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Phys. Rev. D **90**, 114008 — Published 3 December 2014

DOI: 10.1103/PhysRevD.90.114008

The baryon vector current in the chiral and $1/N_c$ expansions

Rubén Flores-Mendieta*[1] and José L. Goity[1,2]

[1] *Theory Center, Thomas Jefferson National Accelerator Facility, Newport News, Virginia 23606, USA*

[2] *Department of Physics, Hampton University, Hampton, Virginia 23668, USA*

(Dated: November 4, 2014)

Abstract

The baryon vector current is computed at one-loop order in large-N_c baryon chiral perturbation theory, where N_c is the number of colors. Loop graphs with octet and decuplet intermediate states are systematically incorporated into the analysis and the effects of the decuplet-octet mass difference and $SU(3)$ flavor symmetry breaking are accounted for, giving the full result to order $\mathcal{O}\left(p^2\right)$ in the chiral expansion. There are large-N_c cancellations between different one-loop graphs as a consequence of the large-N_c spin-flavor symmetry of QCD baryons. The results are compared against the available experimental data through several fits in order to extract information about the unknown parameters. The large-N_c baryon chiral perturbation theory predictions are in very good agreement both with the expectations from the $1/N_c$ expansion and with the experimental data. The effect of $SU(3)$ flavor symmetry breaking for the $|\Delta S| = 1$ vector current form factors $f_1(0)$ results in a reduction by a few percent with respect to the corresponding $SU(3)$ symmetric values.

PACS numbers: 12.39.Fe,11.15.Pg,13.40.Em,12.38.Bx

Keywords:

* On sabbatical leave from Instituto de Física, Universidad Autónoma de San Luis Potosí, San Luis Potosí, S.L.P. México

I. INTRODUCTION

Baryon semileptonic decays (BSD) have served as key source of information and tests of the weak interactions, and through the strictures of $SU(3)$ and chiral symmetries also of the strong interactions. Super-allowed nuclear β decay provides the most accurate determination of the Cabibbo angle, and hyperon semileptonic decays (HSD) provide key information on chiral $SU(3) \times SU(3)$ symmetry and its breaking by the quark masses, and also give access to independent determinations of the CKM matrix element $|V_{us}|$.

BSD, denoted here by $B_1(p_1) \to B_2(p_2) + e^-(p_\ell) + \overline{\nu}_e(p_\nu)$, are described by the effective Hamiltonian

$$H_W = \frac{G_F}{\sqrt{2}} L^\alpha J_\alpha + \text{H.c.}, \tag{1}$$

where L_α and J_α are the leptonic and hadronic weak currents, respectively, which possess the $V - A$ structure of the weak interactions, and G_F is the Fermi constant. The leptonic current is given by

$$L^\alpha = \overline{\psi}_e \gamma^\alpha (1 - \gamma_5) \psi_{\nu_e} + \overline{\psi}_\mu \gamma^\alpha (1 - \gamma_5) \psi_{\nu_\mu}, \tag{2}$$

and the hadronic current is $J_\alpha = V_\alpha - A_\alpha$, where

$$V_\alpha = V_{ud}\, \overline{u} \gamma_\alpha d + V_{us}\, \overline{u} \gamma_\alpha s, \tag{3}$$

and

$$A_\alpha = V_{ud}\, \overline{u} \gamma_\alpha \gamma_5 d + V_{us}\, \overline{u} \gamma_\alpha \gamma_5 s. \tag{4}$$

V_α and A_α are the weak vector and axial-vector currents, respectively, and V_{ud} and V_{us} are elements of the CKM matrix. The matrix elements of J_α between spin-1/2 baryon states have the most general forms:

$$\langle B_2 | V_\alpha | B_1 \rangle \;=\; V_{\text{CKM}}\, \overline{u}_{B_2}(p_2) \left[f_1(q^2) \gamma_\alpha + \frac{f_2(q^2)}{M_{B_1}} \sigma_{\alpha\beta} q^\beta + \frac{f_3(q^2)}{M_{B_1}} q_\alpha \right] u_{B_1}(p_1), \tag{5}$$

and

$$\langle B_2 | A_\alpha | B_1 \rangle \;=\; V_{\text{CKM}}\, \overline{u}_{B_2}(p_2) \left[g_1(q^2) \gamma_\alpha + \frac{g_2(q^2)}{M_{B_1}} \sigma_{\alpha\beta} q^\beta + \frac{g_3(q^2)}{M_{B_1}} q_\alpha \right] \gamma_5 u_{B_1}(p_1), \tag{6}$$

where $q \equiv p_1 - p_2$ is the four-momentum transfer, u_{B_1} and $\overline{u}_{B_2}$ are the Dirac spinors of the decaying and emitted baryons, respectively, and V_{CKM} stands for V_{ud} or V_{us}, as the case may be. Here the metric and γ-matrix conventions of Ref. [1] are used.

The matrix elements (5) and (6) are characterized by three form factors each, $f_i(q^2)$ and $g_i(q^2)$, respectively, where the weak decays probe their charged components. Additional information is of course obtained from the EM current, which is not discussed here. As a shorthand notation, $f_i \equiv f_i(0)$ and $g_i \equiv g_i(0)$ will be used hereafter. For the leading form factors, $f_1(0) = g_V$ and $g_1(0) = g_A$ are also used. The latter couplings are related by Cabibbo's theory, with the further generalization to six quarks by Kobayashi and Maskawa.

At the present level of experimental accuracy on BSD, only the form factors $f_1(q^2)$ and $f_2(q^2)$ of the vector current and $g_1(q^2)$ and $g_2(q^2)$ of the axial vector current are involved in electron modes, whereas the $f_3(q^2)$ and $g_3(q^2)$ contributions can be neglected because of the small factor m_e^2 that comes along with them. At a more detailed level, the q^2-dependence of the leading form factors can be parametrized in a dipole form whereas the q^2-dependence of f_2 and g_2 can be neglected due to the q factor already present in the matrix elements (5) and (6).

In the limit of exact flavor $SU(3)$ symmetry f_1 and f_2 are predicted in terms of the EM form factors of p and n via $SU(3)$ transformations. The g_2 form factor for diagonal matrix elements of hermitian currents vanishes by hermiticity and time-reversal invariance. Therefore, $SU(3)$ symmetry yields $g_2 = 0$ in the symmetry limit. Finally, g_1 is given in terms of the familiar couplings F and D.

The decay widths driven by vector and axial vector currents do not interfere, thus, $\Gamma = \Gamma_V + \Gamma_A$. The determination of $|V_{us}|$ and the mentioned form factors can be extracted from the total decay rate R, and, to a high degree of precision, R must include radiative corrections. The actual expression for R reads,

$$R = R^0 \left(1 + \frac{\alpha}{\pi}\Phi\right), \tag{7}$$

where R^0 is the uncorrected decay rate and model-independent radiative corrections are encoded in the term $(\alpha/\pi)\Phi$ [1]. R^0 is a quadratic function of the form factors and can be written in the most general form as[1]

$$R^0 = |V_{\mathrm{CKM}}|^2 \left(\sum_{i\leq j=1}^{6} a_{ij}^R \, f_i f_j + \sum_{i\leq j=1}^{6} b_{ij}^R \, (f_i \lambda_{f_j} + f_j \lambda_{f_i}) \right), \tag{8}$$

[1] Strictly speaking, the model-dependence of radiative corrections can be absorbed into the leading form factors f_1 and g_1 [1] so Eq. (7) should be written in terms of f_1' and g_1'. Actually, these primed form factors are the ones accessible to experiment.

where the dipole parametrizations assumed for all form factors introduce six slope parameters λ_{f_i}. For the sake of shortening Eq. (8), $g_1 = f_4$, $g_2 = f_5$, $g_3 = f_6$, $\lambda_{g_1} = \lambda_{f_4}$, $\lambda_{g_2} = \lambda_{f_5}$, and $\lambda_{g_3} = \lambda_{f_6}$ have been momentarily redefined. The analytic expressions for R^0 in HSD can be found in Ref. [2]. The short distance contributions of radiative corrections, given by the factor S_{ew}, can be accounted for in the usual way by defining an effective weak coupling constant.

The $|\Delta S| = 1$ form factors f_1 satisfy the Ademollo-Gatto (AG) theorem, which states that the $SU(3)$ symmetry breaking (SB) corrections to their $SU(3)$ limit values are proportional to $(m_s - \hat{m})^2$. One must note that this does not mean the corrections are $\mathcal{O}(p^4)$ in the chiral expansion. As it happens with $K_{\ell 3}$ decays [3, 4], the dominant such corrections are non-analytic in quark masses and stem from the chiral loop contributions. Those corrections, if expanded in $(m_s - \hat{m})$ will behave as the AG theorem requires but with small denominators proportional to quark masses, and therefore the non-analytic corrections are $\mathcal{O}(p^2)$. The analytic contributions are of course $\mathcal{O}(p^4)$ and beyond the accuracy of the calculation in this work. Therefore, the dominant $SU(3)$ SB corrections to f_1 calculated here are ultraviolet finite and well defined.

In this work, the formalism of the $1/N_c$ expansion combined with HBChPT is used to calculate the one-loop corrections to the baryon vector currents. The approach has been successfully applied to compute flavor-**27** baryon mass splittings [5], baryon axial-vector couplings [6, 7] and baryon magnetic moments [8, 9], as well as to the study of Lattice QCD results for baryon masses and axial couplings [10, 11]. Here its applicability is extended to the analysis of one-loop corrections to the baryon vector current operator.

Consistency with the $1/N_c$ expansion requires that the baryon decuplet be also included with specific couplings. Here it is shown how to carry out the calculation following the strictures of the $1/N_c$ expansion, which imposes relations between the various couplings involved. The present work will give the $SU(3)$ SB corrections to the vector current at the leading order of the breaking, i.e. $\mathcal{O}(p^2)$, and represents an important step towards a more accurate calculation where the first sub-leading $SU(3)$ SB effects are also included. Thus the approximations involved, which will be discussed in more detail later, are the following: (i) The $SU(3)$ breaking mass splittings in the baryon propagators involved in the loop are disregarded; it will be shown that such effects are of sub-leading order in the chiral expansion. (ii) The calculation involves the mass splittings between octet and decuplet baryons; in the present work the $SU(3)$ SB in those splittings are ignored as per (i). The $SU(3)$ SB corrections to (i) and (ii) will be studied in detail in future work as they will contribute to sub-leading $SU(3)$ SB effects. (iii) The one-loop correction, as discussed below, is proportional to $A^{ia} \otimes A^{ib}$, where A^{ia} is the axial vector current operator. The $1/N_c$ expansion of A^{ia} is truncated at the physical value $N_c = 3$, so in the correction there appear up to six-body operators, which are suppressed by $1/N_c^4$ factors. Working out to this order is two-fold. First, the operator reductions are doable; secondly, the complete expressions will allow a rigorous comparison with chiral perturbation theory results order by order. Knowing that the chiral and $1/N_c$ expansions do not commute, an expansion scheme can be implemented, such as the low scale or ξ expansion discussed recently in [10]. This will be presented in the mentioned future work. The present work will serve as a reference mark for the effects of those improvements.

The earliest computations in baryon chiral perturbation theory (BChPT) of the vector form factors for HSD were performed in the works by Krause [12] and by Anderson and Luty [13]. In both works the baryonic degrees of freedom involve only the spin 1/2 octet. Reference [12] presents the calculation in relativistic BChPT to $\mathcal{O}(p^2)$, while [13] works in HBChPT and performs a (partially complete) $\mathcal{O}(p^3)$ computation[2]. An analysis which is closest to the one in the present work is the one by Villadoro [14], where HBChPT including both octet and decuplet baryon degrees of freedom is used. That analysis includes (partially) up to $\mathcal{O}(p^3)$ corrections corresponding to subleading in $1/M_B$ terms. Other works using covariant BChPT with the IR regularization are by Lacour, Kubis and Meissner [15], where a calculation to $\mathcal{O}(p^3)$ is performed with only the octet baryons as active degrees of freedom, and by Geng, Martin-Camalich and Vicente-Vacas [16], where in the same framework also the decuplet baryons are included. Finally, a calculation using the $1/N_c$ expansion was performed in Ref. [2], which however does not include chiral loop contributions. Since the one-loop contributions are of upmost importance for the $SU(3)$ breaking corrections to the f_1, the present work will provide those contributions. Indeed, due to the AG theorem, the tree level contributions to those corrections must be $\mathcal{O}(p^4)$, while at one-loop there are non-analytic contributions, consistent with AG, which are $\mathcal{O}(p^2)$, and thus dominant over the ones in Ref. [2].

[2] This reference has a sign mistake in the tadpole contributions, as pointed out in [14].

A comparison of the works mentioned above reveal open issues in the different approaches vis-à-vis the $SU(3)$ breaking effects in the form factor f_1, which can be summarized as follows: i) the leading corrections $\mathcal{O}(p^2)$ calculated with the inclusion of the decuplet in [14] and [16] are in agreement concerning the sign, and also with the results in the present work. However, the numerical values have significant discrepancies between the calculation in HBChPT and the one in covariant BChPT. The calculations without the decuplet [13, 15] give $\mathcal{O}(p^2)$ results which are very different with each other and the ones with the decuplet included. In particular some of the signs of the corrections are different. These disagreements at $\mathcal{O}(p^2)$ are partly related to the approach being used, and are a strong motivation for further investigation on establishing which is the most realistic one. ii) the second big problem is with the $\mathcal{O}(p^3)$ corrections, which in all works where they have been evaluated turn out to be inordinately large and positive in general, leading in most cases to a sign reversal of the full correction. Since the calculations to that order are basically well defined, because no counterterms are required, at face value this would represent a breakdown of the low energy expansion. Except for the references where the decuplet is explicitly included, the issue of inconsistency with the $1/N_c$ expansion could play a role in that power counting problem. However, the works including the decuplet also report large $\mathcal{O}(p^3)$ corrections, which may indicate that the issue is even more profound. Since the $\mathcal{O}(p^3)$ corrections will not be evaluated in the present work, this problem will be further investigated in future work.

The impact of $SU(3)$ breaking effects in form factors for the extraction of $|V_{us}|$ from hyperon decays has been studied in Refs. [17, 18]. Of particular interest in that regard is the best possible determination of the form factors f_1 for extracting $|V_{us}|$ through the product $|f_1 V_{us}|$, which can be determined rather precisely from observables (see for example Ref. [18]).

Important inroads are being made by Lattice QCD calculations of form factors [19–22]. These and future results are very significant, as one will be able to test explicitly the behavior of the vector form factor with the quark masses, and in particular understand more accurately the $SU(3)$ SB effects on f_1, helping clarify the issues mentioned above. In the most recent calculation of f_1 in the transitions $\Xi^0 \to \Sigma^+$ and $\Sigma^- \to n$ Ref. [21], the $SU(3)$ breaking tends to produce a reduction in f_1 which is very similar to what is found in the $\mathcal{O}(p^2)$ calculations in the present work (see Sec. V for details), and contradict the $\mathcal{O}(p^3)$

calculations mentioned earlier.

The main motivation of the present work is to provide a computation of the $SU(3)$ breaking corrections to the form factor f_1 in a framework of the chiral expansion which is consistent with the $1/N_c$ expansion of QCD. As it is well known from the consistency constraints imposed by the large-N_c limit of QCD for baryons [23], it is necessary to include on an equal footing in the effective theory the octet and the decuplet baryons, as demanded by the emergent spin-flavor symmetry ($SU(6)$) in the large-N_c limit which is a consequence of that consistency. The interplay of these multiplets in chiral loops is often necessary to restore the correct N_c power counting, by producing exact cancellations of otherwise power counting violating contributions. As discussed later, this is indeed the case for some of the one loop contributions to $SU(3)$ breaking in f_1. The present work provides the complete $\mathcal{O}(p^2)$ corrections to f_1, leaving to a next stage the full calculation at $\mathcal{O}(p^3)$. The constraints of the $1/N_c$ expansion manifest themselves in the effective Lagrangian through the $SU(6)$ relations between the pseudoscalar octet and the octet and decuplet baryons, and through the $\mathcal{O}(1/N_c)$ hyperfine mass splitting between octet and decuplet. Since the known phenomenological couplings satisfy approximately those constraints, the numerical results obtained in the work by Villadoro [14] should be expected to be approximately matched by the results obtained here.

This article is organized as follows. In Sec. II some general aspects of baryon chiral perturbation theory in the $1/N_c$ expansion are provided. In Sec. III the tree-level contribution of the baryon vector current is dealt with as a prelude to discuss in Sec. IV the one-loop correction, where each Feynman diagram is individually discussed in detail. In Sec. V a numerical analysis is performed to compare the resultant theoretical expressions against the experimental information through several different least-squared fits. In Sec. VI the summary and concluding remarks are given. This work is complemented by three appendices. In Appendix A all the analytical results of the loop integrals that appear in the calculation are provided. In Appendix B the baryon operator reductions performed are listed; this way in Appendix C some useful formulas are given in a compact form.

II. BARYON CHIRAL PERTURBATION THEORY IN THE $1/N_c$ EXPANSION

The $1/N_c$ expansion for baryons has been discussed in detail in Refs. [5, 24, 25], thus this section only provides a brief summary, introducing notations and conventions. In the large-N_c limit, the lowest-lying baryons are given by the completely symmetric spin-flavor representation of N_c quarks $SU(2N_f)$ [24, 26]. Under $SU(2) \times SU(N_f)$, this representation decomposes into a tower of baryon flavor representations with spins $J = 1/2, 3/2, \dots, N_c/2$, where the states with vanishing strangeness satisfy $I = J$. This tower is degenerate in the large-N_c limit, and the hyperfine mass splittings Δ between states with spin J of $\mathcal{O}(N_c^0)$ are $\mathcal{O}(1/N_c)$. In general, corrections to the large-N_c limit of observables are expressed in terms of $1/N_c$ suppressed operators [24], which leads to the $1/N_c$ expansion of QCD. Note however that there are also non-analytic dependencies on the ratios m_π/Δ which are not captured by the expansion in operators, but which emerge from the finite pieces of loop corrections in the chiral expansion, as discussed below.

When a QCD operator is considered, for the purpose of its matrix elements between the ground state spin-flavor multiplet of baryon states, it can be represented by a series of effective operators organized in a power series in $1/N_c$. The $1/N_c$ expansion of a QCD m-body quark operator acting can then be expressed as follows [25]

$$\mathcal{O}_{\text{QCD}}^{m\text{-body}} = \sum_{n=m}^{N_c} \sum_{i=1}^{i_n} c_n^i \frac{1}{N_c^{n-m}} \mathcal{O}_n^i \,, \tag{9}$$

where the $\mathcal{O}_n^i$ constitute a complete set of linearly independent effective n-body operators. These operators are represented by products of n spin-flavor generators J^i, T^a and G^{ia}, and the $c_n^i(1/N_c)$ are unknown coefficients which have an expansion, possibly non-analytic due to loop effects, in $1/N_c$ beginning at order unity. These effective coefficients are determined by the QCD dynamics, and are obtainable through phenomenological analysis or in certain cases also Lattice QCD.

Among the most relevant QCD operators studied in the $1/N_c$ expansion are the Hamiltonian (baryon masses) [25, 27], axial [6, 7, 10, 23, 28] and vector [29] currents and magnetic moments [8, 9, 28].

The expansion for the baryon mass operator is given by [25]

$$\mathcal{M} = m_0^{0,\mathbf{1}} N_c \mathbb{1} + \sum_{n=1}^{N_c-1} \frac{m_n^{0,\mathbf{1}}}{N_c^{2n-1}} J^{2n} + SU(3) \text{ breaking operators} \,, \tag{10}$$

where the coefficients $m_n^{0,\mathbf{1}}$ are $\mathcal{O}(\Lambda_{QCD})$. The first term in Eq. (10) represents the overall spin-independent mass of the baryon spin-flavor multiplet and the remaining spin-dependent terms constitute $\mathcal{M}_{\mathrm{HF}}$, where HF stands for hyperfine. The $SU(3)$ breaking pieces are omitted here as they are not needed in the present work; they have been given in Ref. [27].

In the limit of exact $SU(3)$ flavor symmetry, the $1/N_c$ expansion of the baryon axial vector current, can be written as [25]

$$A^{ia} = a_1 G^{ia} + \sum_{n=2}^{N_c} b_n \frac{1}{N_c^{n-1}} \mathcal{D}_n^{ia} + \sum_{n=3}^{N_c} c_n \frac{1}{N_c^{n-1}} \mathcal{O}_n^{ia}, \tag{11}$$

where the coefficients a_1, b_n and c_n are of order unity and the leading operators that come along with them read[3]

$$\mathcal{D}_2^{ia} = J^i T^a, \tag{12}$$

$$\mathcal{D}_3^{ia} = \{J^i, \{J^j, G^{ja}\}\}, \tag{13}$$

$$\mathcal{O}_3^{ia} = \{J^2, G^{ia}\} - \frac{1}{2}\{J^i, \{J^j, G^{ja}\}\}. \tag{14}$$

Higher order operators are constructed from the previous ones by anticommuting them with J^2. The operators $\mathcal{D}_n^{ia}$ and $\mathcal{O}_n^{ia}$ have non-vanishing matrix elements only between states of equal and different spin, respectively, so they are referred to as diagonal and off-diagonal operators. The axial currents enter in the present calculation via the pseudoscalar-baryon couplings in the one-loop diagrams, and up to the considered chiral order of the calculation there is no need to include the $SU(3)$ SB corrections to them. For details on those effects, see [7] and references therein.

An interesting feature of the large-N_c counting scheme is the determination of the N_c dependence of the matrix elements of the generators J^i, T^a and G^{ia}. The baryon matrix elements of J^i for the low-lying baryons in the $SU(6)$ representation are of order unity. The N_c dependence of the matrix elements of T^a and G^{ia} is by far more subtle because it depends on the component a and on the initial and final baryon states. Specifically, for baryons with strangeness $\mathcal{O}(N_c^0)$ the matrix elements of T^a $(a = 1, 2, 3)$ and G^{i8} are $\mathcal{O}(N_c^0)$; the matrix elements of T^a and G^{ia} $(a = 4, 5, 6, 7)$ are $\mathcal{O}\left(\sqrt{N_c}\right)$; and the matrix elements of T^8 and G^{ia} $(a = 1, 2, 3)$ are $\mathcal{O}(N_c)$ [25]. For concreteness, the naive estimate that matrix elements of T^a

[3] The 2-body operator $\mathcal{O}_2^{ia} = \epsilon^{ijk}\{J^j, G^{ka}\}$ and the higher order operators $\mathcal{O}_{2m+2}^{ia} = \{J^2, \mathcal{O}_{2m}^{ia}\}$ $(m = 1, 2, \ldots)$ are even under time reversal so they do not contribute to A^{ia}.

and G^{ia} are both $\mathcal{O}(N_c)$, which is the largest they can be, will be implemented here. This estimate is legitimate provided the analysis is restricted to the lowest-lying baryon states, namely, those states that make up the **56** dimensional representation of $SU(6)$.

The scaling of the baryon masses proportional to N_c implies that an expansion in $1/N_c$ naturally leads to a formulation of the effective theory in the framework of heavy baryon chiral perturbation theory (HBChPT) [30]. In addition, and as mentioned earlier, the $SU(2N_f)$ dynamical spin-flavor symmetry in large-N_c requires that the ground state baryons appear in a multiplet of such symmetry, namely the totally symmetric one with N_c boxes in the Young tableaux. The chiral Lagrangian can be then constructed to satisfy the strictures of chiral symmetry and spin-flavor symmetry, with the breaking of these symmetries expanded in a Taylor series in quark masses and $1/N_c$ respectively [5].

In the baryon rest frame, the combined HBChPT and $1/N_c$ expansion effective Lagrangian at lowest order is given by [5, 10, 31]:

$$\mathcal{L}_{\mathbf{B}}^{(1)} = \mathbf{B}^\dagger \left(iD_0 + \mathring{g}_A u^{ia} G^{ia} - \frac{m_2^{0,\mathbf{1}}}{N_c} - \frac{C_{HF}}{N_c} J^2 - \frac{c_1}{2} N_c \, \chi_+ \right) \mathbf{B}, \tag{15}$$

where $\mathbf{B}$ is the symmetric spin-flavor baryon multiplet with states $J = 1/2, \cdots, N_c/2$, and G^{ia} are the spin-flavor generators of $SU(6)$ with matrix elements are $\mathcal{O}(N_c)$, where i are spatial indices and a are $SU(3)$ flavor indices. The Goldstone boson pseudoscalar octet π^a resides in the unitary matrix

$$u \equiv \exp\left(\frac{i\pi^a \lambda^a}{2F_0}\right), \tag{16}$$

where F_0 is the pion decay constant in the chiral limit, which for the purpose of the present work can be taken to be $F_0 = F_\pi = 93$ MeV. The chiral operators in the Lagrangian are

$$u^\mu = i(u^\dagger(\partial^\mu - i(v^\mu + a^\mu))u - u(\partial^\mu - i(v^\mu - a^\mu))u^\dagger) = -\frac{1}{F_0}\partial^i \pi^a \lambda^a + \cdots, \tag{17}$$

which gives $u^{ia} = (1/2)\,\mathrm{Tr}\,(\lambda^a u^i)$, and the covariant derivative $D_\mu = \partial_\mu - i\Gamma_\mu$ with

$$\Gamma_\mu = \frac{i}{2}(u^\dagger(\partial_\mu - i(v_0 + a_0))u + u(\partial_\mu - i(v_\mu - a_\mu))u^\dagger). \tag{18}$$

v_μ are the sources coupling to the vector currents, namely $v_\mu = v_\mu^a T^a/2$, and similarly a_μ are sources coupling to the axial vector currents, and the quark masses reside in χ_+. The low energy constants $m_2^{0,\mathbf{1}}$, $\mathring{g}_A$, C_{HF}, and c_1 are $\mathcal{O}(N_c^0)$. As defined here, and at lowest order, the leading order axial coupling $\mathring{g}_A$ (to be later denotation by a_1) is related to the one of the nucleon at $N_c = 3$ by $\mathring{g}_A = \frac{6}{5}\, g_A$, where $g_A = 1.27$ is the well known nucleon axial coupling.

At the lowest order the meson-baryon couplings are all fixed by $\mathring{g}_A$, being entirely determined by the corresponding axial couplings through the underlying Goldberger-Treiman relation. The commonly used axial vector couplings are then given by $F = \mathring{g}_A/3$, $D = \mathring{g}_A/2$, $\mathcal{C} = -\mathring{g}_A$ and $\mathcal{H} = -3\mathring{g}_A/2$. Deviations from these values are due to effects $\mathcal{O}(1/N_c)$.

The vector current is affected by the $SU(3)$ SB effects at higher orders in the chiral expansion. The effects stemming from tree contributions appear in the chiral Lagrangian at $\mathcal{O}(p^3)$ for the magnetic components and for the corresponding charges, which are of the main interest in this work, at $\mathcal{O}(p^5)$, which is beyond the order needed in this work. Thus, for the present calculations only the above displayed Lagrangian is needed, to which the terms that correspond to $1/N_c$ corrections will be added. In particular higher order in $1/N_c$ corrections to the pseudoscalar-baryon couplings, i.e. the F, D, $\mathcal{C}$ and $\mathcal{H}$ couplings, through the corresponding corrections to the axial currents will be included. This will serve the purpose of determining how important such corrections are for the weak decays as well as their impact on the strong decays, which are also included in the fits.

III. THE BARYON VECTOR CURRENT AT TREE LEVEL

In order to set the stage, at this point it is convenient to outline the expansions involved in the relevant form factors in Eqs. (5) and (6). In the rest frame of the decaying baryon, the dominant contribution to the matrix elements of the vector current is the corresponding charge term given by f_1, which is $\mathcal{O}(p^0 \times N_c^0)$. The sub-leading terms involve (i) the recoil piece of the convection current, which is $\mathcal{O}(q/M_B)$, where q is the momentum transfer through the current which is $q \sim M_{B_2} - M_{B_1} \sim m_s = \mathcal{O}(p^2)$, thus the recoil term is $\mathcal{O}(p^2/N_c)$, (ii) the weak magnetism terms from the term proportional to f_2 and from the spin component proportional to f_1, are respectively $\mathcal{O}(qN_c/\Lambda_{QCD})$ and $\mathcal{O}(q/M_B)$, and thus $\mathcal{O}(p^2 N_c)$ and $\mathcal{O}(p^2/N_c)$ respectively, (iii) the term proportional to f_3 vanishes in the $SU(3)$ symmetry limit, and is therefore proportional to $(m_s - \hat{m})q_\mu = \mathcal{O}(p^4)$. A similar discussion can be done for the axial vector current, where (i) the term proportional to g_1 gives matrix elements $\mathcal{O}(p^0 N_c)$ for the spatial components of the current and $\mathcal{O}(q/M_B) = \mathcal{O}(p^2/N_c)$ for the time component, (ii) the term proportional to g_3 is highly suppressed as $\mathcal{O}(q^2/M_B) = \mathcal{O}(p^4/N_c)$, and (iii) g_2 vanishes in the limit of $SU(3)$ symmetry.

At $q^2 = 0$ the baryon matrix elements for the vector current in the limit of exact $SU(3)$

symmetry are simply given by the matrix elements of the associated charge or $SU(3)$ generator. Therefore, to all orders in the $1/N_c$ expansion [29]:

$$V^{0a} = T^a. \tag{19}$$

Due to the AG theorem, tree-level corrections to the $|\Delta S| = 1$ matrix elements, first appear to $\mathcal{O}(p^4)$, which is beyond the order considered in this work.

The matrix elements of V^{0a} between $SU(6)$ baryon states in the limit of exact $SU(3)$ symmetry are listed in the first row of Table I for the $|\Delta S| = 1$ processes of interest. These particular form factors will be referred to as $f_1^{SU(3)}$. When $SU(3)$ breaking is taken into account, the matrix elements of V^{0a} will be again given by the matrix elements of T^a but now multiplied by a factor which is given by the corresponding ratio $f_1/f_1^{SU(3)}$, as worked out in what follows.

IV. ONE-LOOP CORRECTIONS TO THE BARYON VECTOR CURRENT

$SU(3)$ flavor SB will be considered in the exact isospin limit. As mentioned earlier, the leading $SU(3)$ flavor SB corrections to the vector currents occur at one-loop order in the chiral expansion. Previous works focused on computing one-loop corrections to other baryon static properties [6–9] will provide some feedback, so a close parallelism with them will be kept. Also, results of those works are used in the global analysis involving both weak and strong decays in Sec. V.

The one-loop corrections to the baryon vector current operator are displayed in Fig. 1. All these graphs can be written as the product of a baryon operator times a flavor tensor which results from the loop integral. Let us recall that the pion-baryon vertex is proportional to g_A/F_π; in the large-N_c limit, $g_A \propto N_c$ and $F_\pi \propto \sqrt{N_c}$, so the pion-baryon vertex scales as $\sqrt{N_c}$. Although the N_c dependence of each diagram can be deduced straightforwardly from the naive N_c counting rule, the group theoretical structure for $N_c = 3$ will be rigorously computed here. As for the loop integrals, they have a non-analytic dependence on m_q. The appropriate combination of diagrams, however, yields corrections that respect the AG theorem. The overall one-loop correction is thus $\mathcal{O}((m_s - \hat{m})^2)$ when expanded in a Taylor series in the mass difference, as mentioned in the introduction.

At this point it is convenient to spell out the general chiral and $1/N_c$ power countings.

Since the transitions involved are only those with initial and final baryons in the octet, the energy transfer through the current $q^0 \sim M_{\mathcal{B}} - M_{\mathcal{B}'}$ which, as mentioned earlier, is a quantity of $\mathcal{O}(p^2)$ in the chiral expansion. On the other hand the decuplet–octet HF mass splittings Δ have a piece $\mathcal{O}(1/N_c)$ plus an $SU(3)$ SB contribution $\propto (m_s - \hat{m}) = \mathcal{O}(p^2)$. If one works in the linked power counting where $1/N_c = \mathcal{O}(p)$ [10] or ξ expansion, one concludes that the heavy baryon propagator can be Taylor expanded in the $SU(3)$ breaking mass shifts. Also the loop contributions can be expanded in powers of q^0. Thus, the dominant $SU(3)$ SB effects on the one-loop corrections stem from the mass differences of the π, K and η mesons involved, with the $SU(3)$ SB effects in the baryon masses playing a sub-leading role, appearing with an additional suppression factor $\mathcal{O}((m_s - \hat{m})/\Lambda_\chi)$.

The N_c power counting of the contributions to $f_1/f_1^{SU(3)}$ due to the one-loop diagrams in Fig. 1 are summarized as follows: (i) diagram (a) gives naively an $\mathcal{O}(N_c)$ contribution due to the $\mathcal{O}\left(\sqrt{N_c}\right)$ of the meson-baryon vertices, but the algebra boils down to giving a commutator of those vertices, which is actually suppressed by a factor $1/N_c^2$ with respect to the naive expectation. This would mean that (a) is $1/N_c$; it is however a bit more subtle than that: the hyperfine mass splitting between octet and decuplet gives rise to additional terms beyond the mentioned commutator, which become the actual dominant contribution $\mathcal{O}(N_c^0)$ (see Section 1 for the details). Diagram (a) is therefore consistent with N_c power counting. It should also be emphasized that removing the decuplet in the loop leads to a contribution which violates the power counting, and thus the result is incompatible with the $1/N_c$ expansion. Diagram (b) is actually $\mathcal{O}(N_c)$, and the N_c power counting is recovered once the wave function renormalization factor is included, leading to a contribution $\mathcal{O}(N_c^0)$. This is the same cancellation that takes place in general for any one-loop contribution to an operator which attaches to the baryon propagator. In this case also, removing the decuplet leads to the wrong power counting. Finally, diagrams (c) and (d) are both $\mathcal{O}(1/N_c)$ as they are just proportional to $1/F_\pi^2 = \mathcal{O}(1/N_c)$. In the chiral power counting, all diagrams are $\mathcal{O}(p^2)$.

The starting point in the analysis of the $SU(3)$ flavor SB is that it transforms as a flavor octet. The $SU(3)$ SB correction to the baryon vector current is then obtained from the tensor product of the vector current itself and the perturbation, which both transform as $(0, \mathbf{8})$. Let us also keep in mind that the tensor product of two octet representations can be separated into an antisymmetric and a symmetric product, $(\mathbf{8} \times \mathbf{8})_A$ and $(\mathbf{8} \times \mathbf{8})_S$,

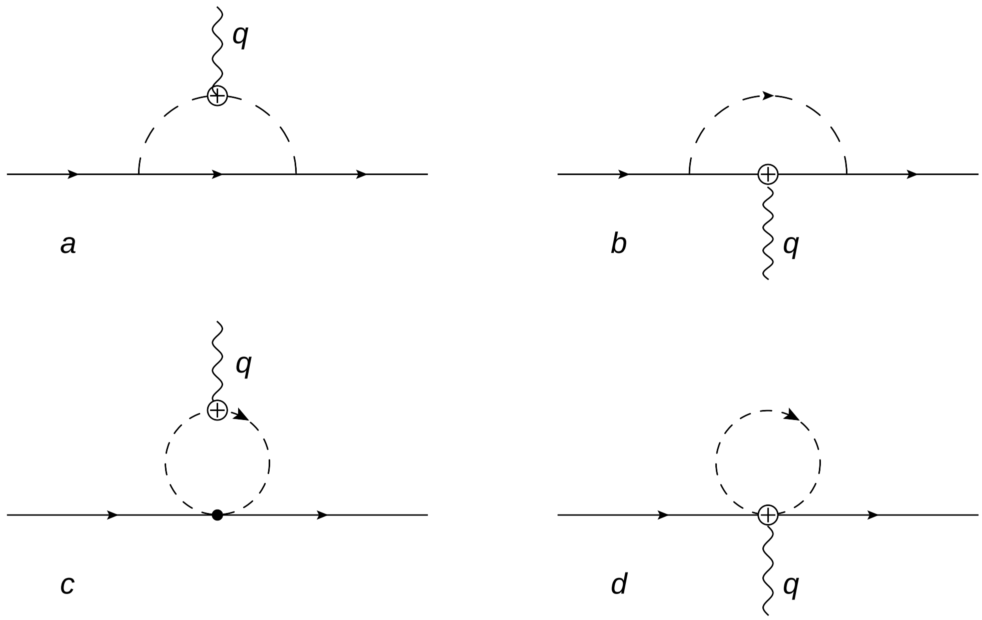

FIG. 1: Feynman diagrams which yield one-loop corrections to the baryon vector current. Dashed lines and solid lines denote mesons and baryons, respectively. The inner solid lines in (a) and (b) can also denote decuplet baryons. Although the wavefunction renormalization graphs are not displayed, they nevertheless have been included in the analysis.

respectively, which can be written as [25]

$$(\mathbf{8}\times\mathbf{8})_A = \mathbf{8}+\mathbf{10}+\overline{\mathbf{10}}, \tag{20a}$$

$$(\mathbf{8}\times\mathbf{8})_S = \mathbf{1}+\mathbf{8}+\mathbf{27}. \tag{20b}$$

The one-loop SB corrections to the baryon vector current will therefore fall in the $SU(2)\times SU(3)$ representations $(0,\mathbf{1})$, $(0,\mathbf{8})$, $(0,\mathbf{8})$, $(0,\mathbf{10}+\overline{\mathbf{10}})$, and $(0,\mathbf{27})$. Let us proceed to analyze each one of them separately.

A. Figure 1(a)

The one-loop contribution to the baryon vector current arising from the Feynman diagram of Fig. 1(a) can be written as

$$\delta V^c_{(\mathrm{a})} = \sum_{\mathsf{j}} A^{ia}\mathcal{P}_{\mathsf{j}}A^{ib}P^{abc}(\Delta_{\mathsf{j}}). \tag{21}$$

Here A^{ia} and A^{jb} are used at the meson-baryon vertices; $\mathcal{P}_{\mathsf{j}}$ is the baryon projector for spin $J=\mathsf{j}$ [5]

$$\frac{i\mathcal{P}_{\mathsf{j}}}{k^0-\Delta_{\mathsf{j}}}, \tag{22}$$

which satisfies by definition

$$\mathcal{P}_{\mathsf{j}}^2 = \mathcal{P}_{\mathsf{j}}, \tag{23a}$$

$$\mathcal{P}_{\mathsf{j}}\mathcal{P}_{\mathsf{j}'} = 0, \qquad \mathsf{j}\neq\mathsf{j}', \tag{23b}$$

and Δ_{j} stands for the difference of the hyperfine mass splittings between the intermediate baryon with spin $J=\mathsf{j}$ and the external baryon, namely,

$$\Delta_{\mathsf{j}} = \mathcal{M}_{\mathrm{HF}}|_{J^2=\mathsf{j}(\mathsf{j}+1)} - \mathcal{M}_{\mathrm{HF}}|_{J^2=\mathsf{j}_{\mathrm{ext}}(\mathsf{j}_{\mathrm{ext}}+1)}. \tag{24}$$

Notice that as only octet to octet weak transitions are of interest, the external baryons have $J=1/2$. In Eq. (30) the sum over spin j has been explicitly indicated whereas the sums over repeated spin and flavor indices are understood. In this work $\mathsf{j}=1/2$

The general expressions for $\mathcal{P}_{\mathsf{j}}$ and Δ_{j} have been introduced in Ref. [5]. For the lowest-lying baryons,

$$\mathcal{P}_{\frac12} = -\frac13\left(J^2-\frac{15}{4}\right), \tag{25a}$$

$$\mathcal{P}_{\frac32} = \frac13\left(J^2-\frac34\right), \tag{25b}$$

along with

$$\Delta_{\frac12} = \begin{cases} 0, & \mathsf{j}_{\mathrm{ext}}=\frac12, \\ -\Delta, & \mathsf{j}_{\mathrm{ext}}=\frac32, \end{cases} \tag{26a}$$

$$\Delta_{\frac32} = \begin{cases} \Delta, & \mathsf{j}_{\mathrm{ext}}=\frac12, \\ 0, & \mathsf{j}_{\mathrm{ext}}=\frac32, \end{cases} \tag{26b}$$

and

$$\Delta = \frac{3}{N_c} m_2^{0,\mathbf{1}}, \tag{27}$$

where $m_2^{0,\mathbf{1}}$ is the leading coefficient of the $1/N_c$ expansion of the baryon mass operator (10). It is important to remark that expressions (25)–(27) have been truncated at the physical value $N_c=3$.

On the other hand, $P^{abc}(\Delta_{\rm j})$ is an antisymmetric tensor which can be expressed as

$$P^{abc}(\Delta_{\rm j}) = A_{\mathbf{8}}(\Delta_{\rm j}) i f^{acb} + A_{\mathbf{10}+\overline{\mathbf{10}}}(\Delta_{\rm j}) i (f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8}), \tag{28}$$

where f^{acb} and $f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8}$ break $SU(3)$as $\mathbf{8}$ and $\mathbf{10}+\overline{\mathbf{10}}$, respectively. The integral over the loop, $I_a(m_1, m_2, \Delta_{\rm j}, \mu; 0)$, is contained in the tensor $P^{abc}(\Delta_{\rm j})$ through

$$A_{\mathbf{8}}(\Delta_{\rm j}) = \frac{1}{2}[I_a(m_\pi, m_K, \Delta_{\rm j}, \mu; 0) + I_a(m_K, m_\eta, \Delta_{\rm j}, \mu; 0)], \tag{29a}$$

$$A_{\mathbf{10}+\overline{\mathbf{10}}}(\Delta_{\rm j}) = -\frac{\sqrt{3}}{2}[I_a(m_\pi, m_K, \Delta_{\rm j}, \mu; 0) - I_a(m_K, m_\eta, \Delta_{\rm j}, \mu; 0)]. \tag{29b}$$

The explicit expression for $I_a(m_\pi, m_K, \Delta_{\rm j}, \mu; 0)$ is given in Eq. (A3)

Thus, the full contribution to the baryon vector current operator from Fig. 1(a) can be cast into the form

$$\delta V^c_{(a)} = \mathcal{P}_{\frac12} A^{ia} \mathcal{P}_{\frac12} A^{ib} \mathcal{P}_{\frac12} P^{abc}(0) + \mathcal{P}_{\frac12} A^{ia} \mathcal{P}_{\frac32} A^{ib} \mathcal{P}_{\frac12} P^{abc}(\Delta). \tag{30}$$

Naively, it could be expected $\delta V^c_{(a)}$ to be $\mathcal{O}(N_c)$: two factors of the pion-baryon vertex g_A/F_π would yield a factor N_c. However, the operator $A^{ia}\mathcal{P}_{\rm j}A^{jb}$ can be decomposed as $\alpha A^{ia}A^{ib} + \beta A^{ia}J^2A^{ib}$, where α and β are some coefficients. Next, $f^{acb}A^{ia}A^{ib}$ can be rewritten as $(1/2)f^{acb}\{A^{ia}, A^{ib}\} + (1/2)f^{acb}[A^{ia}, A^{ib}]$; the anticommutator vanishes whereas the commutator of an n-body operator with and m-body operator is an $(n+m-1)$-operator. Therefore, $f^{acb}A^{ia}A^{ib}$ is $\mathcal{O}(N_c)$. For $f^{acb}A^{ia}J^2A^{ib}$ the relation

$$\begin{aligned} A^{ia}J^2A^{ib} &= \frac{1}{2}\{J^2, A^{ia}A^{ib}\} + \frac{1}{4}[[A^{ia}, J^2], A^{ib}] + \frac{1}{4}[A^{ia}, [J^2, A^{ib}]] + \frac{1}{4}\{[A^{ia}, J^2], A^{ib}\} \\ &\quad + \frac{1}{4}\{A^{ia}, [J^2, A^{ib}]\}, \end{aligned} \tag{31}$$

can be used to verify that $f^{acb}A^{ia}J^2A^{ib}$ is also $\mathcal{O}(N_c)$. In consequence, $\delta V^a_{(a)}$ is $\mathcal{O}(N_c^0)$, or equivalently, $1/N_c$ times the tree level value, which is $\mathcal{O}(N_c)$. In actual calculations, there will appear up to eight-body operators in the operator products on the right-hand side of Eq. (30) if the $1/N_c$ expansion of A^{ia} is truncated at the physical value $N_c = 3$. Because the operator basis is complete [25], the reduction, although long and tedious, is doable.

The way these operator reductions are performed can be better seen through a sample calculation. For the $if^{acb}A^{ia}A^{ib}$ piece, using the form of A^{ia} of (11) truncated at $N_c = 3$, one finds,

$$if^{acb}A^{ia}A^{ib} = a_1^2 if^{acb}G^{ia}G^{ib} + \frac{1}{N_c}a_1b_2 if^{acb}G^{ia}\mathcal{D}_2^{ib} + \ldots + \frac{1}{N_c^4}c_3^2 if^{acb}\mathcal{O}_3^{ia}\mathcal{O}_3^{ib}, \tag{32}$$

where only some contributions are displayed for simplicity. Computing the leading order piece is straightforward by using the $SU(6)$ commutation relations [25], namely,

$$if^{acb}G^{ia}G^{ib} = \frac{i}{2}f^{acb}[G^{ia},G^{ib}] = \frac{i}{2}f^{acb}\left(\frac{i}{4}\delta^{ii}f^{abe}T^e\right) = \frac{3}{8}N_fT^c. \tag{33}$$

The computation of all subleading pieces (at the order worked here) is possible by systematically using the $SU(6)$ commutation relations along with some operator identities. The full reductions are listed in Appendix B for the sake of completeness. The N_c dependence is explicitly kept.

Gathering together partial results, the various contributions from Fig. 1(a) can be organized as

$$if^{acb}A^{ia}A^{ib} = \sum_{n=1}^{7} a_n^{\mathbf{8}} S_n^c, \tag{34}$$

and

$$if^{acb}A^{ia}J^2A^{ib} = \sum_{n=1}^{7} \overline{a}_n^{\mathbf{8}} S_n^c, \tag{35}$$

for the octet contribution, and

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})A^{ia}A^{ib} = \sum_{n=1}^{13} b_n^{\mathbf{10}+\overline{\mathbf{10}}} O_n^c, \tag{36}$$

and

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})A^{ia}J^2A^{ib} = \sum_{n=1}^{13} \overline{b}_n^{\mathbf{10}+\overline{\mathbf{10}}} O_n^c, \tag{37}$$

for the $\mathbf{10}+\overline{\mathbf{10}}$ contribution. The coefficients $a_n^{\mathbf{8}}$, $\overline{a}_n^{\mathbf{8}}$, $b_n^{\mathbf{10}+\overline{\mathbf{10}}}$ and $\overline{b}_n^{\mathbf{10}+\overline{\mathbf{10}}}$ are listed in full in Appendix C. The corresponding operator bases are:

$$\begin{aligned}
&S_1^c = T^c, && S_2^c = \{J^r,G^{rc}\}, && S_3^c = \{J^2,T^c\}, \\
&S_4^c = \{J^2,\{J^r,G^{rc}\}\}, && S_5^c = \{J^2,\{J^2,T^c\}\}, && S_6^c = \{J^2,\{J^2,\{J^r,G^{rc}\}\}\}, \\
&S_7^c = \{J^2,\{J^2,\{J^2,T^c\}\}\},
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
&O_1^c = d^{c8e}T^e, && O_2^c = d^{c8e}\{J^r,G^{re}\}, \\
&O_3^c = d^{c8e}\{J^2,T^e\}, && O_4^c = \{T^c,\{J^r,G^{r8}\}\}, \\
&O_5^c = \{T^8,\{J^r,G^{rc}\}\}, && O_6^c = d^{c8e}\{J^2,\{J^r,G^{re}\}\}, \\
&O_7^c = d^{c8e}\{J^2,\{J^2,T^e\}\}, && O_8^c = \{J^2,\{T^c,\{J^r,G^{r8}\}\}\}, \\
&O_9^c = \{J^2,\{T^8,\{J^r,G^{rc}\}\}\}, && O_{10}^c = d^{c8e}\{J^2,\{J^2,\{J^r,G^{re}\}\}\}, \\
&O_{11}^c = d^{c8e}\{J^2,\{J^2,\{J^2,T^e\}\}\}, && O_{12}^c = \{J^2,\{J^2,\{T^c,\{J^r,G^{r8}\}\}\}\}, \\
&O_{13}^c = \{J^2,\{J^2,\{T^8,\{J^r,G^{rc}\}\}\}\}.
\end{aligned} \tag{39}$$

TABLE I: Matrix elements of baryon operators: singlet case.

	Λp	$\Sigma^- n$	$\Xi^-\Lambda$	$\Xi^-\Sigma^0$	$\Xi^0\Sigma^+$
$\langle S_1^c\rangle$	$-\sqrt{\frac{3}{2}}$	-1	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}$	1
$\langle S_2^c\rangle$	$-\frac{3}{2}\sqrt{\frac{3}{2}}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{\frac{3}{2}}$	$\frac{5}{2\sqrt{2}}$	$\frac{5}{2}$
$\langle S_3^c\rangle$	$-\frac{3}{2}\sqrt{\frac{3}{2}}$	$-\frac{3}{2}$	$\frac{3}{2}\sqrt{\frac{3}{2}}$	$\frac{3}{2\sqrt{2}}$	$\frac{3}{2}$
$\langle S_4^c\rangle$	$-\frac{9}{4}\sqrt{\frac{3}{2}}$	$\frac{3}{4}$	$\frac{3}{4}\sqrt{\frac{3}{2}}$	$\frac{15}{4\sqrt{2}}$	$\frac{15}{4}$
$\langle S_5^c\rangle$	$-\frac{9}{4}\sqrt{\frac{3}{2}}$	$-\frac{9}{4}$	$\frac{9}{4}\sqrt{\frac{3}{2}}$	$\frac{9}{4\sqrt{2}}$	$\frac{9}{4}$
$\langle S_6^c\rangle$	$-\frac{27}{8}\sqrt{\frac{3}{2}}$	$\frac{9}{8}$	$\frac{9}{8}\sqrt{\frac{3}{2}}$	$\frac{45}{8\sqrt{2}}$	$\frac{45}{8}$
$\langle S_7^c\rangle$	$-\frac{27}{8}\sqrt{\frac{3}{2}}$	$-\frac{27}{8}$	$\frac{27}{8}\sqrt{\frac{3}{2}}$	$\frac{27}{8\sqrt{2}}$	$\frac{27}{8}$

The matrix elements of the operators S_n and O_n between baryon octet states are listed in tables I and II for completeness.

All the pieces of the one-loop contribution (30) for the process $\Lambda \to p$ can be put together to illustrate how the approach works for concreteness. In terms of the operator coefficients introduced in Eq. (11), at $N_c = 3$ one gets

$$\left[\frac{f_1^{(\mathrm{a})}}{f_1^{SU(3)}}\right]_{\Lambda p} = \left[\frac{17}{16}a_1^2 + \frac{3}{8}a_1b_2 + \frac{17}{24}a_1b_3 + \frac{1}{16}b_2^2 + \frac{1}{8}b_2b_3 + \frac{17}{144}b_3^2\right] I_a(m_\pi, m_K, 0, \mu; 0) + \left[\frac{9}{16}a_1^2 + \frac{3}{8}a_1b_2 + \frac{3}{8}a_1b_3 + \frac{1}{16}b_2^2 + \frac{1}{8}b_2b_3 + \frac{1}{16}b_3^2\right] I_a(m_\eta, m_K, 0, \mu; 0) + \left[-\frac{1}{2}a_1^2 - \frac{1}{2}a_1c_3 - \frac{1}{8}c_3^2\right] I_a(m_\pi, m_K, \Delta, \mu; 0). \tag{40}$$

Similar expressions can be found for the rest of the processes of interest. In order to display the relative N_c dependence of the different terms, in this expression and similar ones that will follow, one simply replaces $b_n \to (3/N_c)^{n-1} b_n$ and similarly for c_n.

B. Figure 1(b)

The correction to the baryon vector current arising from Fig. 1(b), along with the corresponding wave function renormalization graphs not displayed but nevertheless accounted

TABLE II: Matrix elements of baryon operators: octet case.

	Λp	$\Sigma^- n$	$\Xi^-\Lambda$	$\Xi^-\Sigma^0$	$\Xi^0\Sigma^+$
$\langle O_1^c\rangle$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{6}}$	$-\frac{1}{2\sqrt{3}}$
$\langle O_2^c\rangle$	$\frac{3}{4\sqrt{2}}$	$-\frac{1}{4\sqrt{3}}$	$-\frac{1}{4\sqrt{2}}$	$-\frac{5}{4\sqrt{6}}$	$-\frac{5}{4\sqrt{3}}$
$\langle O_3^c\rangle$	$\frac{3}{4\sqrt{2}}$	$\frac{\sqrt{3}}{4}$	$-\frac{3}{4\sqrt{2}}$	$-\frac{1}{4}\sqrt{\frac{3}{2}}$	$-\frac{\sqrt{3}}{4}$
$\langle O_4^c\rangle$	$\frac{3}{4\sqrt{2}}$	$-\frac{3\sqrt{3}}{4}$	$-\frac{15}{4\sqrt{2}}$	$-\frac{1}{4}\sqrt{\frac{3}{2}}$	$-\frac{\sqrt{3}}{4}$
$\langle O_5^c\rangle$	$-\frac{9}{4\sqrt{2}}$	$\frac{\sqrt{3}}{4}$	$-\frac{3}{4\sqrt{2}}$	$-\frac{5}{4}\sqrt{\frac{3}{2}}$	$-\frac{5\sqrt{3}}{4}$
$\langle O_6^c\rangle$	$\frac{9}{8\sqrt{2}}$	$-\frac{\sqrt{3}}{8}$	$-\frac{3}{8\sqrt{2}}$	$-\frac{5}{8}\sqrt{\frac{3}{2}}$	$-\frac{5\sqrt{3}}{8}$
$\langle O_7^c\rangle$	$\frac{9}{8\sqrt{2}}$	$\frac{3\sqrt{3}}{8}$	$-\frac{9}{8\sqrt{2}}$	$-\frac{3}{8}\sqrt{\frac{3}{2}}$	$-\frac{3\sqrt{3}}{8}$
$\langle O_8^c\rangle$	$\frac{9}{8\sqrt{2}}$	$-\frac{9\sqrt{3}}{8}$	$-\frac{45}{8\sqrt{2}}$	$-\frac{3}{8}\sqrt{\frac{3}{2}}$	$-\frac{3\sqrt{3}}{8}$
$\langle O_9^c\rangle$	$-\frac{27}{8\sqrt{2}}$	$\frac{3\sqrt{3}}{8}$	$-\frac{9}{8\sqrt{2}}$	$-\frac{15}{8}\sqrt{\frac{3}{2}}$	$-\frac{15\sqrt{3}}{8}$
$\langle O_{10}^c\rangle$	$\frac{27}{16\sqrt{2}}$	$-\frac{3\sqrt{3}}{16}$	$-\frac{9}{16\sqrt{2}}$	$-\frac{15}{16}\sqrt{\frac{3}{2}}$	$-\frac{15\sqrt{3}}{16}$
$\langle O_{11}^c\rangle$	$\frac{27}{16\sqrt{2}}$	$\frac{9\sqrt{3}}{16}$	$-\frac{27}{16\sqrt{2}}$	$-\frac{9}{16}\sqrt{\frac{3}{2}}$	$-\frac{9\sqrt{3}}{16}$
$\langle O_{12}^c\rangle$	$\frac{27}{16\sqrt{2}}$	$-\frac{27\sqrt{3}}{16}$	$-\frac{135}{16\sqrt{2}}$	$-\frac{9}{16}\sqrt{\frac{3}{2}}$	$-\frac{9\sqrt{3}}{16}$
$\langle O_{13}^c\rangle$	$-\frac{81}{16\sqrt{2}}$	$\frac{9\sqrt{3}}{16}$	$-\frac{27}{16\sqrt{2}}$	$-\frac{45}{16}\sqrt{\frac{3}{2}}$	$-\frac{45\sqrt{3}}{16}$

for in the analysis, can be written as [cf. Eq. (14) of Ref. [7]]

$$\begin{aligned}\delta V^c_{\mathrm{(b)}} &= \frac{1}{2}[A^{ja},[A^{jb},V^c]]Q^{ab}_{(1)} - \frac{1}{2}\{A^{ja},[V^c,[\mathcal{M},A^{jb}]]\}Q^{ab}_{(2)} \\ &+ \frac{1}{6}\left([A^{ja},[[\mathcal{M},[\mathcal{M},A^{jb}]],V^c]] - \frac{1}{2}[[\mathcal{M},A^{ja}],[[\mathcal{M},A^{jb}],V^c]]\right)Q^{ab}_{(3)} + \ldots \quad ,\end{aligned} \tag{41}$$

where A^{ja} and A^{jb} represent the meson-baryon vertices, V^c denotes the insertion of the baryon vector current operator and $\mathcal{M}$ is the baryon mass operator. $Q^{ab}_{(n)}$ is a symmetric tensor which encodes the loop integral; it decomposes into flavor singlet, flavor **8** and flavor **27** representations as [5]

$$Q^{ab}_{(n)} = I^{(n)}_{b,\mathbf{1}}\delta^{ab} + I^{(n)}_{b,\mathbf{8}}d^{ab8} + I^{(n)}_{b,\mathbf{27}}\left[\delta^{a8}\delta^{b8} - \frac{1}{8}\delta^{ab} - \frac{3}{5}d^{ab8}d^{888}\right], \tag{42}$$

where

$$I_{b,\mathbf{1}}^{(n)} = \frac{1}{8}\left[3I_b^{(n)}(m_\pi,0,\mu)+4I_b^{(n)}(m_K,0,\mu)+I_b^{(n)}(m_\eta,0,\mu)\right], \tag{43a}$$

$$I_{b,\mathbf{8}}^{(n)} = \frac{2\sqrt{3}}{5}\left[\frac{3}{2}I_b^{(n)}(m_\pi,0,\mu)-I_b^{(n)}(m_K,0,\mu)-\frac{1}{2}I_b^{(n)}(m_\eta,0,\mu)\right], \tag{43b}$$

$$I_{b,\mathbf{27}}^{(n)} = \frac{1}{3}I_b^{(n)}(m_\pi,0,\mu)-\frac{4}{3}I_b^{(n)}(m_K,0,\mu)+I_b^{(n)}(m_\eta,0,\mu). \tag{43c}$$

Here $I_b^{(n)}(m,0,\mu)$ represents the degeneracy limit $\Delta \to 0$ of the general function $I_b^{(n)}(m,\Delta,\mu)$, defined as [31]

$$I_b^{(n)}(m,\Delta,\mu) \equiv \frac{\partial^n I_b(m,\Delta,\mu)}{\partial \Delta^n}, \tag{44}$$

where the function $I_b(m,\Delta,\mu)$ is given in Eq. (A6).

The expansion contained in Eq. (41) was derived for the baryon axial vector current in Ref. [31]; here that result is extended to the baryon vector current taking advantage of the fact that both currents transform as flavor octets so one can reach the very same conclusions in the discussion presented in Ref. [31]. Naively, one would expect the double commutator alone in (41) to be $\mathcal{O}(N_c^3)$: one factor of N_c from each baryon current. However, there are large-N_c cancellations between the Feynman diagrams of Fig. 1(b) provided that all baryon states in a complete multiplet of the large-N_c $SU(6)$ spin-flavor symmetry are included in the sum over intermediate states and that the axial coupling ratios predicted by this spin-flavor symmetry are used. Thus it can be proved that the double commutator in (41) is at most $\mathcal{O}(N_c)$. The same behavior is observed in the second contribution in (41), so it can be concluded that $\delta V_{\text{(b)}}^c$ is $\mathcal{O}(N_c^0)$ and is of the *same order* as $\delta V_{\text{(a)}}^c$.

The final form of $\delta V_{\text{(b)}}^c$ can be organized as

$$\begin{aligned}\delta V_{\text{(b)}}^c = {} & \sum_{n=1}^{7}\left(c_n^{\mathbf{1}} S_n^c I_{b,\mathbf{1}}^{(1)} + d_n^{\mathbf{1}} S_n^c I_{b,\mathbf{1}}^{(2)} S_n^c + e_n^{\mathbf{1}} S_n^c I_{b,\mathbf{1}}^{(3)}\right) + \sum_{n=1}^{13}\left(c_n^{\mathbf{8}} O_n^c I_{b,\mathbf{8}}^{(1)} + d_n^{\mathbf{8}} O_n^c I_{b,\mathbf{8}}^{(2)} + e_n^{\mathbf{8}} O_n^c I_{b,\mathbf{8}}^{(3)}\right) \\ & + \sum_{n=1}^{9}\left(c_n^{\mathbf{27}} T_n^c I_{b,\mathbf{27}}^{(1)} + d_n^{\mathbf{27}} T_n^c I_{b,\mathbf{27}}^{(2)} + e_n^{\mathbf{27}} T_n^c I_{b,\mathbf{27}}^{(3)}\right) + \ldots,\end{aligned} \tag{45}$$

where the coefficients $c_n^{\mathbf{r}}$, $d_n^{\mathbf{r}}$ and $e_n^{\mathbf{r}}$ and given in Appendix C. While the singlet and octet

TABLE III: Matrix elements of baryon operators: **27** case.

	Λp	$\Sigma^- n$	$\Xi^- \Lambda$	$\Xi^- \Sigma^0$	$\Xi^0 \Sigma^+$
$\langle T_1^c \rangle$	$-\frac{3}{4}\sqrt{\frac{3}{2}}$	$-\frac{3}{4}$	$\frac{3}{4}\sqrt{\frac{3}{2}}$	$\frac{3}{4\sqrt{2}}$	$\frac{3}{4}$
$\langle T_2^c \rangle$	$-\frac{9}{8}\sqrt{\frac{3}{2}}$	$\frac{3}{8}$	$\frac{3}{8}\sqrt{\frac{3}{2}}$	$\frac{15}{8\sqrt{2}}$	$\frac{15}{8}$
$\langle T_3^c \rangle$	$-\frac{9}{8}\sqrt{\frac{3}{2}}$	$-\frac{9}{8}$	$\frac{9}{8}\sqrt{\frac{3}{2}}$	$\frac{9}{8\sqrt{2}}$	$\frac{9}{8}$
$\langle T_4^c \rangle$	0	$-\frac{3}{2}$	$\frac{3}{2}\sqrt{\frac{3}{2}}$	$-\frac{3}{\sqrt{2}}$	-3
$\langle T_5^c \rangle$	$-\frac{27}{16}\sqrt{\frac{3}{2}}$	$\frac{9}{16}$	$\frac{9}{16}\sqrt{\frac{3}{2}}$	$\frac{45}{16\sqrt{2}}$	$\frac{45}{16}$
$\langle T_6^c \rangle$	$-\frac{27}{16}\sqrt{\frac{3}{2}}$	$-\frac{27}{16}$	$\frac{27}{16}\sqrt{\frac{3}{2}}$	$\frac{27}{16\sqrt{2}}$	$\frac{27}{16}$
$\langle T_7^c \rangle$	0	$-\frac{9}{4}$	$\frac{9}{4}\sqrt{\frac{3}{2}}$	$-\frac{9}{2\sqrt{2}}$	$-\frac{9}{2}$
$\langle T_8^c \rangle$	$-\frac{81}{32}\sqrt{\frac{3}{2}}$	$-\frac{81}{32}$	$\frac{81}{32}\sqrt{\frac{3}{2}}$	$\frac{81}{32\sqrt{2}}$	$\frac{81}{32}$
$\langle T_9^c \rangle$	0	$-\frac{27}{8}$	$\frac{27}{8}\sqrt{\frac{3}{2}}$	$-\frac{27}{4\sqrt{2}}$	$-\frac{27}{4}$

operator bases are listed in Eqs. (38) and (39), respectively, the **27** operator basis is

$$
\begin{aligned}
T_1^c &= f^{a8e} f^{8eg} T^g, & T_2^c &= f^{a8e} f^{8eg} \{J^r, G^{rg}\}, \\
T_3^c &= f^{a8e} f^{8eg} \{J^2, T^g\}, & T_4^c &= \epsilon^{ijk} f^{a8e} \{G^{ke}, \{J^i, G^{j8}\}\}, \\
T_5^c &= f^{a8e} f^{8eg} \{J^2, \{J^r, G^{rg}\}\}, & T_6^c &= f^{a8e} f^{8eg} \{J^2, \{J^2, T^g\}\}, \\
T_7^c &= \epsilon^{ijk} f^{a8e} \{J^2, \{G^{ke}, \{J^i, G^{j8}\}\}\}, & T_8^c &= f^{a8e} f^{8eg} \{J^2, \{J^2, \{J^2, T^g\}\}\}, \\
T_9^c &= \epsilon^{ijk} f^{a8e} \{J^2, \{J^2, \{G^{ke}, \{J^i, G^{j8}\}\}\}\}.
\end{aligned}
\tag{46}
$$

The corresponding matrix elements are given in Table III. The singlet and octet pieces should be subtracted off the **27** piece to have a truly **27** contribution.

The contribution of $\langle \delta V_{(\mathrm{b})}^c \rangle$ to f_1 can be readily computed. Keeping the $\Lambda \to p$ process as an example, the contribution reads

$$\left[\frac{f_1^{(\mathrm{b})}}{f_1^{SU(3)}}\right]_{\Lambda p}$$
$$= \left[\frac{9}{32}a_1^2 + \frac{3}{16}a_1b_2 + \frac{17}{48}a_1b_3 - \frac{1}{4}a_1c_3 + \frac{1}{32}b_2^2 + \frac{1}{16}b_2b_3 + \frac{17}{288}b_3^2 - \frac{1}{16}c_3^2\right] I_b^{(1)}(m_\pi, 0, \mu)$$
$$+ \left[\frac{9}{16}a_1^2 + \frac{3}{8}a_1b_2 + \frac{13}{24}a_1b_3 - \frac{1}{4}a_1c_3 + \frac{1}{16}b_2^2 + \frac{1}{8}b_2b_3 + \frac{13}{144}b_3^2 - \frac{1}{16}c_3^2\right] I_b^{(1)}(m_K, 0, \mu)$$
$$+ \left[\frac{9}{32}a_1^2 + \frac{3}{16}a_1b_2 + \frac{3}{16}a_1b_3 + \frac{1}{32}b_2^2 + \frac{1}{16}b_2b_3 + \frac{1}{32}b_3^2\right] I_b^{(1)}(m_\eta, 0, \mu)$$
$$+ \left[-\frac{3}{4}a_1^2 - \frac{3}{4}a_1c_3 - \frac{3}{16}c_3^2\right] \frac{\Delta}{3} I_b^{(2)}(m_\pi, 0, \mu) + \left[-\frac{3}{4}a_1^2 - \frac{3}{4}a_1c_3 - \frac{3}{16}c_3^2\right] \frac{\Delta}{3} I_b^{(2)}(m_K, 0, \mu)$$
$$+ \left[-\frac{9}{8}a_1^2 - \frac{9}{8}a_1c_3 - \frac{9}{32}c_3^2\right] \frac{\Delta^2}{9} I_b^{(3)}(m_\pi, 0, \mu) + \left[-\frac{9}{8}a_1^2 - \frac{9}{8}a_1c_3 - \frac{9}{32}c_3^2\right] \frac{\Delta^2}{9} I_b^{(3)}(m_K, 0, \mu) + \ldots \tag{47}$$

Equations (40) and (47) are now added together to get

$$\left[\frac{f_1^{(\mathrm{a})} + f_1^{(\mathrm{b})}}{f_1^{SU(3)}}\right]_{\Lambda p} = \left[\frac{17}{32}a_1^2 + \frac{3}{16}a_1b_2 + \frac{17}{48}a_1b_3 + \frac{1}{32}b_2^2 + \frac{1}{16}b_2b_3 + \frac{17}{288}b_3^2\right] H(m_\pi, m_K)$$
$$+ \left[\frac{9}{32}a_1^2 + \frac{3}{16}a_1b_2 + \frac{3}{16}a_1b_3 + \frac{1}{32}b_2^2 + \frac{1}{16}b_2b_3 + \frac{1}{32}b_3^2\right] H(m_K, m_\eta)$$
$$+ \left[-\frac{1}{4}a_1^2 - \frac{1}{4}a_1c_3 - \frac{1}{16}c_3^2\right] K(m_\pi, m_K, \Delta), \tag{48}$$

where

$$H(m_1, m_2) \equiv 2I_a(m_1, m_2, 0, \mu; 0) + I_b^{(1)}(m_1, 0, \mu) + I_b^{(1)}(m_2, 0, \mu), \tag{49}$$

and

$$\begin{aligned} K(m_1, m_2, \Delta) \;&\equiv\; 2I_a(m_1, m_2, \Delta, \mu; 0) + I_b^{(1)}(m_1, 0, \mu) + I_b^{(1)}(m_2, 0, \mu) \\ &\quad + \left[I_b^{(2)}(m_1, 0, \mu) + I_b^{(2)}(m_2, 0, \mu)\right] \Delta \\ &\quad + \left[I_b^{(3)}(m_1, 0, \mu) + I_b^{(3)}(m_2, 0, \mu)\right] \frac{\Delta^2}{2} + \ldots \\ &= 2I_a(m_1, m_2, \Delta, \mu; 0) + I_b^{(1)}(m_1, \Delta, \mu) + I_b^{(1)}(m_2, \Delta, \mu). \end{aligned} \tag{50}$$

The final form of $K(m_1, m_2, \Delta)$ recovers the full form of the function $I_b^{(1)}(m_1, \Delta, \mu)$, which was originally expanded in a power series in Δ in Eq. (41). This is a remarkable result.

On the other hand, the explicit form of the function $H(m_1, m_2)$ becomes

$$H(m_1, m_2) = \frac{1}{16\pi^2 F_\pi^2}\left[-\frac{1}{2}(m_1^2 + m_2^2) + \frac{m_1^2 m_2^2}{m_1^2 - m_2^2}\ln\frac{m_1^2}{m_2^2}\right], \tag{51}$$

which is ultraviolet finite. The function $K(m_1, m_2, \Delta)$ can be easily constructed from $I_a(m_1, m_2, \Delta, \mu; 0)$ and $I_b^{(1)}(m, \Delta, \mu)$ given in Eqs. (A3) and (A6), respectively; the explicit expression will not be provided here. However, some important properties of this function are

1. $\lim_{\Delta \to 0} K(m_1, m_2, \Delta) = H(m_1, m_2)$,
2. $\lim_{\Delta \to \infty} K(m_1, m_2, \Delta) = 0$.

Property (2) above has some interesting physical implications. The present calculation exploits the near degeneracy between octet and decuplet baryons. For instance, in the loop integral $I_b(m, \Delta, \mu)$, Eq. (A6), the full functional dependence on the ratio m_π/Δ has been retained. This ratio does not have to be small necessarily because the conditions for HBChPT to be valid are $m_\pi \ll \Lambda_\chi$ and $\Delta \ll \Lambda_\chi$. In the chiral limit $\Delta \gg m_\pi$ so the decuplet cannot contribute to the non-analytical corrections for octet processes since these corrections come from infrared divergences. The decuplet thus decouples in the large-N_c limit and property (2) holds.

A further aim of the approach can be achieved by rewriting the results in terms of the $SU(3)$ invariant couplings D, F and $\mathcal{C}$ introduced in HBChPT [30, 32]. These couplings are related to the $1/N_c$ expansion coefficients a_1, b_2, b_3, and c_3 at $N_c = 3$ as follows:

$$D = \frac{1}{2}a_1 + \frac{1}{6}b_3, \tag{52a}$$

$$F = \frac{1}{3}a_1 + \frac{1}{6}b_2 + \frac{1}{9}b_3, \tag{52b}$$

$$\mathcal{C} = -a_1 - \frac{1}{2}c_3, \tag{52c}$$

$$\mathcal{H} = -\frac{3}{2}a_1 - \frac{3}{2}b_2 - \frac{5}{2}b_3. \tag{52d}$$

In the large-N_c limit the standard $SU(6)$ ratios $D : F : C : H = 1 : \frac{2}{3} : -2 : -3$ result. In the canonical example worked out so far, substituting Eqs. (52) into Eq. (48) yields

$$\begin{aligned}\left[\frac{f_1^{(\mathrm{a})} + f_1^{(\mathrm{b})}}{f_1^{SU(3)}}\right]_{\Lambda p} &= \frac{1}{8}(9D^2 + 6DF + 9F^2)H(m_\pi, m_K) \\ &+ \frac{1}{8}(D^2 + 6DF + 9F^2)H(m_K, m_\eta) - \frac{1}{4}\mathcal{C}^2 K(m_\pi, m_K, \Delta),\end{aligned} \tag{53}$$

which exactly matches the ones obtained within (H)BChPT: When the decuplet fields are not explicitly retained in the effective theory but integrated out, this result agrees with those presented in Refs. [12, 13, 15]. When the decuplet fields are retained, there is a full agreement with the ones presented in Ref. [14] (in that reference $F_\pi = 131$ MeV is used). Moreover, it can be shown that

$$K(m_p, m_q, \Delta) = \frac{4}{3}\left(G_{pq} - \frac{3}{8}H_{pq}\right), \tag{54}$$

where the functions H_{pq} and G_{pq} are given in Eqs. (22) and (31) of that reference, respectively.

Note that the coupling $\mathcal{H}$ does not appear in the corrections to the vector currents, but it does in the corrections to the axial currents. Its determination is addressed in the analysis below.

C. Figure 1(c)

The tadpole diagrams of Figs. 1(c) and 1(d) can be easily computed within the combined approach. These diagrams do not depend on the coefficients of the $1/N_c$ expansion of A^{ia}.

The loop graph 1(c) can be written as

$$\delta V^c_{\text{(c)}} = -f^{cae} f^{beg} T^g R^{ab}, \tag{55}$$

where

$$R^{ab} = \frac{1}{2}\left[I_c(m_\pi, m_K, \mu; 0) + I_c(m_K, m_\eta, \mu; 0)\right]\delta^{ab}, \tag{56}$$

where the loop integral $I_c(m_1, m_2, \mu; q^2)$ in the $q^2 \to 0$ limit is given in Eq. (A20) of Appendix A. This contribution breaks $SU(3)$ as a flavor singlet.

D. Figure 1(d)

The Feynman diagram of Fig. 1(d) is given by

$$\delta V^c_{\text{(d)}} = -\frac{1}{2}\left[T^a, \left[T^b, V^c\right]\right] S^{ab}, \tag{57}$$

where S^{ab} has the very same structure as $P^{ab}_{(n)}$ of Eq. (42), namely,

$$S^{ab} = I_{d,\mathbf{1}}\delta^{ab} + I_{d,\mathbf{8}}d^{ab8} + I_{d,\mathbf{27}}\left[\delta^{a8}\delta^{b8} - \frac{1}{8}\delta^{ab} - \frac{3}{5}d^{ab8}d^{888}\right], \tag{58}$$

where

$$I_{d,\mathbf{1}} = \frac{1}{8}\left[3I_d(m_\pi,\mu)+4I_d(m_K,\mu)+I_d(m_\eta,\mu)\right], \tag{59a}$$

$$I_{d,\mathbf{8}} = \frac{2\sqrt{3}}{5}\left[\frac{3}{2}I_d(m_\pi,\mu)-I_d(m_K,\mu)-\frac{1}{2}I_d(m_\eta,\mu)\right], \tag{59b}$$

$$I_{d,\mathbf{27}} = \frac{1}{3}I_d(m_\pi,\mu)-\frac{4}{3}I_d(m_K,\mu)+I_d(m_\eta,\mu). \tag{59c}$$

The integral over the loop is given in Appendix A, (A22). The different flavor contributions in Eq. (57) read

(1) Flavor singlet contribution

$$[T^a,[T^a,V^c]] = N_f V^c. \tag{60}$$

(2) Flavor octet contribution

$$d^{ab8}[T^a,[T^b,V^c]] = \frac{N_f}{2}d^{c8e}V^e. \tag{61}$$

(3) Flavor **27** contribution

$$[T^8,[T^8,V^c]] = f^{c8e}f^{8eg}V^g. \tag{62}$$

The straightforward combination of loop corrections 1(c) and 1(d), for the $\Lambda \to p$ process, yields

$$\left[\frac{f_1^{(\mathrm{c})}+f_1^{(\mathrm{d})}}{f_1^{SU(3)}}\right]_{\Lambda p} = \frac{3}{8}\left[H(m_\pi,m_K)+H(m_K,m_\eta)\right]. \tag{63}$$

Equation (63) agrees with the results derived in Refs. [12, 14, 15] but differs in a global sign with respect to the expression presented in Ref. [13].

It is also interesting to remark that Eq. (63) contributes at the same order in N_c as Eq. (48). This assertion can be proved numerically.

E. Total one-loop correction to the baryon vector current

The baryon vector current operator V^c including one-loop corrections can be organized in a single expression as

$$V^c+\delta V^c = V^c+\delta V^c_{(\mathrm{a})}+\delta V^c_{(\mathrm{b})}+\delta V^c_{(\mathrm{c})}+\delta V^c_{(\mathrm{d})}, \tag{64}$$

where $\delta V^c_{(\mathrm{a})}$, $\delta V^c_{(\mathrm{b})}$, $\delta V^c_{(\mathrm{c})}$, and $\delta V^c_{(\mathrm{d})}$ are given by Eqs. (30), (41), (55), and (57). In the large-N_c counting, each correction is suppressed at least by a factor $1/N_c$ with respect to the tree-level operator V^c. The loop contributions expanded in $(m_s - \hat{m})$ satisfy the AG theorem.

The matrix elements of the operator $V^c + \delta V^c$ give the actual values of the vector form factors f_1 as defined in HSD. The full expressions for the processes observed are

$$\begin{aligned}
&\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Lambda p} \\
&= 1 + \left[\frac{3}{8} + \frac{17}{32}a_1^2 + \frac{3}{16}a_1b_2 + \frac{17}{48}a_1b_3 + \frac{1}{32}b_2^2 + \frac{1}{16}b_2b_3 + \frac{17}{288}b_3^2\right] H(m_\pi, m_K) \\
&\quad + \left[\frac{3}{8} + \frac{9}{32}a_1^2 + \frac{3}{16}a_1b_2 + \frac{3}{16}a_1b_3 + \frac{1}{32}b_2^2 + \frac{1}{16}b_2b_3 + \frac{1}{32}b_3^2\right] H(m_K, m_\eta) \\
&\quad + \left[-\frac{1}{4}a_1^2 - \frac{1}{4}a_1c_3 - \frac{1}{16}c_3^2\right] K(m_\pi, m_K, \Delta),
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Sigma^- n} \\
&= 1 + \left[\frac{3}{8} - \frac{7}{32}a_1^2 - \frac{1}{16}a_1b_2 - \frac{7}{48}a_1b_3 + \frac{1}{32}b_2^2 - \frac{1}{48}b_2b_3 - \frac{7}{288}b_3^2\right] H(m_\pi, m_K) \\
&\quad + \left[\frac{3}{8} + \frac{1}{32}a_1^2 - \frac{1}{16}a_1b_2 + \frac{1}{48}a_1b_3 + \frac{1}{32}b_2^2 - \frac{1}{48}b_2b_3 + \frac{1}{288}b_3^2\right] H(m_K, m_\eta) \\
&\quad + \left[\frac{1}{2}a_1^2 + \frac{1}{2}a_1c_3 + \frac{1}{8}c_3^2\right] K(m_\pi, m_K, \Delta) + \left[\frac{1}{4}a_1^2 + \frac{1}{4}a_1c_3 + \frac{1}{16}c_3^2\right] K(m_K, m_\eta, \Delta) \quad ,
\end{aligned} \tag{66}$$

$$\begin{aligned}
&\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^- \Lambda} \\
&= 1 + \left[\frac{3}{8} + \frac{9}{32}a_1^2 + \frac{1}{16}a_1b_2 + \frac{3}{16}a_1b_3 + \frac{1}{32}b_2^2 + \frac{1}{48}b_2b_3 + \frac{1}{32}b_3^2\right] H(m_\pi, m_K) \\
&\quad + \left[\frac{3}{8} + \frac{1}{32}a_1^2 + \frac{1}{16}a_1b_2 + \frac{1}{48}a_1b_3 + \frac{1}{32}b_2^2 + \frac{1}{48}b_2b_3 + \frac{1}{288}b_3^2\right] H(m_K, m_\eta) \\
&\quad + \left[\frac{1}{4}a_1^2 + \frac{1}{4}a_1c_3 + \frac{1}{16}c_3^2\right] K(m_K, m_\eta, \Delta),
\end{aligned} \tag{67}$$

$$\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Sigma^0}$$
$$= 1 + \left[\frac{3}{8} + \frac{17}{32}a_1^2 + \frac{5}{16}a_1b_2 + \frac{17}{48}a_1b_3 + \frac{1}{32}b_2^2 + \frac{5}{48}b_2b_3 + \frac{17}{288}b_3^2\right] H(m_\pi, m_K)$$
$$+ \left[\frac{3}{8} + \frac{25}{32}a_1^2 + \frac{5}{16}a_1b_2 + \frac{25}{48}a_1b_3 + \frac{1}{32}b_2^2 + \frac{5}{48}b_2b_3 + \frac{25}{288}b_3^2\right] H(m_K, m_\eta) \quad (68)$$
$$+ \left[-\frac{1}{4}a_1^2 - \frac{1}{4}a_1c_3 - \frac{1}{16}c_3^2\right] K(m_\pi, m_K, \Delta) + \left[-\frac{1}{2}a_1^2 - \frac{1}{2}a_1c_3 - \frac{1}{8}c_3^2\right] K(m_K, m_\eta, \Delta),$$

and
$$\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Sigma^-n} = \left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Sigma^0p}, \qquad \left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^0\Sigma^+} = \left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Sigma^0}, \quad (69)$$
where the latter are isospin relations.

A full crosscheck of the above expressions has been performed with their counterparts obtained within (heavy) baryon chiral perturbation theory [12–15], according to the guidelines described above. The results agree order by order.

It is interesting to notice that the contribution of diagrams (c) and (d) to the ratios $f_1/f_1^{SU(3)}$ is the same for all ratios.

V. NUMERICAL ANALYSIS

An analysis of the available experimental data [33] can be performed by using the results obtained here. In previous works [6, 7] a number of fits have been carried out to determine the baryon axial couplings, which are given by the matrix elements of the baryon axial current operator $A^{kc} + \delta A^{kc}$. For octet baryons, the axial vector couplings are g_1 normalized in such a way that $g_1 \sim 1.27$ for neutron β decay. For decuplet baryons, the axial vector couplings are denoted by g, which are extracted via Goldberger-Treiman relations from the widths of the strong decays of decuplet to octet baryons and pions [28].

The effects related to $SU(3)$ SB are contained in δA^{kc} in two ways: On the one hand, at tree level, all relevant operators which explicitly break $SU(3)$ at leading order are included; this contribution is loosely referred to as perturbative SB. On the other hand, in the one-loop corrections, $SU(3)$ SB is accounted for implicitly, since the loop integrals depend on the π, K and η masses.

The operator δA^{kc} has been built up in a systematic way. In Ref. [6], δA^{kc} was constituted only by one-loop corrections within the combined approach, while in Ref. [7], a more refined calculation was performed to include the effects of perturbative $SU(3)$ SB corrections and the effects of the baryon decuplet–octet mass splitting. The corrected axial vector current operator actually used in the numerical analysis reads

$$\begin{aligned} A^{kc} + \delta A^{kc}_{\rm SB} + \delta A^{kc}_{\rm 1L} \; = \; & a_1 G^{kc} + b_2 \frac{1}{N_c} \mathcal{D}_2^{kc} + b_3 \frac{1}{N_c^2} \mathcal{D}_3^{kc} + c_3 \frac{1}{N_c^2} \mathcal{O}_3^{kc} + \Bigg[d_1 d^{c8e} G^{ke} + d_2 \frac{1}{N_c} d^{c8e} \mathcal{D}_2^{ke} \\ & + d_3 \frac{1}{N_c} \left(\{G^{kc}, T^8\} - \{G^{k8}, T^c\} \right) + d_4 \frac{1}{N_c} \left(\{G^{kc}, T^8\} + \{G^{k8}, T^c\} \right) \Bigg] \\ & + \delta A^{kc}_{\rm 1L}, \end{aligned} \tag{70}$$

where $\delta A^{kc}_{\rm SB}$ is the correction that arises from perturbative SB and $\delta A^{kc}_{\rm 1L}$ is the one-loop correction. Note that the loop corrections are renormalized by the counter terms corresponding to the coefficients a_i, b_i, and c_i. Minimal subtraction is used with renormalization scale μ. Equation (70) was parametrized in Ref. [7] in such a way that flavor SB took place entirely in the non-zero strangeness sector only. This involves however a bias, namely that $g_1 = F + D$ for neutron β decay *even in the presence of* $SU(3)$ SB, which corresponds to a constraint on the counter term coefficients. That bias is avoided here by instead taking into account $SU(3)$ SB in the axial couplings throughout.

The scope of the numerical analyses performed in Refs. [6, 7] within these two scenarios was limited to determining only g_1 and g, because the f_1's were given at their $SU(3)$ symmetric values, $f_1^{SU(3)}$, in view of the AG theorem. The present analysis, however, is uniquely positioned in the sense that, on the same footing as g_1, the one-loop corrections to f_1 within large-N_c chiral perturbation theory have been computed, including the effects of a non-zero baryon decuplet–octet mass splitting. Thus, the pattern of $SU(3)$ SB for f_1, which will be referred to as $f_1/f_1^{SU(3)}$ hereafter, can be evaluated.

The available experimental information for octet baryons is given in terms of the decay rates R, the ratios g_1/f_1, the angular correlation coefficients $\alpha_{e\nu}$, and the spin-asymmetry coefficients α_e, α_ν, α_B, A, and B. All eight decay rates and all six possible g_1/f_1 ratios have been measured (the ratios g_1/f_1 for $\Sigma^\pm \to \Lambda$ semileptonic decays are undefined). A summary of this experimental information can be found in Table II of Ref. [7], along with a detailed discussion about how this information can be matched with the one listed in Ref. [33]. That discussion is not repeated here. For decuplet baryons, the axial couplings

g for the processes $\Delta \to N\pi$, $\Sigma^* \to \Lambda\pi$, $\Sigma^* \to \Sigma\pi$, and $\Xi^* \to \Xi\pi$ are given in Table IX of that reference as well. For the purposes of the present work, the experimental information is arranged into three sets: R and g_1/f_1 constitute set 1; R, g_1/f_1 and g constitute set 2; and g_1/f_1 and g constitute set 3. The latter can be enriched by adding two more pieces of information: the g_1 couplings for the $\Sigma^\pm \to \Lambda$ semileptonic processes, which can be obtained from their respective decay rates through a standard procedure.[4] The values found are $g_1 = 0.619 \pm 0.077$ and $g_1 = 0.597 \pm 0.014$ for $\Sigma^+ \to \Lambda e^+ \nu$ and $\Sigma^- \to \Lambda e^- \overline{\nu}$, respectively. In passing, it is worth mentioning that set 3 is also particularly interesting because g_1 and g are related in the large-N_c limit; in actual numerical analyses, the fits that include g yield more stable solutions [7].

There are eight parameters to be determined in the analysis, all of them affecting directly the g_1's. Four of them, a_1, b_2, b_3, and c_3 arise from the $1/N_c$ expansion of A^{kc} alone, Eq. (11), and the remaining four, $d_1, \ldots, d_4$, come from perturbative SB, according to the discussion provided in Sec. V.B. of Ref. [7]. For definiteness, the physical masses of the mesons and baryons listed in Ref. [33] are used, along with $\Delta = 0.231$ GeV, $F_\pi = 93$ MeV, and $\mu = 1$ GeV. Also, the suggested values of the CKM matrix elements V_{ud} and V_{us} are used as inputs.

Without further ado, a fit where $SU(3)$ SB corrections to both f_1 and g_1 (to the respective orders considered here) enter into play can be performed using data set 3, which is equivalent to using the data about g_1/f_1 and g. An analysis under this circumstance has some implications. First, it has been pointed out that both g_1 and g are related in the large-N_c limit, so for a consistent analysis they should be present simultaneously. Also, the new output can be contrasted with the equivalent one obtained in Ref. [7]. But most importantly, the use of the axial couplings only will allow one to check whether the predicted decay rates and asymmetry and spin-angular correlation coefficients agree with the experimental ones. This may be a crucial test of this approach. The eight-parameter fit for 12 pieces of information yields the results labeled as Fit 1 in Table IV. First the limit $f_1 = f_1^{SU(3)}$ is used (case a) to subsequently add SB effects in f_1 (case b). Some interesting features emerge from this analysis. First, the $a_1, \ldots, c_3$ parameters are order one, which completely agrees with expectations. Besides, the SB parameters $d_1, \ldots, d_4$ are roughly suppressed by

[4] Radiative corrections and a dipole parametrization of the axial vector form factors are two key considerations [1].

TABLE IV: Best-fit parameters for the fit presented in this work. The pertinent values of the equivalent $SU(3)$ couplings D, F, $\mathcal{C}$, and $\mathcal{H}$ are also listed. The quoted errors come from the fit only and do not include any theoretical uncertainty.

	Fit 1a	Fit 1b
Data set	3	3
SB in f_1	×	✓
SB in g_1	✓	✓
a_1	0.89(0.15)	0.95(0.14)
b_2	−1.03(0.19)	−1.10(0.19)
b_3	1.18(0.15)	1.10(0.09)
c_3	1.18(0.17)	1.07(0.15)
d_1	0.52(0.12)	0.62(0.13)
d_2	−0.56(0.25)	−0.57(0.24)
d_3	0.38(0.05)	0.39(0.05)
d_4	−0.05(0.08)	−0.06(0.08)
D	0.64(0.05)	0.66(0.05)
F	0.26(0.01)	0.25(0.01)
$\mathcal{C}$	−1.48(0.07)	−1.48(0.07)
$\mathcal{H}$	−2.74(0.27)	−2.50(0.17)
F/D	0.40(0.03)	0.39(0.02)
$3F-D$	0.13(0.04)	0.10(0.04)
χ^2/dof	5.6/4	5.5/4

a factor of $\epsilon \sim 0.3$ with respect to the leading ones, which is consistent with first-order SB. However, what is also worth mentioning is that the $SU(3)$ invariant couplings D, F, $\mathcal{C}$, and $\mathcal{H}$ reach values which are in good agreement with expectations (the coupling $\mathcal{H}$ still remains a little high, but possesses the correct sign). On the other hand, Fit 1b deserves special attention because it is where the effects of SB in f_1 are evaluated. With the best-fit parameters, the corresponding $SU(3)$ SB pattern of f_1 is displayed in Table V. This SB pattern suggest a systematic decrease between 3.4 and 4.8% in the f_1 values with respect to their $SU(3)$-symmetric values in all the channels considered, which is in perfect agreement with the expectation from second-order SB dictated by the AG theorem.

Armed with the vector and axial couplings from Fit 1b, the integrated observables for BSD can be estimated. The overall behavior of Fit 1b is excellent in the sense that the predicted observables are in very good agreement with their experimental counterparts. There is no need to present new tables to display such small discrepancies.

TABLE V: Relative contributions to the leading form factor f_1 for non-vanishing Δ obtained in the present work. The $\Xi^0\Sigma^+$ and $\Xi^-\Sigma^0$ values are related by isospin symmetry. The correction $\delta^{(2)}$ computed in other works is also listed.

Process	$\frac{f_1^{(a)}+f_1^{(b)}}{f_1^{SU(3)}}$	$\frac{f_1^{(c)}+f_1^{(d)}}{f_1^{SU(3)}}$	$\frac{f_1}{f_1^{SU(3)}}$	$\frac{f_1}{f_1^{SU(3)}}-1$	$\delta^{(2)}$		
					Ref. [14]	Ref. [15]	Ref. [16]
Λp	−0.026	−0.022	0.952	−0.048	−0.080	−0.097	−0.031
$\Sigma^- n$	−0.013	−0.022	0.966	−0.034	−0.024	0.008	−0.022
$\Xi^-\Lambda$	−0.025	−0.022	0.953	−0.047	−0.063	−0.063	−0.029
$\Xi^-\Sigma^0$	−0.016	−0.022	0.962	−0.038	−0.076	−0.094	−0.030

A variant of Fit 1b consists in removing all subleading corrections from f_1 and keeping the a_1 contribution only. There are no significant changes in the best-fit parameters. The pattern of SB in f_1 varies between $\pm 1\%$ but the total χ^2 remains practically unaltered. One however notices that there is significant sensitivity to contributions which depend on b_2, b_3 and c_3, for instance up to 2% for the case of b_3. Since these are $1/N_c$ suppressed contributions, this may indicate a slow convergence. This is also indicative of the sensitivity of the corrections to the values used for the couplings F, D and H, which is a source of the disagreements mentioned earlier between different calculations.

Following the lines of Refs. [14–16], the chiral corrections to the vector form factor can be parametrized as

$$f_1 = f_1^{SU(3)}(1 + \delta^{(2)} + \ldots), \tag{71}$$

where $\delta^{(2)}$ is the leading $SU(3)$-breaking loop correction $\mathcal{O}(p^2)$ and the dots stand for higher chiral corrections computed in Refs. [14–16]. References [14] and [16] include dynamical octet and decuplet contributions. A numerical comparison of $\delta^{(2)}$ is displayed in Table V for the sake of completeness. The comparison is acceptable for Refs. [14, 16]. In order to compare with Ref. [15] on an equal footing, a fit using data set 1 should be done, dropping all the decuplet effects. This falls in the context of BChPT without decuplet dynamical degrees of freedom. In this case, the parameters that enter are a_1 and b_2, along with the four d_i's. The analysis yields $a_1 = 0.93 \pm 0.01$, $b_2 = -0.01 \pm 0.07$, $d_1 = 0.14 \pm 0.05$, $d_2 = 1.01 \pm 0.31$,

$d_3 = 0.31 \pm 0.06$, and $d_4 = -0.15 \pm 0.07$, with $\chi^2 = 16/8$ dof. The leading vector form factors reduce their $SU(3)$ symmetric values by 5% for $\Lambda \to p$, $\Xi^- \to \Sigma^0$ and $\Xi^0 \to \Sigma^+$ processes, and by 1.1% and 3.6% for $\Sigma^- \to n$ and $\Xi^- \to \Lambda$ processes, respectively. In other words, in this very last case also an overall decrease in the symmetry pattern is also observed.

On the other hand, Lattice QCD results constitute another source to compare with. Certainly, a direct comparison is meaningful only when relevant issues such as chiral extrapolation [34] are addressed. In this regard, a comparison with the most recent unquenched Lattice QCD results in Ref. [21], where a chiral extrapolation of $f_1/f_1^{SU(3)}$ for $\Xi^0 \to \Sigma^+$ and $\Sigma^- \to n$ gives 0.974(5) and 0.982(8) respectively, which is in qualitative agreement with the results in Table V. In the earlier quenched calculations [19, 20] the errors are too large to definitely conclude about the signs of those corrections. Since quenching should give subleading in $1/N_c$ effects, it would be important as a test to have those calculations revisited.

To close this section, it should be pointed out that the SB pattern of f_1 observed here opposes the one observed in Refs. [2, 29], obtained within the $1/N_c$ expansion alone. The analysis presented there can be repeated by using the updated experimental information about HSD (the data on the $\Xi^0 \to \Sigma^+$ semileptonic decay was not available by that time) and the current determination of $|V_{us}|$. The analysis yields a systematic reduction of the SB pattern with respect to the one obtained on that reference. For instance, $f_1/f_1^{SU(3)}$ for $\Lambda \to p$ semileptonic decay is now slightly lower than one. This last remark leads to a final comment. One cannot yet consider the theoretical issues as closed. It is most important that within the same combined approach used to calculate the f_1's to $\mathcal{O}(p^2)$, higher $\mathcal{O}(p^3)$ corrections be also computed. Naively those corrections are expect to be 30% of the corrections $\mathcal{O}(p^2)$. As discussed earlier, it is particularly important to investigate them because of the large $\mathcal{O}(p^3)$ corrections found in the previous works.

VI. SUMMARY AND CONCLUSIONS

The objective of the present work was to perform a calculation of the $SU(3)$ breaking corrections for the f_1 form factor for different channels to $\mathcal{O}(p^2)$ in the chiral expansion and consistent with the strictures imposed by the $1/N_c$ expansion. The results are predictions, as they are not affected by unknown parameters: they are entirely given in terms of known low energy constants. It should be emphasized that having a prediction of this kind based on a framework that respects QCD throughout is very important.

With the purpose of checking with known results in which decuplet baryons were included with phenomenological couplings [14], higher order terms in $1/N_c$ were included to reproduce those (axial vector) couplings and determine the agreement of the results with those obtained in that reference. The results were also confronted with the experimental observables for BSD and the strong decays of the decuplet baryons. Two different fits were carried out in order to elucidate the relative importance of the various effects, where the summary is presented in Table IV. The following conclusions can be derived from those results:

a) The $1/N_c$ corrections to the axial vector currents are very important. These are reflected in the deviation of the relations between the couplings F, D, $\mathcal{C}$ and $\mathcal{H}$ which hold in the $SU(6)$ limit. Both, the strong decuplet to octet strong transitions as well as the weak decays are sensitive to those sub-leading corrections.

b) The effects of $SU(3)$ SB in f_1 are calculable at the order considered here and turn out to be about -5%. The hyperon weak decay observables at the current degree of accuracy are not sensitive to those effects, endorsing the same claim made by Cabibbo, Swallow and Winston [17]. It is noted that the $1/N_c$ corrections to the axial currents which determine the vertices in the loop-diagrams do not affect significantly the correction to f_1.

c) The effects of $SU(3)$ SB on the axial vector couplings g_1 are on the other hand very important, as shown in Table V. The octet pieces of the SB are the dominant ones with magnitudes up to 0.3, while the 27-plet pieces are much smaller, at most 20% of the octet ones and in most cases much smaller than that. Because of the small tree level value of the axial coupling of the transition $\Xi^-\Lambda$, the subleading corrections, which include the SB effects, turn out to be larger than the leading term. For the other cases the subleading corrections do not exceed the expected 30% of the leading order value.

d) In the calculation of the $SU(3)$ SB corrections to f_1 it is noted that the inclusion of the subleading in $1/N_c$ corrections to the meson-baryon couplings produce small deviations, and to the current level of accuracies they are unnoticeable. Similarly, any $SU(3)$ breaking effects on those couplings turn out to be insignificant: they are of higher order in the chiral expansion, but they were evaluated in order to check their insignificance.

e) Perhaps the most important reason for accurate calculations of HSD is to provide an additional accurate extraction of $|V_{us}|$. At present the ratio of $K_{\ell 2}$ to $\pi_{\ell 2}$ decay together with the ratio F_K/F_π from Lattice QCD and $|V_{ud}|$ from super-allowed β decay, and the $K_{\ell 3}$ decays give the most accurate determinations. The smallness of $|V_{ub}|$ means that $|V_{us}|$ is very close to the unitarity limit. A test of unitarity the CKM matrix requires as accurate as possible results for $|V_{us}|$, for which the increase in precision from HSD would be welcome. This however will require further experimental progress in the determination of the various HSD parameters.

The natural next step in the study of the BSD in the present framework is the calculation in the combined framework of the ξ expansion to $\mathcal{O}(\xi^3)$. This is the next order beyond the one presented here. While such a complete calculation for the axial currents is already available for two flavors, it needs to be implemented for three flavors and also for the vector currents. This will be the objective of future work.

To close this article, it is worth quoting a sentence found in Ref. [35]: "It will take a lot more work to see whether the $1/N_c$ expansion can be combined with baryon chiral perturbation theory to analyze baryon properties in a systematic and controlled expansion." Two decades later, one can claim that this task is indeed possible.

Acknowledgments

This work was supported in part by DOE Contract No. DE-AC05-06OR23177 under which JSA operates the Thomas Jefferson National Accelerator Facility (RFM and JLG), and by the National Science Foundation (USA) through grant PHY-1307413 (JLG), and by Consejo Nacional de Ciencia y Tecnología and Fondo de Apoyo a la Investigación (Universidad Autónoma de San Luis Potosí), Mexico (RFM).

Appendix A: Loop integrals

The integrals over the loops displayed in Fig. 1 are fully discussed in this section and the most general results needed in the present analysis are provided for the sake of completeness.

First, for the Feynman diagram displayed in Fig. 1(a), the loop integral can be written in the most general way as

$$J^{\mu}_{ij}(m_1,m_2,\Delta,\mu;q^2)$$
$$= \frac{i}{F_\pi^2}(\mu^2)^{\frac{4-d}{2}} \int \frac{d^dk}{(2\pi)^d} \frac{(2k+q)^\mu k_i (k+q)_j}{(k^2-m_1^2+i\varepsilon)[(k+q)^2-m_2^2+i\varepsilon](p^0-k^0-\Delta+i\varepsilon)}, \quad \text{(A1)}$$

where m_1 and k and m_2 and $k+q$ denote respectively the masses and four-momenta of the mesons in the loop, q is the four-momentum transfer, Δ is the baryon decuplet-octet mass difference, and $d = 4-\epsilon$ to use dimensional regularization with scale μ. Due to the Lorentz structure of $J^{\mu}_{ij}(m_1,m_2,\Delta,\mu;q^2)$, it can be separated into temporal $J^{0}_{ij}(m_1,m_2,\Delta,\mu;q^2)$ and spatial $J^{k}_{ij}(m_1,m_2,\Delta,\mu;q^2)$ components. The former, which is the one needed here, can be decomposed as $J_A(m_1,m_2,\Delta,\mu;q^2)\delta_{ij} + J_B(m_1,m_2,\Delta,\mu;q^2)q_iq_j$. In the $q^2 \to 0$ limit,

$$I_a(m_1,m_2,\Delta,\mu;0) \equiv \lim_{q^2\to 0} J_A(m_1,m_2,\Delta,\mu;q^2), \quad \text{(A2)}$$

where I_a is the integral associated to the one-loop correction to the baryon vector current of Fig. 1(a) at zero recoil. Without further ado, the resultant expression reads,

$$\begin{aligned}
&32\pi^2 F_\pi^2 I_a(m_1,m_2,\Delta,\mu;0) \\
&\quad = -(m_1^2+m_2^2-4\Delta^2)\lambda_\epsilon - \frac{3}{2}(m_1^2+m_2^2) + \frac{28}{3}\Delta^2 \\
&\qquad + \frac{1}{3(m_1^2-m_2^2)}\left[(3m_1^4-12m_1^2\Delta^2+8\Delta^4)\ln\frac{m_1^2}{\mu^2} - (3m_2^4-12m_2^2\Delta^2+8\Delta^4)\ln\frac{m_2^2}{\mu^2}\right] \\
&\qquad + \frac{8}{3}\frac{\Delta}{m_1^2-m_2^2} \times \begin{cases}
2(m_1^2-\Delta^2)^{3/2}\left[\frac{\pi}{2} - \tan^{-1}\left[\frac{\Delta}{\sqrt{m_1^2-\Delta^2}}\right]\right] \\
-2(m_2^2-\Delta^2)^{3/2}\left[\frac{\pi}{2} - \tan^{-1}\left[\frac{\Delta}{\sqrt{m_2^2-\Delta^2}}\right]\right], & |\Delta| < m_1 < m_2 \\
-(\Delta^2-m_1^2)^{3/2}\ln\left[\frac{\Delta-\sqrt{\Delta^2-m_1^2}}{\Delta+\sqrt{\Delta^2-m_1^2}}\right] \\
-2(m_2^2-\Delta^2)^{3/2}\left[\frac{\pi}{2} - \tan^{-1}\left[\frac{\Delta}{\sqrt{m_2^2-\Delta^2}}\right]\right], & m_1 < |\Delta| < m_2 \\
-(\Delta^2-m_1^2)^{3/2}\ln\left[\frac{\Delta-\sqrt{\Delta^2-m_1^2}}{\Delta+\sqrt{\Delta^2-m_1^2}}\right] \\
+(\Delta^2-m_2^2)^{3/2}\ln\left[\frac{\Delta-\sqrt{\Delta^2-m_2^2}}{\Delta+\sqrt{\Delta^2-m_2^2}}\right], & m_1 < m_2 < |\Delta|
\end{cases}
\end{aligned} \quad \text{(A3)}$$

where

$$\lambda_\epsilon = \frac{2}{\epsilon} - \gamma + \ln(4\pi), \quad \text{(A4)}$$

with $\gamma \simeq 0.577216$ the Euler constant. Without loss of generality, the condition $m_1 < m_2$ has been assumed in order to get the above result.

Now, the correction arising from the Feynman diagram displayed in Fig. 1(b) is given in terms of the derivatives of the basic loop integral [31]

$$\delta^{ij} I_b(m,\Delta,\mu) = \frac{i}{F_\pi^2}(\mu^2)^{\frac{4-d}{2}} \int \frac{d^dk}{(2\pi)^d} \frac{-k^i k^j}{(k^0-\Delta+i\varepsilon)(k^2-m^2+i\varepsilon)}. \tag{A5}$$

An explicit calculation yields[5].

$$\begin{aligned} 24\pi^2 F_\pi^2 I_b(m,\Delta,\mu) \;=\; & -\Delta\left[\Delta^2-\frac{3}{2}m^2\right]\lambda_\epsilon + \Delta\left[\Delta^2-\frac{3}{2}m^2\right]\ln\frac{m^2}{\mu^2} - \frac{8}{3}\Delta^3 + \frac{7}{2}\Delta m^2 \\ & + \begin{cases} 2(m^2-\Delta^2)^{3/2}\left[\frac{\pi}{2} - \tan^{-1}\left[\frac{\Delta}{\sqrt{m^2-\Delta^2}}\right]\right], & |\Delta| < m \\ -(\Delta^2-m^2)^{3/2}\ln\left[\frac{\Delta-\sqrt{\Delta^2-m^2}}{\Delta+\sqrt{\Delta^2-m^2}}\right], & |\Delta| > m. \end{cases} \end{aligned} \tag{A6}$$

From this function it follows that

$$\begin{aligned} 16\pi^2 F_\pi^2 I_b^{(1)}(m,\Delta,\mu) \;=\; & (m^2-2\Delta^2)\left[\lambda_\epsilon + 1 - \ln\frac{m^2}{\mu^2}\right] - 2\Delta^2 \\ & - \begin{cases} 4\Delta\sqrt{m^2-\Delta^2}\left[\frac{\pi}{2} - \tan^{-1}\left[\frac{\Delta}{\sqrt{m^2-\Delta^2}}\right]\right], & |\Delta| < m \\ 2\Delta\sqrt{\Delta^2-m^2}\ln\left[\frac{\Delta-\sqrt{\Delta^2-m^2}}{\Delta+\sqrt{\Delta^2-m^2}}\right], & |\Delta| > m. \end{cases} \end{aligned} \tag{A7}$$

$$\begin{aligned} 4\pi^2 F_\pi^2 I_b^{(2)}(m,\Delta,\mu) \;=\; & -\Delta\left[\lambda_\epsilon + 1 - \ln\frac{m^2}{\mu^2}\right] \\ & + \begin{cases} -\frac{m^2-2\Delta^2}{\sqrt{m^2-\Delta^2}}\left[\frac{\pi}{2} - \tan^{-1}\left[\frac{\Delta}{\sqrt{m^2-\Delta^2}}\right]\right], & |\Delta| < m \\ \frac{m^2-2\Delta^2}{2\sqrt{\Delta^2-m^2}}\ln\left[\frac{\Delta-\sqrt{\Delta^2-m^2}}{\Delta+\sqrt{\Delta^2-m^2}}\right], & |\Delta| > m. \end{cases} \end{aligned} \tag{A8}$$

$$\begin{aligned} 4\pi^2 F_\pi^2 I_b^{(3)}(m,\Delta,\mu) \;=\; & -\lambda_\epsilon - \frac{\Delta^2}{m^2-\Delta^2} + \ln\frac{m^2}{\mu^2} \\ & + \begin{cases} \frac{\Delta(3m^2-2\Delta^2)}{(m^2-\Delta^2)^{3/2}}\left[\frac{\pi}{2} - \tan^{-1}\left[\frac{\Delta}{\sqrt{m^2-\Delta^2}}\right]\right], & |\Delta| < m \\ \frac{\Delta(3m^2-2\Delta^2)}{2(\Delta^2-m^2)^{3/2}}\ln\left[\frac{\Delta-\sqrt{\Delta^2-m^2}}{\Delta+\sqrt{\Delta^2-m^2}}\right], & |\Delta| > m. \end{cases} \end{aligned} \tag{A9}$$

[5] Here the sign in front of the term $\frac{7}{2}\Delta m^2$ in the function $I_b(m,\Delta,\mu)$ has been fixed. The opposite sign, which is incorrect, was used in Refs. [6–9, 31].

Therefore, the function $I_b(m,\Delta,\mu)$ and its derivatives in the $\Delta\to 0$ limit follow accordingly; they read

$$I_b(m,0,\mu) = \frac{m^3}{24\pi^2 F_\pi^2}, \tag{A10}$$

$$I_b^{(1)}(m,0,\mu) = \frac{m^2}{16\pi^2 F_\pi^2}\left[\lambda_\epsilon + 1 - \ln\frac{m^2}{\mu^2}\right], \tag{A11}$$

$$I_b^{(2)}(m,0,\mu) = -\frac{m}{8\pi F_\pi^2}, \tag{A12}$$

and

$$I_b^{(3)}(m,0,\mu) = \frac{1}{4\pi^2 F_\pi^2}\left[-\lambda_\epsilon + \ln\frac{m^2}{\mu^2}\right]. \tag{A13}$$

Next, for the Feynman diagrams displayed in Figs. 1(c) and 1(d), it is useful to introduce the scalar function

$$L_{r,m} = i\mu^{4-d}\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^{2r}}{(\ell^2-\beta+i\varepsilon)^m}, \tag{A14}$$

where r and m are integers and β is an independent function of ℓ^2. An explicit calculation yields

$$L_{r,m} = \frac{(-1)^{r-m+1}}{16\pi^2}\frac{\Gamma(r+2-\frac{\epsilon}{2})\Gamma(-r+m-2+\frac{\epsilon}{2})}{\Gamma(m)\Gamma(2-\frac{\epsilon}{2})}(4\pi\mu^2)^{\frac{\epsilon}{2}}\beta^{r-m+2-\frac{\epsilon}{2}}, \tag{A15}$$

Now, the loop integral of Fig. 1(c) is given in chiral perturbation theory by

$$I^\alpha(m_1,m_2,\mu;q^2) = \frac{i}{F_\pi^2}\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{(2\not{k}-\not{q})q^\alpha}{[(k-q)^2-m_2^2+i\varepsilon](k^2-m_1^2+i\varepsilon)}. \tag{A16}$$

I^α will have a piece proportional to γ^α and another one proportional to q^α. The former is the one related to the vector form factor $f_1(q^2)$.

By using the conventional Feynman method to combine denominators, it is easy to see that the contribution of I^α proportional to γ^α can be written as

$$I_c(m_1,m_2,\mu;q^2) = \frac{1}{F_\pi^2}\int_0^1 dx\frac{2}{4-\epsilon}L_{1,2}, \tag{A17}$$

where $L_{1,2}$ can be obtained from Eq. (A15) with

$$\beta = -q^2x(1-x) + m_2^2x + m_1^2(1-x). \tag{A18}$$

A standard calculation yields

$$\begin{aligned}
16\pi^2 F_\pi^2 I_c(m_1,m_2,\mu;q^2) &= \frac{1}{6}(q^2-3m_1^2-3m_2^2)\lambda_\epsilon - \frac{1}{12}(q^2-3m_1^2-3m_2^2)\left[\ln\frac{m_1^2}{\mu^2}+\ln\frac{m_2^2}{\mu^2}\right] \\
&+ \frac{4}{9}q^2 - \frac{7}{6}(m_1^2+m_2^2) + \frac{(m_1^2-m_2^2)^2}{6q^2} - \frac{m_1^4-m_2^4}{4q^2}\ln\frac{m_2^2}{m_1^2} \\
&+ \frac{(m_1^2-m_2^2)^3+[q^2(q^2-2m_1^2-2m_2^2)+(m_1^2-m_2^2)^2]^{3/2}}{12(q^2)^2}\ln\frac{m_2^2}{m_1^2} \\
&+ \frac{[q^2(q^2-2m_1^2-2m_2^2)+(m_1^2-m_2^2)^2]^{3/2}}{6(q^2)^2} \\
&\times \ln\left[\frac{-q^2-m_1^2+m_2^2+\sqrt{q^2(q^2-2m_1^2-2m_2^2)+(m_1^2-m_2^2)^2}}{q^2-m_1^2+m_2^2+\sqrt{q^2(q^2-2m_1^2-2m_2^2)+(m_1^2-m_2^2)^2}}\right].
\end{aligned} \tag{A19}$$

In the $q^2 \to 0$ limit, $I_c(m_1,m_2,\mu;q^2)$ reduces to

$$32\pi^2 F_\pi^2 I_c(m_1,m_2,\mu;0) = -(m_1^2+m_2^2)\lambda_\epsilon - \frac{3}{2}(m_1^2+m_2^2) + \frac{1}{m_1^2-m_2^2}\left[m_1^4\ln\frac{m_1^2}{\mu^2} - m_2^4\ln\frac{m_2^2}{\mu^2}\right]. \tag{A20}$$

Finally, for the Feynman diagram displayed in Fig. 1(d), the integral over the loop is

$$\begin{aligned}
I_d(m,\mu) &= \frac{i}{F_\pi^2}\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2+i\varepsilon} \\
&= \frac{1}{F_\pi^2}L_{0,1},
\end{aligned} \tag{A21}$$

where $\beta = m^2$ in this case. A straightforward calculation yields

$$I_d(m,\mu) = \frac{m^2}{16\pi^2 F_\pi^2}\left[-\lambda_\epsilon - 1 + \ln\frac{m^2}{\mu^2}\right]. \tag{A22}$$

Appendix B: Reduction of baryon operators

The full list of operator reductions performed in the current analysis is presented in this appendix. For $N_c = 3$, there appeared operator products containing up to eight-body operators for which the reductions turned out to be quite involved. Due to the fact that for any $SU(6)$ representation polynomials in the spin-flavor generators J^i, T^a and G^{ia} form a complete set of operators, the reductions were always possible. Apart from using well-known decompositions among operators, a particularly useful identity was also used, namely,

$$[T^a, X^b] = i f^{abc} X^c \ ,$$

where X^b stands for *any* spin-0 or spin-1 flavor octet. For instance,

$$[T^a, A^{ib}] = if^{abc}A^{ic} \ ,$$

where A^{ib} is the axial-vector current operator (11), or

$$[T^a, [J^2, A^{ib}]] = if^{abc}[J^2, A^{ic}] \ ,$$

or

$$d^{abe}[T^c, \{J^2, \{T^a, T^b\}\}] = if^{ceg}d^{gde}\{J^2, \{T^d, T^e\}\} \ ,$$

to name but a few.

For computational ease, the second and third summands of Eq. (41) can be respectively rewritten as

$$\{A^{ja}, [T^c, [J^2, A^{jb}]]\} = if^{cbe}\{A^{ja}, [J^2, A^{je}]\}$$

and

$$\begin{aligned}
&[A^{ja}, [[J^2, [J^2, A^{jb}]], T^c]] - \frac{1}{2}[[J^2, A^{ja}], [[J^2, A^{jb}], T^c]] \\
&= \frac{3}{2}if^{bce}[[J^2, A^{je}], [J^2, A^{ja}]] - if^{bce}[J^2, [[J^2, A^{je}], A^{ja}]] \ ,
\end{aligned}$$

where

$$if^{bce}[J^2, [[J^2, A^{je}], A^{ja}]]F^{ab} = 0 \ ,$$

for $F^{ab} = \delta^{ab}$, d^{ab8}, or $\delta^{a8}\delta^{b8}$.

The operator reductions performed, for arbitrary N_c and N_f, are listed below. These expressions are to be evaluated at the physical values $N_f = N_c = 3$.

1. $if^{acb}A^{ia}A^{ib}$

$$if^{acb}G^{ia}G^{ib} = \frac{3}{8}N_f T^c, \tag{B1}$$

$$if^{acb}(G^{ia}\mathcal{D}_2^{ib} + \mathcal{D}_2^{ia}G^{ib}) = \frac{1}{2}N_f\{J^r, G^{rc}\}, \tag{B2}$$

$$if^{acb}(G^{ia}\mathcal{D}_3^{ib} + \mathcal{D}_3^{ia}G^{ib}) = (N_c + N_f)\{J^r, G^{rc}\} + \frac{1}{2}(N_f - 2)\{J^2, T^c\}, \tag{B3}$$

$$if^{acb}(G^{ia}\mathcal{O}_3^{ib} + \mathcal{O}_3^{ia}G^{ib}) = \frac{3}{2}N_fT^c - \frac{3}{2}(N_c+N_f)\{J^r,G^{rc}\} + \frac{1}{2}(N_f+3)\{J^2,T^c\}, \tag{B4}$$

$$if^{acb}\mathcal{D}_2^{ia}\mathcal{D}_2^{ib} = \frac{1}{4}N_f\{J^2,T^c\}, \tag{B5}$$

$$if^{acb}(\mathcal{D}_2^{ia}\mathcal{D}_3^{ib} + \mathcal{D}_3^{ia}\mathcal{D}_2^{ib}) = N_f\{J^2,\{J^r,G^{rc}\}\}, \tag{B6}$$

$$if^{acb}(\mathcal{D}_2^{ia}\mathcal{O}_3^{ib} + \mathcal{O}_3^{ia}\mathcal{D}_2^{ib}) = 0, \tag{B7}$$

$$if^{acb}\mathcal{D}_3^{ia}\mathcal{D}_3^{ib} = (N_c+N_f)\{J^2,\{J^r,G^{rc}\}\} + \frac{1}{2}(N_f-2)\{J^2,\{J^2,T^c\}\}, \tag{B8}$$

$$if^{acb}(\mathcal{D}_3^{ia}\mathcal{O}_3^{ib} + \mathcal{O}_3^{ia}\mathcal{D}_3^{ib}) = 0, \tag{B9}$$

$$\begin{aligned} if^{acb}\mathcal{O}_3^{ia}\mathcal{O}_3^{ib} \ = \ & \frac{3}{2}N_fT^c - \frac{3}{2}(N_c+N_f)\{J^r,G^{rc}\} + \frac{1}{2}(4N_f+3)\{J^2,T^c\} \\ & - \frac{5}{4}(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\} + \frac{1}{4}(N_f+5)\{J^2,\{J^2,T^c\}\}. \end{aligned} \tag{B10}$$

2. $if^{acb}A^{ia}J^2A^{ib}$

$$if^{acb}G^{ia}J^2G^{ib} = \frac{3}{4}N_fT^c - \frac{1}{2}(N_c+N_f)\{J^r,G^{rc}\} + \frac{1}{16}(3N_f+8)\{J^2,T^c\}, \tag{B11}$$

$$if^{acb}(G^{ia}J^2\mathcal{D}_2^{ib} + \mathcal{D}_2^{ia}J^2G^{ib}) = \frac{1}{4}N_f\{J^2,\{J^r,G^{rc}\}\}, \tag{B12}$$

$$if^{acb}(G^{ia}J^2\mathcal{D}_3^{ib} + \mathcal{D}_3^{ia}J^2G^{ib}) = \frac{1}{2}(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\} + \frac{1}{4}(N_f-2)\{J^2,\{J^2,T^c\}\}, \tag{B13}$$

$$if^{acb}(G^{ia}J^2\mathcal{O}_3^{ib}+\mathcal{O}_3^{ia}J^2G^{ib}) = 3N_fT^c-3(N_c+N_f)\{J^r,G^{rc}\}+\frac{1}{4}(13N_f+12)\{J^2,T^c\} - \frac{7}{4}(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\}+\frac{1}{4}(N_f+7)\{J^2,\{J^2,T^c\}\} , \tag{B14}$$

$$if^{acb}\mathcal{D}_2^{ia}J^2\mathcal{D}_2^{ib}=\frac{1}{8}N_f\{J^2,\{J^2,T^c\}\}, \tag{B15}$$

$$if^{acb}(\mathcal{D}_2^{ia}J^2\mathcal{D}_3^{ib}+\mathcal{D}_3^{ia}J^2\mathcal{D}_2^{ib})=\frac{1}{2}N_f\{J^2,\{J^2,\{J^r,G^{rc}\}\}\}, \tag{B16}$$

$$if^{acb}(\mathcal{D}_2^{ia}J^2\mathcal{O}_3^{ib}+\mathcal{O}_3^{ia}J^2\mathcal{D}_2^{ib})=0, \tag{B17}$$

$$if^{acb}\mathcal{D}_3^{ia}J^2\mathcal{D}_3^{ib}=\frac{1}{2}(N_c+N_f)\{J^2,\{J^2,\{J^r,G^{rc}\}\}\}+\frac{1}{4}(N_f-2)\{J^2,\{J^2,\{J^2,T^c\}\}\}, \tag{B18}$$

$$if^{acb}(\mathcal{D}_3^{ia}J^2\mathcal{O}_3^{ib}+\mathcal{O}_3^{ia}J^2\mathcal{D}_3^{ib})=0, \tag{B19}$$

$$if^{acb}\mathcal{O}_3^{ia}J^2\mathcal{O}_3^{ib} = 3N_fT^c-3(N_c+N_f)\{J^r,G^{rc}\}+\frac{1}{4}(25N_f+12)\{J^2,T^c\} -\frac{19}{4}(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\}+\frac{1}{4}(11N_f+19)\{J^2,\{J^2,T^c\}\} -\frac{9}{8}(N_c+N_f)\{J^2,\{J^2,\{J^r,G^{rc}\}\}\}+\frac{1}{8}(N_f+9)\{J^2,\{J^2,\{J^2,T^c\}\}\}. \tag{B20}$$

3. $i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})A^{ia}A^{ib}$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})G^{ia}G^{ib}=0, \tag{B21}$$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})(G^{ia}\mathcal{D}_2^{ib}+\mathcal{D}_2^{ia}G^{ib})=0, \tag{B22}$$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})(G^{ia}\mathcal{D}_3^{ib}+\mathcal{D}_3^{ia}G^{ib}) = \{T^c,\{J^r,G^{r8}\}\}-\{T^8,\{J^r,G^{rc}\}\}, \quad \text{(B23)}$$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})(G^{ia}\mathcal{O}_3^{ib}+\mathcal{O}_3^{ia}G^{ib}) = -\frac{3}{2}\{T^c,\{J^r,G^{r8}\}\}+\frac{3}{2}\{T^8,\{J^r,G^{rc}\}\}, \quad \text{(B24)}$$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})\mathcal{D}_2^{ia}\mathcal{D}_2^{ib} = 0, \quad \text{(B25)}$$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})(\mathcal{D}_2^{ia}\mathcal{D}_3^{ib}+\mathcal{D}_3^{ia}\mathcal{D}_2^{ib}) = 0, \quad \text{(B26)}$$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})(\mathcal{D}_2^{ia}\mathcal{O}_3^{ib}+\mathcal{O}_3^{ia}\mathcal{D}_2^{ib}) = 0, \quad \text{(B27)}$$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})\mathcal{D}_3^{ia}\mathcal{D}_3^{ib} = \{J^2,\{T^c,\{J^r,G^{r8}\}\}\}-\{J^2,\{T^8,\{J^r,G^{rc}\}\}\}, \quad \text{(B28)}$$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})(\mathcal{D}_3^{ia}\mathcal{O}_3^{ib}+\mathcal{O}_3^{ia}\mathcal{D}_3^{ib}) = 0, \quad \text{(B29)}$$

$$\begin{aligned} i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})\mathcal{O}_3^{ia}\mathcal{O}_3^{ib} = & -\frac{3}{2}\{T^c,\{J^r,G^{r8}\}\}+\frac{3}{2}\{T^8,\{J^r,G^{rc}\}\} \\ & -\frac{5}{4}\{J^2,\{T^c,\{J^r,G^{r8}\}\}\}+\frac{5}{4}\{J^2,\{T^8,\{J^r,G^{rc}\}\}\}. \end{aligned} \quad \text{(B30)}$$

4. $i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})A^{ia}J^2A^{ib}$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})G^{ia}J^2G^{ib} = -\frac{1}{2}\{T^c,\{J^r,G^{r8}\}\}+\frac{1}{2}\{T^8,\{J^r,G^{rc}\}\}, \quad \text{(B31)}$$

$$i(f^{aec}d^{be8}-f^{bec}d^{ae8}-f^{abe}d^{ec8})(G^{ia}J^2\mathcal{D}_2^{ib}+\mathcal{D}_2^{ia}J^2G^{ib}) = 0, \quad \text{(B32)}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})(G^{ia}J^2\mathcal{D}_3^{ib} + \mathcal{D}_3^{ia}J^2G^{ib})$$
$$= \frac{1}{2}\{J^2,\{T^c,\{J^r,G^{r8}\}\}\} - \frac{1}{2}\{J^2,\{T^8,\{J^r,G^{rc}\}\}\}, \tag{B33}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})(G^{ia}J^2\mathcal{O}_3^{ib} + \mathcal{O}_3^{ia}J^2G^{ib})$$
$$= -3\{T^c,\{J^r,G^{r8}\}\} + 3\{T^8,\{J^r,G^{rc}\}\} - \frac{7}{4}\{J^2,\{T^c,\{J^r,G^{r8}\}\}\}$$
$$+ \frac{7}{4}\{J^2,\{T^8,\{J^r,G^{rc}\}\}\}, \tag{B34}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})\mathcal{D}_2^{ia}J^2\mathcal{D}_2^{ib} = 0, \tag{B35}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})(\mathcal{D}_2^{ia}J^2\mathcal{D}_3^{ib} + \mathcal{D}_3^{ia}J^2\mathcal{D}_2^{ib}) = 0, \tag{B36}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})(\mathcal{D}_2^{ia}J^2\mathcal{O}_3^{ib} + \mathcal{O}_3^{ia}J^2\mathcal{D}_2^{ib}) = 0, \tag{B37}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})\mathcal{D}_3^{ia}J^2\mathcal{D}_3^{ib}$$
$$= \frac{1}{2}\{J^2,\{J^2,\{T^c,\{J^r,G^{r8}\}\}\}\} - \frac{1}{2}\{J^2,\{J^2,\{T^8,\{J^r,G^{rc}\}\}\}\}, \tag{B38}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})(\mathcal{D}_3^{ia}J^2\mathcal{O}_3^{ib} + \mathcal{O}_3^{ia}J^2\mathcal{D}_3^{ib}) = 0, \tag{B39}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})\mathcal{O}_3^{ia}J^2\mathcal{O}_3^{ib}$$
$$= -3\{T^c,\{J^r,G^{r8}\}\} + 3\{T^8,\{J^r,G^{rc}\}\} - \frac{19}{4}\{J^2,\{T^c,\{J^r,G^{r8}\}\}\}$$
$$+ \frac{19}{4}\{J^2,\{T^8,\{J^r,G^{rc}\}\}\} - \frac{9}{8}\{J^2,\{J^2,\{T^c,\{J^r,G^{r8}\}\}\}\}$$
$$+ \frac{9}{8}\{J^2,\{J^2,\{T^8,\{J^r,G^{rc}\}\}\}\}. \tag{B40}$$

5. $[A^{ia},[A^{ia},T^c]]$

$$[G^{ia},[G^{ia},T^c]] = \frac{3}{4}N_fT^c, \tag{B41}$$

$$[G^{ia},[\mathcal{D}_2^{ia},T^c]] + [\mathcal{D}_2^{ia},[G^{ia},T^c]] = N_f\{J^r,G^{rc}\}, \tag{B42}$$

$$[G^{ia},[\mathcal{D}_3^{ia},T^c]] + [\mathcal{D}_3^{ia},[G^{ia},T^c]] = 2(N_c+N_f)\{J^r,G^{rc}\} + (N_f-2)\{J^2,T^c\}, \tag{B43}$$

$$[G^{ia},[\mathcal{O}_3^{ia},T^c]] + [\mathcal{O}_3^{ia},[G^{ia},T^c]] = 3N_fT^c - 3(N_c+N_f)\{J^r,G^{rc}\} + (N_f+3)\{J^2,T^c\}, \tag{B44}$$

$$[\mathcal{D}_2^{ia},[\mathcal{D}_2^{ia},T^c]] = \frac{1}{2}N_f\{J^2,T^c\}, \tag{B45}$$

$$[\mathcal{D}_2^{ia},[\mathcal{D}_3^{ia},T^c]] + [\mathcal{D}_3^{ia},[\mathcal{D}_2^{ia},T^c]] = 2N_f\{J^2,\{J^r,G^{rc}\}\}, \tag{B46}$$

$$[\mathcal{D}_2^{ia},[\mathcal{O}_3^{ia},T^c]] + [\mathcal{O}_3^{ia},[\mathcal{D}_2^{ia},T^c]] = 0, \tag{B47}$$

$$[\mathcal{D}_3^{ia},[\mathcal{D}_3^{ia},T^c]] = 2(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\} + (N_f-2)\{J^2,\{J^2,T^c\}\}, \tag{B48}$$

$$[\mathcal{D}_3^{ia},[\mathcal{O}_3^{ia},T^c]] + [\mathcal{O}_3^{ia},[\mathcal{D}_3^{ia},T^c]] = 0, \tag{B49}$$

$$\begin{aligned}[\mathcal{O}_3^{ia},[\mathcal{O}_3^{ia},T^c]] \; = \; & 3N_fT^c - 3(N_c+N_f)\{J^r,G^{rc}\} + (4N_f+3)\{J^2,T^c\} \\ & -\frac{5}{2}(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\} + \frac{1}{2}(N_f+5)\{J^2,\{J^2,T^c\}\}.\end{aligned} \tag{B50}$$

6. $d^{ab8}[A^{ia},[A^{ib},T^c]]$

$$d^{ab8}[G^{ia},[G^{ib},T^c]] = \frac{3}{8}N_f d^{c8e}T^e, \tag{B51}$$

$$d^{ab8}([G^{ia},[\mathcal{D}_2^{ib},T^c]] + [\mathcal{D}_2^{ia},[G^{ib},T^c]]) = \frac{1}{2}N_f d^{c8e}\{J^r,G^{re}\}, \tag{B52}$$

$$\begin{aligned} d^{ab8}([G^{ia},[\mathcal{D}_3^{ib},T^c]] + [\mathcal{D}_3^{ia},[G^{ib},T^c]]) \ = \ & (N_c+N_f)d^{c8e}\{J^r,G^{re}\} - \{T^c,\{J^r,G^{r8}\}\} \\ & +\{T^8,\{J^r,G^{rc}\}\} + \frac{1}{2}(N_f-2)d^{c8e}\{J^2,T^e\} \ , \end{aligned} \tag{B53}$$

$$\begin{aligned} d^{ab8}([G^{ia},[\mathcal{O}_3^{ib},T^c]] + [\mathcal{O}_3^{ia},[G^{ib},T^c]]) \ = \ & \frac{3}{2}N_f d^{c8e}T^e + \frac{3}{2}\{T^c,\{J^r,G^{r8}\}\} - \frac{3}{2}\{T^8,\{J^r,G^{rc}\}\} \\ & +\frac{1}{2}(N_f+3)d^{c8e}\{J^2,T^e\} - \frac{3}{2}(N_c+N_f)d^{c8e}\{J^r,G^{re}\} \ , \end{aligned} \tag{B54}$$

$$d^{ab8}[\mathcal{D}_2^{ia},[\mathcal{D}_2^{ib},T^c]] = \frac{1}{4}N_f d^{c8e}\{J^2,T^e\}, \tag{B55}$$

$$d^{ab8}([\mathcal{D}_2^{ia},[\mathcal{D}_3^{ib},T^c]] + [\mathcal{D}_3^{ia},[\mathcal{D}_2^{ib},T^c]]) = N_f d^{c8e}\{J^2,\{J^r,G^{re}\}\}, \tag{B56}$$

$$d^{ab8}([\mathcal{D}_2^{ia},[\mathcal{O}_3^{ib},T^c]] + [\mathcal{O}_3^{ia},[\mathcal{D}_2^{ib},T^c]]) = 0, \tag{B57}$$

$$\begin{aligned} d^{ab8}[\mathcal{D}_3^{ia},[\mathcal{D}_3^{ib},T^c]] \ = \ & (N_c+N_f)d^{c8e}\{J^2,\{J^r,G^{re}\}\} - \{J^2,\{T^c,\{J^r,G^{r8}\}\}\} \\ & +\{J^2,\{T^8,\{J^r,G^{rc}\}\}\} + \frac{1}{2}(N_f-2)d^{c8e}\{J^2,\{J^2,T^e\}\}, \end{aligned} \tag{B58}$$

$$d^{ab8}([\mathcal{D}_3^{ia},[\mathcal{O}_3^{ib},T^c]] + [\mathcal{O}_3^{ia},[\mathcal{D}_3^{ib},T^c]]) = 0, \tag{B59}$$

$$d^{ab8}[\mathcal{O}_3^{ia},[\mathcal{O}_3^{ib},T^c]] = \frac{3}{2}N_f d^{c8e}T^e - \frac{3}{2}(N_c+N_f)d^{c8e}\{J^r,G^{re}\} + \frac{3}{2}\{T^c,\{J^r,G^{r8}\}\} - \frac{3}{2}\{T^8,\{J^r,G^{rc}\}\} + \frac{1}{2}(4N_f+3)d^{c8e}\{J^2,T^e\} + \frac{5}{4}\{J^2,\{T^c,\{J^r,G^{r8}\}\}\} - \frac{5}{4}\{J^2,\{T^8,\{J^r,G^{rc}\}\}\} + \frac{1}{4}(N_f+5)d^{c8e}\{J^2,\{J^2,T^e\}\} - \frac{5}{4}(N_c+N_f)d^{c8e}\{J^2,\{J^r,G^{re}\}\}. \tag{B60}$$

7. $[A^{i8},[A^{i8},T^c]]$

$$[G^{i8},[G^{i8},T^c]] = \frac{3}{4}f^{c8e}f^{8eg}T^g, \tag{B61}$$

$$[G^{i8},[\mathcal{D}_2^{i8},T^c]] + [\mathcal{D}_2^{i8},[G^{i8},T^c]] = f^{c8e}f^{8eg}\{J^r,G^{rg}\}, \tag{B62}$$

$$[G^{i8},[\mathcal{D}_3^{i8},T^c]] + [\mathcal{D}_3^{i8},[G^{i8},T^c]] = 3f^{c8e}f^{8eg}T^g + f^{c8e}f^{8eg}\{J^2,T^g\} - 2\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\}, \tag{B63}$$

$$[G^{i8},[\mathcal{O}_3^{i8},T^c]] + [\mathcal{O}_3^{i8},[G^{i8},T^c]] = -\frac{3}{2}f^{c8e}f^{8eg}T^g + f^{c8e}f^{8eg}\{J^2,T^g\} + 3\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\}, \tag{B64}$$

$$[\mathcal{D}_2^{i8},[\mathcal{D}_2^{i8},T^c]] = \frac{1}{2}f^{c8e}f^{8eg}\{J^2,T^g\}, \tag{B65}$$

$$[\mathcal{D}_2^{i8},[\mathcal{D}_3^{i8},T^c]] + [\mathcal{D}_3^{i8},[\mathcal{D}_2^{i8},T^c]] = 2f^{c8e}f^{8eg}\{J^2,\{J^r,G^{rg}\}\}, \tag{B66}$$

$$[\mathcal{D}_2^{i8},[\mathcal{O}_3^{i8},T^c]] + [\mathcal{O}_3^{i8},[\mathcal{D}_2^{i8},T^c]] = 0, \tag{B67}$$

$$[\mathcal{D}_3^{i8},[\mathcal{D}_3^{i8},T^c]] = 3f^{c8e}f^{8eg}\{J^2,T^g\} + f^{c8e}f^{8eg}\{J^2,\{J^2,T^g\}\} - 2\epsilon^{ijk}f^{c8e}\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\}, \tag{B68}$$

$$[\mathcal{D}_3^{i8},[\mathcal{O}_3^{i8},T^c]]+[\mathcal{O}_3^{i8},[\mathcal{D}_3^{i8},T^c]]=0, \tag{B69}$$

$$\begin{aligned}[\mathcal{O}_3^{i8},[\mathcal{O}_3^{i8},T^c]] \ = \ & -\frac{3}{2}f^{c8e}f^{8eg}T^g+\frac{1}{4}f^{c8e}f^{8eg}\{J^2,T^g\}+3\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\} \\ & +\frac{1}{2}f^{c8e}f^{8eg}\{J^2,\{J^2,T^g\}\}+\frac{5}{2}\epsilon^{ijk}f^{c8e}\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\}.\end{aligned} \tag{B70}$$

8. $if^{cae}\{A^{ia},[J^2,A^{ie}]\}$

$$if^{cae}\{G^{ia},[J^2,G^{ie}]\}=-\frac{3}{2}N_fT^c+(N_c+N_f)\{J^r,G^{rc}\}-\{J^2,T^c\}, \tag{B71}$$

$$if^{cae}\{\mathcal{D}_2^{ia},[J^2,G^{ie}]\}=0, \tag{B72}$$

$$if^{cae}\{\mathcal{D}_3^{ia},[J^2,G^{ie}]\}=0, \tag{B73}$$

$$\begin{aligned}& if^{cae}(\{\mathcal{O}_3^{ia},[J^2,G^{ie}]\}+\{G^{ia},[J^2,\mathcal{O}_3^{ie}]\}) \\ & \quad =-6N_fT^c+6(N_c+N_f)\{J^r,G^{rc}\}-(5N_f+6)\{J^2,T^c\}+2(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\} \\ & \qquad -2\{J^2,\{J^2,T^c\}\},\end{aligned} \tag{B74}$$

$$if^{cae}\{\mathcal{D}_2^{ia},[J^2,\mathcal{O}_3^{ie}]\}=0, \tag{B75}$$

$$if^{cae}\{\mathcal{D}_3^{ia},[J^2,\mathcal{O}_3^{ie}]\}=0, \tag{B76}$$

$$\begin{aligned}if^{cae}\{\mathcal{O}_3^{ia},[J^2,\mathcal{O}_3^{ie}]\} \ = \ & -6N_fT^c+6(N_c+N_f)\{J^r,G^{rc}\}-(11N_f+6)\{J^2,T^c\} \\ & +8(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\}-\frac{1}{2}(7N_f+16)\{J^2,\{J^2,T^c\}\} \\ & +(N_c+N_f)\{J^2,\{J^2,\{J^r,G^{rc}\}\}\}-\{J^2,\{J^2,\{J^2,T^c\}\}\}.\end{aligned} \tag{B77}$$

9. $id^{ab8}f^{cbe}\{A^{ia},[J^2,A^{ie}]\}$

$$id^{ab8}f^{cbe}\{G^{ia},[J^2,G^{ie}]\} = -\frac{3}{4}N_f d^{c8e}T^e + \frac{1}{2}(N_c+N_f)d^{c8e}\{J^r,G^{re}\} - \frac{1}{2}\{T^c,\{J^r,G^{r8}\}\} + \frac{1}{2}\{T^8,\{J^r,G^{rc}\}\} - \frac{1}{2}d^{c8e}\{J^2,T^e\}, \tag{B78}$$

$$id^{ab8}f^{cbe}\{\mathcal{D}_2^{ia},[J^2,G^{ie}]\} = 0, \tag{B79}$$

$$id^{ab8}f^{cbe}\{\mathcal{D}_3^{ia},[J^2,G^{ie}]\} = 0, \tag{B80}$$

$$\begin{aligned} &id^{ab8}f^{cbe}(\{\mathcal{O}_3^{ia},[J^2,G^{ie}]\} + \{G^{ia},[J^2,\mathcal{O}_3^{ie}]\}) \\ &= -3N_f d^{c8e}T^e + 3(N_c+N_f)d^{c8e}\{J^r,G^{re}\} - 3\{T^c,\{J^r,G^{r8}\}\} + 3\{T^8,\{J^r,G^{rc}\}\} \\ &\quad - \frac{1}{2}(5N_f+6)d^{c8e}\{J^2,T^e\} + (N_c+N_f)d^{c8e}\{J^2,\{J^r,G^{re}\}\} - \{J^2,\{T^c,\{J^r,G^{r8}\}\}\} \\ &\quad + \{J^2,\{T^8,\{J^r,G^{rc}\}\}\} - d^{c8e}\{J^2,\{J^2,T^e\}\}, \end{aligned} \tag{B81}$$

$$id^{ab8}f^{cbe}\{\mathcal{D}_2^{ia},[J^2,\mathcal{O}_3^{ie}]\} = 0, \tag{B82}$$

$$id^{ab8}f^{cbe}\{\mathcal{D}_3^{ia},[J^2,\mathcal{O}_3^{ie}]\} = 0, \tag{B83}$$

$$\begin{aligned} &id^{ab8}f^{cbe}\{\mathcal{O}_3^{ia},[J^2,\mathcal{O}_3^{ie}]\} \\ &= -3N_f d^{c8e}T^e + 3(N_c+N_f)d^{c8e}\{J^r,G^{re}\} - 3\{T^c,\{J^r,G^{r8}\}\} + 3\{T^8,\{J^r,G^{rc}\}\} \\ &\quad - \frac{1}{2}(11N_f-6)d^{c8e}\{J^2,T^e\} + 4(N_c+N_f)d^{c8e}\{J^2,\{J^r,G^{re}\}\} - 4\{J^2,\{T^c,\{J^r,G^{r8}\}\}\} \\ &\quad + 4\{J^2,\{T^8,\{J^r,G^{rc}\}\}\} - \frac{1}{4}(7N_f+16)d^{c8e}\{J^2,\{J^2,T^e\}\} \\ &\quad + \frac{1}{2}(N_c+N_f)d^{c8e}\{J^2,\{J^2,\{J^r,G^{re}\}\}\} - \frac{1}{2}\{J^2,\{J^2,\{T^c,\{J^r,G^{r8}\}\}\}\} \\ &\quad + \frac{1}{2}\{J^2,\{J^2,\{T^8,\{J^r,G^{rc}\}\}\}\} - \frac{1}{2}d^{c8e}\{J^2,\{J^2,\{J^2,T^e\}\}\}, \end{aligned} \tag{B84}$$

10. $if^{c8e}\{A^{i8},[J^2,A^{ie}]\}$

$$if^{c8e}\{G^{i8},[J^2,G^{ie}]\} = -\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\}, \tag{B85}$$

$$if^{c8e}\{\mathcal{D}_2^{i8},[J^2,G^{ie}]\} = 0, \tag{B86}$$

$$if^{c8e}\{\mathcal{D}_3^{i8},[J^2,G^{ie}]\} = 0, \tag{B87}$$

$$\begin{aligned} &if^{c8e}(\{\mathcal{O}_3^{i8},[J^2,G^{ie}]\} + \{G^{i8},[J^2,\mathcal{O}_3^{ie}]\}) \\ &\quad = 3f^{c8e}f^{8eg}T^g - 2f^{c8e}f^{8eg}\{J^2,T^g\} - 6\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\} \\ &\qquad - 2\epsilon^{ijk}f^{c8e}\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\}, \end{aligned} \tag{B88}$$

$$if^{c8e}\{\mathcal{D}_2^{i8},[J^2,\mathcal{O}_3^{ie}]\} = 0, \tag{B89}$$

$$if^{c8e}\{\mathcal{D}_3^{i8},[J^2,\mathcal{O}_3^{ie}]\} = 0, \tag{B90}$$

$$\begin{aligned} &if^{c8e}\{\mathcal{O}_3^{i8},[J^2,\mathcal{O}_3^{ie}]\} \\ &\quad = 3f^{c8e}f^{8eg}T^g + f^{c8e}f^{8eg}\{J^2,T^g\} - 6\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\} - 2f^{c8e}f^{8eg}\{J^2,\{J^2,T^g\}\} \\ &\qquad - 8\epsilon^{ijk}f^{c8e}\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\} - \epsilon^{ijk}f^{c8e}\{J^2,\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\}\}. \end{aligned} \tag{B91}$$

11. $if^{ace}[[J^2,A^{ie}],[J^2,A^{ia}]]$

$$if^{ace}[[J^2,G^{ie}],[J^2,G^{ia}]] = 3N_fT^c - 3(N_c+N_f)\{J^r,G^{rc}\} + (N_f+3)\{J^2,T^c\}, \tag{B92}$$

$$\begin{aligned} if^{ace}[[J^2,G^{ie}],[J^2,\mathcal{O}_3^{ia}]] \;=\; & 6N_fT^c - 6(N_c+N_f)\{J^r,G^{rc}\} + (8N_f+6)\{J^2,T^c\} \\ & - 5(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\} + (N_f+5)\{J^2,\{J^2,T^c\}\}\;, \end{aligned} \tag{B93}$$

$$if^{ace}[[J^2,\mathcal{O}_3^{ie}],[J^2,G^{ia}]] = 6N_fT^c-6(N_c+N_f)\{J^r,G^{rc}\}+(8N_f+6)\{J^2,T^c\} \\ -5(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\}+(N_f+5)\{J^2,\{J^2,T^c\}\}\,, \tag{B94}$$

$$\begin{aligned} &if^{ace}[[J^2,\mathcal{O}_3^{ie}],[J^2,\mathcal{O}_3^{ia}]] \\ &= 12N_fT^c-12(N_c+N_f)\{J^r,G^{rc}\}+4(7N_f+3)\{J^2,T^c\}-22(N_c+N_f)\{J^2,\{J^r,G^{rc}\}\} \\ &\quad +(15N_f+22)\{J^2,\{J^2,T^c\}\}-7(N_c+N_f)\{J^2,\{J^2,\{J^r,G^{rc}\}\}\} \\ &\quad +(N_f+7)\{J^2,\{J^2,\{J^2,T^c\}\}\}, \end{aligned} \tag{B95}$$

12. $id^{ab8}f^{bce}[[J^2,A^{ie}],[J^2,A^{ia}]]$

$$id^{ab8}f^{bce}[[J^2,G^{ie}],[J^2,G^{ia}]] = \frac{3}{2}N_fd^{c8e}T^e-\frac{3}{2}(N_c+N_f)d^{c8e}\{J^r,G^{re}\}+\frac{3}{2}\{T^c,\{J^r,G^{r8}\}\} \\ -\frac{3}{2}\{T^8,\{J^r,G^{rc}\}\}+\frac{1}{2}(N_f+3)d^{c8e}\{J^2,T^e\}, \tag{B96}$$

$$\begin{aligned} &id^{ab8}f^{bce}[[J^2,G^{ie}],[J^2,\mathcal{O}_3^{ia}]] \\ &= 3N_fd^{c8e}T^e-3(N_c+N_f)d^{c8e}\{J^r,G^{re}\}+3\{T^c,\{J^r,G^{r8}\}\}-3\{T^8,\{J^r,G^{rc}\}\} \\ &\quad +(4N_f+3)d^{c8e}\{J^2,T^e\}+\frac{5}{2}\{J^2,\{T^c,\{J^r,G^{r8}\}\}\}-\frac{5}{2}\{J^2,\{T^8,\{J^r,G^{rc}\}\}\} \\ &\quad +\frac{1}{2}(N_f+5)d^{c8e}\{J^2,\{J^2,T^e\}\}-\frac{5}{2}(N_c+N_f)d^{c8e}\{J^2,\{J^r,G^{re}\}\}, \end{aligned} \tag{B97}$$

$$\begin{aligned} &id^{ab8}f^{bce}[[J^2,\mathcal{O}_3^{ie}],[J^2,G^{ia}]] \\ &= 3N_fd^{c8e}T^e-3(N_c+N_f)d^{c8e}\{J^r,G^{re}\}+3\{T^c,\{J^r,G^{r8}\}\}-3\{T^8,\{J^r,G^{rc}\}\} \\ &\quad +(4N_f+3)d^{c8e}\{J^2,T^e\}+\frac{5}{2}\{J^2,\{T^c,\{J^r,G^{r8}\}\}\}-\frac{5}{2}\{J^2,\{T^8,\{J^r,G^{rc}\}\}\} \\ &\quad +\frac{1}{2}(N_f+5)d^{c8e}\{J^2,\{J^2,T^e\}\}-\frac{5}{2}(N_c+N_f)d^{c8e}\{J^2,\{J^r,G^{re}\}\}, \end{aligned} \tag{B98}$$

$$
\begin{aligned}
&id^{ab8}f^{bce}[[J^2,\mathcal{O}_3^{ie}],[J^2,\mathcal{O}_3^{ia}]] \\
&\quad = -6f^{c8e}f^{8eg}T^g - 5f^{c8e}f^{8eg}\{J^2,T^g\} + 12\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\} \\
&\qquad + \frac{9}{2}f^{c8e}f^{8eg}\{J^2,\{J^2,T^g\}\} + 22\epsilon^{ijk}f^{c8e}\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\} \\
&\qquad + f^{c8e}f^{8eg}\{J^2,\{J^2,\{J^2,T^g\}\}\} + 7\epsilon^{ijk}f^{c8e}\{J^2,\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\}\},
\end{aligned} \tag{B99}
$$

13. $if^{8ce}[[J^2,A^{ie}],[J^2,A^{i8}]]$

$$
if^{8ce}[[J^2,G^{ie}],[J^2,G^{i8}]] = -\frac{3}{2}f^{c8e}f^{8eg}T^g + f^{c8e}f^{8eg}\{J^2,T^g\} + 3\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\}, \tag{B100}
$$

$$
\begin{aligned}
if^{8ce}[[J^2,G^{ie}],[J^2,\mathcal{O}_3^{i8}]] = {} & -3f^{c8e}f^{8eg}T^g + \frac{1}{2}f^{c8e}f^{8eg}\{J^2,T^g\} + f^{c8e}f^{8eg}\{J^2,\{J^2,T^g\}\} \\
& + 6\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\} + 5\epsilon^{ijk}f^{c8e}\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\},
\end{aligned} \tag{B101}
$$

$$
\begin{aligned}
if^{8ce}[[J^2,\mathcal{O}_3^{ie}],[J^2,G^{i8}]] = {} & -3f^{c8e}f^{8eg}T^g + \frac{1}{2}f^{c8e}f^{8eg}\{J^2,T^g\} + f^{c8e}f^{8eg}\{J^2,\{J^2,T^g\}\} \\
& + 6\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\} + 5\epsilon^{ijk}f^{c8e}\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\},
\end{aligned} \tag{B102}
$$

$$
\begin{aligned}
&if^{8ce}[[J^2,\mathcal{O}_3^{ie}],[J^2,\mathcal{O}_3^{i8}]] \\
&\quad = -6f^{c8e}f^{8eg}T^g - 5f^{c8e}f^{8eg}\{J^2,T^g\} + 12\epsilon^{ijk}f^{c8e}\{G^{ke},\{J^i,G^{j8}\}\} \\
&\qquad + \frac{9}{2}f^{c8e}f^{8eg}\{J^2,\{J^2,T^g\}\} + 22\epsilon^{ijk}f^{c8e}\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\} \\
&\qquad + f^{c8e}f^{8eg}\{J^2,\{J^2,\{J^2,T^g\}\}\} + 7\epsilon^{ijk}f^{c8e}\{J^2,\{J^2,\{G^{ke},\{J^i,G^{j8}\}\}\}\}.
\end{aligned} \tag{B103}
$$

Appendix C: Operator coefficients

The several operator products involved in the analysis can be cast into rather compact forms. They can be written as summations involving an operator coefficient times a corresponding operator belonging to the $SU(3)$ flavor representations **1**, **8** and **27**, listed in Eqs. (38), (39) and (46) respectively.

The compact expressions are listed as follows.

$$if^{acb}A^{ia}A^{ib} = \sum_{n=1}^{7} a_n^{\mathbf{8}} S_n^c, \tag{C1}$$

where

$$\begin{aligned}
a_1^{\mathbf{8}} &= \frac{3N_f}{8}a_1^2 + \frac{3N_f}{2N_c^2}a_1c_3 + \frac{3N_f}{2N_c^4}c_3^2 \,, \\
a_2^{\mathbf{8}} &= \frac{N_f}{2N_c}a_1b_2 + \frac{N_c+N_f}{N_c^2}a_1b_3 - \frac{3(N_c+N_f)}{2N_c^2}a_1c_3 - \frac{3(N_c+N_f)}{2N_c^4}c_3^2 \,, \\
a_3^{\mathbf{8}} &= \frac{N_f-2}{2N_c^2}a_1b_3 + \frac{N_f+3}{2N_c^2}a_1c_3 + \frac{N_f}{4N_c^2}b_2^2 + \frac{4N_f+3}{2N_c^4}c_3^2 \,, \\
a_4^{\mathbf{8}} &= \frac{N_f}{N_c^3}b_2b_3 + \frac{N_c+N_f}{N_c^4}b_3^2 - \frac{5(N_c+N_f)}{4N_c^4}c_3^2 \,, \\
a_5^{\mathbf{8}} &= \frac{N_f-2}{2N_c^4}b_3^2 + \frac{N_f+5}{4N_c^4}c_3^2 \,, \\
a_6^{\mathbf{8}} &= 0 \,, \\
a_7^{\mathbf{8}} &= 0 \,.
\end{aligned}$$

$$if^{acb}A^{ia}J^2A^{ib} = \sum_{n=1}^{7} \overline{a}_n^{\mathbf{8}} S_n^c, \tag{C2}$$

where

$$\begin{aligned}
\overline{a}_1^{\mathbf{8}} &= \frac{3N_f}{4}a_1^2 + \frac{3N_f}{N_c^2}a_1c_3 + \frac{3N_f}{N_c^4}c_3^2 \,, \\
\overline{a}_2^{\mathbf{8}} &= -\frac{N_c+N_f}{2}a_1^2 - \frac{3(N_c+N_f)}{N_c^2}a_1c_3 - \frac{3(N_c+N_f)}{N_c^4}c_3^2 \,, \\
\overline{a}_3^{\mathbf{8}} &= \frac{3N_f+8}{16}a_1^2 + \frac{13N_f+12}{4N_c^2}a_1c_3 + \frac{25N_f+12}{4N_c^4}c_3^2 \,, \\
\overline{a}_4^{\mathbf{8}} &= \frac{N_f}{4N_c}a_1b_2 + \frac{N_c+N_f}{2N_c^2}a_1b_3 - \frac{7(N_c+N_f)}{4N_c^2}a_1c_3 - \frac{19(N_c+N_f)}{4N_c^4}c_3^2 \,, \\
\overline{a}_5^{\mathbf{8}} &= \frac{N_f-2}{4N_c^2}a_1b_3 + \frac{N_f+7}{4N_c^2}a_1c_3 + \frac{N_f}{8N_c^2}b_2^2 + \frac{11N_f+19}{4N_c^4}c_3^2 \,, \\
\overline{a}_6^{\mathbf{8}} &= \frac{N_f}{2N_c^3}b_2b_3 + \frac{N_c+N_f}{2N_c^4}b_3^2 - \frac{9(N_c+N_f)}{8N_c^4}c_3^2 \,, \\
\overline{a}_7^{\mathbf{8}} &= \frac{N_f-2}{4N_c^4}b_3^2 + \frac{N_f+9}{8N_c^4}c_3^2 \,.
\end{aligned}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})A^{ia}A^{ib} = \sum_{n=1}^{13} b_n^{\mathbf{10}+\overline{\mathbf{10}}} O_n^c, \tag{C3}$$

where

$$\begin{aligned}
b_1^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \,, \\
b_2^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \,, \\
b_3^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \,, \\
b_4^{\mathbf{10}+\overline{\mathbf{10}}} &= \frac{1}{N_c^2}a_1b_3 - \frac{3}{2N_c^2}a_1c_3 - \frac{3}{2N_c^4}c_3^2 \,, \\
b_5^{\mathbf{10}+\overline{\mathbf{10}}} &= -\frac{1}{N_c^2}a_1b_3 + \frac{3}{2N_c^2}a_1c_3 + \frac{3}{2N_c^4}c_3^2 \,, \\
b_6^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \,, \\
b_7^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \,, \\
b_8^{\mathbf{10}+\overline{\mathbf{10}}} &= \frac{1}{N_c^4}b_3^2 - \frac{5}{4N_c^4}c_3^2 \,, \\
b_9^{\mathbf{10}+\overline{\mathbf{10}}} &= -\frac{1}{N_c^4}b_3^2 + \frac{5}{4N_c^4}c_3^2 \,, \\
b_{10}^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \,, \\
b_{11}^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \,, \\
b_{12}^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \,, \\
b_{13}^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \,.
\end{aligned}$$

$$i(f^{aec}d^{be8} - f^{bec}d^{ae8} - f^{abe}d^{ec8})A^{ia}J^2A^{ib} = \sum_{n=1}^{13} \bar{b}_n^{\mathbf{10}+\overline{\mathbf{10}}} O_n^c, \tag{C4}$$

where

$$
\begin{aligned}
\bar{b}_1^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \, , \\
\bar{b}_2^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \, , \\
\bar{b}_3^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \, , \\
\bar{b}_4^{\mathbf{10}+\overline{\mathbf{10}}} &= -\frac{1}{2}a_1^2 - \frac{3}{N_c^2}a_1c_3 - \frac{3}{N_c^4}c_3^2 \, , \\
\bar{b}_5^{\mathbf{10}+\overline{\mathbf{10}}} &= \frac{1}{2}a_1^2 + \frac{3}{N_c^2}a_1c_3 + \frac{3}{N_c^4}c_3^2 \, , \\
\bar{b}_6^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \, , \\
\bar{b}_7^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \, , \\
\bar{b}_8^{\mathbf{10}+\overline{\mathbf{10}}} &= \frac{1}{2N_c^2}a_1b_3 - \frac{7}{4N_c^2}a_1c_3 - \frac{19}{4N_c^4}c_3^2 \, , \\
\bar{b}_9^{\mathbf{10}+\overline{\mathbf{10}}} &= -\frac{1}{2N_c^2}a_1b_3 + \frac{7}{4N_c^2}a_1c_3 + \frac{19}{4N_c^4}c_3^2 \, , \\
\bar{b}_{10}^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \, , \\
\bar{b}_{11}^{\mathbf{10}+\overline{\mathbf{10}}} &= 0 \, , \\
\bar{b}_{12}^{\mathbf{10}+\overline{\mathbf{10}}} &= \frac{1}{2N_c^4}b_3^2 - \frac{9}{8N_c^4}c_3^2 \, , \\
\bar{b}_{13}^{\mathbf{10}+\overline{\mathbf{10}}} &= -\frac{1}{2N_c^4}b_3^2 + \frac{9}{8N_c^4}c_3^2 \, .
\end{aligned}
$$

$$
\frac{1}{2}[A^{ia},[A^{ia},T^c]] = \sum_{n=1}^{7} c_n^{\mathbf{1}} S_n^c, \tag{C5}
$$

where

$$
\begin{aligned}
c_1^{\mathbf{1}} &= \frac{3N_f}{8}a_1^2 + \frac{3N_f}{2N_c^2}a_1c_3 + \frac{3N_f}{2N_c^4}c_3^2 \, , \\
c_2^{\mathbf{1}} &= \frac{N_f}{2N_c}a_1b_2 + \frac{N_c+N_f}{N_c^2}a_1b_3 - \frac{3(N_c+N_f)}{2N_c^2}a_1c_3 - \frac{3(N_c+N_f)}{2N_c^4}c_3^2 \, , \\
c_3^{\mathbf{1}} &= \frac{N_f-2}{2N_c^2}a_1b_3 + \frac{N_f+3}{2N_c^2}a_1c_3 + \frac{N_f}{4N_c^2}b_2^2 + \frac{4N_f+3}{2N_c^4}c_3^2 \, , \\
c_4^{\mathbf{1}} &= \frac{N_f}{N_c^3}b_2b_3 + \frac{N_c+N_f}{N_c^4}b_3^2 - \frac{5(N_c+N_f)}{4N_c^4}c_3^2 \, , \\
c_5^{\mathbf{1}} &= \frac{N_f-2}{2N_c^4}b_3^2 + \frac{N_f+5}{4N_c^4}c_3^2 \, , \\
c_6^{\mathbf{1}} &= 0 \, , \\
c_7^{\mathbf{1}} &= 0 \, .
\end{aligned}
$$

$$-\frac{1}{2}\{A^{ja},[V^c,[\mathcal{M},A^{ja}]]\} = \sum_{n=1}^{7} d_n^{\mathbf{1}} S_n^c, \tag{C6}$$

where

$$\begin{aligned}
d_1^{\mathbf{1}} &= \left(\frac{3N_f}{4}a_1^2 + \frac{3N_f}{N_c^2}a_1c_3 + \frac{3N_f}{N_c^4}c_3^2\right)\frac{\Delta}{N_c}\,, \\
d_2^{\mathbf{1}} &= \left(-\frac{1}{2}(N_c+N_f)a_1^2 - \frac{3(N_c+N_f)}{N_c^2}a_1c_3 - \frac{3(N_c+N_f)}{N_c^4}c_3^2\right)\frac{\Delta}{N_c}\,, \\
d_3^{\mathbf{1}} &= \left(\frac{1}{2}a_1^2 + \frac{5N_f+6}{2N_c^2}a_1c_3 + \frac{11N_f+6}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c}\,, \\
d_4^{\mathbf{1}} &= \left(-\frac{N_c+N_f}{N_c^2}a_1c_3 - \frac{4(N_c+N_f)}{N_c^4}c_3^2\right)\frac{\Delta}{N_c}\,, \\
d_5^{\mathbf{1}} &= \left(\frac{1}{N_c^2}a_1c_3 + \frac{7N_f+16}{4N_c^4}c_3^2\right)\frac{\Delta}{N_c}\,, \\
d_6^{\mathbf{1}} &= \left(-\frac{N_c+N_f}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c}\,, \\
d_7^{\mathbf{1}} &= \left(\frac{1}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c}\,.
\end{aligned}$$

$$\frac{1}{6}\left([A^{ja},[[\mathcal{M},[\mathcal{M},A^{ja}]],V^c]] - \frac{1}{2}[[\mathcal{M},A^{ja}],[[\mathcal{M},A^{ja}],V^c]]\right) = \sum_{n=1}^{7} e_n^{\mathbf{1}} S_n^c, \tag{C7}$$

where

$$\begin{aligned}
e_1^{\mathbf{1}} &= \left(\frac{3N_f}{4}a_1^2 + \frac{3N_f}{N_c^2}a_1c_3 + \frac{3N_f}{N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\,, \\
e_2^{\mathbf{1}} &= \left(-\frac{3}{4}(N_c+N_f)a_1^2 - \frac{3(N_c+N_f)}{N_c^2}a_1c_3 - \frac{3(N_c+N_f)}{N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\,, \\
e_3^{\mathbf{1}} &= \left(\frac{1}{4}(N_f+3)a_1^2 + \frac{4N_f+3}{N_c^2}a_1c_3 + \frac{7N_f+3}{N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\,, \\
e_4^{\mathbf{1}} &= \left(-\frac{5(N_c+N_f)}{2N_c^2}a_1c_3 - \frac{11(N_c+N_f)}{2N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\,, \\
e_5^{\mathbf{1}} &= \left(\frac{N_f+5}{2N_c^2}a_1c_3 + \frac{15N_f+22}{4N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\,, \\
e_6^{\mathbf{1}} &= \left(-\frac{7(N_c+N_f)}{4N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\,, \\
e_7^{\mathbf{1}} &= \left(\frac{N_f+7}{4N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\,.
\end{aligned}$$

$$\frac{1}{2}d^{ab8}[A^{ja},[A^{jb},V^c]] = \sum_{n=1}^{13} c_n^{\mathbf{8}} O_n^c, \tag{C8}$$

where

$$
\begin{aligned}
c_1^{\mathbf{8}} &= \frac{3N_f}{16}a_1^2 + \frac{3N_f}{4N_c^2}a_1c_3 + \frac{3N_f}{4N_c^4}c_3^2 \, , \\
c_2^{\mathbf{8}} &= \frac{N_f}{4N_c}a_1b_2 + \frac{N_c+N_f}{2N_c^2}a_1b_3 - \frac{3(N_c+N_f)}{4N_c^2}a_1c_3 - \frac{3(N_c+N_f)}{4N_c^4}c_3^2 \, , \\
c_3^{\mathbf{8}} &= \frac{N_f-2}{4N_c^2}a_1b_3 + \frac{N_f+3}{4N_c^2}a_1c_3 + \frac{N_f}{8N_c^2}b_2^2 + \frac{4N_f+3}{4N_c^4}c_3^2 \, , \\
c_4^{\mathbf{8}} &= -\frac{1}{2N_c^2}a_1b_3 + \frac{3}{4N_c^2}a_1c_3 + \frac{3}{4N_c^4}c_3^2 \, , \\
c_5^{\mathbf{8}} &= \frac{1}{2N_c^2}a_1b_3 - \frac{3}{4N_c^2}a_1c_3 - \frac{3}{4N_c^4}c_3^2 \, , \\
c_6^{\mathbf{8}} &= \frac{N_f}{2N_c^3}b_2b_3 + \frac{N_c+N_f}{2N_c^4}b_3^2 - \frac{5(N_c+N_f)}{8N_c^4}c_3^2 \, , \\
c_7^{\mathbf{8}} &= \frac{N_f-2}{4N_c^4}b_3^2 + \frac{N_f+5}{8N_c^4}c_3^2 \, , \\
c_8^{\mathbf{8}} &= -\frac{1}{2N_c^4}b_3^2 + \frac{5}{8N_c^4}c_3^2 \, , \\
c_9^{\mathbf{8}} &= \frac{1}{2N_c^4}b_3^2 - \frac{5}{8N_c^4}c_3^2 \, , \\
c_{10}^{\mathbf{8}} &= 0 \, , \\
c_{11}^{\mathbf{8}} &= 0 \, , \\
c_{12}^{\mathbf{8}} &= 0 \, , \\
c_{13}^{\mathbf{8}} &= 0 \, .
\end{aligned}
$$

$$
-\frac{1}{2}d^{ab8}\{A^{ja},[V^c,[\mathcal{M},A^{jb}]]\} = \sum_{n=1}^{13} d_n^{\mathbf{8}} O_n^c, \tag{C9}
$$

where

$$
\begin{aligned}
d_1^{\mathbf{8}} &= \left(\frac{3N_f}{8}a_1^2 + \frac{3N_f}{2N_c^2}a_1c_3 + \frac{3N_f}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_2^{\mathbf{8}} &= \left(-\frac{1}{4}(N_c+N_f)a_1^2 - \frac{3(N_c+N_f)}{2N_c^2}a_1c_3 - \frac{3(N_c+N_f)}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_3^{\mathbf{8}} &= \left(\frac{1}{4}a_1^2 + \frac{5N_f+6}{4N_c^2}a_1c_3 + \frac{11N_f+6}{4N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_4^{\mathbf{8}} &= \left(\frac{1}{4}a_1^2 + \frac{3}{2N_c^2}a_1c_3 + \frac{3}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_5^{\mathbf{8}} &= \left(-\frac{1}{4}a_1^2 - \frac{3}{2N_c^2}a_1c_3 - \frac{3}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_6^{\mathbf{8}} &= \left(-\frac{N_c+N_f}{2N_c^2}a_1c_3 - \frac{2(N_c+N_f)}{N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_7^{\mathbf{8}} &= \left(\frac{1}{2N_c^2}a_1c_3 + \frac{7N_f+16}{8N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_8^{\mathbf{8}} &= \left(\frac{1}{2N_c^2}a_1c_3 + \frac{2}{N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_9^{\mathbf{8}} &= \left(-\frac{1}{2N_c^2}a_1c_3 - \frac{2}{N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_{10}^{\mathbf{8}} &= \left(-\frac{N_c+N_f}{4N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_{11}^{\mathbf{8}} &= \left(\frac{1}{4N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_{12}^{\mathbf{8}} &= \left(\frac{1}{4N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ , \\
d_{13}^{\mathbf{8}} &= \left(-\frac{1}{4N_c^4}c_3^2\right)\frac{\Delta}{N_c} \ .
\end{aligned}
$$

$$
\frac{1}{6}d^{ab8}\left([A^{ja},[[\mathcal{M},[\mathcal{M},A^{jb}]],V^c]] - \frac{1}{2}[[\mathcal{M},A^{ja}],[[\mathcal{M},A^{jb}],V^c]]\right) = \sum_{n=1}^{13} e_n^{\mathbf{8}} O_n^c, \tag{C10}
$$

where

$$
\begin{aligned}
e_1^{\mathbf{8}} &= \left(\frac{3N_f}{8}a_1^2 + \frac{3N_f}{2N_c^2}a_1c_3 + \frac{3N_f}{2N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_2^{\mathbf{8}} &= \left(-\frac{3}{8}(N_c+N_f)a_1^2 - \frac{3(N_c+N_f)}{2N_c^2}a_1c_3 - \frac{3(N_c+N_f)}{2N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_3^{\mathbf{8}} &= \left(\frac{1}{8}(N_f+3)a_1^2 + \frac{4N_f+3}{2N_c^2}a_1c_3 + \frac{7N_f+3}{2N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_4^{\mathbf{8}} &= \left(\frac{3}{8}a_1^2 + \frac{3}{2N_c^2}a_1c_3 + \frac{3}{2N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_5^{\mathbf{8}} &= \left(-\frac{3}{8}a_1^2 - \frac{3}{2N_c^2}a_1c_3 - \frac{3}{2N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_6^{\mathbf{8}} &= \left(-\frac{5(N_c+N_f)}{4N_c^2}a_1c_3 - \frac{11(N_c+N_f)}{4N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_7^{\mathbf{8}} &= \left(\frac{N_f+5}{4N_c^2}a_1c_3 + \frac{15N_f+22}{8N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_8^{\mathbf{8}} &= \left(\frac{5}{4N_c^2}a_1c_3 + \frac{11}{4N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_9^{\mathbf{8}} &= \left(-\frac{5}{4N_c^2}a_1c_3 - \frac{11}{4N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_{10}^{\mathbf{8}} &= \left(-\frac{7(N_c+N_f)}{8N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_{11}^{\mathbf{8}} &= \left(\frac{N_f+7}{8N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_{12}^{\mathbf{8}} &= \left(\frac{7}{8N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ , \\
e_{13}^{\mathbf{8}} &= \left(-\frac{7}{8N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2} \ .
\end{aligned}
$$

$$
\frac{1}{2}[A^{j8},[A^{j8},V^c]] = \sum_{n=1}^{9} c_n^{\mathbf{27}} T_n^c, \tag{C11}
$$

where

$$\begin{aligned}
c_1^{\mathbf{27}} &= \frac{3}{8}a_1^2 + \frac{3}{2N_c^2}a_1b_3 - \frac{3}{4N_c^2}a_1c_3 - \frac{3}{4N_c^4}c_3^2 \, , \\
c_2^{\mathbf{27}} &= \frac{1}{2N_c}a_1b_2 \, , \\
c_3^{\mathbf{27}} &= \frac{1}{2N_c^2}a_1b_3 + \frac{1}{2N_c^2}a_1c_3 + \frac{1}{4N_c^2}b_2^2 + \frac{3}{2N_c^4}b_3^2 + \frac{1}{8N_c^4}c_3^2 \, , \\
c_4^{\mathbf{27}} &= -\frac{1}{N_c^2}a_1b_3 + \frac{3}{2N_c^2}a_1c_3 + \frac{3}{2N_c^4}c_3^2 \, , \\
c_5^{\mathbf{27}} &= \frac{1}{N_c^3}b_2b_3 \, , \\
c_6^{\mathbf{27}} &= \frac{1}{2N_c^4}b_3^2 + \frac{1}{4N_c^4}c_3^2 \, , \\
c_7^{\mathbf{27}} &= -\frac{1}{N_c^4}b_3^2 + \frac{5}{4N_c^4}c_3^2 \, , \\
c_8^{\mathbf{27}} &= 0 \, , \\
c_9^{\mathbf{27}} &= 0 \, .
\end{aligned}$$

$$-\frac{1}{2}\{A^{j8},[V^c,[\mathcal{M},A^{j8}]]\} = \sum_{n=1}^{9} d_n^{\mathbf{27}} T_n^c, \tag{C12}$$

where

$$\begin{aligned}
d_1^{\mathbf{27}} &= \left(-\frac{3}{2N_c^2}a_1c_3 - \frac{3}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c} \, , \\
d_2^{\mathbf{27}} &= 0 \, , \\
d_3^{\mathbf{27}} &= \left(\frac{1}{N_c^2}a_1c_3 - \frac{1}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c} \, , \\
d_4^{\mathbf{27}} &= \left(\frac{1}{2}a_1^2 + \frac{3}{N_c^2}a_1c_3 + \frac{3}{N_c^4}c_3^2\right)\frac{\Delta}{N_c} \, , \\
d_5^{\mathbf{27}} &= 0 \, , \\
d_6^{\mathbf{27}} &= \left(\frac{1}{N_c^4}c_3^2\right)\frac{\Delta}{N_c} \, , \\
d_7^{\mathbf{27}} &= \left(\frac{1}{N_c^2}a_1c_3 + \frac{4}{N_c^4}c_3^2\right)\frac{\Delta}{N_c} \, , \\
d_8^{\mathbf{27}} &= 0 \, , \\
d_9^{\mathbf{27}} &= \left(\frac{1}{2N_c^4}c_3^2\right)\frac{\Delta}{N_c} \, .
\end{aligned}$$

$$\frac{1}{6}\left([A^{j8},[[\mathcal{M},[\mathcal{M},A^{j8}]],V^c]] - \frac{1}{2}[[\mathcal{M},A^{j8}],[[\mathcal{M},A^{j8}],V^c]]\right) = \sum_{n=1}^{9} e_n^{\mathbf{27}} T_n^c, \tag{C13}$$

where

$$
\begin{aligned}
e_1^{\mathbf{27}} &= \left(-\frac{3}{8}a_1^2 - \frac{3}{2N_c^2}a_1c_3 - \frac{3}{2N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\ , \\
e_2^{\mathbf{27}} &= 0\ , \\
e_3^{\mathbf{27}} &= \left(\frac{1}{4}a_1^2 + \frac{1}{4N_c^2}a_1c_3 - \frac{5}{4N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\ , \\
e_4^{\mathbf{27}} &= \left(\frac{3}{4}a_1^2 + \frac{3}{N_c^2}a_1c_3 + \frac{3}{N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\ , \\
e_5^{\mathbf{27}} &= 0\ , \\
e_6^{\mathbf{27}} &= \left(\frac{1}{2N_c^2}a_1c_3 + \frac{9}{8N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\ , \\
e_7^{\mathbf{27}} &= \left(\frac{5}{2N_c^2}a_1c_3 + \frac{11}{2N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\ , \\
e_8^{\mathbf{27}} &= \left(\frac{1}{4N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\ , \\
e_9^{\mathbf{27}} &= \left(\frac{7}{4N_c^4}c_3^2\right)\frac{\Delta^2}{N_c^2}\ .
\end{aligned}
$$

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