -\frac{1}{\sqrt{\Delta}} \left[X + \frac{a^2 \sin^2\theta (\Pi_c - \Pi_s)^2 (r_+ - r_-)(r - r_+)}{[(\Pi_c - \Pi_s) r_+ + 2m\Pi_s]^2} \right] . \tag{2.28}$$

which is manifestly negative for $r > r_+$ and thus their is no super-radiance.

2.2 Massless Wave Equation and Quasi-Normal Modes

The massless scalar wave equation for the multi-charge black hole metric (2.1) is separable and the solutions expressible in terms of spheroidal functions of θ [2, 1]. The radial function may be expressed in terms of solutions of a confluent form of Heun's equation which has two regular singular points and an irregular singular point at infinity.

For the subtracted geometry metric (2.12) the massless scalar wave equation is also separable and of a specific form:

$$e^{-i\omega t} e^{in\phi} P^n_l(\theta) \chi(x)\,, \tag{2.29}$$

where $P^n_l(\theta)$ is an associated Legendre polynomial, the solution of the unit two-sphere S^2 Laplacian with eigenvalues $l(l+1)$, $l = 0, 1, \ldots$ and $n = \pm l, \pm(l-1), \ldots$

The radial equation takes the form [2, 1]:

$$\Big[\frac{\partial}{\partial x}(x^2 - \frac{1}{4})\frac{\partial}{\partial x} + \frac{1}{4(x - \frac{1}{2})}(\frac{\omega}{\kappa_+} - n\frac{\Omega_+}{\kappa_+})^2 - \frac{1}{4(x + \frac{1}{2})}(\frac{\omega}{\kappa_-} - n\frac{\Omega_-}{\kappa_-})^2 - l(l+1)\Big]\chi(x) = 0\,, \tag{2.30}$$

where

$$x = \frac{r - \frac{1}{2}(r_+ + r_-)}{r_+ - r_-}\,, \tag{2.31}$$

is designed so that the two horizons $r_\pm$ are at $x = \pm\frac{1}{2}$.

Due to (2.10) rotating solutions have the property:

$$\frac{\Omega_+}{\kappa_+} = \frac{\Omega_-}{\kappa_-}\,, \tag{2.32}$$

and thus the solutions to (2.30) depend only on one ratio $\Omega_+\kappa_+^{-1}$, only.

Solutions which are ingoing on the future horizon must be regular at $U = 0$ in Kruskal-Szekeres coordinates and this implies [2, 1, 5]

$$\begin{aligned}\chi(x) &= (x+\frac{1}{2})^{-(l+1)}(\frac{x-\frac{1}{2}}{x+\frac{1}{2}})^{-i(\omega-n\Omega_+)\frac{\beta_H}{4\pi}} \\ &\times\ F(l+1-i\frac{\beta_R\omega-2n\beta_H\Omega_+}{4\pi}, l+1-i\frac{\beta_L\omega}{4\pi}, 1-i\frac{\beta_H(\omega-n\Omega_+)}{2\pi}; \frac{x-\frac{1}{2}}{x+\frac{1}{2}})\,,\end{aligned} \tag{2.33}$$

where

$$\frac{\beta_H}{2\pi} = \frac{1}{\kappa_+}\,, \quad \frac{\beta_R}{2\pi} = \frac{1}{\kappa_+} + \frac{1}{\kappa_-}\,, \quad \frac{\beta_L}{2\pi} = \frac{1}{\kappa_+} - \frac{1}{\kappa_-}\,. \tag{2.34}$$

Near the outer horizon $r^\star \to -\infty$, $(x-\frac{1}{2})(x+\frac{1}{2})^{-1} \to e^{2\kappa_+ r^\star}$ and so

$$\chi(x) \approx e^{-i(\omega-n\Omega_+)r^\star} F(l+1-i\frac{\beta_R\omega-2n\beta_H\Omega_+}{4\pi}, l+1-i\frac{\beta_L\omega}{4\pi}, 1-i\frac{\beta_H(\omega-n\Omega_+)}{2\pi}; e^{2\kappa_+ r^\star})\,. \tag{2.35}$$

In Kruskal-Szekeres coordinates therefore

$$e^{-i\omega t}e^{in\phi}\chi(x) \approx e^{in\phi_+}V^{-i\frac{\omega-n\Omega_+}{\kappa_+}}(1+\dots)\,, \tag{2.36}$$

where the ellipses denote a power series in UV which is convergent in a neighbourhood of the future horizon $U = 0$.

At large x [2, 1]

$$\begin{aligned}\chi(x) &\approx x^{-(l+1)}\frac{\Gamma(1-i\frac{\beta_H(\omega-n\Omega_+)}{2\pi})\Gamma(-2l-1)}{\Gamma(-l-i\frac{\beta_L\omega}{4\pi})\Gamma(-l-i\frac{\omega\beta_R-2n\beta_H\Omega_+}{4\pi})} \\ &+\ x^l\frac{\Gamma(1-i\frac{\omega\beta_H(\omega-n\Omega_+)}{2\pi})\Gamma(2l+1)}{\Gamma(l+1-i\frac{\beta_L\omega}{4\pi})\Gamma(l+1-i\frac{\omega\beta_R-2n\beta_H\Omega_+}{4\pi})}\,.\end{aligned} \tag{2.37}$$

In order that χ be finite at spatial infinity, we must set

$$\begin{aligned} i\omega\frac{\beta_L}{4\pi} &= l+1+N_L\,, \\ \text{or} \qquad i\frac{\omega\beta_R-2n\beta_H\Omega_+}{4\pi} &= l+1+N_R\,,\end{aligned} \tag{2.38}$$

where $N_{L,R} = 0, 1, \dots$ This gives remarkably simple formulae for the frequencies of the quasi-normal modes

$$\begin{aligned} \omega &= -\frac{i}{2m(\Pi_c-\Pi_s)}(1+l+N_L)\,, \\ \text{or} \qquad \omega &= -\frac{i\sqrt{m^2-a^2}}{2m^2(\Pi_c+\Pi_s)}(1+l+N_R) + \frac{a}{2m^2(\Pi_c+\Pi_s)}n\,.\end{aligned} \tag{2.39}$$

Both frequencies result in damped modes, with the under-damped branch exhibiting oscillatory behaviour and the damping absent in the extremal limit $a \to m$. The specific asymmetry in frequencies of the two branches, resulting in the oscillatory behaviour of the under-damped branch only, is due to the special relationship between ratios (2.32). It is intriguing that the expressions are no more complex than those in the Kerr case [25]. In particular, eq. (2.39) agrees with eq. (0.28) of [25] which was obtained for the subtracted geometry of the neutral Kerr solution, i.e. the case with $\delta_i = 0$, and thus $\Pi_c = 1$ and $\Pi_s = 0$.

The subtracted geometry has a remarkable property that in the near-BPS limit ($m \to 0$, $a \to 0$, $\delta_i \to \infty$, with $me^{2\delta_i}$ and ma^{-1} finite) the near-horizon geometry of such black holes and their subtracted geometry are the same. As a consequence, the quasi-normal modes of the near-BPS black holes and those of their subtracted geometry are the same[4].

3 Subtracted Magnetised Geometry

The original subtracted Melvin metric was derived in [16] as a scaling limit of magnetised STU black holes. It describes a generalization of the (static) subtracted geometry, parameterised by an additional magnetic field parameter β_4 which is associated with the magnetic component of the Kaluza-Klein gauge field $\mathcal{A}_2$. The full solution is given in the Appendix 5.2.

Remarkably, one may cast this metric in the same form as the rotating subtracted metric (2.12), which we display again

$$ds^2 = \sqrt{\Delta}\frac{X}{F^2}\Big(-dt^2 + \frac{F^2 dr^2}{X^2}\Big) + \sqrt{\Delta} d\theta^2 + \frac{F^2 \sin^2\theta}{\sqrt{\Delta}}(d\phi + W dt)^2 \,, \tag{3.1}$$

[4]We are grateful for Shahar Hod for pointing out to us after the appearance of [25] that if one specialises to the near-BPS case of slowly rotating ($a \ll m$) Kerr-Newman black holes then $\beta_R \simeq 2\beta_H$ and the family of modes given by eq. (11) of [26] have identical frequencies to those of the second family of modes in eq. (0.28) of [25] and hence to the second family of (2.39) of this paper. The first family of (2.39) in this limit corresponds to negative imaginary frequencies whose absolute values are much larger than those of the second family, and thus this (ultra-damped) branch did not appear in [26].

where now

$$
\begin{aligned}
X &= r^2 - 2mr\,,\\
F^2 &= (2m)^3\left[(\Pi_c^2-\Pi_s^2)r+(2m)\Pi_s^2\right]\,,\\
W &= -\frac{16m^4\Pi_s\Pi_c\beta_4}{F^2}\,,\\
\Delta &= F^2+(2m)^6\beta_4^2(\Pi_c^2-\Pi_s^2)^2\sin^2\theta\,.
\end{aligned}
\tag{3.2}
$$

This is effectively a generalization of the static subtracted geometry with the magnetic field parameter β_4 introducing a specific spatial rotation. The metric has two horizons

$$
r_+ = 2m\,,\quad r_- = 0\,. \tag{3.3}
$$

The inverse surface gravities of the inner and outer horizon are determined by

$$
\frac{1}{\kappa_+}=\frac{2F(r_+)}{r_+-r_-}=4m\Pi_c\,,\quad \frac{1}{\kappa_-}=\frac{2F(r_-)}{r_+-r_-}=4m\Pi_s\,, \tag{3.4}
$$

and are the same as the inverse surface gravities for the static subtracted geometry, i.e. (2.9) with $a=0$. The angular velocities at the inner and outer horizon are are given by

$$
\Omega_+=-W(r_+)=\beta_4\frac{\Pi_s}{\Pi_c}\,,\quad \Omega_-=-W(r_-)=\beta_4\frac{\Pi_c}{\Pi_s}\,. \tag{3.5}
$$

Note that in this case the ratios

$$
\frac{\Omega_+}{\kappa_+}=4m\beta_4\Pi_s\,,\quad \frac{\Omega_-}{\kappa_-}=4m\beta_4\Pi_c\,, \tag{3.6}
$$

are different, and now the radial part of the massless scalar wave equation (2.30) depends on both independent ratios.

3.1 Kruskal-Szekeres Coordinates for Subtracted Magnetised Geometry

The retarded and advanced co-rotating Eddington-Finkelstein coordinates are of the same form as in (2.18) and the Killing vector l^+ (2.21) again coincides with the null generator on the horizon.

We introduce the Kruskal-Szekeres coordinates (2.23) which yield (2.24) and the metric (3.1) takes the form (2.25). In the vicinity of the outer horizon $r\sim 2m$ one obtains $-UV\sim(r-2m)$, and thus the metric (2.25) is regular and analytic there.

We calculate the length squared of the co-rotating Killing vector l^+ (2.21)

$$
\begin{aligned}
g^{\alpha\beta}l^+_\alpha l^+_\beta &= -\frac{\sqrt{\Delta_s}}{F}\frac{(r-2m)}{(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2+8m^3\beta_4^2\sin^2\theta(\Pi_c^2-\Pi_s^2)^2}\\
&\quad\times\left[(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2\right]\times\left[r+8m^3\beta_4^2\sin^2\theta\frac{1}{\Pi_c^2}(\Pi_c^2-\Pi_s^2)^2\right]\,,
\end{aligned}
\tag{3.7}
$$

which is negative outside the horizon, $r > 2m$. Thus, this geometry is stable with no super-radiance.

3.2 Massless Wave Equation and Quasi-Normal Modes

The massless wave equation is again separable with the same wave function Ansatz as (2.29). The radial wave equation can be cast in the same form as (2.30) with the inverse surface gravities (3.4) and angular velocities (3.5).

Solutions which are ingoing on the future horizon must be regular at $U = 0$ in Kruskal-Szekeres coordinates and this implies that [2, 1, 5]

$$\chi(x) = (x + \frac{1}{2})^{-(l+1)} (\frac{x - \frac{1}{2}}{x + \frac{1}{2}})^{-i(\omega - n\Omega_+)\frac{\beta_H}{4\pi}}$$
$$F(l + 1 - i\frac{\beta_R\omega - n(\beta_H\Omega_+ + \beta_-\Omega_-)}{4\pi}, l + 1 - i\frac{\beta_L\omega - n(\beta_H\Omega_+ - \beta_-\Omega_-)}{4\pi}, 1 - i\frac{\beta_H(\omega - n\Omega_+)}{2\pi}; \frac{x - \frac{1}{2}}{x + \frac{1}{2}}),$$

where again

$$\frac{\beta_H}{2\pi} = \frac{1}{\kappa_+}, \quad \frac{\beta_-}{2\pi} = \frac{1}{\kappa_-} \quad \frac{\beta_R}{2\pi} = \frac{1}{\kappa_+} + \frac{1}{\kappa_-}, \quad \frac{\beta_L}{2\pi} = \frac{1}{\kappa_+} - \frac{1}{\kappa_-}. \tag{3.8}$$

Near the outer horizon $r^\star \to -\infty$, $(x - \frac{1}{2})(x + \frac{1}{2})^{-1} \to e^{2\kappa_+ r^\star}$ and so

$$\chi(x) \approx e^{-i(\omega - n\Omega_+)r^\star}$$
$$F(l + 1 - i\frac{\beta_R\omega - n(\beta_H\Omega_+ + \beta_-\Omega_-)}{4\pi}, l + 1 - i\frac{\beta_L\omega - n(\beta_H\Omega_+ - \beta_-\Omega_-)}{4\pi}, 1 - i\frac{\beta_H(\omega - n\Omega_+)}{2\pi}; e^{2\kappa_+ r^\star}).$$

In Kruskal-Szekeres coordinates therefore

$$e^{-i\omega t}e^{in\phi}\chi(x) \approx e^{in\phi_+}V^{-i\frac{\omega - n\Omega_+}{\kappa_+}}(1 + \dots) \tag{3.9}$$

where the ellipses denote a power series in UV which is convergent in a neighbourhood of the future horizon $U = 0$.

At large x [2, 1]

$$\chi(x) \approx x^{-(l+1)}\frac{\Gamma(1 - i\frac{\beta_H(\omega - n\Omega_+)}{2\pi})\Gamma(-2l - 1)}{\Gamma(-l - i\frac{\beta_L\omega - n(\beta_H\Omega_+ - \beta_-\Omega_-)}{4\pi})\Gamma(-l - i\frac{\omega\beta_R - n(\beta_H\Omega_+ + \beta_-\Omega_-)}{4\pi})}$$
$$+ x^l\frac{\Gamma(1 - i\frac{\omega\beta_H(\omega - n\Omega_+)}{2\pi})\Gamma(2l + 1)}{\Gamma(l + 1 - i\frac{\beta_L\omega - n(\beta_H\Omega_+ - \beta_-\Omega_-)}{4\pi})\Gamma(l + 1 - i\frac{\omega\beta_R - n(\beta_H\Omega_+ + \beta_-\Omega_-)}{4\pi})}. \tag{3.10}$$

In order that χ be finite at spatial infinity, we must set

$$i\left(\frac{\omega\beta_L}{4\pi} - n\frac{\beta_H\Omega_+ - \beta_-\Omega_-}{4\pi}\right) = l + 1 + N_L,$$
$$\text{or} \qquad i\left(\frac{\omega\beta_R}{4\pi} - n\frac{\beta_H\Omega_+ + \beta_-\Omega_-}{4\pi}\right) = l + 1 + N_R, \tag{3.11}$$

where $N_{L,R} = 0, 1, \dots$ This gives remarkably simple and symmetric formulae for the frequencies of the quasi-normal modes

$$\begin{aligned} \omega &= -\frac{i}{2m(\Pi_c - \Pi_s)}(1 + l + N_L) - n\beta_4\,, \\ \text{or} \qquad \omega &= -\frac{i}{2m(\Pi_c + \Pi_s)}(1 + l + N_R) + n\beta_4\,. \end{aligned} \tag{3.12}$$

Both frequencies result in damped modes with a symmetric shift in advanced and retarded oscillatory behaviour due to the magnetic field parameter β_4.

An interesting observation can be made here about the magnetic field parameter in the above quasi-normal modes. According to the Bohr's correspondence principle, the frequency of oscillation of a classical system is equivalent to the frequency of transition of the corresponding quantum system. Guided by this principle, in [17], some observations were made which indicate that the real part of the quasi-normal modes is related to the quantized area spectrum of the quantum black hole. In our case the real part of the quasi-normal modes is related in a very simple way to the magnetic field parameter, thus making it easy to see how turning on the magnetic field affects the area spectrum of the quantum black hole.

4 Lifted Geometries and Quasi-Normal Modes

In Appendix 5.3 we derive the explicit lift of the subtracted geometries on a circle of size $2\pi R$ and parameterised by a coordinate z. The five-dimensional geometry is locally BTZ $\times$ S^2 with the BTZ coordinates denoted by $\{t_3, r_3, \phi_3\}$ and the S^2 coordinates denoted by $\{\theta, \bar{\phi}\}$. The explicit transformation between $\{t, r, \theta, \phi, z\}$ coordinates, and the BTZ$\times S^2$ coordinates is given in the Appendix 5.3, too. The BTZ metric (5.29) can also be cast into local AdS$_3$ metric (5.39), parameterised by coordinates $\{T, \rho, \Phi\}$. The explicit transformation between the BTZ and the local AdS$_3$ coordinates is given in Appendix 5.4, following [18, 19]. The radius of AdS$_3$ is ℓ and the radius of S^2 is $\frac{\ell}{2}$. Specifically, $\ell = 4m(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}$.

Since for this five-dimensional geometry the wave equation for the minimally coupled massive scalar field is separable and exactly solvable, this allows us to study explicitly the quasi-normal modes directly in five dimensions. Furthermore, the scalar field wave function can be expanded in terms of Kaluza-Klein modes, parameterised by a quantised wave number k along the circle direction z. We can therefore study the quasi-normal modes for each Kaluza-Klein mode by solving directly the wave equation in five dimensions for the complete tower of Kaluza-Klein states, i.e. we do not have to resort to solving a complicated equation for each Kaluza-Klein mode separately.

The wave equation for a massive, minimally coupled scalar field Φ in the local $\mathrm{AdS}_3 \times S^2$ background is separable and solved with the Ansatz

$$\mathbf{\Phi} = e^{-i\bar{\omega}T} e^{i\bar{k}\Phi} e^{in\bar{\phi}} \, P_l^n(\cos\theta)\, \chi(\rho)\,. \tag{4.1}$$

$P_l^n(\cos\theta)$, the associated Legendre function, is a solution for the Laplacian of the unit two-sphere S^2 with eigenvalues $l(l+1)$. Here $n = 0, \pm 1, \pm 2 ... \pm l$ and l is a non-negative integer. Again, $\{T, \Phi, \rho\}$ and $\{\theta, \bar{\phi}\}$ parameterise the local AdS_3 and S^2 coordinates, respectively. Furthermore, in our context the radius of AdS_3 is ℓ and that of S^2 is $\frac{\ell}{2}$ where we have $\ell = 4m(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}$ (see Appendix 5.3).

The metric, describing a local AdS_3 (5.39)

$$ds^2_{AdS_3} = \ell^2\,(-\sinh^2\rho\, dT^2 + d\rho^2 + \cosh^2\rho\, d\Phi^2)\,. \tag{4.2}$$

has the Laplacian

$$\Box_{AdS_3} = \partial_\rho^2 + \frac{2\cosh(2\rho)}{\sinh(2\rho)}\partial_\rho - \frac{1}{\sinh^2\rho}\partial_T^2 + \frac{1}{\cosh^2\rho}\partial_\Phi^2\,, \tag{4.3}$$

and enters the five-dimensional Klein-Gordon equation equation in the following form:

$$[\ell^2\,(\Box_{AdS_3} - 4l(l+1)) - M_5^2]\mathbf{\Phi} = 0 \tag{4.4}$$

Note again that $4\ell^2 l(l+1)$ is the eigenvalue of the two-sphere S^2 Laplacian with the two-sphere radius $\frac{\ell}{2}$. For the Ansatz (4.1) this equation becomes

$$\Big[\ell^2\big(\partial_\rho^2 + \frac{2\cosh(2\rho)}{\sinh(2\rho)}\partial_\rho + \frac{\bar{\omega}^2}{\sinh^2\rho} - \frac{\bar{k}^2}{\cosh^2\rho} - 4l(l+1)\big) - M_5^2\Big])\chi(\rho) = 0\,. \tag{4.5}$$

The solution, corresponding to the incoming wave at the outer horizon, is

$$\begin{aligned}\chi(\rho) &= (x + \frac{1}{2})^{-(\bar{l}+1)}\big(\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\big)^{-i\frac{\bar{\omega}}{2}} \\ &F(\bar{l}+1 - i\frac{(\bar{\omega}+\bar{k})}{2}, \bar{l}+1-i\frac{(\bar{\omega}-\bar{k})}{2}, 1 - i\bar{\omega}; \tanh^2\rho)\,.\end{aligned} \tag{4.6}$$

Here we have introduced

$$\bar{l}(\bar{l}+1) \equiv l(l+1) + \frac{M_5^2}{4\ell^2}\,. \tag{4.7}$$

While the analysis can be completed for massive minimally coupled five-dimensional scalars, in the following we will focus on massless ones, i.e. taking $M_5 = 0$ and thus $\bar{l} = l$. The only quantitative difference in the analysis for massive five-dimensional scalars is that the expressions below involve a change $l \to \bar{l} > l$, and thus a shift in the quasi-normal frequencies.

At this point we relate the respective local AdS$_3$ and S^2 coordinates $\{T, \Phi, \rho\}$ and $\{\theta, \bar{\phi}\}$ to $\{t, r, \theta, \phi, z\}$. This can be done by first employing Appendix 5.3, where the explicit lift to the BTZ $\times S^2$ and the map to the BTZ and S^2 coordinates is given, and then employing Appendix 5.4, where the transformation between the BTZ and local AdS$_3$ coordinates is provided. The result for the subtracted rotating geometry is

$$\begin{aligned} T &= \frac{4\sqrt{m^2 - a^2}}{\ell^3}\left(\frac{t}{\kappa_+} - \frac{z}{\kappa_-}\right), \\ \Phi &= \frac{4\sqrt{m^2 - a^2}}{\ell^3}\left(\frac{z}{\kappa_+} - \frac{t}{\kappa_-}\right), \end{aligned} \tag{4.8}$$

and

$$\cosh^2 \rho = x + \frac{1}{2}, \quad \sinh^2 \rho = x - \frac{1}{2}, \tag{4.9}$$

where x is defined in (2.31), i.e. $x = \left[r - \frac{1}{2}(r_+ + r_-)\right](r_+ - r_-)^{-1}$. Furthermore, for S^2 coordinates, θ is unchanged and the azimuthal angle $\bar{\phi}$ is related to ϕ as in (5.28):

$$\bar{\phi} = \phi - \frac{16ma(\Pi_c - \Pi_s)}{\ell^3}(z + t). \tag{4.10}$$

The 2π periodicity of $\bar{\phi}$ is ensured if $16ma(\Pi_c - \Pi_s)\ell^{-3} = a(2m)^{-2}(\Pi_c + \Pi_s)^{-1}$ is quantized in units of R^{-1}.

The radial equation (4.5) can be cast in the following form:

$$\left[\partial_x(x^2 - \frac{1}{4})\partial_x + \frac{\bar{\omega}^2}{4(x - \frac{1}{2})} - \frac{\bar{k}^2}{4(x + \frac{1}{2})} - l(l+1)\right]\chi(x) = 0\,. \tag{4.11}$$

The above coordinate transformations allow us to relate the quantum numbers in the Ansatz (4.1) to those of the standard Kaluza-Klein Ansatz:[5]

$$\mathbf{\Phi} = e^{-i\omega t} e^{ikz} e^{in\phi}\, P_l^n(\cos\theta)\,\chi(r)\,. \tag{4.12}$$

Namely, equating the two Ansätze (4.1) and (4.12), and employing the coordinate transformations (4.8) and (4.10) yields the following transformation between quantum numbers $\{\bar{\omega}, \bar{k}\}$ and $\{\omega, k\}$:

$$\bar{\omega} = \frac{\omega}{\kappa_+} - \frac{k}{\kappa_-} - n\frac{\Omega_+}{\kappa_+}, \quad \bar{k} = -\frac{\omega}{\kappa_-} + \frac{k}{\kappa_+} + n\frac{\Omega_+}{\kappa_+}, \tag{4.13}$$

and n unchanged.

[5]By abuse of notation we use above the same radial function notation.

For the subtracted magnetised geometry the expressions for (4.8) are the same, but with $a = 0$ and static expressions for inverse surface gravities (3.4), i.e. $\kappa_+^{-1} = 4m\Pi_c$ and $\kappa_-^{-1} = 4m\Pi_s$. The azimuthal angle is shifted due to the magnetic field β_4 as in (5.32):

$$\bar{\phi} = \phi - \beta_4 z\,. \tag{4.14}$$

Note that 2π periodicity of the S^2 azimuthal angle $\bar{\phi}$ is ensured if the magnetic field parameter β_4 is quantised in units of R^{-1}.

As a consequence, the transformation between the quantum numbers $\{\bar{\omega}, \bar{k}\}$ and $\{\omega, k\}$ is

$$\bar{\omega} = \frac{\omega}{\kappa_+} - \frac{k + n\beta_4}{\kappa_-}\,, \quad \bar{k} = -\frac{\omega}{\kappa_-} + \frac{k + n\beta_4}{\kappa_+}\,, \tag{4.15}$$

and again, n unchanged.

These general expressions now allow us to recover results for the massless four-dimensional field with vanishing wave number $k = 0$. For the subtracted rotating geometry one obtains

$$\bar{\omega} = \frac{\omega}{\kappa_+} - n\frac{\Omega_+}{\kappa_+}\,, \quad \bar{k} = -\frac{\omega}{\kappa_-} + n\frac{\Omega_+}{\kappa_+}\,, \tag{4.16}$$

just as in Section 2. Similarly for the magnetised subtracted geometry:

$$\bar{\omega} = \frac{\omega}{\kappa_+} - \frac{n\beta_4}{\kappa_-}\,, \quad \bar{k} = -\frac{\omega}{\kappa_-} + \frac{n\beta_4}{\kappa_+}\,, \tag{4.17}$$

in agreement with Section 3.

We can also study massive Kaluza-Klein modes with the wave number $k \neq 0$, which is quantised in units of R^{-1}, where R is the radius of the circle S^1. Those are massive four-dimensional particles with mass $m_4 \propto k$, and they are charged under the Kaluza-Klein U(1) gauge symmetry with the charge $k = q$ (see Appendix 5.5). Their quasi-normal modes can be determined completely analogously to massless modes in Sections 2 and 3.

The solution (4.6), corresponding to the incoming wave at the outer horizon, is required to be finite at a large x, which is achieved for

$$\frac{\bar{\omega} + \bar{k}}{2} = -i(1 + l + N_L)\,, \quad \text{or} \quad \frac{\bar{\omega} - \bar{k}}{2} = -i(1 + l + N_R)\,, \tag{4.18}$$

where $l = 0, 1, \ldots,$ and $N_L = 0, 1, \ldots$ or $N_R = 0, 1, \ldots$ This constrains a specific combination of ω and k. In the rotating case we have

$$\begin{aligned} &\omega = -\frac{i}{2m(\Pi_c - \Pi_s)}(1 + l + N_L) + k\,, \\ \text{or} \qquad &\omega = -\frac{i\sqrt{m^2 - a^2}}{2m^2(\Pi_c + \Pi_s)}(1 + l + N_R) + \frac{a}{2m^2(\Pi_c + \Pi_s)}n - k\,. \end{aligned} \tag{4.19}$$

In the subtracted magnetised case we obtain

$$\begin{aligned} \omega &= -\frac{i}{2m(\Pi_c - \Pi_s)}(1 + l + N_L) + n\beta_4 + k\,, \\ \text{or} \quad \omega &= -\frac{i}{2m(\Pi_c + \Pi_s)}(1 + l + N_R) - n\beta_4 - k\,. \end{aligned} \tag{4.20}$$

Again, we obtained two branches of damped quasi-normal modes, both with oscillatory behaviour symmetrically advanced and retarded by $n\beta_4 + k$.

It is interesting to point out that the solution (4.6) for massive modes with $k \neq 0$ has a regular, analytic behaviour near the outer horizon, after one has made a gauge transformation $\chi(x) \to e^{ik\mathcal{A}_{2t+}t}\chi(x)$, where $\mathcal{A}_{2t+} = (2m)^4\Pi_c\Pi_s F^{-2}(r_+)$ is the time component of the Kaluza-Klein gauge potential $\mathcal{A}_2$ (5.4) or (5.24), evaluated at the outer horizon r_+. Namely, we obtain

$$\begin{aligned} e^{ik\mathcal{A}_{2t+}t}e^{-i\omega t}e^{in\phi}\chi(x) &\approx e^{ik\mathcal{A}_{2t+}t}e^{-i(\omega-n\Omega_+)t}e^{in\phi_+}e^{-i\bar{\omega}\kappa_+ r*}(1+\cdots) \\ &\approx e^{in\phi_+}V^{-i\frac{\omega-n\Omega_+}{\kappa_+}+i\frac{k}{\kappa_-}}(1+\dots)\,, \end{aligned} \tag{4.21}$$

where we wrote the final expression in terms of Kruskal-Szekeres coordinates, and the ellipses denote a power series in UV which is convergent in a neighbourhood of the future horizon $U = 0$.

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5 Appendices

5.1 Subtracted Rotating Geometry with Sources

In [9] it was shown that the subtracted geometry (2.1), (2.2), (2.7) for four-charge rotating black hole is a solution of the equations of motion for the STU Lagrangian, describing the bosonic part of the N=2 supergravity Lagrangian coupled to three vector super-multiplets:

$$\begin{aligned}\mathcal{L}_4 =\, & R\,{*\mathbf{1}} - \frac{1}{2}{*d\varphi_i} \wedge d\varphi_i - \frac{1}{2}e^{2\varphi_i}\,{*d\chi_i} \wedge d\chi_i - \frac{1}{2}e^{-\varphi_1}\,(e^{\varphi_2-\varphi_3}\,{*F_1} \wedge F_1 \\ & + e^{\varphi_2+\varphi_3}\,{*F_2} \wedge F_2 + e^{-\varphi_2+\varphi_3}\,{*\mathcal{F}_1} \wedge \mathcal{F}_1 + e^{-\varphi_2-\varphi_3}\,{*\mathcal{F}_2} \wedge \mathcal{F}_2) \\ & - \chi_1\,(F_1 \wedge \mathcal{F}_1 + F_2 \wedge \mathcal{F}_2)\,, \end{aligned} \tag{5.1}$$

where the index i labelling the dilatons φ_i and axions χ_i ranges over $1 \leq i \leq 3$. The four U(1) field strengths can be written in terms of potentials as

$$\begin{aligned} F_1 &= dA_1 - \chi_2\, d\mathcal{A}_2\,, \\ F_2 &= dA_2 + \chi_2\, d\mathcal{A}_1 - \chi_3\, dA_1 + \chi_2\,\chi_3\, d\mathcal{A}_2\,, \\ \mathcal{F}_1 &= d\mathcal{A}_1 + \chi_3\, d\mathcal{A}_2\,, \\ \mathcal{F}_2 &= d\mathcal{A}_2\,. \end{aligned}$$

The three axio-scalar fields and the four U(1) gauge potentials can be formally obtained as a scaling limit of a certain black hole solution (for details, see [9]), resulting in

$$\chi_1 = \chi_2 = \chi_3 = -\frac{2ma(\Pi_c - \Pi_s)\cos\theta}{Q^2}, \quad e^{\varphi_1} = e^{\varphi_2} = e^{\varphi_3} = \frac{Q^2}{\sqrt{\Delta}}\,, \tag{5.2}$$

and the gauge potentials $A_1 = A_2 = A_3 \equiv A$ for gauge field strengths $*F_1 = F_2 = *\mathcal{F}_1 \equiv F$ and $\mathcal{A}_2$ for $\mathcal{F}_2$ are of the following form:

$$\begin{aligned} A =\; & -\frac{r}{Q}dt + \frac{(2m)^2a^2[2m\Pi_s^2 - r(\Pi_c - \Pi_s)^2]\cos^2\theta}{Q\Delta}dt \\ & - \frac{2m\, a(\Pi_c - \Pi_s)\sin^2\theta}{Q}\left(1 + \frac{(2m)^2a^2(\Pi_c - \Pi_s)^2\cos^2\theta}{\Delta}\right)d\phi\,, \end{aligned} \tag{5.3}$$

$$\mathcal{A}_2 = \frac{Q^3[(2m)^2\Pi_c\Pi_s + a^2(\Pi_c - \Pi_s)^2\cos^2\theta]}{2m(\Pi_c^2 - \Pi_s^2)\Delta}\,dt + \frac{Q^3 2m\, a(\Pi_c - \Pi_s)\sin^2\theta}{\Delta}\,d\phi\,, \tag{5.4}$$

where

$$Q = 2m(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}\epsilon^{-\frac{1}{3}} \equiv \tfrac{1}{2}\ell\epsilon^{-\frac{1}{3}}, \quad \text{as} \quad \epsilon \to 0\,. \tag{5.5}$$

and again, Δ defined as in (2.7):

$$\Delta_0 \to \Delta = (2m)^3 r(\Pi_c^2 - \Pi_s^2) + (2m)^4\Pi_s^2 - (2m)^2(\Pi_c - \Pi_s)^2a^2\cos^2\theta\,. \tag{5.6}$$

The (formally infinite) factors of Q can in principle be removed from gauge potentials by removing corresponding factors from scalar fields. However, when lifting the scaling limit solution to five dimensions, it is useful to keep this scaling factor explicit; in the final five-dimensional metric an overall factor is not relevant.

5.2 Subtracted Magnetised Geometry with Sources

The magnetised solution of the static STU black hole was obtained in [16] and is of the form:

$$ds_4^2 = H\,[-r(r-2m)dt^2 + \frac{r_1r_2r_3r_4}{r(r-2m)}\,dr^2 + r_1r_2r_3r_4 d\theta^2\,] + H^{-1}\,\sin^2\theta\,(d\phi - \tilde{\omega}dt)^2\,. \quad (5.7)$$

Here

$$r_i = r + 2ms_i^2\,, \quad (5.8)$$

and we shall use the notation $s_i = \sinh\delta_i$ and $c_i = \cosh\delta_i$, with $i = 1, 2, 3, 4$. The function $\tilde{\omega}$ is given by

$$\tilde{\omega} = \sum_{i=1}^{4}\Big[-\frac{q_i\,\beta_i}{r_i} + \frac{q_i\,\Xi_i\,[r_i + (r-2m)\cos^2\theta]r}{r_i}\Big]\,, \quad (5.9)$$

where

$$q_i = 2ms_ic_i\,, \qquad \Xi_i = \frac{\beta_1\beta_2\beta_3\beta_4}{\beta_i}\,, \qquad \beta_i = \tfrac{1}{2}B_i\,, \quad (5.10)$$

and B_i $(i = 1, 2, 3, 4)$ denote the external magnetic field strengths for each of the four gauge fields. Finally, the function H is given by

$$H = \frac{\sqrt{\bar{\Delta}}}{\sqrt{r_1r_2r_3r_4}}\,, \quad (5.11)$$

where

$$\begin{aligned}\bar{\Delta} \,=\, & 1 + \sum_i \frac{\beta_i^2 r_1r_2r_3r_4}{r_i^2}\,\sin^2\theta + 2[\beta_3\beta_4q_1q_2 + \cdots]\cos^2\theta + [\beta_3^2\,\beta_4^2\,R_1^2\,R_2^2 + \cdots] \\ & -2(\prod_j \beta_jr_j)\,\sum_i \frac{q_i^2}{r_i^2}\,\sin^2\theta\cos^2\theta + [2\beta_2\beta_3\beta_4^2q_2q_3\,R_1^2 + \cdots]\cos^2\theta + \prod_i \beta_i^2\,R_i^2 \\ & + r_1r_2r_3r_4\,\sum_i \frac{\Xi_i^2\,R_i^2}{r_i^2}\,\sin^2\theta + [2\beta_1\beta_2\beta_3^2\beta_4^2q_3q_4\,R_1^2\,R_2^2 + \cdots]\cos^2\theta\,,\end{aligned} \quad (5.12)$$

and we have defined

$$R_i^2 = r_i^2\,\sin^2\theta + q_i^2\,\cos^2\theta\,. \quad (5.13)$$

The Kaluza-Klein gauge field here is given by

$$\mathcal{A}_2 = \Big[\frac{q_4}{r_4} - \sum_{i=1}^{3}\frac{r\,q_i\,\beta_1\beta_2\beta_3\,[r_i + (r-2m)\cos^2\theta]}{\beta_i\,r_i}\Big]\,dt - \sigma_4\,(d\phi - \tilde{\omega}dt)\,, \quad (5.14)$$

where $\sigma_4 = \tilde{\sigma_4}\bar{\Delta}^{-1}$, and

$$
\begin{aligned}
\tilde{\sigma}_4 =\ & \frac{\beta_4 r_1 r_2 r_3}{r_4}\sin^2\theta + (\beta_1 q_2 q_3 + \cdots)\cos^2\theta + \beta_4(\beta_1^2 R_2^2 R_3^2 + \cdots) \\
& +2\beta_4(\beta_2\beta_3 q_2 q_3 R_1^2 + \cdots)\cos^2\theta + q_4[\beta_1^2(\beta_2 q_2 R_3^2 + \beta_3 q_3 R_2^2) + \cdots]\cos^2\theta \\
& +4\beta_1\beta_2\beta_3 q_1 q_2 q_3 q_4 \cos^4\theta \\
& -\frac{\beta_1\beta_2\beta_3 q_4^2 r_1 r_2 r_3}{r_4}\sin^2\theta\cos^2\theta - \beta_1\beta_2\beta_3\, r_4\left(\frac{q_1^2 r_2 r_3}{r_1} + \cdots\right)\sin^2\theta\cos^2\theta \\
& +\beta_1\beta_2\beta_3(\beta_2\beta_3 q_2 q_3 R_1^2 + \cdots)R_4^2\cos^2\theta + \beta_4 r_4\left[\frac{\beta_2^2\beta_3^2 r_2 r_3}{r_1}\,R_1^4 + \cdots\right]\sin^2\theta \\
& +2\beta_1\beta_2\beta_3\beta_4 q_4(\beta_1 q_1 R_2^2 R_3^2 + \cdots)\cos^2\theta + \beta_4\beta_1^2\beta_2^2\beta_3^2 R_1^2 R_2^2 R_3^2 R_4^2\,.
\end{aligned}
\tag{5.15}
$$

The dilaton field is given by

$$e^{\varphi_1} = \frac{Y_1}{\sqrt{\bar{\Delta}\, r_1 r_2 r_3 r_4}}\,, \tag{5.16}$$

where

$$
\begin{aligned}
Y_1 =\ & r_1 r_3(1 + 2\beta_1\beta_3 q_2 q_4\cos^2\theta + \beta_1^2\beta_3^2 R_2^2 R_4^2) \\
& + r_2 r_4(\beta_1^2 R_3^2 + \beta_3^2 R_1^2 + 2\beta_1\beta_3 q_1 q_3\cos^2\theta)\,.
\end{aligned}
\tag{5.17}
$$

For explicit expressions of all the fields see [16]. Note however in order to have the same sign for the gauge fields of the rotating and magnetised geometries, we have changed an overall sign for the gauge fields relative to [16].

The Scaling Limit

The subtracted geometry can be obtained by taking a scaling limit of the above magnetised electric black holes, analogously to the rotating case. The limit can be implemented by means of the scalings

$$
\begin{aligned}
& m \to m\,\epsilon\,, \qquad r = r\,\epsilon\,, \qquad t \to t\,\epsilon^{-1}\,, \qquad \beta_i \to \beta_i\,\epsilon\,, \quad i = 1,2,3,4\,, \\
& \sinh^2\delta_4 \to \frac{\Pi_s^2}{\Pi_c^2 - \Pi_s^2}\,, \qquad \sinh^2\delta_i \to (\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}\,\epsilon^{-\frac{4}{3}}\,, \quad i = 1,2,3\,,
\end{aligned}
\tag{5.18}
$$

where ϵ is then sent to zero. In particular, this gives

$$(d\phi - \tilde{\omega}dt) \longrightarrow d\phi - (\beta_1 + \beta_2 + \beta_3)dt - \frac{2m\beta_4\,\Pi_c\Pi_s}{(\Pi_c^2 - \Pi_s^2)r + 2m\Pi_s^2}\,dt\,, \tag{5.19}$$

and

$$\bar{\Delta} \longrightarrow 1 + \frac{(2m)^3\beta_4^2(\Pi_c^2 - \Pi_s^2)^2\sin^2\theta}{(\Pi_c^2 - \Pi_s^2)r + 2m\Pi_s^2}\,, \quad r_1 r_2 r_3 r_4 \longrightarrow (2m)^3\left[(\Pi_c^2 - \Pi_s^2)r + 2m\Pi_s^2\right]\,. \tag{5.20}$$

The quantities β_1, β_2 and β_3 are removed by a gauge transformation $\phi \longrightarrow \phi+(\beta_1+\beta_2+\beta_3)t$. We shall assume from now on that this transformation has been performed. The final metric can be cast in the following form:

$$ds^2 = \sqrt{\Delta}\frac{X}{F^2}\Big(-dt^2 + \frac{F^2 dr^2}{X^2}\Big) + \sqrt{\Delta}d\theta^2 + \frac{F^2 \sin^2\theta}{\sqrt{\Delta}}(d\phi + W dt)^2 \,, \tag{5.21}$$

where

$$\begin{aligned} X &= r^2 - 2mr \,, \\ F^2 &= (2m)^3 \left[(\Pi_c^2 - \Pi_s^2) r + (2m)\Pi_s^2\right] \,, \\ W &= -\frac{16 m^4 \Pi_s \Pi_c \beta_4}{F^2} \,, \\ \Delta &= F^2 + (2m)^6 \beta_4^2 (\Pi_c^2 - \Pi_s^2)^2 \sin^2\theta \,. \end{aligned} \tag{5.22}$$

The dilation fields are of the form:

$$e^{\phi_1} = e^{\varphi_2} = e^{\varphi_3} = \frac{Q^2}{\sqrt{\Delta}} \,, \tag{5.23}$$

and the axion fields vanish. The Kaluza-Klein U(1) gauge field becomes

$$\mathcal{A}_2 = \frac{Q^3 2m \Pi_c \Pi_s}{(\Pi_c^2 - \Pi_s^2) F^2} dt - \frac{Q^3 (2m)^3 \beta_4 (\Pi_c^2 - \Pi_s^2)\sin^2\theta}{\Delta}(d\phi + W dt) \,. \tag{5.24}$$

Note that at the horizon the combination $\phi + W(r_+)t = \phi_+$, and thus the second term in (5.24) becomes the ϕ_+ component of the Kaluza-Klein gauge potential. The remaining three gauge potentials become identified and are of the form (5.3) by setting $a = 0$.

One can of course remove Q in the scalar and gauge fields via a gauge transformation. However, it is useful to keep it in the discussion of the lift and at the end remove the overall scaling parameter ϵ.

5.3 Subtracted Geometry Lifted to Five Dimensions

We now provide a lift of the subtracted rotating geometry to five-dimensions[6]. The five-dimensional metric for the scaling limit takes the form:

$$ds_5^2 = e^{\varphi_1} ds_4^2 \, + e^{-2\varphi_1}(dz + \mathcal{A}_2)^2 \,, \tag{5.25}$$

where we have to implement the scaling $z \to z\epsilon^{-1}$. This metric takes the form:

$$ds_5^2 = \epsilon^{-\frac{2}{3}}(ds_{S^2}^2 + ds_{BTZ}^2) \,, \tag{5.26}$$

[6]Partial results were provided in [5, 9]. Here we take particular care of the dimensions and of the periodicities of metric coordinates.

where

$$ds^2_{S^2} = \tfrac{1}{4}\ell^2\left(d\theta^2 + \sin^2\theta d\bar{\phi}^2\right)\,, \tag{5.27}$$

with

$$\bar{\phi} = \phi - \frac{16ma(\Pi_c - \Pi_s)}{\ell^3}(z+t)\,, \tag{5.28}$$

and

$$ds^2_{BTZ} = -\frac{(r_3^2 - r_{3+}^2)(r_3^2 - r_{3-}^2)}{\ell^2\, r_3^2}dt_3^2 + \frac{\ell^2 r_3^2}{(r_3^2 - r_{3+}^2)(r_3^2 - r_{3-}^2)}\,dr_3^2 + r_3^2(d\phi_3 + \frac{r_{3+}r_{3-}}{\ell r_3^2}\,dt_3)^2\,, \tag{5.29}$$

where

$$\begin{aligned} \phi_3 &= \frac{z}{R}\,, \\ t_3 &= \frac{\ell}{R}\,t\,, \\ r_3^2 &= \frac{16(2mR)^2}{\ell^4}\left[2m(\Pi_c^2 - \Pi_s^2)r + (2m)^2\Pi_s^2 - a^2(\Pi_c - \Pi_s)^2\right]\,. \end{aligned} \tag{5.30}$$

Here, R is the radius of the circle S^1 and $\ell = 4m(\Pi_c^2 - \Pi_s^2)^{\frac{1}{3}}$ is the radius of the AdS_3. Furthermore

$$r_{3\pm} = \frac{8mR}{\ell^2}\left[m(\Pi_c + \Pi_s) \pm \sqrt{m^2 - a^2}(\Pi_c - \Pi_s)\right]\,. \tag{5.31}$$

The periodicity of z coordinate is $2\pi R$, and thus the angular coordinate ϕ_3 has the correct periodicity of 2π. Note also that the 2π periodicity of $\bar{\phi}$ is ensured if $16ma(\Pi_c - \Pi_s)\ell^{-3} = a(2m)^{-2}(\Pi_c + \Pi_s)^{-1}$ is quantized in units of R^{-1}.

The lifted geometry is indeed locally $\mathrm{AdS}_3 \times S^2$ with the radius of AdS_3 equal to ℓ and the radius of S^2 equal to $\frac{\ell}{2}$.

Subtracted Magnetised Geometry

This geometry also lifts to (5.26) where now $\bar{\phi}$ in (5.27) is defined as[7]

$$\bar{\phi} = \phi - \beta_4\, z\,, \tag{5.32}$$

and we set in all expressions above $a = 0$, i.e. the BTZ coordinates are related to $\{t, r, z\}$ as in (5.30) with $a = 0$. (Obviously, $\beta_4 = 0$ corresponds to the lift of the static subtracted geometry.) Note that the shift requires that β_4 be quantized in units of R^{-1}, in order for $\bar{\phi}$ to have the correct periodicity of 2π.

[7]It was observed in [28] that such a shift produces a magnetic field for the Kaluza-Klein U(1) gauge potential and thus a four-dimensional geometry in a Kaluza-Klein magnetic field.

5.4 Relation of the BTZ Black Hole Coordinates to the AdS$_3$ Coordinates

According to [18, 19] AdS$_3$ is the quadric

$$u^2 + v^2 - x^2 - y^2 = \ell^2 \,, \tag{5.33}$$

in $\mathbb{E}^{2,2}$ with the metric induced from

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \,. \tag{5.34}$$

In a local patch we have the embedding

$$u = \sqrt{A(r)} \cosh \Phi = \ell \cosh \rho \cosh \Phi \,, \tag{5.35}$$

$$x = \sqrt{A(r)} \sinh \Phi = \ell \cosh \rho \sinh \Phi \,, \tag{5.36}$$

$$y = \sqrt{B(r)} \cosh T = \ell \sinh \rho \cosh T \,, \tag{5.37}$$

$$v = \sqrt{B(r)} \sinh T = \ell \sinh \rho \sinh T \,. \tag{5.38}$$

The metric is of the form:

$$ds^2_{AdS_3} = \ell^2 \, (-\sinh^2 \rho \, dT^2 + d\rho^2 + \cosh^2 \rho \, d\Phi^2) \,. \tag{5.39}$$

The relationship to the BTZ metric coordinates and parameters introduced in the Appendix 5.3 (eqs.(5.29,5.30)) is

$$A(r) = \ell^2 \frac{r_3^2 - r_{3-}^2}{r_{3+}^2 - r_{3-}^2} \,, \qquad B(r) = \ell^2 \frac{r_3^2 - r_{3+}^2}{r_{3+}^2 - r_{3-}^2} \,, \tag{5.40}$$

$$T = \frac{r_{3+} t_3 - r_{3-} \ell \phi_3}{\ell^2} \qquad \Phi = \frac{r_{3+} \ell \phi_3 - r_{3-} t_3}{\ell^2} \,, \tag{5.41}$$

where $r_{3\pm}$ is defined in (5.31).

Note that a shift in T is a boost in the Minkowski $v-y$ plane and a shift in Φ corresponds to a boost in the Minkowski $u-x$ plane. Since ϕ_3 of the BTZ metric (5.29) is periodic with period 2π, the coordinates $\{T, \Phi\}$ must be identified under the composition of two discrete boosts:

$$\big(T, \Phi\big) \;\rightarrow\; \big(T - \frac{2\pi r_{3-}}{\ell}, \Phi + \frac{2\pi r_{3+}}{\ell}\big) \,. \tag{5.42}$$

5.5 Kaluza-Klein Reduction of the Scalar Wave Equation

The five-dimenskonal Kaluza-Klein metric Ansatz

$$ds_5^2 = e^{\phi_1} \gamma_{\alpha\beta} dx^\alpha dx^\beta + e^{-2\phi_1} (dz + \mathcal{A}_{2\alpha} dx^\alpha)^2 \,, \tag{5.43}$$

where $\{\alpha, \beta\} = 0, 1, 2, 3$, results in the five-dimensional wave equation given by

$$\nabla^\alpha \nabla_\alpha \mathbf{\Phi} - \nabla^\alpha \mathcal{A}_{2\alpha} \partial_z \mathbf{\Phi} - 2\mathcal{A}_2{}^\alpha \nabla_\alpha \partial_z \mathbf{\Phi} + (\mathcal{A}_2)^2 \partial_z^2 \mathbf{\Phi} = -e^{\phi_1} \partial_z^2 \mathbf{\Phi}\,. \tag{5.44}$$

If we make the assumption that $\mathbf{\Phi}$ is separable in term of a four-dimensional wave function and a function of the fifth coordinate z:

$$\mathbf{\Phi}(x^\alpha, z) = \mathbf{\Phi}(x^\alpha) e^{if(z)}\,, \tag{5.45}$$

we can rewrite the above equation as

$$\gamma^{\alpha\beta} \left(\nabla_\alpha - i(\partial_z f)\mathcal{A}_{2\alpha}\right) \left(\nabla_\beta - i(\partial_z f)\mathcal{A}_{2\beta}\right) \mathbf{\Phi}(x^\alpha) = (\partial_z f)^2 e^{\phi_1} \mathbf{\Phi}(\mathbf{x}^\alpha)\,. \tag{5.46}$$

For the compactification on a circle S^1 with radius $2\pi R$, the above equation is solved with the Ansatz for $f(z) = kz$, where the wave number k is quantised in units of R^{-1}. The remaining effective four-dimensional wave equation can then be interpreted as the Klein-Gordon equation of the four-dimensional charged particle with a charge $q = k$ and an effective mass $\propto k$ which is modulated by the scalar field e^{ϕ_1}:

$$m_{eff}^2 = k^2 e^{\phi_1}\,. \tag{5.47}$$

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