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Relativistic corrections to Higgs-boson decays to quarkonia

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Abstract

We improve the theoretical predictions for the decays of the Higgs boson to an S-wave vector quarkonium plus a photon by calculating the relativistic correction of order v^2, where v is the heavy-quark velocity in the quarkonium rest frame. Our numerical results are given for the J/ψ and $\Upsilon(nS)$ channels, with $n = 1, 2, 3$. The numerical results include a previously calculated correction of order α_s and summations, to all orders in α_s, of leading logarithms of m_H^2/m_Q^2, where m_H is the Higgs-boson mass and m_Q is the heavy-quark mass. These QCD corrections apply to the contribution of leading order in v and to part of the order-v^2 correction. For the remainder of the order-v^2 correction, we sum leading logarithms of m_H/m_Q through order α_s^2. These refinements reduce the theoretical uncertainties in the direct-production amplitudes for $H \to J/\psi + \gamma$ and $H \to \Upsilon(1S) + \gamma$ by approximately a factor of three and open the door to improved determinations at the LHC of the Higgs-boson Yukawa couplings to the charm and bottom quarks.

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I. INTRODUCTION

A primary activity of the LHC program is the exploration of the properties of the Higgs boson, which was discovered over two years ago by the ATLAS and CMS collaborations [1, 2]. Currently, only couplings to gauge bosons and third-generation fermions are measured directly [3, 4]. The couplings that are fixed through the well measured diboson decays of the Higgs are determined at the 20–30% level. No deviations from the predictions of the Standard Model (SM) have been observed.

While the possibility of measuring the Higgs-boson couplings to muons at the high-luminosity LHC (HL-LHC) has been studied [5–7], the couplings of the Higgs boson to first- and second-generation quarks are *terra incognita*. They are only weakly constrained by the inclusive Higgs-boson production cross sections, yet they can deviate significantly from their SM values in numerous theories of new physics. It was long thought to be impossible to measure these couplings, owing to the severe experimental difficulties that are inherent in reconstructing the signal and isolating it from the background.

Recent work has demonstrated that there is hope to determine the Yukawa couplings of first- and second-generation quarks at future runs of the LHC. Much of this renewed interest has arisen because of the realization that exclusive decays of the Higgs boson to vector mesons can probe its couplings to light quarks. The resulting final states are relatively clean experimentally, and the theoretical predictions are also under control. The first manifestation of this idea was the discovery that decays of the Higgs boson to an S-wave vector quarkonium plus a photon ($H \to V + \gamma$) provide opportunities to determine the $Hc\bar{c}$ and $Hb\bar{b}$ couplings [8].[1] (Here, $c(b)$ and $\bar{c}(\bar{b})$ denote a charm (bottom) quark and charm (bottom) antiquark.) While the $Hc\bar{c}$ coupling might be probed at the LHC by making use of charm-tagging techniques [10], its phase must be determined through processes that involve quantum interference effects, such as the decay $H \to J/\psi + \gamma$.

It is our intention in this paper to refine the theoretical prediction for the $H \to V + \gamma$ processes, where $V = J/\psi$ or $\Upsilon(nS)$, with $n = 1, 2, 3$. These modes feature clean experimental signatures in which a high-transverse-momentum lepton pair recoils against a photon. They proceed through two distinct mechanisms.

[1] It has also been realized that decays to light mesons might be used to map out the structure of Yukawa couplings of the Higgs boson to first- and second-generation quarks [9].

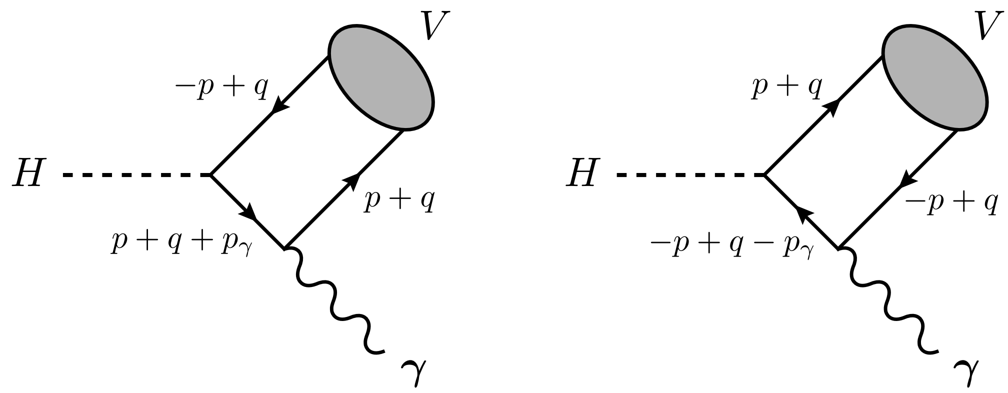

FIG. 1: The Feynman diagrams for the direct amplitude for $H \to V + \gamma$ at order α_s^0. The shaded blob represents the quarkonium wave function. The momenta that are adjacent to the heavy-quark lines are defined in the text.

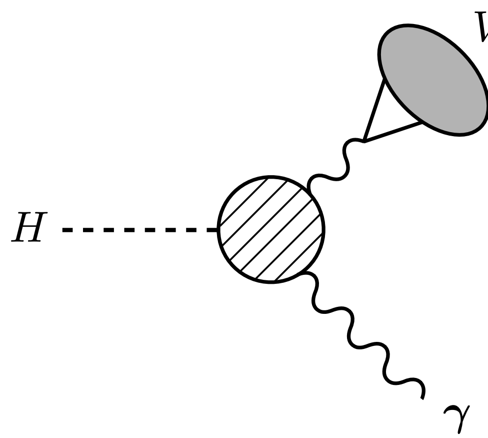

FIG. 2: The Feynman diagram for the indirect amplitude for $H \to V + \gamma$. The hatched circle represents top-quark or W-boson loops and the shaded blob represents the quarkonium wave function.

- In the *direct process*, the Higgs boson decays into a heavy-quark-antiquark ($Q\bar{Q}$) pair, one of which radiates a photon before forming a quarkonium with the other element of the pair.

- In the *indirect process*, the Higgs boson decays through a top-quark loop or a vector-boson loop to a γ and a γ^* (virtual photon). The γ^* then decays into a vector quarkonium.

The Feynman diagrams for the direct and indirect processes are shown in Figs. 1 and 2, respectively. It is the quantum interference between these two processes that provides phase information about the $Hc\bar{c}$ and $Hb\bar{b}$ couplings. The interference is destructive. In the case of the decay to the Υ, the destructive interference is nearly complete, and so the rate is very sensitive to the $Hb\bar{b}$ coupling.

The indirect decay amplitudes are determined at percent-level accuracy. The partial amplitude for the Higgs-boson decay to $\gamma\gamma^*$ can be inferred from calculations of the $H\gamma\gamma$ rate [11, 12]. The coupling of the quarkonium to a virtual photon is known from the decay rate of the quarkonium to a lepton pair.

The largest theoretical uncertainty in the direct amplitude for $H \to J/\psi + \gamma$ and, consequently, in the decay rate, arises from uncalculated relativistic corrections. These corrections take into account the relative motion of the Q and $\bar{Q}$ in the quarkonium. They are nominally of order v^2, where v is the RMS velocity of the Q or $\bar{Q}$ in the quarkonium rest frame. $v^2 \approx 25\%$ for the J/ψ and $v^2 \approx 10\%$ for the Υ.

In this paper, we compute the relativistic corrections to the direct amplitudes for the processes $H \to J/\psi + \gamma$ and $H \to \Upsilon(nS) + \gamma$ and some corrections of order $v^2\alpha_s$, where α_s is the strong coupling. We also include some corrections involving leading logarithms of m_H^2/m_Q^2 that are of order v^2 and of higher orders in α_s. (Here, m_H is the Higgs-boson mass and m_Q is the heavy-quark mass.)

The remainder of this paper is organized as follows. In Sec. II, we use the methods of nonrelativistic QCD (NRQCD) factorization [13] to compute the relativistic corrections to $H \to V + \gamma$. These corrections can also be computed, in the limit $m_V/m_H \to 0$, where m_V is the quarkonium mass, by making use of light-cone methods [14, 15]. We carry out the light-cone calculation of the relativistic corrections in Sec. III. The light-cone computation allows us to take advantage of existing calculations of corrections of next-to-leading order in α_s and is a convenient framework in which to compute logarithms of m_H^2/m_Q^2. We give numerical results for the decay rates in Sec. V and summarize our findings in Sec. VI.

II. NRQCD CALCULATION

In this section we compute relativistic corrections to the direct amplitude for $H \to V + \gamma$ by making use of the standard methods of NRQCD factorization [13]. We begin by considering the amplitude for $H \to Q\bar{Q} + \gamma$, where the $Q\bar{Q}$ pair is in a color-singlet, spin-triplet S-wave state. We take the Higgs-boson, Q, $\bar{Q}$, and γ momenta to be p_H, $p_1 = p + q$,

$p_2 = p - q$, and p_γ, respectively. These momenta satisfy the following relations:

$$\begin{aligned} p_H &= 2p + p_\gamma, & p\cdot q &= 0, & p_H^2 &= m_H^2, \\ p_1^2 &= m_Q^2, & p_2^2 &= m_Q^2, & p_\gamma^2 &= 0, \\ p^2 &= E^2, & E^2 &\equiv m_Q^2 - q^2 \equiv m_Q^2(1+v^2). \end{aligned} \tag{1}$$

In the $Q\bar{Q}$ rest frame, $p = (E, \mathbf{0})$ and $q = (0, \boldsymbol{q})$.

We take the polarization of the γ to be ϵ_γ and we take the spin polarization of the $Q\bar{Q}$ pair to be $\epsilon(\lambda)$, where λ is the polarization state. The color-singlet, spin-triplet projector, correct to all orders in v, is given by [16]

$$\Pi_3(p_1, p_2, \lambda) = \frac{1}{8\sqrt{2}E^2(E+m_Q)}(\not{p}_2 - m_Q)\not{\epsilon}^*(\lambda)(\not{p}_1 + \not{p}_2 + 2E)(\not{p}_1 + m_Q) \otimes \frac{\mathbf{1}}{\sqrt{N_c}}, \tag{2}$$

where $\mathbf{1}$ is the unit color matrix and $N_c = 3$ is the number of colors.

The $H \to Q\bar{Q} + \gamma$ amplitude arises from two Feynman diagrams, which are shown in Fig. 1. For a color-singlet, spin-triplet $Q\bar{Q}$ pair, it is given by

$$\begin{aligned} i\mathcal{M}_{\rm dir}[Q\bar{Q}(\text{triplet})] = -iee_Q\kappa_Q m_Q(\sqrt{2}G_F)^{\frac{1}{2}}\text{Tr}\Bigg\{ \Bigg[&\frac{(-\not{p} + \not{q} - \not{p}_\gamma + m_Q)\not{\epsilon}_\gamma^*}{(p-q+p_\gamma)^2 - m_Q^2 + i\varepsilon} \\ &+ \frac{\not{\epsilon}_\gamma^*(\not{p} + \not{q} + \not{p}_\gamma + m_Q)}{(p+q+p_\gamma)^2 - m_Q^2 + i\varepsilon}\Bigg] \Pi_3(p+q, p-q, \lambda)\Bigg\}, \end{aligned} \tag{3}$$

where the trace is over the gamma and the color matrices, e is the electromagnetic coupling, G_F is the Fermi weak coupling, e_Q is the fractional heavy-quark charge, and κ_Q is an adjustable factor in the $HQ\bar{Q}$ coupling. $\kappa_Q = 1$ in the SM.

Owing to charge-conjugation symmetry, the two contributions in Eq. (3) differ only by a change of sign of q. We obtain the S-wave contribution by averaging over the angles of $\boldsymbol{q}$ in the $Q\bar{Q}$ rest frame. In that average, contributions that are odd in q vanish. Hence, we can write the spin-triplet, S-wave amplitude as

$$i\mathcal{M}[Q\bar{Q}(^3S_1)] = -2iee_Q\kappa_Q m_Q\big(\sqrt{2}G_F\big)^{\frac{1}{2}}\int_{\hat{\boldsymbol{q}}}\text{Tr}\left[\Pi_3(p+q, p-q, \lambda)\frac{(-\not{p} + \not{q} - \not{p}_\gamma + m_Q)\not{\epsilon}_\gamma^*}{(p-q+p_\gamma)^2 - m_Q^2 + i\varepsilon}\right], \tag{4}$$

where a factor of 2 takes into account both contributions in Eq. (3) and the symbol $\int_{\hat{\boldsymbol{q}}}$ denotes the average over the direction of $\hat{\boldsymbol{q}} \equiv \boldsymbol{q}/|\boldsymbol{q}|$ in the rest frame of V:

$$\int_{\hat{\boldsymbol{q}}} \equiv \int \frac{d\Omega_{\hat{\boldsymbol{q}}}}{4\pi}. \tag{5}$$

Evaluation of the trace in Eq. (4) gives

$$
\begin{aligned}
i\mathcal{M}_{\rm dir}[Q\bar{Q}({}^3S_1)] = -2iee_Q\kappa_Q m_Q(\sqrt{2}G_F)^{1/2}\int_{\hat{\boldsymbol{q}}}\frac{-\sqrt{N_c}}{2\sqrt{2}E^2[(p-q+p_\gamma)^2-m_Q^2+i\varepsilon]} \\
\times\left[\frac{(m_H^2+4E^2+8Em_Q)}{E+m_Q}\,\epsilon_\gamma^*\cdot q\,\epsilon^*\cdot q-\frac{4p_\gamma\cdot q}{E+m_Q}\,\epsilon_\gamma^*\cdot p\,\epsilon^*\cdot q\right. \\
\left.-8E\epsilon_\gamma^*\cdot p\,\epsilon^*\cdot q+4m_Q\epsilon_\gamma^*\cdot p\,\epsilon^*\cdot p_\gamma-(m_H^2-4E^2)m_Q\epsilon_\gamma^*\cdot\epsilon^*\right]. \quad (6)
\end{aligned}
$$

We can write the quark-propagator denominator as $2(p-q)\cdot p_\gamma$. Then, the amplitude in Eq. (4) contains the tensor integrals

$$I = \int_{\hat{\boldsymbol{q}}}\frac{p\cdot p_\gamma}{(p-q)\cdot p_\gamma}, \quad (7a)$$

$$I^\mu = \int_{\hat{\boldsymbol{q}}}\frac{p\cdot p_\gamma}{(p-q)\cdot p_\gamma}\,q^\mu, \quad (7b)$$

$$I^{\mu\nu} = \int_{\hat{\boldsymbol{q}}}\frac{p\cdot p_\gamma}{(p-q)\cdot p_\gamma}\,q^\mu q^\nu. \quad (7c)$$

Because $q\cdot p = 0$, the tensor integrals I^μ and $I^{\mu\nu}$ must be orthogonal to p: $I^\mu p_\mu = 0$, $I^{\mu\nu}p_\mu = 0$. Therefore, it is convenient to define the four-vector

$$\bar{p}_\gamma \equiv p_\gamma - \frac{p_\gamma\cdot p}{p^2}p, \quad (8)$$

which is orthogonal to p. From the orthogonality of I^μ and $I^{\mu\nu}$ to p, it follows that I^μ must be proportional to $\bar{p}_\gamma^\mu$ and that $I^{\mu\nu}$ must be a linear combination of $-g^{\mu\nu}+(p^\mu p^\nu)/p^2$ and $\bar{p}_\gamma^\mu\bar{p}_\gamma^\nu$. A straightforward analysis then shows that

$$I = L(\delta) \equiv \frac{1}{2\delta}\log\frac{1+\delta}{1-\delta}, \quad (9a)$$

$$I^\mu = \frac{4E^2(1-I)}{m_H^2-4E^2}\,\bar{p}_\gamma^\mu, \quad (9b)$$

$$I^{\mu\nu} = \frac{E^2-m_Q^2 I}{2}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{p^2}\right)+\frac{8E^2[(m_Q^2+2E^2)I-3E^2]}{(m_H^2-4E^2)^2}\,\bar{p}_\gamma^\mu\bar{p}_\gamma^\nu, \quad (9c)$$

where

$$\delta = \frac{\sqrt{-q^2}}{E} = \frac{|\boldsymbol{q}|}{E} = \frac{v}{\sqrt{1+v^2}}. \quad (9d)$$

Now the amplitude can be written as

$$i\mathcal{M}_{\rm dir}[Q\bar{Q}({}^3S_1)] = i\mathcal{M}_{\rm dir}^{(0)}[Q\bar{Q}({}^3S_1)]R(v^2), \quad (10a)$$

where

$$i\mathcal{M}_{\rm dir}^{(0)}[Q\bar{Q}({}^3S_1)] = iee_Q\kappa_Q\big(\sqrt{2}G_F\big)^{\frac{1}{2}}\sqrt{2N_c}\left(-\epsilon^*\cdot\epsilon_\gamma^*+\frac{\epsilon^*\cdot p_\gamma\,p\cdot\epsilon_\gamma^*}{p_\gamma\cdot p}\right), \quad (10b)$$

is the amplitude in order v^0, and the factor $R(v^2)$, which contains the relativistic corrections, is given by

$$R(v^2) = \frac{m_Q}{2E^2}\left\{\frac{E^2 + m_Q(2E+m_Q)L(\delta)}{E+m_Q} + \frac{8E[E^2 - m_Q^2 L(\delta)]}{m_H^2 - 4E^2}\right\}. \tag{10c}$$

The invariance under electromagnetic gauge transformations is manifest in the last factor in Eq. (10b). In a physical gauge in the H rest frame, $p\cdot\epsilon_\gamma = 0$, and the last term in the last factor in Eq. (10b) vanishes. Hence, the expression in Eq. (10b) is independent of v.[2]

Now we can obtain the physical amplitude by carrying out the standard matching procedure between NRQCD and full QCD [13]. That is, we write $i\mathcal{M}_{\rm dir}$ in terms of NRQCD long-distance matrix elements (LDMEs) and determine the corresponding short-distance coefficients by comparing the NRQCD expression, evaluated in the $Q\bar{Q}(^3S_1)$ state, with Eq. (10a). Having determined the short-distance coefficients, we obtain the physical amplitude by evaluating the NRQCD LDMEs in the physical quarkonium state. We find that the direct amplitude for $H\to V+\gamma$ is given by

$$i\mathcal{M}_{\rm dir}[H\to V+\gamma] = \sqrt{2m_V}\phi_0\, i\mathcal{M}^{(0)}_{\rm dir}[H\to V+\gamma]\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial}{\partial v^2}\right)^n R(v^2)\bigg|_{v=0}\langle v^{2n}\rangle, \tag{11a}$$

where

$$i\mathcal{M}^{(0)}_{\rm dir}[H\to V+\gamma] \equiv iee_Q\kappa_Q(\sqrt{2}G_F)^{\frac{1}{2}}\sqrt{2N_c}\left(-\epsilon_V^*\cdot\epsilon_\gamma^* + \frac{\epsilon_V^*\cdot p_\gamma\, p_V\cdot\epsilon_\gamma^*}{p_\gamma\cdot p_V}\right), \tag{11b}$$

and p_V, m_V, and ϵ_V are the momentum, mass, and polarization of the quarkonium.[3] The quantity $\langle v^{2n}\rangle$ is given by a ratio of NRQCD LDMEs:

$$\langle v^{2n}\rangle = \frac{1}{m_Q^{2n}}\frac{\langle V(\boldsymbol{\epsilon})|\psi^\dagger(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}})^{2n}\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi|0\rangle}{\langle V(\boldsymbol{\epsilon})|\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi|0\rangle}. \tag{11c}$$

ϕ_0 is the quarkonium wave function at the origin, which is given by

$$\phi_0 = \frac{1}{\sqrt{2N_c}}\langle V(\boldsymbol{\epsilon})|\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi|0\rangle. \tag{11d}$$

---

[2] One can also see that the expression (10b) is independent of v from the fact that the v dependence of the four-vector p is contained in a factor that is common to all of the components of p. That factor cancels in the expression (10b).

[3] Owing to the denominator factors p and p_V in the expressions in Eqs. (10b) and (11b), the corresponding NRQCD LDMEs contain nonlocal operators. One can avoid the appearance of these nonlocal operators in the matching procedure by working in a physical gauge, in which $p\cdot\epsilon_\gamma = p_V\cdot\epsilon_\gamma = 0$, so that the second term in parentheses in Eqs. (10b) and (11b) vanishes. These terms can then be restored by requiring the final expression to be manifestly gauge invariant.

In the LDMEs, ψ is the two-component (Pauli) spinor field that annihilates a heavy quark and χ is the two-component spinor field that annihilates a heavy antiquark. The factor $\sqrt{2m_V}$ in Eq. (11a) arises from the relativistic normalization of the quarkonium state. In this factor and in the phase space, we choose m_V to be the physical quarkonium mass, rather than the mass of the $Q\bar{Q}$ state ($2E$).

In Eq. (11a), we have neglected contributions from LDMEs that involve factors of the gauge field. These contributions first appear in order v^4. In this paper, we work through order v^2. Retaining only contributions through order v^2 in Eq. (11a), we obtain

$$\begin{aligned} i\mathcal{M}_{\rm dir}[H \to V + \gamma] &= \sqrt{2m_V}\phi_0\, i\mathcal{M}^{(0)}_{\rm dir}[H \to V + \gamma] \left[1 - \frac{3m_H^2 - 28m_Q^2}{6(m_H^2 - 4m_Q^2)}\langle v^2\rangle + O(\langle v^4\rangle)\right] \\ &\approx \sqrt{2m_V}\phi_0\, i\mathcal{M}^{(0)}_{\rm dir}[H \to V + \gamma] \left[1 - \frac{1}{2}\langle v^2\rangle + O(\langle v^4\rangle)\right], \end{aligned} \tag{12}$$

where we have dropped contributions of higher order in m_Q^2/m_H^2 in the last line. Our result for the order-v^0 amplitude in Eq. (12) agrees with those in Refs. [8, 17].

We can assess the convergence of the v expansion for the class of LDMEs in Eq. (11a) by making use of the generalized Gremm-Kapustin relation [18]

$$\langle v^{2n}\rangle = \langle v^2\rangle^n, \tag{13}$$

which holds for dimensionally regulated LDMEs up to corrections of relative order v^2. Taking $\langle v^2\rangle = 0.20$, which is the approximate value for the J/ψ,[4] we find that the full expression in Eq. (11a) gives a relativistic correction of -8.8%, while the order-v^2 expression in Eq. (12) gives a relativistic correction of -10%. The difference between these corrections, 1.2%, is smaller than the nominal relative size of an order-v^4 correction, indicating that the v expansion is converging well. In fact, from the analytic structure of $R(v^2)$, we can see that the radius of convergence of the series in v^2 is unity.

[4] Note that the ratio of LDMEs $\langle v^2\rangle$ is different from the quantity v^2 that was mentioned earlier. v^2 is the average of $\boldsymbol{q}^2/m_Q^2$ over the square of the quarkonium wave function in the quarkonium rest frame. $\langle v^2\rangle$ can be significantly different from v^2, in part because the numerator LDME of $\langle v^2\rangle$ contains a linear ultraviolet divergence that is subtracted in dimensional regularization.

III. LIGHT-CONE CALCULATION

One can also compute the direct amplitude $i\mathcal{M}_{\rm dir}[H \to V+\gamma]$ in the light-cone approach. In leading twist, the computation is accurate up to corrections of order m_Q^2/m_H^2. Our motivation for examining the light-cone approach is two-fold: (1) we wish to make contact with the order-α_s light-cone calculation of $i\mathcal{M}_{\rm dir}[H \to V+\gamma]$ in Ref. [19]; (2) the light-cone formalism is a convenient one in which to compute logarithms of m_H^2/m_Q^2.

A. Light-cone direct amplitude

Let us now derive the light-cone amplitude for the direct process at order α_s^0 and at leading twist, that is, at leading order in $1/m_H$. We work implicitly in the H rest frame and neglect m_Q in comparison with m_H. Hence, p is lightlike, and we take p to be in the $-$ light-cone direction. The $H \to V+\gamma$ amplitude for the direct process is

$$\begin{aligned} i\mathcal{M}_{\rm dir}^{\rm LC}[H \to V+\gamma] = &-iee_Q\kappa_Q m_Q\big(\sqrt{2}G_F\big)^{\frac{1}{2}}\int\frac{d^4q}{(2\pi)^4}\,\langle V|\bar{Q}(p+q) \\ &\times\left[\frac{-\not{p}+\not{q}-\not{p}_\gamma}{(p-q+p_\gamma)^2+i\varepsilon}\,\not{\epsilon}_\gamma^* + \not{\epsilon}_\gamma^*\frac{\not{p}+\not{q}+\not{p}_\gamma}{(p+q+p_\gamma)^2+i\varepsilon}\right]Q(p-q)|0\rangle, \end{aligned} \tag{14}$$

where we have set $m_Q=0$, except in the $HQ\bar{Q}$ coupling. It is understood that the integration over the transverse components of q is dimensionally regulated. The scale of the dimensional regularization ultimately sets the scale of the light-cone distribution amplitude (LCDA).

Using $(\not{p}-\not{q})Q(p-q) = \bar{Q}(p+q)(\not{p}+\not{q}) = 0$, we obtain

$$\begin{aligned} i\mathcal{M}_{\rm dir}^{\rm LC}[H \to V+\gamma] &= -\frac{iee_Q\kappa_Q m_Q\big(\sqrt{2}G_F\big)^{\frac{1}{2}}}{p_\gamma\cdot p_V}\int\frac{d^4q}{(2\pi)^4}\,\langle V|\bar{Q}(p+q)\left(\frac{-\not{p}_\gamma\not{\epsilon}_\gamma^*}{1-x}+\frac{\not{\epsilon}_\gamma^*\not{p}_\gamma}{1+x}\right) \\ &\quad\times Q(p-q)|0\rangle \\ &= -iee_Q\kappa_Q m_Q\big(\sqrt{2}G_F\big)^{\frac{1}{2}}\frac{\epsilon_\gamma^{*\mu}p_\gamma^\nu}{p_\gamma\cdot p_V}\int\frac{d^4q}{(2\pi)^4}\,\langle V|\bar{Q}(p+q)\frac{[\gamma_\mu,\gamma_\nu]}{1-x^2}Q(p-q)|0\rangle. \end{aligned} \tag{15}$$

Here, we have followed the light-cone effective-field-theory procedure. That is, we have set $q=xp$, neglecting q^+ and $\boldsymbol{q}_\perp$, in the expression between $\bar{Q}$ and Q, which is proportional to the hard-scattering amplitude. However, we have retained q^+ and $\boldsymbol{q}_\perp$ nonzero in the other factors, which are proportional to the quarkonium wave function. In the last line, we have used the fact that $\epsilon_\gamma^*\cdot p_\gamma = 0$.

The LCDA $\phi(x)$ is defined by

$$\frac{1}{2}\langle V|\bar{Q}(z)[\gamma^\mu,\gamma^\nu][z,-z]Q(-z)|0\rangle = f_V(\epsilon_V^{*\,\mu}p_V^\nu - \epsilon_V^{*\,\nu}p_V^\mu)\int_{-1}^{+1} dx\, e^{ip^- zx}\phi(x), \tag{16}$$

where z lies along the + light-cone direction. The gauge link $[z,-z]$, which makes the nonlocal operator gauge invariant, is given by

$$[z,-z] = P\exp\left[ig_s\int_{-z}^{+z} dx\, A^+(x)\right], \tag{17}$$

where $g_s = \sqrt{4\pi\alpha_s}$, $A^\mu = A_a^\mu T^a$ is a matrix-valued gluon field A_a^μ with the color index $a = 1$, $2, \cdots, N_c^2 - 1$, T^a is the generator of the fundamental representation of SU(3) color, and P denotes path ordering. The gauge link vanishes in our case because we are working at order α_s^0. (More generally, the gauge link vanishes in the light-cone gauge $A^+ = 0$.) It follows from the definition (16) that

$$i\mathcal{M}_{\rm dir}^{\rm LC}[H\to V+\gamma] = \frac{i}{2}ee_Q\kappa_Q m_Q\big(\sqrt{2}G_F\big)^{\frac{1}{2}} f_V\left(-\epsilon_V^*\cdot\epsilon_\gamma^* + \frac{\epsilon_V^*\cdot p_\gamma\, p\cdot\epsilon_\gamma^*}{p_\gamma\cdot p}\right)\int_{-1}^{+1} dx\, T_0(x)\phi(x), \tag{18}$$

where

$$T_0(x) = \frac{4}{1-x^2} \tag{19}$$

is the hard-scattering kernel at leading order in α_s. The result in Eq. (18) agrees with the corresponding expression in Ref. [20].

B. Decay constant f_V

Next, we wish to determine the decay constant f_V. Setting $z = 0$ in Eq. (16) and imposing the normalization condition

$$\int_{-1}^{+1} dx\, \phi(x) = 1, \tag{20}$$

we obtain

$$\langle V|\bar{Q}[\gamma^\mu,\gamma^\nu]Q|0\rangle = 2f_V(\epsilon_V^{*\,\mu}p_V^\nu - \epsilon_V^{*\,\nu}p_V^\mu). \tag{21}$$

We can evaluate the matrix element on the left side of Eq. (21) in terms of NRQCD LDMEs by making use of the procedure that we followed in Sec. II. The result is

$$\begin{aligned}
\langle Q\bar{Q}(^3S_1)|\bar{Q}[\gamma^\mu,\gamma^\nu]Q|0\rangle &= \int_{\hat{\boldsymbol{q}}} \mathrm{Tr}\left[\Pi_3(p+q,p-q,\lambda)(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\right] \\
&= \frac{2\sqrt{2N_c}(E+2m_Q)}{3E^2}(\epsilon^{\mu *}p^\nu-\epsilon^{\nu *}p^\mu) \\
&= 2F(v^2)\frac{\sqrt{2N_c}}{m_Q}(\epsilon^{\mu *}p^\nu-\epsilon^{\nu *}p^\mu), \qquad (22)
\end{aligned}$$

where

$$F(v^2)=\frac{m_Q(E+2m_Q)}{3E^2}=\frac{2+\sqrt{1+v^2}}{3(1+v^2)}=1-\frac{5}{6}v^2+O(v^4). \qquad (23)$$

Then, carrying out the NRQCD matching procedure, we obtain

$$\langle V|\bar{Q}[\gamma^\mu,\gamma^\nu]Q|0\rangle = \frac{\sqrt{2N_c}\sqrt{2m_V}}{m_Q}\phi_0(\epsilon_V^{\mu *}p_V^\nu-\epsilon_V^{\nu *}p_V^\mu)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial}{\partial v^2}\right)^n F(v^2)\Big|_{v^2=0}\langle v^{2n}\rangle. \qquad (24)$$

Inserting this result into Eq. (21), we find that

$$f_V=\frac{\sqrt{2N_c}\sqrt{2m_V}}{2m_Q}\phi_0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial}{\partial v^2}\right)^n F(v^2)\Big|_{v^2=0}\langle v^{2n}\rangle. \qquad (25)$$

Hence, from Eq. (18), we see that

$$\begin{aligned}
i\mathcal{M}_{\mathrm{dir}}^{\mathrm{LC}}[H\to V+\gamma] &= \sqrt{2m_V}\phi_0\, i\mathcal{M}_{\mathrm{dir}}^{(0)}[H\to V+\gamma]\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial}{\partial v^2}\right)^n F(v^2)\Big|_{v=0}\langle v^{2n}\rangle \\
&\quad\times\frac{1}{4}\int_{-1}^{+1}dx\,T_0(x)\phi(x) \\
&= \sqrt{2m_V}\phi_0\, i\mathcal{M}_{\mathrm{dir}}^{(0)}[H\to V+\gamma]\left[1-\frac{5}{6}\langle v^2\rangle+O(\langle v^4\rangle)\right] \\
&\quad\times\frac{1}{4}\int_{-1}^{+1}dx\,T_0(x)\phi(x). \qquad (26)
\end{aligned}$$

C. Relativistic corrections

Some of the relativistic corrections in the direct amplitude for $H\to V+\gamma$ are apparent in the factor $F(v^2)$ in Eq. (26). There are additional relativistic corrections that come from the integral over x in Eq. (26). We make them manifest by carrying out a formal expansion of $\phi(x)$ about $x=0$:

$$\phi(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\langle x^k\rangle}{k!}\delta^{(k)}(x), \qquad (27)$$

where $\delta^{(k)}(x)$ is the kth derivative of the Dirac delta function. Then, using the fact that $\phi(x)$ is an even function of x, we find that

$$\begin{aligned}\int_{-1}^{+1} dx\, T_0(x)\phi(x) &= 4\sum_{k=0}^{\infty}\int_{-1}^{+1} dx\, x^{2k}\phi(x) = 4\sum_{k=0}^{\infty}\langle x^{2k}\rangle \\ &= 4 + \frac{4}{3}\langle v^2\rangle + O(\langle v^4\rangle),\end{aligned} \tag{28}$$

where

$$\langle x^n\rangle = \int_{-1}^{+1} dx\, x^n \phi(x), \tag{29}$$

and we have used the relation [21, 22]

$$\langle x^2\rangle = \frac{1}{3}\langle v^2\rangle, \tag{30}$$

which holds for S-wave quarkonia, up to corrections of order $\langle v^4\rangle$. Then, from Eqs. (23), (26), and (30), we have

$$i\mathcal{M}_{\rm dir}^{\rm LC}[H\to V+\gamma] = \sqrt{2m_V}\phi_0\, i\mathcal{M}_{\rm dir}^{(0)}[H\to V+\gamma]\Big[1 - \frac{1}{2}\langle v^2\rangle + O(\langle v^4\rangle)\Big], \tag{31}$$

in agreement with the last line of Eq. (12).

D. Evolution of the LCDA

The LCDA depends on a scale μ. If we employ dimensional regularization to define and renormalize the LCDA, then μ is the scale that is associated with the dimensional regularization. The evolution with respect to μ is governed by the equation [14]

$$\mu^2\frac{\partial}{\partial\mu^2}\phi(x,\mu) = C_F\frac{\alpha_s(\mu)}{4\pi}\int_{-1}^{1} dy\, V_T(x,y)\phi(y,\mu), \tag{32}$$

where

$$V_T(x,y) = V_0(x,y) - \frac{1-x}{1-y}\theta(x-y) - \frac{1+x}{1+y}\theta(y-x), \tag{33a}$$

$$V_0(x,y) = V_{\rm BL}(x,y) - \delta(x-y)\int_{-1}^{1} dz\, V_{\rm BL}(z,x), \tag{33b}$$

$$V_{\rm BL}(x,y) = \frac{1-x}{1-y}\left(1+\frac{2}{x-y}\right)\theta(x-y) + \frac{1+x}{1+y}\left(1+\frac{2}{y-x}\right)\theta(y-x). \tag{33c}$$

The evolution equation (32) is usually solved by expanding $\phi(x)$ in eigenfunctions of the evolution kernel V_T. That approach is discussed in Appendix A. It was used in Ref. [19] to obtain a summation of the leading logarithms of m_H^2/m_Q^2 to all orders in α_s for the 1 term in T_0. As is explained in Appendix A, this approach fails to give a convergent expression for physical values of m_H and m_Q for the x^2 term in T_0. Therefore, we compute the logarithms of m_H^2/m_Q^2 for the x^2 term in T_0 by solving the evolution equation perturbatively.

The solution of Eq. (32) through order α_s^2 is given by [20]

$$\begin{aligned}\phi(x,\mu) = \phi(x,\mu_0) + &\left[C_F\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right]\left[1+\frac{\beta_0}{2}\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right]\\ &\times\int_{-1}^{1}dy\; V_T(x,y)\phi(y,\mu_0) + \frac{1}{2}\left[C_F\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right]^2\\ &\times\int_{-1}^{1}dy\int_{-1}^{1}dz\; V_T(x,y)V_T(y,z)\phi(z,\mu_0) + O(\alpha_s^3), \end{aligned} \tag{34}$$

where $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$. We can compute $\int_{-1}^{1} dx\, T_0(x)\phi(x,\mu)$ by making use of Eq. (27) and the following integrals from Ref. [20]:

$$f_1(y) = \int_{-1}^{1}dx\; T_0(x)V_T(x,y) = \frac{4}{1-y^2}\left(3+2\log\frac{1-y^2}{4}\right), \tag{35a}$$

$$\begin{aligned}f_2(z) &= \int_{-1}^{1}dx\int_{-1}^{1}dy\; T_0(x)V_T(x,y)V_T(y,z)\\ &= \frac{4}{1-z^2}\left[9+12\log\frac{1-z^2}{4}+4\left(\log^2\frac{1+z}{2}+\log^2\frac{1-z}{2}\right)\right].\end{aligned} \tag{35b}$$

The result is

$$\begin{aligned}\int_{-1}^{1}dx\,T_0(x)\phi(x,\mu) = 4\sum_{k=0}^{\infty}\langle x^{2k}\rangle + &\left[C_F\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right]\left[1+\frac{\beta_0}{2}\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right]\\ &\times\sum_{k=0}^{\infty}\frac{f_1^{(2k)}(0)}{(2k)!}\langle x^{2k}\rangle + \frac{1}{2}\left[C_F\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right]^2\sum_{k=0}^{\infty}\frac{f_2^{(2k)}(0)}{(2k)!}\langle x^{2k}\rangle\\ &+O(\alpha_s^3),\end{aligned} \tag{36}$$

where, of course, this expression contains only the leading logarithmic term in each order in α_s. Using

$$f_1(0) = 4(3-4\log 2), \tag{37a}$$

$$f_1^{(2)}(0) = 8(1-4\log 2), \tag{37b}$$

$$f_2(0) = 4(9-24\log 2+8\log^2 2), \tag{37c}$$

$$f_2^{(2)}(0) = 8(5-16\log 2+8\log^2 2), \tag{37d}$$

we obtain

$$\int_{-1}^{1} dx\, T_0(x)\phi(x,\mu) = 4c_0(\mu) + 4c_2(\mu)\langle x^2\rangle + O(\langle x^4\rangle), \tag{38}$$

where

$$\begin{aligned} c_0(\mu) &= 1 + C_F\left[\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right](3-4\log 2) + C_F\left[\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right]^2 \\ &\quad\times\left[C_F\left(\frac{9}{2}-12\log 2+4\log^2 2\right)+\beta_0\left(\frac{3}{2}-2\log 2\right)\right]+O(\alpha_s^3), \end{aligned} \tag{39a}$$

$$\begin{aligned} c_2(\mu) &= 1 + C_F\left[\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right](1-4\log 2) + C_F\left[\frac{\alpha_s(\mu)}{4\pi}\log\frac{\mu^2}{\mu_0^2}\right]^2 \\ &\quad\times\left[C_F\left(\frac{5}{2}-8\log 2+4\log^2 2\right)+\beta_0\left(\frac{1}{2}-2\log 2\right)\right]+O(\alpha_s^3). \end{aligned} \tag{39b}$$

These series converge rapidly. The α_s^2 term in c_2 is about 6% for $\mu_0 = m_c$ and about 4% for $\mu_0 = m_b$.

IV. SUMMARY OF CORRECTIONS TO THE DIRECT AMPLITUDE THROUGH ORDER v^2

Now let us summarize the corrections through order v^2 that we use in computing the direct amplitude in this paper. Our calculations of the direct amplitude are carried out through order v^2 and at leading order in m_Q^2/m_H^2. They are based on the expression in the second equality of Eq. (26). We expand the LCDA according to Eq. (27).

The $\delta(x)$ term in Eq. (27) was taken into account in Ref. [19]. There, the coefficient $c_0(\mu)$ in Eq. (38) was computed to all orders in α_s. These leading logarithms from the evolution of the LCDA were combined with additional leading logarithms of m_H^2/m_Q^2 that arise from the running of m_Q in the $HQ\bar{Q}$ coupling:[5]

$$F_{HQ\bar{Q}}(\mu) = [\alpha_s(\mu_0)/\alpha_s(\mu)]^{-3C_F/\beta_0}. \tag{40}$$

Finally, the all-orders sums of logarithms were combined with a fixed-order light-cone calculation of the amplitude through order α_s. The order α_s logarithm of m_H^2/m_Q^2 that is

[5] The logarithms in $F_{HQ\bar{Q}}$ are much more important numerically than the logarithms in c_0 because of cancellations that make the coefficients of the logarithms in c_0 small. That is not the case for the logarithms in c_2, which are comparable numerically to the logarithms in $F_{HQ\bar{Q}}$.

contained in the all-orders sum was subtracted from this fixed-order calculation in order to avoid double counting. The complete correction factor for the direct amplitude, relative to the order-α_s^0 contribution, is given in Eq. (78) of Ref. [19]. In that expression, the LCDA and the $HQ\bar{Q}$ coupling are evolved from $2m_Q$ to m_H. We evolve from m_Q to m_H, instead.[6] Therefore, we modify the expression in Eq. (78) of in Ref. [19] by making the replacement

$$-2\log 2\log\frac{m_H^2}{4m_Q^2}\to -2\log 2\log\frac{m_H^2}{m_Q^2} \tag{41}$$

in the last term of that equation. (We have also corrected an obvious typo: $\log[2(1-\kappa)]\to\log[2(\kappa-1)]$.) We denote this modified version of the expression in Eq. (78) of Ref. [19] by g_{SV}.

For the $\delta^{(2)}(x)$ term in Eq. (27), we include the factor $c_2(\mu)$ in Eq. (39) and the factor $F_{HQ\bar{Q}}(\mu)$ in Eq. (40). These take into account the leading logarithms of m_H^2/m_Q^2 from the evolution of the LCDA through order α_s^2 and the leading logarithms from the running of the $HQ\bar{Q}$ coupling to all orders in α_s, respectively. A fixed-order calculation at order α_s is not available for the $\delta^{(2)}(x)$ term in Eq. (27).

The complete expression for the direct amplitude that we use in our numerical calculations is then

$$i\mathcal{M}_{\rm dir}^{\rm calc}[H\to V+\gamma]=\sqrt{2m_V}\phi_0 i\mathcal{M}_{\rm dir}^{(0)}[H\to V+\gamma]\left[\left(1-\frac{5}{6}\langle v^2\rangle\right)g_{SV}+\frac{1}{3}\langle v^2\rangle c_2(\mu)F_{HQ\bar{Q}}(\mu)\right]. \tag{42}$$

As we have mentioned, in computing g_{SV}, $c_2(\mu)$, and $F_{HQ\bar{Q}}$ in this expression, we evolve from m_Q to m_H. When $m_Q=m_b$, we carry out the evolution with $n_f=5$. When $m_Q=m_c$, we carry out the evolution in two steps: one from m_c to m_b, with $n_f=4$, and another from m_b to m_H, with $n_f=5$.

V. DECAY RATE

In this section we compute numerical results for the rates for $H\to J/\psi+\gamma$ and $H\to\Upsilon+\gamma$. First we write the direct amplitude in Eq. (42) as

$$\mathcal{M}_{\rm dir}^{\rm calc}=\mathcal{A}_{\rm dir}\left(-\epsilon_V^*\cdot\epsilon_\gamma^*+\frac{\epsilon_V^*\cdot p_\gamma\, p_V\cdot\epsilon_\gamma^*}{p_\gamma\cdot p_V}\right). \tag{43}$$

[6] The logarithms in the LCDA are collinear logarithms, whose natural cutoff is m_Q. The logarithms in the running mass vanish when $\mu=m_Q$.

The indirect amplitude is given by [8]

$$\mathcal{M}_{\rm ind} = \mathcal{A}_{\rm ind}\left(-\epsilon_V^*\cdot\epsilon_\gamma^* + \frac{\epsilon_V^*\cdot p_\gamma\, p_V\cdot\epsilon_\gamma^*}{p_\gamma\cdot p_V}\right), \tag{44}$$

where

$$\mathcal{A}_{\rm ind} = \frac{g_{V\gamma}\sqrt{4\pi\alpha(m_V)}m_H}{m_V^2}\left[16\pi\frac{\alpha(m_V)}{\alpha(0)}\Gamma(H\to\gamma\gamma)\right]^{\frac{1}{2}}, \tag{45}$$

and $g_{V\gamma}$ can be written in terms of the width of V into leptons [8]:

$$g_{V\gamma} = -\frac{e_Q}{|e_Q|}\left[\frac{3m_V^3\Gamma(V\to l^+l^-)}{4\pi\alpha^2(m_V)}\right]^{\frac{1}{2}}. \tag{46}$$

We remind the reader that $g_{V\gamma}$, as computed in Eq. (46), already contains all of the corrections of higher order in α_s and v that would appear in the NRQCD expression for the indirect rate [8, 23]. Note that both $\mathcal{A}_{\rm dir}$ and $\mathcal{A}_{\rm ind}$ have dimensions of mass and are normalized differently than in Ref. [8]. We have neglected a small phase in $\mathcal{A}_{\rm ind}$ that is about 0.005. We have dropped terms in Eq. (45) that are proportional to m_V^2 divided by combinations of m_H^2, m_t^2, m_Z^2, or m_W^2. The calculation of such terms in Ref. [8] was incomplete in that it did not include the full set of diagrams that is needed for electroweak gauge invariance.

The sum over the polarizations of the photon and the quarkonium is given by

$$\sum_{\rm pol}\left|-\epsilon_V^*\cdot\epsilon_\gamma^* + \frac{\epsilon_V^*\cdot p_\gamma\, p_V\cdot\epsilon_\gamma^*}{p_\gamma\cdot p_V}\right|^2 = 2, \tag{47}$$

where we have used

$$\sum_{\gamma\ \rm pol}\epsilon_\gamma^{\mu *}\epsilon_\gamma^\nu = -g^{\mu\nu}, \tag{48a}$$

$$\sum_{V\ \rm pol}\epsilon_V^{\mu *}\epsilon_V^\nu = -g^{\mu\nu} + \frac{p_V^\mu p_V^\nu}{p_V^2}. \tag{48b}$$

We then find that the decay rate is

$$\Gamma(H\to V+\gamma) = 2\frac{1}{2m_H}\frac{m_H^2-m_V^2}{8\pi m_H^2}|\mathcal{A}_{\rm dir}+\mathcal{A}_{\rm ind}|^2, \tag{49}$$

where the first factor comes from the polarization sum, the second factor comes from relativistic normalization of the Higgs-boson state, and the third factor comes from the phase space.

Now let us comment on the choices of scales for the electromagnetic coupling α. In the direct amplitude, the photon is on shell, and so we take $e = \sqrt{4\pi\alpha(0)}$. In the indirect amplitude we use $\alpha(m_V)$ to compute $g_{V\gamma}$ from the V leptonic width. We also use $e = \sqrt{4\pi\alpha(m_V)}$ for the couplings of the virtual photon and $e = \sqrt{4\pi\alpha(0)}$ for the coupling of the real photon. We have compensated for the fact that $\Gamma(H \to \gamma\gamma)$ was computed using $e = \sqrt{4\pi\alpha(0)}$. The couplings in the indirect amplitude are shown explicitly in Eqs. (45) and (46). Note that the dependences on $\alpha(m_V)$ cancel in the indirect amplitude. We use the following value of α:

$$\alpha(0) = 1/137.036. \tag{50}$$

In evaluating Eq. (49), we take m_Q to be the pole mass in order to maintain consistency with the one-loop corrections to the direct amplitude that we include. We obtain the numerical value of the pole mass by making use of the one-loop expression that relates the pole mass to the modified minimal subtraction ($\overline{\text{MS}}$) mass. This procedure has the effect of replacing the pole mass with the $\overline{\text{MS}}$ mass in the expressions through one-loop order and avoids the issue that the pole mass does not have a definite value, owing to the presence of an infrared renormalon in its definition. We use

$$m_c = 1.483 \pm 0.029 \text{ GeV}, \tag{51a}$$

$$m_b = 4.580 \pm 0.033 \text{ GeV}. \tag{51b}$$

Interpolating the results in Ref. [24] (J/ψ) and in Ref. [25] (Υ) for the values of m_Q that we use, we obtain

$$\phi_0^2(J/\psi) = \quad 0.0729 \pm 0.0109 \text{ GeV}^3, \tag{52a}$$

$$\langle v^2 \rangle(J/\psi) = \quad 0.201 \pm 0.064, \tag{52b}$$

$$\phi_0^2[\Upsilon(1S)] = \quad 0.512 \pm 0.035 \text{ GeV}^3, \tag{52c}$$

$$\langle v^2 \rangle[\Upsilon(1S)] = -0.00920 \pm 0.00348, \tag{52d}$$

$$\phi_0^2[\Upsilon(2S)] = \quad 0.271 \pm 0.019 \text{ GeV}^3, \tag{52e}$$

$$\langle v^2 \rangle[\Upsilon(2S)] = \quad 0.0905 \pm 0.0100, \tag{52f}$$

$$\phi_0^2[\Upsilon(3S)] = \quad 0.213 \pm 0.015 \text{ GeV}^3, \tag{52g}$$

$$\langle v^2 \rangle[\Upsilon(3S)] = \quad 0.157 \pm 0.017. \tag{52h}$$

We take $m_H = 125.9 \pm 0.4$ GeV, and we obtain $\Gamma(H \to \gamma\gamma) = 9.565 \times 10^{-6}$ GeV from the values of the Higgs-boson total width and branching fraction to $\gamma\gamma$ in Refs. [11, 12].

We estimate the uncertainties in the indirect amplitude along the lines that were suggested in footnote 2 of Ref. [8]. In $\Gamma(H \to \gamma\gamma)$, we take the uncertainty from uncalculated higher-order corrections to be 1% and the uncertainties that arise from the uncertainties in the top-quark mass m_t and the W-boson mass m_W to be 0.022% and 0.024%, respectively. We take the uncertainties in the leptonic decay widths to be 2.5% for the J/ψ and 1.3% for the Υ. We estimate the uncertainties in the indirect amplitude from uncalculated mass corrections to be m_V^2/m_H^2. We have not included the effects of the uncertainty in m_H, as it is expected that that uncertainty will be significantly reduced in Run II of the LHC.

The uncertainties in the direct amplitude arise primarily from the uncertainties in ϕ_0, $\langle v^2 \rangle$, and uncalculated corrections of order α_s^2, $\alpha_s v^2$, and order v^4. We estimate the order-α_s^2 correction to be 2%, the order-$\alpha_s v^2$ correction to be 5% for the J/ψ and 1.5% for the Υ, and the order-v^4 correction to be 9% for the J/ψ and 1% for the Υ. The uncertainties in the direct amplitude that arise from the uncertainties in m_c and m_b are 0.6% in the case of the J/ψ and 0.1% in the case of the Υ, and so they are negligible in comparison with the other uncertainties in the direct amplitude.

Our results for the widths are[7]

$$\Gamma(H \to J/\psi + \gamma) = \left|(11.9 \pm 0.2) - (1.04 \pm 0.14)\kappa_c\right|^2 \times 10^{-10}\ \text{GeV}, \tag{53a}$$

$$\Gamma[H \to \Upsilon(1S) + \gamma] = \left|(3.33 \pm 0.03) - (3.49 \pm 0.15)\kappa_b\right|^2 \times 10^{-10}\ \text{GeV}, \tag{53b}$$

$$\Gamma[H \to \Upsilon(2S) + \gamma] = \left|(2.18 \pm 0.03) - (2.48 \pm 0.11)\kappa_b\right|^2 \times 10^{-10}\ \text{GeV}, \tag{53c}$$

$$\Gamma[H \to \Upsilon(3S) + \gamma] = \left|(1.83 \pm 0.02) - (2.15 \pm 0.10)\kappa_b\right|^2 \times 10^{-10}\ \text{GeV}. \tag{53d}$$

The SM values for the widths ($\kappa_Q = 1$) are

$$\Gamma_{\text{SM}}(H \to J/\psi + \gamma) = 1.17^{+0.05}_{-0.05} \times 10^{-8}\ \text{GeV}, \tag{54a}$$

$$\Gamma_{\text{SM}}[H \to \Upsilon(1S) + \gamma] = 2.56^{+7.30}_{-2.56} \times 10^{-12}\ \text{GeV}, \tag{54b}$$

$$\Gamma_{\text{SM}}[H \to \Upsilon(2S) + \gamma] = 8.46^{+7.79}_{-5.35} \times 10^{-12}\ \text{GeV}, \tag{54c}$$

$$\Gamma_{\text{SM}}[H \to \Upsilon(3S) + \gamma] = 10.25^{+7.33}_{-5.45} \times 10^{-12}\ \text{GeV}. \tag{54d}$$

Using $\Gamma(H) = 4.195^{+0.164}_{-0.159} \times 10^{-3}$ GeV [26], we obtain the following results for the branching

[7] We do not include results for the $\psi(2S)$ because a value for $\langle v^2 \rangle[\psi(2S)]$ does not exist in the literature and because it is likely that v^2 for the $\psi(2S)$ is so large that the theoretical uncertainties in the width would be very large.

fractions in the SM:

$$\mathcal{B}_{\rm SM}(H \to J/\psi + \gamma) = 2.79^{+0.16}_{-0.15} \times 10^{-6}, \tag{55a}$$

$$\mathcal{B}_{\rm SM}[H \to \Upsilon(1S) + \gamma] = 6.11^{+17.41}_{-6.11} \times 10^{-10}, \tag{55b}$$

$$\mathcal{B}_{\rm SM}[H \to \Upsilon(2S) + \gamma] = 2.02^{+1.86}_{-1.28} \times 10^{-9}, \tag{55c}$$

$$\mathcal{B}_{\rm SM}[H \to \Upsilon(3S) + \gamma] = 2.44^{+1.75}_{-1.30} \times 10^{-9}. \tag{55d}$$

In comparison with the results in Ref. [8], the coefficient of κ_c has been reduced by about 30%, and the coefficient of κ_b has been reduced by about 12%. In the case of the coefficient of κ_c, the reduction arises as follows: a reduction of 11% from the relativistic corrections; a reduction of 18% from summing logarithms by evolving from the scale m_c, rather than from the scale $2m_c$, and from using a variable flavor number, rather than a fixed flavor number $n_f = 3$; a reduction of 3% from using $\alpha(0)$, rather than $\alpha(m_H/2)$ for the electromagnetic coupling of the on-shell quark. In the case of the coefficient of κ_b, the reduction arises as follows: a reduction of 0% from the relativistic corrections; a reduction of 9% from summing logarithms by evolving from the scale m_b, rather than from the scale $2m_b$, and from using $n_f = 5$, rather than $n_f = 3$; a reduction of 3% from using $\alpha(0)$, rather than $\alpha(m_H/2)$ for the electromagnetic coupling of the on-shell quark. In addition, there are changes in the coefficients of κ_c and κ_b of less than 1% that come from changes in the values of m_c, m_b, and m_H.

VI. SUMMARY AND DISCUSSION

In this paper, we have calculated relativistic corrections to the direct decay amplitude that appears in the Higgs-boson width $\Gamma(H \to V + \gamma)$, where V is a J/ψ or an $\Upsilon(nS)$ state with $n = 1, 2, 3$.

Using NRQCD factorization methods, we have calculated corrections to all orders in the heavy-quark velocity v for NRQCD LDMEs of the form in Eq. (11c), keeping the exact dependence on the ratio of the heavy-quark mass m_Q to the Higgs-boson mass m_H. The result of this calculation is given in Eq. (11a), where $R(v^2)$ is given in Eq. (10c).

Using light-cone methods, we have calculated relativistic corrections through order v^2 at the leading order in m_Q^2/m_H^2. In the light-cone method, the corrections in order v^2 arise from both the x^0 term and the x^2 term in the hard-scattering kernel $T_0(x)$, where x is the light-cone momentum fraction. In the case of the corrections that arise from the x^0 term, we have applied existing corrections of order α_s and corrections from a summation of leading logarithms of m_H^2/m_Q^2 to all orders in α_s [19]. In the case of the corrections that arise from the x^2 term, we have computed and applied corrections from leading logarithms of m_H^2/m_Q^2. We have computed leading logarithms from the running of the $HQ\bar{Q}$ coupling to all orders in α_s and leading logarithms from the evolution of the LCDA through order α_s^2. Leading logarithmic corrections of order α_s^3 and higher are estimated to contribute at the level of about 1%. The complete result from applying these various corrections is given in Eq. (42). We used this result in our numerical calculations.

Our numerical results for the widths $\Gamma(H \to J/\psi + \gamma)$ and $\Gamma(H \to \Upsilon(nS) + \gamma)$ are given in Eqs. (53) and (54), where κ_Q in Eq. (53) parametrizes the deviation of the $HQ\bar{Q}$ coupling from the SM value. In comparison with the results in Ref. [8], the coefficient of κ_c has been reduced by about 30%, and the coefficient of κ_b in $\Gamma(H \to \Upsilon(1S) + \gamma)$ has been reduced by about 12%. The relativistic corrections themselves contribute only about 11% and 0% of this reduction, respectively. The bulk of the reduction comes from the use of a different procedure for summing leading logarithms of m_H^2/m_Q^2, namely evolving from the scale m_Q, rather than from the scale $2m_Q$ and using a variable flavor number, rather than $n_f = 3$. The relativistic corrections are very small in the $\Upsilon(1S)$ case owing to a cancellation in the corresponding dimensionally regulated NRQCD LDME that makes $\langle v^2 \rangle$ anomalously small. We note that, for SM couplings, the destructive interference between the direct and indirect amplitudes is less complete in the $\Upsilon(2S)$ and $\Upsilon(3S)$ channels than in the $\Upsilon(1S)$ channel, and, hence, the SM rates are larger in the former channels.

More significant than the changes in the values of the coefficients of κ_Q are the changes in the theoretical uncertainties for those coefficients. Relative to the uncertainties that were given in Ref. [8], they have been reduced by about a factor of 3.3 for the coefficient of κ_c and by about a factor of 2.8 for the coefficient of κ_b in $\Gamma(H \to \Upsilon(1S) + \gamma)$.

In the case of the channel $H \to J/\psi + \gamma$, our values for the decay rate indicate that it should be possible to collect a sample of about 50 events in a high-luminosity run at the LHC [8]. This would imply a statistical error in the measurement of $\Gamma(H \to J/\psi + \gamma)$ of 14% and a statistical error in the determination of κ_c of about 40%. The latter error is comparable to the theoretical uncertainty in the coefficient of κ_c that existed in the absence of a calculation of relativistic corrections. The inclusion of the relativistic corrections that we have calculated reduces that uncertainty to about 16% and opens the door to determinations of the $Hc\bar{c}$ coupling at higher levels of precision.

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Appendix A: Eigenfunction evolution

In this appendix, we solve the LCDA evolution equation (32) in terms of the eigenfunctions of the evolution kernel V_T.

The kernel $V_T(x,y)$ has eigenfunctions

$$G_n(x) = \frac{1-x^2}{4} C_n^{3/2}(x), \tag{A1}$$

where $C_n^{3/2}(x)$ is a Gegenbauer polynomial. The eigenfunctions satisfy[8]

$$\frac{1}{2}\int_{-1}^{1} dy V_T(x,y) G_n(y) = -\gamma_n G_n(x), \tag{A2}$$

[8] See, for example, Ref. [20].

where the eigenvalues γ_n are given by

$$\gamma_n = \frac{1}{2} + 2\sum_{j=2}^{n+1}\frac{1}{j}. \tag{A3}$$

Following Ref. [20], we find a formal solution by writing

$$\phi(x,\mu) = \sum_n \phi_n(\mu) G_n(x), \tag{A4}$$

where the $\phi_n(\mu)$ can be found by using the orthogonality of the Gegenbauer polynomials:

$$\phi_n(\mu) = \frac{2(2n+3)}{(n+1)(n+2)}\int_{-1}^{1} dx\, C_n^{3/2}(x)\phi(x,\mu). \tag{A5}$$

The amplitude $i\mathcal{M}$ is proportional to $\int_{-1}^{1} dx\, T_0(x)\phi(x,\mu)$. Using Eq. (A4), we can write

$$\begin{aligned}\int_{-1}^{1} dx\, T_0(x)\phi(x,\mu) &= \sum_{n=0}^{\infty}\phi_n(\mu)\int_{-1}^{1} dx\, C_n^{3/2}(x)\\ &= 2\sum_{n=0}^{\infty}\phi_{2n}(\mu) = 2\sum_{n=0}^{\infty}\phi_{2n}(\mu_0)\left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{d_{2n}},\end{aligned} \tag{A6}$$

where we have used the facts that $\int_{-1}^{1} dx\, C_{2n}^{3/2}(x) = 2$ and $\int_{-1}^{1} dx\, C_{2n+1}^{3/2}(x) = 0$ for n a nonnegative integer, and we have defined $d_{2n} \equiv 2C_F\gamma_{2n}/\beta_0$, with $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$.

In order to find the coefficients $\phi_{2n}(\mu_0)$ from Eq. (A5), we expand $\phi(x,\mu_0)$ formally, using Eq. (27). For n a nonnegative integer, we have

$$\begin{aligned}\phi_{2n}(\mu_0) &= \frac{2(4n+3)}{(2n+1)(2n+2)}\sum_{k=0}^{\infty}\frac{\langle x^{2k}\rangle}{(2k)!}\frac{d^{2k}}{dx^{2k}}C_{2n}^{3/2}(0)\\ &= \frac{2(4n+3)}{(2n+1)(2n+2)}\sum_{k=0}^{\infty}\frac{\langle x^{2k}\rangle}{(2k)!}(4k+1)!!C_{2(n-k)}^{(4k+3)/2}(0)\\ &= \frac{2(4n+3)}{(2n+1)(2n+2)}\sum_{k=0}^{\infty}(-1)^{n-k}\langle x^{2k}\rangle\frac{(2n+2k+1)!!}{(2k)!(2n-2k)!!}.\end{aligned} \tag{A7}$$

Here, we have used the recurrence relation

$$\frac{d}{dx}C_n^{\lambda/2}(x) = \lambda C_{n-1}^{(\lambda+2)/2}(x) \tag{A8}$$

and the values of the Gegenbauer polynomials at zero argument

$$C_{2n+1}^{\lambda/2}(0) = 0, \tag{A9a}$$

$$C_{2n}^{\lambda/2}(0) = \frac{(-1)^n\Gamma(n+\frac{\lambda}{2})}{n!\Gamma(\frac{\lambda}{2})} = \frac{(-1)^n}{(2n)!!}\frac{(\lambda+2n-2)!!}{(\lambda-2)!!}. \tag{A9b}$$

Taking into account the effect of the running of the $HQ\bar{Q}$ coupling, (that is, the running of the quark mass), whose anomalous dimension is $-3C_F$, we can write

$$\int_{-1}^{1} dx\, T_0(x)\phi(x,\mu) = 4\sum_{k=0}^{\infty} c_{2k}(\mu)\langle x^{2k}\rangle, \tag{A10}$$

where

$$c_{2k}(\mu) = \sum_{n=0}^{\infty} \frac{(-1)^{n-k}(4n+3)}{(2n+1)(2n+2)} \frac{(2n+2k+1)!!}{(2k)!(2n-2k)!!} \left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{d_{2n}+3C_F/\beta_0} \tag{A11}$$

contains all of the leading logarithms of m_H^2/m_Q^2. The expression for $c_0(\mu)$ reproduces the expression in Eq. (58) of Ref. [19].

Note that, for large n, the nth term of $c_{2k}(\mu)$ is equal to

$$\begin{aligned} &(-1)^{n-k} n^{2k-1} \frac{(2n+1)!!}{(2n)!!} \frac{2^{2k}}{(2k)!} \left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{d_{2n}+3C_F/\beta_0} \\ &\sim (-1)^{n-k} \frac{2^{2k+1}}{\sqrt{\pi}(2k)!} n^{2k-1/2} \left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{(4C_F/\beta_0)(\gamma_{\rm E}+\log 2)} n^{(4C_F/\beta_0)\log[\alpha_s(\mu)/\alpha_s(\mu_0)]}. \end{aligned} \tag{A12}$$

Hence, the series for $c_{2k}(\mu)$ converges if and only if

$$\frac{4C_F}{\beta_0} \log\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} < -2k + \frac{1}{2}. \tag{A13}$$

For $k = 0$, this convergence condition is satisfied for $\mu = m_H$ and $\mu_0 = m_c$ or $\mu_0 = m_b$. However, for $k \geq 1$, it is not satisfied for $\mu = m_H$ and $\mu_0 = m_c$ or $\mu_0 = m_b$.

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