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Viscosity spectral function of a scale invariant non-relativistic fluid from holography

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Abstract

We study the viscosity spectral function of a holographic 2+1 dimensional fluid with Schrödinger symmetry. The model is based on a twisted compactification of $AdS_5 \times S_5$. We numerically compute the spectral function of the stress tensor correlator for all frequencies, and analytically study the limits of high and low frequency. We compute the shear viscosity, the viscous relaxation time, and the quasi-normal mode spectrum in the shear channel. We find a number of unexpected results: The high frequency behavior is governed by a fractional $1/3$ power law, the viscous relaxation time is negative, and the quasi-normal mode spectrum in the shear channel is not doubled.

I. INTRODUCTION

In recent years there has been a great deal of interest in the transport properties of strongly correlated fluids, such as the quark gluon plasma produced in collisions of relativistic heavy ions, the dilute Fermi gas at unitarity, and the strange metal phase of high T_c superconductors [1–5]. One of the central questions regarding these systems is whether the relevant degrees of freedom are quasi-particles, or whether the only possible description is in terms of non-local, holographic, degrees of freedom. This question is difficult to answer based solely on experimental data on the equation of state and the transport coefficients. Instead, it has been suggested that the problem can be addressed through the study of spectral functions, in particular the viscosity spectral function. If the excitations of the fluid are quasi-particles, then the spectral function has a quasi-particle peak at zero frequency. The line shape is approximately Lorentzian, and the width of the quasi-particle peak is related to the relaxation time. In holographic models, on the other hand, the spectral function is essentially featureless for small frequency, and may even have a dip at $\omega = 0$. There is no direct relationship between the viscous relaxation time and the shape of the spectral function. Poles of the retarded correlation function in the complex plane are not related to quasi-particles, but to quasi-normal modes.

In this work we consider the viscosity spectral function in a holographic theory that describes a scale invariant non-relativistic fluid [6–10]. The theory is intended to serve as a model for the dilute Fermi gas at unitarity, which is a scale invariant strongly correlated fluid that can be studied using trapped ultracold atomic gases [11, 12]. Nearly perfect fluidity in the unitary gas was discovered in 2002 [13], and a number of studies have demonstrated [14–16] that near the phase transition between the normal and the superfluid phase the ratio η/s of shear viscosity to entropy density of the unitary gas is close to the Kovtun-Son-Starinets bound $1/(4\pi)$ [17].

For both high and very low temperatures the shear viscosity of the unitary Fermi is dominated by quasi-particles and can be computed using kinetic theory [18, 19]. Kinetic theory has also been used to compute the viscosity spectral function [20], and second order transport coefficients [21]. General constraints on the spectral function are provided by sum rules [22, 23], and the high frequency behavior of the spectral function is governed by the operator product expansion [24, 25]. Model independent constraints at low frequency follow

from hydrodynamics [26, 27], and lattice calculations of the viscosity spectral function have been reported in [28].

In the following we will study the spectral function of a non-relativistic fluid in two spatial dimensions. Holographic models of 2+1 dimensional fluids can be constructed using light-like compactifications of $AdS_5 \times \mathcal{X}$, where \mathcal{X} is a compact manifold [6, 8, 9]. The basic idea can be explained based on the dispersion relation of a massless particle on the light cone, $p^+ = p_\perp^2/(2p^-)$, where $\vec{p}_\perp = (p^x, p^y)$ is the transverse momentum and $p^\pm = p^0 \pm p^z$ are light cone momenta. This dispersion relation exhibits Galilean scaling in 2+1 dimensions if the light-like momentum p^- is discrete. 2+1 dimensional fluids have been studied experimentally, and hydrodynamic behavior was observed by studying the damping of collective modes [29–31]. We note that in 2+1 dimensions the scale invariance of a dilute classical Fermi gas is broken by a quantum anomaly [32], but the effect of the anomaly on transport properties of the fluid is quite small [33, 34].

This paper is organized as follows. The structure of the stress tensor correlation function in fluid dynamics is reviewed in Sect. II. The Galilean invariant metric derived by Herzog et al. [8] and Adams et al. [9] is reviewed in Sect. III, and the spectral function of the stress tensor correlation function is computed in Sect. IV. We also study the low and high frequency limits, and compute the viscous relaxation time. In Sect. V we determine the spectrum of quasi-normal modes. Similar studies have been performed for the $AdS_5 \times S_5$ black hole, which is dual to the $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma. The viscosity spectral function was computed in [35, 36], and the viscous relaxation time was determined in [37]. The quasi-normal mode spectrum can be found in [38–40].

II. FLUID DYNAMICS

The main focus of this study is the retarded correlation function of the stress tensor,

$$(G_R)_{xy,xy}(\omega, \vec{k}) = -i \int dy \int d^2x e^{i(\omega t - \vec{k} \cdot \vec{x})} \Theta(t) \langle [\Pi_{xy}(u, \vec{x}), \Pi_{xy}(0, 0)] \rangle . \quad (1)$$

The low energy, small momentum behavior of this correlation function is dictated by fluid dynamics. In fluid dynamics the stress tensor of a scale invariant fluid is

$$\Pi_{ij} = \rho u_i u_j + P g_{ij} + \delta \Pi_{ij} , \quad (2)$$

where ρ is the mass density, \vec{u} is the fluid velocity, P is the pressure, g_{ij} is the d -dimensional metric, and $\delta\Pi_{ij}$ contains terms that involve gradients of the thermodynamic variables. At first order in the gradient expansion $\delta\Pi_{ij}$ can be written as $\delta\Pi_{ij} = -\eta\sigma_{ij}$ with [41]

$$\sigma_{ij} = \nabla_i u_j + \nabla_j u_i + \dot{g}_{ij} - \frac{2}{d}g_{ij}\langle\sigma\rangle, \quad \langle\sigma\rangle = \nabla \cdot u + \frac{\dot{g}}{2g}, \quad (3)$$

where η is the shear viscosity, ∇_i is a covariant derivative and $g = \det(g_{ij})$. At second order in the gradient expansion the stress tensor is [42]

$$\begin{aligned} \delta\Pi_{ij} = & -\eta\sigma_{ij} + \eta\tau_\pi \left[g_{ik}\dot{\sigma}_j^k + u^k\nabla_k\sigma_{ij} + \frac{2}{d}\langle\sigma\rangle\sigma_{ij} \right] + \lambda_1\sigma_{\langle i}{}^k\sigma_{j\rangle k} + \lambda_2\sigma_{\langle i}{}^k\Omega_{j\rangle k} \\ & + \lambda_3\Omega_{\langle i}{}^k\Omega_{j\rangle k} + \gamma_1\nabla_{\langle i}T\nabla_{j\rangle}T + \gamma_2\nabla_{\langle i}P\nabla_{j\rangle}P + \gamma_3\nabla_{\langle i}T\nabla_{j\rangle}P \\ & + \gamma_4\nabla_{\langle i}\nabla_{j\rangle}T + \gamma_5\nabla_{\langle i}\nabla_{j\rangle}P + \kappa_R R_{\langle ij\rangle}. \end{aligned} \quad (4)$$

Here, $\mathcal{O}_{\langle ij\rangle} = \frac{1}{2}(\mathcal{O}_{ij} + \mathcal{O}_{ji} - \frac{2}{d}\delta_{ij}\mathcal{O}^k{}_k)$ denotes the symmetric traceless part of a tensor \mathcal{O}_{ij} , $\Omega_{ij} = \nabla_i u_j - \nabla_j u_i$ is the vorticity tensor, and R_{ij} is the Ricci tensor.

The low energy expansion of the retarded correlation is given by [26]

$$G_R(\omega, 0) = P - i\eta\omega + \eta\tau_\pi\omega^2 + O(\omega^3), \quad (5)$$

where we have defined $G_R \equiv (G_R)_{xy,xy}$. Equation (5) can be used to determine the shear viscosity η and the viscous relaxation time τ_π . For non-zero momentum k the shear mode is diffusive and the dispersion relation is given by

$$\omega = -i\nu k^2 - i\nu^2\tau_\pi k^4 + \dots, \quad (6)$$

where $\nu = \eta/\rho$. Note that the $O(k^4)$ term is not complete, because it is suppressed by two powers of k relative to the leading order term, and at this level $O(\nabla^3)$ terms in the stress tensor contribute. A popular scheme for implementing second order fluid dynamics, known as the Israel-Stewart method in the case of relativistic fluids [43], is based on promoting the viscous stress tensor $\pi_{ij} \equiv \delta\Pi_{ij}$ to a hydrodynamic variable. We can write

$$\pi_{ij} = -\eta\sigma_{ij} - \tau_\pi \left(g_{ik}\dot{\pi}_j^k + u^k\nabla_k\pi_{ij} + \frac{d+2}{d}\langle\sigma\rangle\pi_{ij} \right) + \frac{\lambda_1}{\eta^2}\pi_{\langle i}{}^k\pi_{j\rangle k} + \dots, \quad (7)$$

where \dots refers to terms proportional to $\lambda_{2,3}, \gamma_{1-3}$ and κ_R . Treating π_{xy} as an independent variable, and solving the equations of linearized fluid dynamics we find one mode with the dispersion relation given in equ. (6), and a second mode described by

$$\omega = -\frac{i}{\tau_\pi}. \quad (8)$$

Note that this mode is outside the low energy regime $\omega \ll \tau_\pi$. Finally, we can also study the dispersion relation of the sound mode. For this purpose we consider fluctuations of the form $\pi_{xx}(x, t)$. We find

$$\omega = c_s k - \frac{i}{2} \Gamma k^2 + \frac{1}{8c_s} \left(8 \left(1 - \frac{1}{d} \right) c_s^2 \nu \tau_\pi - \Gamma^2 \right) k^3 + \dots, \quad (9)$$

where the sound attenuation constant is

$$\Gamma = 2 \left(1 - \frac{1}{d} \right) \nu + \frac{\kappa}{\rho} \left(\frac{1}{c_V} - \frac{1}{c_P} \right). \quad (10)$$

Here, κ is the thermal conductivity, and $c_{V,P}$ denotes the specific heat at constant volume and pressure, respectively. For simplicity, we have dropped $O(k^3)$ terms in equ. (9) that arise from $O(\nabla^2)$ terms in the entropy current [42]. In the Israel-Stewart scheme there is an additional mode with dispersion relation $\omega = -i/\tau_\pi$.

III. GALILEAN INVARIANT ADS/CFT

In order to study a holographic realization of non-relativistic fluid dynamics we consider the metric constructed by Herzog et al. and Adams et al. [8, 9]. We follow the notation of [8]. The five dimensional metric is

$$ds^2 = r^2 k(r)^{-2/3} \left\{ \left[\frac{1-f(r)}{4\beta^2} - r^2 f(r) \right] du^2 + \frac{\beta^2 r_+^4}{r^4} dv^2 - [1+f(r)] du dv \right\} \\ + k(r)^{1/3} \left\{ r^2 d\vec{x}^2 + \frac{dr^2}{r^2 f(r)} \right\}, \quad (11)$$

where u, v are light cone coordinates, $\vec{x} = (x_1, x_2)$, and r is the radial *AdS* coordinate. β is a parameter that is determined the chemical potential, $r = r_+$ is the position of the horizon, and $f(r) = 1 - (r_+/r)^4$. We also define

$$k(r) = 1 + \beta^2 r^2 (1 - f(r)) = 1 + \frac{\beta^2 r_+^4}{r^2}. \quad (12)$$

This metric can be derived from the 10-dimensional metric describing non-extremal D3 branes via a null Melvin twist and Kaluza-Klein reduction. More straightforwardly, we can view equ. (11) as a solution of the equations of motion for the five dimensional action

$$S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left(R - \frac{4}{3} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{4} e^{-8\phi/3} F_{\mu\nu} F^{\mu\nu} - 4A_\mu A^\mu - V(\phi) \right), \quad (13)$$

where the scalar potential is given by

$$V(\phi) = 4e^{2\phi/3} (e^{2\phi} - 4) . \quad (14)$$

The classical solution for the the vector and scalar fields is given by

$$A = \frac{r^2}{k(r)} \left(\frac{1+f(r)}{2} du - \frac{\beta^2 r_+^4}{r^4} dv \right) , \quad e^\phi = \frac{1}{\sqrt{k(r)}} . \quad (15)$$

The Hawking-Bekenstein entropy of the black hole can be computed from the area of the event horizon, and the temperature follows from the surface gravity. We find

$$S = \beta \frac{r_+^3}{4G_5} \Delta v \Delta x_1 \Delta x_2 , \quad T = \frac{r_+}{\pi\beta} . \quad (16)$$

There is a non-zero chemical potential, $\mu = 1/(2\beta^2)$, which is canonically conjugate to the momentum in the compactified v -direction. The equation of state is given $P \sim T^4/\mu^2$. This is an unusual equation of state, but consistent with scale invariance and stability. In particular, one finds $\mathcal{E} = P$, which follows from scale invariance in $2 + 1$ dimensions. In order for the density to be positive the chemical potential has to be negative, similar to a classical gas. The speed of sound is

$$c_s^2 = \left. \frac{\partial P}{\partial \rho} \right|_{s/n} = \frac{|\mu|}{m} . \quad (17)$$

The speed of sound and the pressure go to zero in the limit $T \rightarrow 0$ at constant density. This means that in the zero temperature limit the equation of state of the holographic fluid behaves like a classical gas. In particular, there is no Fermi pressure and the Bertsch parameter [44] is zero.

The dilute Fermi gas has a phase transition to a superfluid state, which in the case of 2+1 dimensional gases is of Berezinskii-Kosterlitz-Thouless (BKT) type. This transition has been observed in a trapped 2+1 dimensional Fermi gas [45], and a holographic model of the BKT transition was studied in [46]. The theory described by equ. (11) does not exhibit a phase transition, but it can serve as a model for the normal phase of a trapped atomic gas. The equation of state determines the density profile of a finite system confined by an external potential $V(x) \simeq \frac{1}{2}m\omega^2 x^2$. In particular, the equation of hydrostatic equilibrium, $\vec{\nabla}P = -n\vec{\nabla}V$ where n is the density, is solved by the local density approximation $\mu(x) = \mu_0 - V(x)$. For $P \sim T^4/\mu^2$ the density of a trapped gas is $n(x) \sim 1/[|\mu_0| + V(x)]^3$, which is a physically reasonable model for a trapped gas.

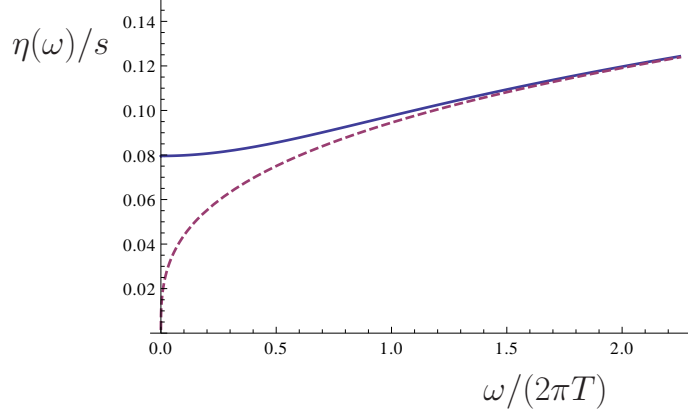


FIG. 1: Viscosity spectral function $\eta(\omega)/s$ in the holographic model constructed in [8, 9]. The dashed curve shows the asymptotic behavior $\eta(\omega) \sim \omega^{1/3}$.

IV. VISCOSITY SPECTRAL FUNCTION

We consider the retarded correlation function of the shear component of the stress tensor

$$(G_R)_{xy,xy}(\omega) = -i \int du \int d^2x e^{i\omega u} \Theta(u) \langle [\Pi_{xy}(u, \vec{x}), \Pi_{xy}(0, 0)] \rangle. \quad (18)$$

In the holographic model this correlation function can be computed by studying fluctuations of the bulk metric of the form $\delta g_x^y = e^{-i\omega u + i\nu v} \chi(\omega, r)$. One can show that n controls the particle number carried by the operator that is being probed. The stress tensor carries zero particle number and $n = 0$. Small fluctuations are governed by the wave equation

$$\frac{f(r)}{r^5} \frac{d}{dr} \left(r^5 f(r) \frac{d\chi}{dr} \right) + V_{\text{eff}}(r) \chi(\omega, r), \quad (19)$$

where $V_{\text{eff}} = f(r)^2 g^{uu}/g^{rr} = (1 - f(r))\beta^2 \omega^2 / r^4$. Using the explicit form of $f(r)$ and defining $\mathfrak{w} = \omega/(2\pi T)$ we can write

$$\chi''(\omega, u_R) - \frac{1 + u_R^2}{f(u_R)u_R} \chi'(\omega, u_R) + \frac{u_R}{f(u_R)^2} \mathfrak{w}^2 \chi(\omega, u_R) = 0, \quad (20)$$

where $u_R = (r_+/r)^2$ and $f(u_R) = 1 - u_R^2$. Near the horizon, $u_R = 1$, this equation is identical to the scalar wave equation in the AdS₅ Schwarzschild background. In particular, the near-horizon solution is of the form $\chi \sim (1 - u_R)^\alpha$ with $\alpha = \pm i\mathfrak{w}/2$. The retarded correlation function is related to the solution that satisfies infalling boundary conditions, $\alpha = -i\mathfrak{w}/2$, near the horizon. We have solved equ. (19) numerically by starting from the

analytical solution near $u_R = 1$,

$$\chi(\omega, u_R) = (1 - u_R)^{-i\nu/2} \left[1 - \frac{\mathfrak{w}}{4(i + \mathfrak{w})} (1 - u_R) + \dots \right], \quad (21)$$

and integrating towards the boundary at $u_R = 0$. The retarded correlation function is given by

$$G_R(\omega) = \frac{\beta r_+^3 \Delta v}{4\pi G_5} \frac{f(u_R) \chi'(\omega, u_R)}{u_R \chi(\omega, u_R)} \Big|_{u_R \rightarrow 0}. \quad (22)$$

where $G_R \equiv (G_R)_{xy,xy}$ and the viscosity spectral function is $\eta(\omega) = \frac{1}{\omega} \text{Im} G_R(\omega)$. The spectral function is shown in Fig. 1. We observe that $\eta(0)/s = 1/(4\pi)$ [8, 9], and that at large frequency the spectral function grows as $\omega^{1/3}$.

A. Low frequency behavior

The behavior of the spectral function at large and small ω can be understood analytically. At low frequency we use the ansatz

$$\chi(\mathfrak{w}, u_R) = (1 - u_R)^{-i\nu/2} [1 + i\mathfrak{w}F(u_R) + \mathfrak{w}^2 G(u_R) + \dots]. \quad (23)$$

Putting this ansatz into the wave equation determines the functions F and G . We find

$$F(u_R) = -\frac{1}{2} \log\left(\frac{1 + u_R}{2}\right), \quad (24)$$

$$G(u_R) = \frac{1}{8} \left[\log\left(\frac{1 + u_R}{2}\right) \right]^2 - \frac{1}{2} \text{Li}_2\left(\frac{1 - u_R}{2}\right) \quad (25)$$

Inserting equ. (23) into the boundary action gives the retarded correlation function

$$G_R(\mathfrak{w}) = -\frac{sT}{2} [i\mathfrak{w} + \mathfrak{w}^2 \log(2) + \dots]. \quad (26)$$

Matching this result to the Kubo relation (5) leads to

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad \tau_\pi = -\frac{\log(2)}{2\pi T}. \quad (27)$$

The sign of the viscous relaxation time is unusual. In $\mathcal{N} = 4$ SUSY Yang Mills theory [37], as well as in weak coupling calculations for dilute gases and the quark gluon plasma the relaxation time is always positive [21, 47]. Equations (6) and (9) show that τ_π determines certain higher order corrections to the dispersion relation of shear and sound modes, and that in the fluid dynamic regime $\omega\tau_\pi \ll 1$ there is no constraint on the sign of τ_π . On the

other hand, if we try to match the correlation function to an Israel-Stewart like scheme, then we find an unstable mode near the ultraviolet cutoff. We note that the $O(\omega^2)$ term in equ. (26) has the same sign as the corresponding term in the $\mathcal{N} = 4$ theory. The difference in sign arises from the Kubo relation, which has an extra term $G_R \sim -\frac{\kappa_R}{2}\omega^2$ in the relativistic case. This term is absent in the non-relativistic theory in both 2+1 and 3+1 dimensions, and it is also absent in a 2+1 dimensional relativistic theory [37].

B. High frequency behavior

The high frequency limit can be studied using a WKB approximation. We define $\psi(\mathfrak{w}, u_R) = \sqrt{(1 - u_R^2)/u_R} \chi(\mathfrak{w}, u_R)$. Then $\psi(\mathfrak{w}, u_R)$ satisfies a Schrödinger-like equation

$$\psi''(\mathfrak{w}, u_R) + \frac{1}{4u_R^2(1 - u_R^2)^2} (-3 + 6u_R^2 + 4\mathfrak{w}^2u_R^3 + u_R^4) \psi(\mathfrak{w}, u_R) = 0. \quad (28)$$

For $\mathfrak{w} \gg 1$ the function $\psi(\mathfrak{w}, u_R)$ is rapidly oscillating in the bulk. We can write

$$\psi(\mathfrak{w}, u_R) \simeq \frac{1}{\sqrt{p(u_R)}} e^{\pm i(S_0(u_R) + \varphi)}, \quad S_0(u_R) = \int^{u_R} p(u'_R) du'_R. \quad (29)$$

with

$$p(u_R) = \frac{\mathfrak{w}\sqrt{u_R}}{1 - u_R^2}, \quad S_0(u_R) = \mathfrak{w} [-\arctan(\sqrt{u_R}) + \operatorname{arctanh}(\sqrt{u_R})]. \quad (30)$$

The two wave functions in equ. (29) are linearly independent. The correct solution is determined by matching to the near horizon solution. Near $u_R = 1$ we have

$$p(u_R) = \frac{\mathfrak{w}}{2(1 - u_R)}, \quad S_0(u_R) = -\frac{\mathfrak{w}}{2} \log(1 - u_R). \quad (31)$$

Comparing the WKB result to the infalling solution $\psi(u) \sim (1 - u_R)^{(1 - i\mathfrak{w})/2}$ picks out the “+” sign in equ. (29). Once the sign is fixed the WKB solution determines a unique boundary wave function. The general solution near $u_R = 0$ is

$$\psi(\mathfrak{w}, u_R) \simeq \frac{1}{\sqrt{u_R}} [c_1 \operatorname{Ai}'((-1)^{1/3} \mathfrak{w}^{2/3} u_R) + c_2 \operatorname{Bi}'((-1)^{1/3} \mathfrak{w}^{2/3} u_R)], \quad (32)$$

where Ai' and Bi' are derivatives of the Airy function. Using the asymptotic behavior of the Airy functions,

$$\operatorname{Ai}'(z) \simeq -\frac{z^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{3/2}}, \quad \operatorname{Bi}'(z) \simeq \frac{z^{1/4}}{\sqrt{\pi}} e^{\frac{2}{3}z^{3/2}}, \quad (33)$$

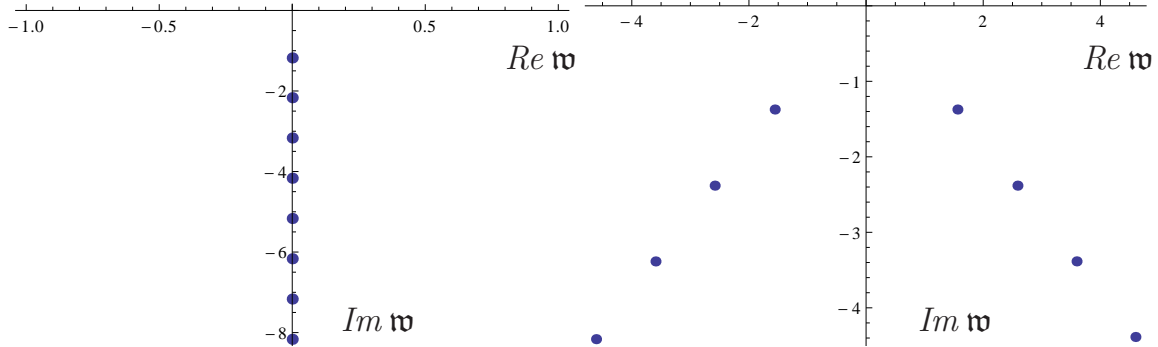


FIG. 2: Complex frequency of quasi-normal modes in the shear channel for zero spatial momentum. The left panel shows results for the light-like compactification studied in this work, and the right panel shows quasi-normal modes of the AdS_5 Schwarzschild black hole [38].

we see that the infalling wave functions matches to the solution proportional to Bi' . The corresponding mode function χ , normalized according to $\chi(\mathfrak{w}, 0) = 1$, is

$$\chi(\mathfrak{w}, u_R) \simeq \frac{\Gamma(1/3)}{3^{1/6} \sqrt{1 - u_R^2}} Bi'((-1)^{1/3} \mathfrak{w}^{2/3} u_R). \quad (34)$$

Inserting this solution into the boundary action gives

$$\frac{\eta(\omega)}{s} \simeq \frac{1}{4\pi} \frac{3^{1/6} \Gamma(1/3)}{2\Gamma(2/3)} \mathfrak{w}^{1/3}. \quad (35)$$

This result is shown as the dashed line in Fig. 1. We observe that the asymptotic behavior sets is rapidly for $\mathfrak{w} \gtrsim 1$. In particular, there are no oscillations around the asymptotic form as is the case in the $\mathcal{N} = 4$ theory [35].

V. QUASI-NORMAL MODES

We have seen that the viscous relaxation time is negative, and that in the context of an Israel-Stewart scheme this results implies the existence of an unstable mode outside the hydrodynamic regime. In order to understand relaxation to the hydrodynamic limit, and to verify that the fluid is indeed stable, we have computed the spectrum of quasi-normal modes in the shear channel. We note that near $u_R = 0$ the solution of the wave equation is of the form

$$\chi(\mathfrak{w}, u_R) \simeq \mathcal{A}(\mathfrak{w})[1 + \dots] + \mathcal{B}(\mathfrak{w})[u_R^2 + \dots], \quad (36)$$

and $G_R(\mathfrak{w}) \sim \mathcal{B}(\mathfrak{w})/\mathcal{A}(\mathfrak{w})$. Finding poles of $G_R(\mathfrak{w})$ in the complex \mathfrak{w} plane corresponds to solutions $\psi(\mathfrak{w}, u_R)$ that satisfy a Dirichlet problem at $u_R = 0$ and infalling boundary conditions at $u_R = 1$. We follow [38] and solve the Dirichlet problem by transforming equ. (20) to the standard form of the Heun differential equation. We define

$$\chi(\mathfrak{w}, 1 - z) = z^{-i\mathfrak{w}/2}(z - 2)^{-\mathfrak{w}/2}y(z). \quad (37)$$

Then $y(z)$ satisfies

$$y''(z) + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-2} \right] y'(z) + \frac{\alpha\beta z - Q}{z(z-1)(z-2)} y(z) = 0, \quad (38)$$

with $\alpha + \beta = \delta + \gamma + \epsilon - 1$ and

$$\alpha = \beta = -\frac{\mathfrak{w}}{2}(1 + i), \quad \gamma = 1 - i\mathfrak{w}, \quad \delta = -1, \quad \epsilon = 1 - \mathfrak{w}, \quad (39)$$

$$Q = -\frac{\mathfrak{w}}{2}(1 - i) + \frac{i}{2}\mathfrak{w}^2. \quad (40)$$

We seek the solution $y(z)$ as a power series expansion

$$y(z) = \sum_n a_n z^n, \quad (41)$$

where the coefficients a_n satisfy a two-term recursion relation

$$a_{n+2} + A_n(\mathfrak{w})a_{n+1} + B_n(\mathfrak{w})a_n = 0, \quad (42)$$

The coefficients A_n and B_n are given by [38]

$$A_n = -\frac{(n+1)(2\delta + \epsilon + 3(n+\gamma)) + Q}{2(n+2)(n+1+\gamma)}, \quad (43)$$

$$B_n = \frac{(n+\alpha)(n+\beta)}{2(n+2)(n+1+\gamma)}. \quad (44)$$

The near horizon behavior implies that $a_0 = 1$ and $a_1 = Q/(2\gamma)$. From the asymptotic behavior of A_n and B_n we see that the series expansion converges at least for $|z| < 1$. We have numerically searched for solutions of the Dirichlet problem $y(1) = 0$. The results are shown in the left panel of Fig. 2. We observe that the quasi-normal are located along the negative imaginary axis. This implies, in particular, that there are no unstable modes. The scale of the first non-hydrodynamic mode is set by the temperature, $\omega_{QNM}^1 \simeq -1.18(2\pi iT)$. For comparison we also show the quasi-normal modes of the AdS_5 black hole computed with the same method (right panel). The AdS_5 black hole quasi-normal modes are governed by the same recursion relation, but with a different governing parameter, $Q = -\frac{\mathfrak{w}}{2}(1 - i) + \frac{i}{2}\mathfrak{w}^2(1 + 2i)$. The difference in Q results in a doubling of the quasi-normal mode spectrum, in agreement with the results obtained in [38].

VI. OUTLOOK

We can compare our results to predictions from kinetic theory and quantum many-body theory. The field theory dual of the gravitational action given in equ. (13) has not been studied, but there are a number of general results that do not depend on details of the underlying field theory. If the fluid has an effective description in terms of quasi-particles then the viscosity spectral function can be studied using kinetic theory. In kinetic theory we find [20, 30]

$$\eta(\omega) = \frac{\eta(0)}{1 + \omega^2 \tau_R^2} \quad (45)$$

where τ_R^{-1} is the lowest eigenvalue of the linearized collision operator. In kinetic theory we also find $\tau_\pi = \tau_R$ [21, 42], which means that the kinetic relaxation time is equal to the relaxation time for viscous stresses. The viscosity $\eta(0)$ of the dilute Fermi gas was computed in [30, 31], and it was shown that kinetic theory extrapolated to the BKT transition is consistent with a shear viscosity to entropy density ratio as small as $1/(4\pi)$. Hydrodynamic fluctuations lead to a logarithmic divergence of the shear viscosity in 2+1 dimensional fluids, $\eta(\omega) \sim \log(T/\omega)$ [26]. This divergence is suppressed in systems with a large number N of internal degrees of freedom.

Kinetic theory is applicable for $\omega < T$. The high frequency tail of the spectral function is determined by the operator product expansion (OPE) [24, 25]. We can write

$$\eta(\omega) = \sum_k \langle \mathcal{O}_k \rangle \frac{1}{\omega^{(\Delta_k - d)/2}}, \quad (46)$$

where d is the number of spatial dimensions, Δ_k is the dimension of the operator \mathcal{O}_k , and we have set the mass $m = 1$. In the dilute Fermi gas the leading operator is $\mathcal{O}_c = \phi^\dagger \phi$ where $\phi = C_0 \epsilon_{\alpha\beta} \psi^\alpha \psi^\beta$ is the difermion operator and C_0 is the four-fermion coupling. The operator \mathcal{O}_c is known as the contact density [48]. It has dimension $\Delta_c = 4$ in both $d = 3$ and $d = 2$ dimensions. This implies that the high frequency tail of the spectral function is $\eta(\omega) \sim 1/\sqrt{\omega}$ in $d = 3$, and $\eta(\omega) \sim 1/\omega$ in $d = 2$. Our result $\eta(\omega) \sim \omega^{1/3}$ corresponds to an operator of dimension $\Delta = 4/3$ in $d = 2$. This result is consistent with the bound $\Delta \geq d/2$ on the scaling dimension of local operators derived in [49]. The bound implies, however, that the leading operator in the OPE cannot be of the form $\mathcal{O}^\dagger \mathcal{O}$, where \mathcal{O} is an operator that carries fermion number. In particular, it cannot be the contact density.

There are a number of questions that remain to be addressed. The first is to understand

the asymptotic behavior of the spectral function. What is the operator in the dual field theory that governs the power law? Within a larger class of holographic models, how is the asymptotic behavior of the spectral function encoded in the geometry? The second set of questions has to do with the unusual sign of the viscous relaxation time. How does this sign manifest itself in the approach to equilibrium? This can be studied, for example, using the fluid-gravity correspondence [50]. Finally, what is the significance of the quasi-normal mode spectrum? In particular, how does the difference between the spectrum for $AdS_5 \times S_5$ black hole and the Galilean model manifest itself in the relaxation towards equilibrium? Ultimately, we are also interested in a broader class of models that realize Galilean invariance without using lightlike compactifications. Proposals in this direction can be found in [51–55]

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