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Enhanced Ultraviolet Cancellations in $\mathcal{N} = 5$ Supergravity at Four Loops

Zvi Bern^a, Scott Davies^a and Tristan Dennen^b

*^aDepartment of Physics and Astronomy
University of California at Los Angeles
Los Angeles, CA 90095-1547, USA*

*^bNiels Bohr International Academy and Discovery Center
The Niels Bohr Institute
Blegdamsvej 17,
DK-2100 Copenhagen Ø, Denmark*

Abstract

We show that the four-loop four-point amplitudes of $\mathcal{N} = 5$ supergravity are ultraviolet finite in four dimensions, contrary to expectations based on supersymmetry and duality-symmetry arguments. We explain why the diagrams of any covariant local formalism cannot manifestly exhibit the necessary cancellations for finiteness but instead require a new type of ultraviolet cancellation that we call an “enhanced cancellation”. We also show that the three-loop four-point amplitudes in $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravity theories display enhanced cancellations. To construct the loop integrand, we use the duality between color and kinematics. We apply standard methods for extracting ultraviolet divergences in conjunction with the FIRE5 integral reduction program to arrive at the four-loop results.

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I. INTRODUCTION

Quantum field theories of gravity are nonrenormalizable by power counting Feynman diagrams. This leads to the widely held belief that all unitary gravity field theories must be ultraviolet divergent at some loop order. Indeed, no known symmetry is powerful enough to render such theories ultraviolet finite. On the other hand, recent years have made it abundantly clear that scattering amplitudes contain hidden symmetries and new structures beyond those expected from Lagrangians. While these are not yet fully understood, they can have profound consequences on ultraviolet properties.

In this paper, we identify a new class of multiloop ultraviolet cancellations that go beyond the ones established by standard-symmetry arguments. We call these *enhanced ultraviolet cancellations*. These are defined as cancellations that cannot be displayed term-by-term in any local covariant diagrammatic formalism. By such a formalism we mean that the poles in the diagram integrands are simply the standard Feynman propagator ones. Using the maximal cut conditions, as defined in Ref. [1], we can identify terms unique to a given diagram which we can then power count. To illustrate enhanced cancellations, we use previous three- and four-loop calculations in $\mathcal{N} = 4$ supergravity [2, 3], as well as new calculations in $\mathcal{N} = 5$ supergravity performed here.

The study of the ultraviolet properties of gravity theories has a rich history, starting with the seminal work of 't Hooft and Veltman [4]. They showed that pure Einstein gravity is finite at one loop, but divergent with the addition of matter, a point on which other papers elaborated as well [5]. Goroff and Sagnotti later showed that at two loops, pure Einstein gravity diverges [6]. With the addition of supersymmetry, the ultraviolet behavior tends to improve: Pure ungauged supergravities are known to have no divergences prior to three loops [7]. However, the consensus reached from studies in the 1980s was that all supergravity theories would likely diverge at the third loop order (see, for example, Ref. [8]), though one can raise the loop order with additional assumptions [9].

The complexity of gravity theories makes it difficult to explicitly check these expectations. This situation was ameliorated by the advent of the unitarity method [10, 11], which makes it possible to directly determine ultraviolet properties of gravity theories at high loop orders. More recently, a new constraint on gauge-theory and gravity amplitudes has been introduced—the duality between color and kinematics found by Carrasco, Johansson and one of the authors (BCJ) [12, 13]—allowing additional new nontrivial computations to be carried out.

For maximally supersymmetric supergravity (in $D = 4$ this is $\mathcal{N} = 8$ supergravity) [14], explicit calculations show that four-point amplitudes are finite at three loops for dimensions $D < 6$ [15, 16] and at four loops for dimensions $D < 11/2$ [17]. In $D = 4$, these ultraviolet cancellations were subsequently understood to follow from supersymmetry and the $E_{7(7)}$ duality symmetry of $\mathcal{N} = 8$ supergravity [18, 19]. A purely supersymmetric explanation has also been developed by Björnsson and Green [20] using a field-theory version of the Berkovits pure spinor formalism [21]. The current consensus based on symmetry considerations is that a $D^8 R^4$ counterterm is valid under all standard symmetries, leading to the expectations of a seven-loop divergence in $D = 4$ and a five-loop divergence in $D = 24/5$.

While technical difficulties have prevented the $\mathcal{N} = 8$ supergravity expectations from being confronted by calculation, there is now evidence that implies even better behavior in this case than that suggested by standard-symmetry arguments: Similar argumentation in half-maximal supergravity leads to predictions of valid counterterms in cases where no

divergences exist. In particular, at three loops, half-maximal $\mathcal{N} = 4$ supergravity [22] is ultraviolet finite in four dimensions [15], while similar supersymmetry and duality-symmetry considerations suggest that it should diverge [19]. (See Ref. [23] for string-theory arguments for finiteness.) In addition, half-maximal pure supergravity in $D = 5$ is ultraviolet finite at two loops, again contrary to symmetry considerations [24].

An important question is whether it is possible that the observed three-loop finiteness of $\mathcal{N} = 4$ supergravity in $D = 4$ can be explained using only arguments based on supersymmetry and duality symmetry. An attempt to find such an explanation relied on the assumption of the existence of an appropriate non-Lorentz covariant off-shell 16-supercharge superspace [25, 26]. Not surprisingly, the assumption carries other consequences as well: In particular, it predicts additional finiteness conditions when matter multiplets are added [26] that directly contradict subsequent explicit calculations [27]. Three-loop finiteness of pure $\mathcal{N} = 4$ supergravity therefore remains unexplained by standard-symmetry arguments. Nevertheless, it remains a key problem to understand the extent to which such arguments can restrict divergences.

To carry out further probes of the ultraviolet properties of supergravity theories, together with Smirnov and Smirnov, we recently computed the four-loop four-point divergence of $\mathcal{N} = 4$ supergravity in $D = 4$ [3], finding that the theory does diverge at four loops. Naively, this might suggest that all supergravity theories should diverge at some sufficiently high loop order. However, when one looks at the details of the divergence, a rather different picture emerges: The divergence appears to be tied to the duality-symmetry anomaly of $\mathcal{N} = 4$ supergravity found by Marcus [28]. The consequences of the anomaly on the amplitudes of $\mathcal{N} = 4$ supergravity have been described in some detail in Ref. [29]. The role of the anomaly implies that divergences of this type should not exist in $\mathcal{N} \geq 5$ supergravity, since these theories have no such analogous anomalies.

In this paper, we identify a subset of terms in $\mathcal{N} = 8$ supergravity that are ultraviolet divergent in four dimensions at seven loops, reproducing the analysis of Björnsson and Green [20] from a different perspective. To identify irreducible terms with poor power counting, we use maximal cuts. The expectation of a seven-loop divergence is also consistent with other standard-symmetry arguments [16–19]. A key question is whether there are nontrivial enhanced cancellations between the divergent terms that then make the amplitude as a whole ultraviolet finite. Unfortunately, the high required loop order makes it unfeasible at present to test for the existence of enhanced cancellations in $\mathcal{N} = 8$ supergravity. Instead, here we demonstrate the presence of enhanced cancellations in $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravities, since they are easier to work with, because the required loop order is lower. As we demonstrate in this paper, enhanced cancellations are responsible for making the four-point $\mathcal{N} = 4$ supergravity amplitudes finite at three loops and $\mathcal{N} = 5$ supergravity amplitudes finite at four loops. We also demonstrate three-loop cancellations in four-point $\mathcal{N} = 5$ supergravity amplitudes beyond those needed for finiteness. Such cancellations are reminiscent of the types of nontrivial cancellations noticed in certain unitarity cuts [30]. The surprising aspect is that no covariant local diagrammatic representation can make these results manifest.

What might be behind enhanced ultraviolet cancellations? In a previous paper with Huang [24], using the duality between color and kinematics, we explicitly tied the enhanced cancellations at two loops in half-maximal supergravity in $D = 5$ to corresponding cancellations in pure Yang-Mills theory that prevent forbidden color factors from appearing in divergences. A key feature is that the ultraviolet cancellations occur between the planar and nonplanar sectors of the theory. This case is particularly simple to analyze in detail be-

cause the supergravity amplitudes are simple linear combinations of Yang-Mills amplitudes even after integration. Unfortunately, the situation beyond two loops is much more complex because different sets of integrals appear in the supergravity case than in the corresponding gauge-theory case.

To carry out our computations, we use the same techniques as those used for three and four loops [2, 3, 27] in $\mathcal{N} = 4$ supergravity. Our computations make use of the many advances in constructing integrands, including the unitarity method [10, 11, 31] and the duality between color and kinematics [12, 13]. While nonplanar integrands cannot be uniquely defined, they can still be integrated to obtain unique results. Once the integrands are constructed, a mass is introduced as an infrared regulator. We then series expand in small external momenta (or equivalently large loop momenta) to focus on ultraviolet singularities [32, 33]. At four loops the resulting vacuum integrals are nontrivial. To deal with them we use FIRE5 [34], which implements the Laporta algorithm [35], to reduce the integrals to a basis set.¹ The basis integrals are known since they are identical to those used in the evaluation of the four-loop QCD β function [36, 37].

This paper is organized as follows. In Sect. II, we summarize the methods used to carry out the calculations. In Sect. III, we review the results of standard-symmetry power counting and show that power counting maximal cuts gives identical results. Then in Sect. IV, we exhibit the enhanced cancellations responsible for ultraviolet finiteness of $\mathcal{N} = 4$ supergravity at three loops [2]. We also present a new three-loop calculation in $\mathcal{N} = 5$ supergravity, pointing out that it too exhibits enhanced cancellations beyond those needed for ultraviolet finiteness. In Sect. V, we demonstrate that the four-point amplitudes of $\mathcal{N} = 5$ supergravity are all ultraviolet finite and again display enhanced cancellations. We present our conclusions in Sect. VI.

II. METHODS

A. Duality between color and kinematics

The duality between color and kinematics and the associated gravity double-copy property [12, 13] make it simple to construct supergravity amplitudes once corresponding gauge-theory amplitudes are arranged into a form that makes the duality manifest. (For a review of this duality and its applications, see Ref. [38].) At loop level the duality remains a conjecture, but we rely only on explicitly constructed forms of $\mathcal{N} = 4$ super-Yang-Mills amplitudes where the duality is manifest [13, 39].

The duality between color and kinematics is usually formulated via graphs with only cubic vertices. Any L -loop m -point gauge-theory amplitude with all particles in the color-adjoint representation can be written in terms of such graphs as

$$\mathcal{A}_m^{L\text{-loop}} = i^L g^{m-2+2L} \sum_{\mathcal{S}_m} \sum_j \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}. \quad (2.1)$$

The sum labeled by j runs over the set of distinct non-isomorphic graphs, while the sum over \mathcal{S}_m is over all $m!$ permutations of external legs. The symmetry factor S_j removes over-counts

¹ We are grateful to Alexander and Volodya Smirnov for carrying out this step for us.

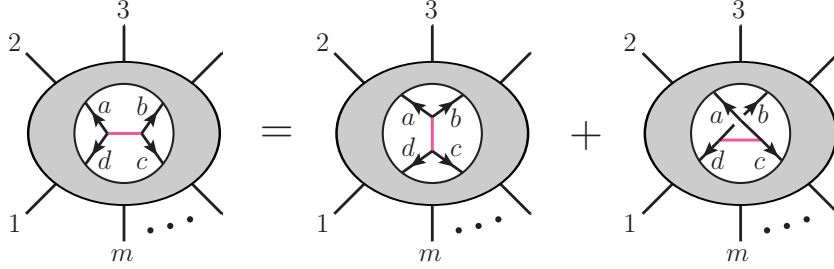


FIG. 1: The basic loop-level Jacobi relation for either color or numerator factors given in Eq. (2.2). The basic identity can be embedded in a diagram at any loop order.

arising from automorphisms of the diagrams. The product in the denominator runs over all Feynman propagators of graph j , and the integrals are over L independent D -dimensional loop momenta. The color factor c_j of graph j is given by dressing every three-vertex with a group-theory structure constant, $\tilde{f}^{abc} = i\sqrt{2}f^{abc}$, while n_j is the kinematic numerator of graph j depending on momenta, polarizations and spinors. So far this representation involves nothing more than absorbing contact-term contributions into graphs with only cubic vertices by multiplying and dividing by appropriate propagators.

The nontrivial part is the all-loop-order conjecture that there exists a form of gauge-theory amplitudes where kinematic numerators satisfy the same algebraic relations as color factors. These are known as BCJ representations of amplitudes. For the theories we discuss in this paper, this amounts to imposing the same Jacobi identities on the kinematic numerators as those satisfied by adjoint-representation color factors:

$$c_i = c_j - c_k \Rightarrow n_i = n_j - n_k, \quad (2.2)$$

where the indices i, j, k denote the diagram to which the color factors and numerators belong. The basic Jacobi identity is illustrated in Fig. 1 embedded in an arbitrary diagram. The numerator factors are also required to have the same antisymmetry properties as color factors. In general, the duality relations (2.2) work only after appropriate nontrivial rearrangements of the amplitudes.

Remarkably, we can obtain corresponding gravity loop integrands simply by replacing color factors in a gauge-theory amplitude by kinematic numerators of a second gauge-theory amplitude where the duality is manifest [12, 13]:

$$c_i \rightarrow \tilde{n}_i. \quad (2.3)$$

Putting in the appropriate gravitational coupling gives us the double-copy form of gravity amplitudes,

$$\mathcal{M}_m^{L\text{-loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{S_m} \sum_j \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}. \quad (2.4)$$

Only one of the two sets of numerators n_j or \tilde{n}_j needs to satisfy the duality relation (2.2) [13, 40]. We note that at tree level ($L = 0$), the duality encodes the Kawai-Lewellen-Tye [41] relations between gauge-theory and gravity amplitudes, as well as nontrivial relations between color-ordered gauge-theory partial amplitudes [12].

B. Construction of $\mathcal{N} = 5$ supergravity amplitudes

In this paper we construct the three- and four-loop four-point $\mathcal{N} = 5$ supergravity integrands using the procedure presented above. We do so by starting with an $\mathcal{N} = 1$ super-Yang-Mills integrand and then replacing the color factors with the BCJ forms of kinematic numerators of $\mathcal{N} = 4$ super-Yang-Mills theory given in Refs. [13, 39]. Similar constructions of less-than-maximal supergravity amplitudes are found in Refs. [2, 3, 24, 27, 42]. We express

$$(\mathcal{N} = 5 \text{ sugra}) : (\mathcal{N} = 4 \text{ sYM}) \otimes (\mathcal{N} = 1 \text{ sYM}), \quad (2.5)$$

where “sugra” and “sYM” are shorthands for, respectively, supergravity and super-Yang-Mills theory. We further decompose the $\mathcal{N} = 5$ amplitudes into a direct sum,

$$\begin{aligned} (\mathcal{N} = 5 \text{ sugra}) : & (\mathcal{N} = 4 \text{ sYM}) \otimes (\mathcal{N} = 0 \text{ sYM}) \\ & \oplus (\mathcal{N} = 4 \text{ sYM}) \otimes \left((\mathcal{N} = 1 \text{ sYM}) \ominus (\mathcal{N} = 0 \text{ sYM}) \right), \end{aligned} \quad (2.6)$$

where “ $\mathcal{N} = 0$ sYM” refers to ordinary pure nonsupersymmetric Yang-Mills theory. The first term in the direct sum is the pure $\mathcal{N} = 4$ supergravity amplitude, while the second term is the difference between the $\mathcal{N} = 5$ and $\mathcal{N} = 4$ supergravity amplitudes. The second term comes from taking the diagrams of pure $\mathcal{N} = 1$ super-Yang-Mills, subtracting out the pure-gluon part, and then replacing the color factors with the BCJ numerators of $\mathcal{N} = 4$ super-Yang-Mills theory. On the $\mathcal{N} = 1$ super-Yang-Mills side, this amounts to separating the pure gluon contributions from the contributions including also gluinos.

Following Refs. [2, 3, 27], we use ordinary Feynman diagrams for the $\mathcal{N} = 0$ and $\mathcal{N} = 1$ super-Yang-Mills integrands. While this might seem to be a poor starting point given the complexity of such diagrams, the BCJ construction (2.4) ensures that only a small fraction of diagrams actually contribute. Whenever an $\mathcal{N} = 4$ super-Yang-Mills diagram vanishes, we do not need to evaluate corresponding diagrams in $\mathcal{N} = 0$ or $\mathcal{N} = 1$ super-Yang-Mills theory. At one, two, three and four loops, the BCJ-satisfying representations of the four-point amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory have only, respectively, 1, 2, 12 and 85 nonvanishing diagrams (up to permutations of external legs). This is already a remarkable simplification, allowing the calculation to proceed.

The decomposition in Eq. (2.6) results in an integrand where the $\mathcal{N} = 4$ supersymmetry cancellations are manifest, but the $\mathcal{N} = 1$ ones are not. While we do not do so here, one could simplify the $\mathcal{N} = 5$ supergravity integrand to make cancellations from all supersymmetries manifest. This could be accomplished by using the unitarity method to systematically move terms between diagrams, subject to maintaining the unitarity cuts. However, as we shall see below, no covariant local representation exists either in $\mathcal{N} = 4$ or $\mathcal{N} = 5$ supergravity that makes manifest the complete set of ultraviolet cancellations that we find. Because of this, there is no obvious way to avoid direct integration to see the cancellations. We consequently call such cancellations enhanced.

C. Extraction of ultraviolet divergences

Once we have an integrand, the next step is to extract the ultraviolet divergences. The procedure that we use has been described in some detail in Ref. [27], so here we only briefly summarize it. To deal with potential ultraviolet divergences we use dimensional

Theory	Counterterm	Loop Order
$D = 4, Q = 32, \mathcal{N} = 8$	$\mathcal{D}^8 R^4$	7
$D = 4, Q = 16, \mathcal{N} = 4$	R^4	3
$D = 4, Q = 20, \mathcal{N} = 5$	$\mathcal{D}^2 R^4$	4
$D = 24/5, Q = 32$	$\mathcal{D}^8 R^4$	5
$D = 5, Q = 16$	R^4	2

TABLE I: Selected valid counterterms based on supersymmetry and duality-symmetry considerations [18–20, 24, 26, 27]. Q is the number of supercharges.

reduction [43]. Rather than evaluate integrals with their full momentum dependence, it is much simpler to series expand the integrands prior to integration in order to pick up only the desired ultraviolet divergences [32]. This procedure introduces new unphysical infrared singularities beyond the standard ones, so one needs an infrared cutoff to separate the ultraviolet divergences from the infrared ones.

An especially good choice for regulating infrared singularities is to introduce a uniform mass into all Feynman propagators prior to expanding in external momenta [44, 45]. For the cases we study in this paper, where there are no lower-loop divergences, the subdivergences should all cancel amongst themselves with the use of this regulator. The uniform mass regulator therefore greatly simplifies the computation since we do not need to compute subdivergences. We have, however, performed extensive checks confirming that they cancel as expected. We note that if the mass regulator were introduced later in the calculation, for example after the expansion in external momenta and tensor integral simplifications, it would ruin the cancellations of subdivergences between different integrals. One would then need to include all subdivergence subtractions to properly remove the regulator dependence, greatly complicating the calculation.

The procedure results in a large number of vacuum integrals. At three loops, evaluating the integrals is straightforward [2, 45], but at four loops it is a more serious challenge. To deal with this, we use the FIRE5 program [34], which is a highly-efficient implementation of integration-by-parts relations [33] using the Laporta algorithm [46]. It allows us to write down any given integral as a linear combination of a small number of so-called master integrals. In our four-loop calculation, the reduction to master integrals is especially nontrivial due to the high powers of numerator loop momenta that occur in gravity. Earlier related calculations already determined the four-loop vacuum master integrals [35–37]; we use the master-integral basis and values given in Ref. [37].

III. POWER COUNTING

A. Review of standard-symmetry power counting

The restrictions supersymmetry and duality symmetry impose on counterterms have been studied in great detail over the years. The most recent power-counting predictions based on symmetry considerations are collected in Table I. In $D = 4$, apparently valid counterterms exist at loop orders $L = 7$ in $\mathcal{N} = 8$ supergravity [18–20], $L = 3$ in $\mathcal{N} = 4$ supergravity [19], and $L = 4$ in $\mathcal{N} = 5$ supergravity [19]. By increasing the space-time dimensions, one can also

lower the loop order at which a potential counterterm can correspond to a divergence. For example, in $D = 24/5$, maximal 32-supercharge supergravity has a valid five-loop counterterm [20]. Similarly, half-maximal 16-supercharge supergravity in $D = 5$ has an apparently valid two-loop counterterm [24, 26, 27]. As explained in Ref. [19], the counterterms listed in Table I cannot be written as full-superspace integrals, but they do appear to be valid under all known standard-symmetry considerations. See also Refs. [25–27] for an attempt to put tighter restrictions on the counterterms and the associated difficulties with doing so.

Björnsson and Green [20] constructed a first-quantized pure spinor formalism useful for power counting. Their formalism exposes all supersymmetry cancellations and gives an identical power count as other recent methods, including those that account for duality symmetry [18, 19]. Their results imply that unless there are some extra nonstandard cancellations beyond those implied by supersymmetry, $\mathcal{N} = 8$ supergravity will diverge at five loops in $D = 24/5$ and at seven loops in $D = 4$, corresponding to the first and fourth rows of Table I. We know that through four loops in $\mathcal{N} = 8$ supergravity, such symmetry-based predictions match the explicitly computed critical dimensions where a divergence first appears [11, 13, 16, 39, 47]. A key question is whether this pattern continues or whether there are cancellations beyond the well-understood ones.

While it is not currently feasible to answer this for $\mathcal{N} = 8$ supergravity, we can answer it for $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravity. From previous work [2], we already know that the three-loop R^4 counterterm of $\mathcal{N} = 4$ supergravity in $D = 4$ listed on the second row of Table I does *not* result in a three-loop divergence. Similarly, half-maximal 16-supercharge supergravity at two loops in $D = 5$ is free of divergences [24]. As we discuss below in some detail for the three-loop $\mathcal{N} = 4$ case, these cancellations are a nontrivial manifestation of enhanced cancellations. However, one may worry that these two cases are special and not representative of a general pattern. In particular, $\mathcal{N} = 4$ supergravity in $D = 4$ has a $U(1)$ anomaly [28] that would not occur in theories with higher supersymmetry. The two-loop $D = 5$ case also has some special features: At two loops the BCJ kinematic numerators of maximally supersymmetric Yang-Mills four-point amplitudes are independent of loop momenta, implying that half-maximal supergravity amplitudes are simple linear combinations of the corresponding pure Yang-Mills ones. To go beyond these special cases, here we study the case of $\mathcal{N} = 5$ supergravity in $D = 4$ to show that there is no divergence associated with the counterterm listed on the third line of Table I. This case is not entangled with any known anomaly. Furthermore, unlike the two-loop case, the kinematic numerators do depend on loop momenta, so the gravity integrals are different from the corresponding Yang-Mills ones.

B. Power counting maximal cuts

In order to describe the phenomenon of enhanced cancellations, we turn to power counting using maximal cuts. The terms selected by a maximal cut are a gauge-invariant set that are unique to a diagram. Using maximal cuts, we can incorporate all supersymmetric cancellations into supergravity power counts using the known power counts of super-Yang-Mills theories. Because all supersymmetric cancellations are accounted for, this gives us a power-counting method equivalent to the one of Björnsson and Green [20].

The maximal cut of a given diagram is obtained by replacing all propagators with on-shell conditions. While the cut conditions set various terms to zero, they do allow us to identify terms with poor behavior, in some cases worse behavior than that of the full amplitude. Once we have selected terms using the maximal cuts, we promote them back to Feynman integrals,

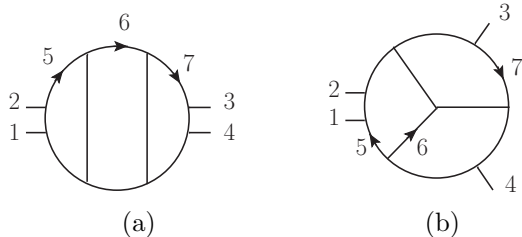


FIG. 2: Three-loop sample diagrams for maximal-cut power counting.

making sure that the obtained representation has the *minimum power count* consistent with the cut. If any term is then found whose ultraviolet behavior is worse than that of the amplitude as a whole, then by definition, we have enhanced cancellations. Of course, some care is required to be sure that we are using a form that has minimum power count but is also consistent with the cut. To be clear, we are defining enhanced cancellations entirely by their integrand power-counting properties and not by cancellations that appear only after integration. We do the power counting in D dimensions, viewing D as arbitrarily large, to not include hidden relations in the cut solutions that might lead to extra cancellations. In this way we maintain D -dimensional covariance.

We will show that the maximal cuts give power counts equivalent to the potential counterterms in Table I. This should not be too surprising given that the maximal cuts are a gauge-invariant subset built from objects that respect all standard symmetries of the amplitudes. As with other power counts, the maximal cuts do not make enhanced cancellations visible because they do not account for nontrivial cancellations between diagrams. Indeed, at a sufficiently high loop order, the amplitudes of *every* supergravity theory necessarily have divergences in individual terms selected by the maximal cuts. To see a better behavior in unitarity cuts, one needs to instead look at cuts that collect together many diagrams so as to allow cancellations between them. The cuts analyzed in Ref. [30] suggesting improved all-loop behavior of the amplitudes are examples of this.

As a warm-up, we first consider maximal cuts in $\mathcal{N} = 8$ supergravity. We consider the diagram in Fig. 2(a) as a simple first example. In $\mathcal{N} = 8$ supergravity, a kinematic numerator consistent with the maximal cuts is given by [15]

$$N_{\mathcal{N}=8 \text{ sugra}}^{3\text{-loop}} = s^5 t u M_4^{\text{tree}}, \quad (3.1)$$

where M_4^{tree} is the four-point gravity tree amplitude, and s, t and u are the standard four-point Mandelstam invariants. The maximal-cut conditions have no effect on this numerator since it is independent of loop momentum. Counting the three D -dimensional loop integrals, no powers of loop momentum in the numerator, and ten propagators gives us the power count,

$$\mathcal{D}_{\mathcal{N}=8 \text{ sugra}}^{3\text{-loop}} \sim \Lambda^{3D-20}, \quad (3.2)$$

where Λ is an ultraviolet cutoff. The critical dimension where an ultraviolet divergence first occurs is thus $D_c = 20/3$ for the maximal-cut terms of this diagram. However, the $\mathcal{N} = 8$ three-loop amplitude also contains worse-behaved terms. We consider instead the diagram in Fig. 2(b). In $\mathcal{N} = 8$ supergravity, a kinematic numerator consistent with the unitarity cuts of this diagram is given in Ref. [16]:

$$N_{\mathcal{N}=8 \text{ sugra}}^{3\text{-loop}} = s^3 t u M_4^{\text{tree}} (l_5 - k_4)^4, \quad (3.3)$$

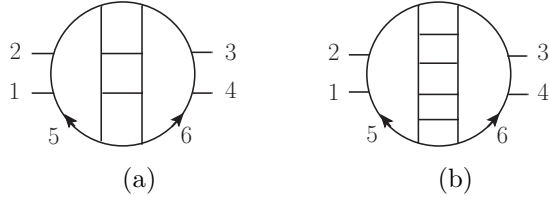


FIG. 3: Sample diagrams for power counting maximal cuts at five and seven loops.

where the momenta correspond to the labels in the diagram. Applying the maximal-cut conditions, we set $l_5^2 = 0$ and obtain the minimal power-counting form,

$$N_{\mathcal{N}=8 \text{ sugra}}^{3\text{-loop}} \Big|_{\text{max. cut}} = s^3 t u M_4^{\text{tree}} (2l_5 \cdot k_4)^2. \quad (3.4)$$

After promoting this back to the numerator of a full three-loop Feynman integral, we count the powers of loop momenta. Counting the three D -dimensional loop integrals, two powers of loop momentum in the numerator, and ten propagators gives us an overall power count for the diagram of

$$\mathcal{D}_{\mathcal{N}=8 \text{ sugra}}^{3\text{-loop}} \sim \Lambda^{3D+2-20}. \quad (3.5)$$

The critical dimension of this contribution is thus $D_c = 6$, which matches the critical dimension obtained from explicit divergence calculations [2, 16]. In this case then, there are no enhanced cancellations.

Next we consider the maximal cut of the five-loop diagram in Fig. 3(a).² The simplest numerator consistent with the diagram's maximal cut in $\mathcal{N} = 8$ supergravity is

$$N_{\mathcal{N}=8 \text{ sugra}}^{5\text{-loop}} \Big|_{\text{max. cut}} = s^5 t u M_4^{\text{tree}} (2l_5 \cdot l_6)^4, \quad (3.6)$$

where the momenta follow the labels of Fig. 3(a). The numerator follows from the rung rule [11, 31]—a rule devised to give the correct iterated two-particle cuts—after dropping terms that vanish with the on-shell conditions $l_i^2 = 0$. Promoting the maximal-cut terms back to numerators of a Feynman integral, we have five D -dimensional loop integrals and sixteen propagators. Together with the numerator (3.6), we then have a power count,

$$\mathcal{D}_{\mathcal{N}=8 \text{ sugra}}^{5\text{-loop}} \sim \Lambda^{5D+8-32}. \quad (3.7)$$

This gives a critical dimension $D_c = 24/5$, matching the Björnsson and Green analysis [20]. After stepping through the diagrams, this turns out to be the worst-behaved contribution. If it were to turn out that the critical dimension of the full amplitude is greater than $24/5$, then by definition there would be enhanced ultraviolet cancellations.

Similarly, we go through the same exercise for the seven-loop diagram shown in Fig. 3(b). In this case, the simplest form of the numerator consistent with the maximal cuts is

$$N_{\mathcal{N}=8 \text{ sugra}}^{7\text{-loop}} \Big|_{\text{max. cut}} = s^5 t u M_4^{\text{tree}} (2l_5 \cdot l_6)^8. \quad (3.8)$$

Together with seven D -dimensional loop integrations and 22 propagators, we obtain the power count,

$$\mathcal{D}_{\mathcal{N}=8 \text{ sugra}}^{7\text{-loop}} \sim \Lambda^{7D+16-44}. \quad (3.9)$$

² The importance of these types of cuts for power counting was first pointed out by Henrik Johansson.

Thus, the critical dimension is $D_c = 4$, again in agreement with other power-counting methods [18–20].

While it is not yet technically feasible to directly study the enhanced cancellations in $\mathcal{N} = 8$ supergravity five- and seven-loop amplitudes, we are able to study them in $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravities. We therefore turn to power counting in these theories.

Consider $\mathcal{N} = 4$ supergravity at three loops. As explained in Ref. [2], the BCJ construction of the integrand is in terms of the 12 diagrams displayed in Fig. 5. To be concrete, we examine diagram (a) in Fig. 2 for $\mathcal{N} = 4$ supergravity. The maximal-cut conditions on the kinematic invariants are

$$\begin{aligned} l_5^2 = l_6^2 = l_7^2 = 0, \quad l_6 \cdot l_7 = 0, \quad l_5 \cdot l_6 = 0, \quad k_2 \cdot l_7 = -\frac{s}{2} - k_1 \cdot l_7, \\ k_3 \cdot l_7 = 0, \quad k_2 \cdot l_6 = -\frac{s}{2} - k_1 \cdot l_6, \quad k_1 \cdot l_5 = -\frac{s}{2}, \quad k_2 \cdot l_5 = 0. \end{aligned} \quad (3.10)$$

Applying these and taking the explicit expression for the numerator of Fig. 2(a) in $\mathcal{N} = 4$ supergravity obtained by the double-copy procedure, we obtain

$$\mathcal{N}_{\mathcal{N}=4}^{(a)3\text{-loop}} \Big|_{\text{sugra}_{\text{max. cut}}} = -64s^3 t A_{\mathcal{N}=4}^{\text{tree}} (\varepsilon_1 \cdot l_5) (\varepsilon_2 \cdot l_5) (\varepsilon_3 \cdot l_7) (\varepsilon_4 \cdot l_7) (l_5 \cdot l_7)^2 + \dots, \quad (3.11)$$

where we kept only those term with the largest powers of loop momenta. The momentum labels are the ones shown in the figure and $A_{\mathcal{N}=4}^{\text{tree}}$ is an $\mathcal{N} = 4$ super-Yang-Mills tree amplitude depending only on the external states and momenta. The ε_i are polarization vectors of gluons. As discussed in Sect. II, the pure $\mathcal{N} = 4$ supergravity states are just the direct product of states of the two gauge theories. The displayed term in Eq. (3.11) is irreducible in that its power count cannot be lowered by imposing the maximal-cut conditions (3.10). Since the term (3.11) is uniquely assigned to the diagram, it is a lower bound on the power count of the diagram. After including the three D -dimensional loop integrals, eight powers of numerator loop momentum and ten propagators, we obtain a power-counting for this diagram,

$$\mathcal{D}_{\mathcal{N}=4}^{(a)3\text{-loop}} \Big|_{\text{sugra}} \sim \Lambda^{3D+8-20}. \quad (3.12)$$

Thus, in $D = 4$ this diagram has divergent terms. As a direct confirmation of this power count, we integrated the irreducible numerator in Eq. (3.11) after putting back the propagators. Indeed, it is ultraviolet divergent as indicated from the power count. This power count agrees with the one based on standard-symmetry arguments [19].

On the other hand, explicit calculations show that the three-loop four-point $\mathcal{N} = 4$ supergravity amplitude is finite [2]. Given that there are divergent terms in Fig. 2(a) that can only cancel against terms that were set to zero by the maximal cut conditions or terms from other diagrams, the finiteness of the amplitude as a whole is a prime example of an enhanced cancellation.

The maximal-cut constraints can sometimes lower the power count of diagrams below their true critical dimension. For example, for the diagram in Fig. 2(b), under the maximal-cut conditions, all $l_i \cdot l_j$ can be made to be no worse than linear in loop momenta. By choosing the minimal resulting power count, this results in an integrand that is ultraviolet finite, even after including an extra power of loop momentum from the $\mathcal{N} = 4$ super-Yang-Mills side of the double copy. Another point is that after integration, it may be possible to combine terms even from a single diagram to get a finite result. In particular, one can imagine taking the numerator of Eq. (3.11) and combining it with a judiciously chosen set of terms that vanish on the cuts to cancel the ultraviolet divergences. However, this is not relevant for

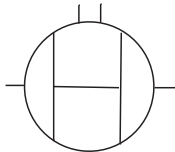


FIG. 4: A four-loop diagram whose maximal-cut power count suggests that $\mathcal{N} = 5$ supergravity should diverge in four dimensions, contrary to the behavior of the four-point amplitude as a whole.

enhanced cancellations which are defined in terms of power counting individual terms at the integrand level. If even a single term in the integrand of a single diagram has a worse power count compared to the actual behavior of the full amplitude and the power count cannot be lowered by maximal-cut conditions, then we have identified enhanced cancellations. We also note that, by using spinor-helicity, we can set the integrated divergence resulting from the diagram in Fig. 2(a) to zero (i.e., the integration results in terms containing $\varepsilon_i \cdot \varepsilon_j$ that can be set to zero by special reference momentum choices). Indeed, we shall do so later to simplify various tables. Of course, this does not change the fact that there is a term in the integrand that has a worse power count than the amplitude as a whole that cannot be set to zero.

The counting for $\mathcal{N} = 5$ supersymmetry is similar except that one should subtract two powers of loop momenta from each numerator because of additional $\mathcal{N} = 1$ supersymmetry cancellations described in Ref. [9]. Taking this into account, for the diagram Fig. 2(a), we obtain a maximal-cut power count in $\mathcal{N} = 5$ supergravity of

$$\mathcal{D}_{\mathcal{N}=5 \text{ sugra}}^{(a) \text{ 3-loop}} \sim \Lambda^{3D+6-20}. \quad (3.13)$$

Thus, the critical dimension from the maximal cut of this diagram is $D_c = 14/3 > 4$, so we expect there to be no obstruction to finding a covariant representation of $\mathcal{N} = 5$ supergravity that is manifestly ultraviolet finite in $D = 4$ at this loop order. Nevertheless, in Sect. IV we will show that on top of the supersymmetric cancellations, there are additional enhanced cancellations beyond those needed for ultraviolet finiteness.

If we repeat the same exercise at four loops for $\mathcal{N} = 4$ supergravity using similar power counting on, for example, the diagram in Fig. 4, we have the behavior,

$$\mathcal{D}_{\mathcal{N}=4 \text{ sugra}}^{4\text{-loop}} \sim \Lambda^{4D+12-26}, \quad (3.14)$$

Here we count four D -dimensional loop integrals, 13 propagators, 10 powers of numerator loop momenta from the Yang-Mills vertices and 2 additional powers of loop momenta from the $\mathcal{N} = 4$ super-Yang-Mills numerator. This means that terms in the diagram in Fig. 4 have a critical dimension of $D_c = 14/4 < 4$ and that in $D = 4$ it is quadratically divergent by power counting.

If we increase the supersymmetry to $\mathcal{N} = 5$ supergravity, as noted above, the extra $\mathcal{N} = 1$ supersymmetry decreases the maximal-cut power count by two powers of loop momentum so that

$$\mathcal{D}_{\mathcal{N}=5 \text{ sugra}}^{4\text{-loop}} \sim \Lambda^{4D+10-26}, \quad (3.15)$$

which corresponds to a critical dimension of $D_c = 4$. Therefore, based on the maximal-cut power counting, we would expect $\mathcal{N} = 5$ supergravity to be logarithmically divergent at four loops. This is consistent with the standard-symmetry power count of Ref. [19], leading to

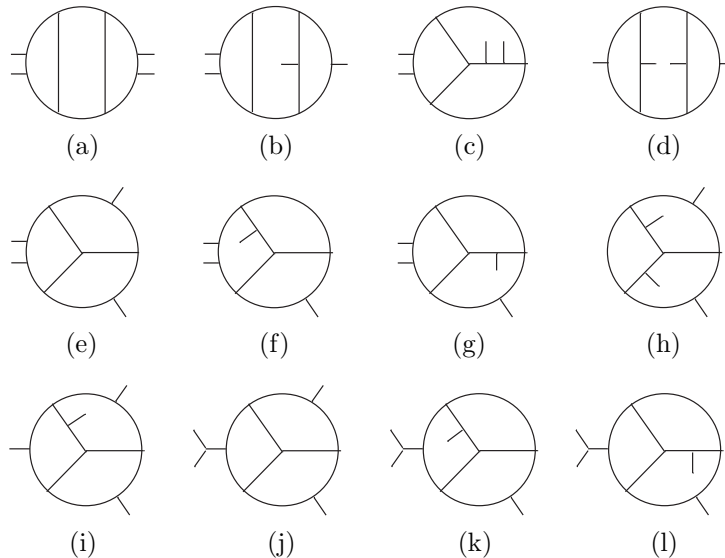


FIG. 5: Contributing three-loop diagrams in $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravity.

an expected counterterm on the third line of Table I. In Sect. V we show that because of enhanced cancellations, the $\mathcal{N} = 5$ four-loop four-point amplitude is, in fact, ultraviolet finite, contrary to these power counts.

IV. THREE LOOPS

As a warm up to our four-loop calculation, we first present the corresponding three-loop calculation in $\mathcal{N} = 5$ supergravity. We follow the same techniques summarized in Sect. II and described in some detail in Ref. [27]. In contrast to $\mathcal{N} = 4$ supergravity, in this case we should be able to construct a covariant integrand that is manifestly ultraviolet finite, bypassing the need for loop integration to demonstrate that it is ultraviolet finite. However, we do not do so here. Instead, we proceed the same way as at four loops by first computing the $\mathcal{N} = 4$ supergravity divergences and then adding in the extra contributions needed in $\mathcal{N} = 5$ supergravity. This allows us to observe enhanced cancellations. In fact, we are able to show finiteness with the enhanced cancellations alone, even without accounting for cancellations arising from the extra supersymmetry in the $\mathcal{N} = 5$ theory compared to the $\mathcal{N} = 4$ theory. Thus, the cancellations are stronger than those required to demonstrate finiteness.

In the calculation, we leave two state-counting parameters to make it simple to switch between various supergravity theories. The first parameter is D_s , which is obtained from contractions of the metric $\eta_{\mu\nu}$ from the Lorentz algebra, while the second is n_f , which counts the number of Majorana fermions added to the pure Yang-Mills side of the double copy. By choosing $D_s = 4$ and $n_f = 0$, we obtain pure $\mathcal{N} = 4$ supergravity. By setting the parameters to $D_s = 4$ and $n_f = 1$, we obtain $\mathcal{N} = 5$ supergravity. We can also obtain results for $\mathcal{N} = 4$ supergravity with n_V matter multiplets by choosing $D_s = 4 + n_V$ and $n_f = 0$, where n_V is the number of internal matter vector multiplets [27].

Our $D = 4$ divergence-calculation results are summarized in Tables II and III with the results corresponding to each graph in Fig. 5 used to organize these calculations. The 12

graph	(divergence) $(4\pi)^6 / ((12)^2 [34]^2 st A^{\text{tree}} (\frac{\kappa}{2})^8)$
(a)–(d)	0
(e)	$\left(-\frac{77}{768} + \frac{85D_s}{768}\right) \frac{1}{\epsilon^3} + \left(\frac{35071}{55296} - \frac{2371D_s}{6912} - \frac{D_s^2}{64}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{11815}{768} + \frac{4367D_s}{768} + \frac{9D_s^2}{32}\right) \text{S2} + \left(-\frac{77}{512} + \frac{85D_s}{512}\right) \zeta_2 + \left(\frac{10627}{2304} - \frac{1705D_s}{576}\right) \zeta_3 - \frac{170275}{110592} + \frac{1381D_s}{1536} - \frac{5D_s^2}{128}\right] \frac{1}{\epsilon}$
(f)	$\left(\frac{397}{2304} - \frac{143D_s}{2304}\right) \frac{1}{\epsilon^3} + \left(\frac{1717}{4608} + \frac{211D_s}{4608}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{1777}{256} + \frac{263D_s}{256}\right) \text{S2} + \left(\frac{397}{1536} - \frac{143D_s}{1536}\right) \zeta_2 + \left(-\frac{649}{288} + \frac{69D_s}{64}\right) \zeta_3 + \frac{685733}{165888} - \frac{116663D_s}{82944}\right] \frac{1}{\epsilon}$
(g)	$\left(-\frac{23}{288} - \frac{65D_s}{1152}\right) \frac{1}{\epsilon^3} + \left(-\frac{7919}{6912} + \frac{4631D_s}{13824} + \frac{D_s^2}{192}\right) \frac{1}{\epsilon^2} + \left[\left(\frac{2447}{96} - \frac{2911D_s}{384} - \frac{3D_s^2}{32}\right) \text{S2} + \left(-\frac{23}{192} - \frac{65D_s}{768}\right) \zeta_2 + \left(-\frac{2173}{768} + \frac{2161D_s}{1152}\right) \zeta_3 - \frac{464957}{165888} + \frac{51515D_s}{82944} + \frac{23D_s^2}{1152}\right] \frac{1}{\epsilon}$
(h)	$-\frac{3}{32} \frac{1}{\epsilon^3} + \left(-\frac{1841}{3072} + \frac{59D_s}{192} - \frac{D_s^2}{48}\right) \frac{1}{\epsilon^2} + \left[\left(\frac{687}{64} - \frac{21D_s}{4} + \frac{3D_s^2}{8}\right) \text{S2} - \frac{9}{64} \zeta_2 + \left(\frac{3347}{2304} - \frac{5D_s}{384}\right) \zeta_3 - \frac{144431}{55296} + \frac{13811D_s}{13824} - \frac{17D_s^2}{288}\right] \frac{1}{\epsilon}$
(i)	$\left(\frac{13}{128} + \frac{D_s}{128}\right) \frac{1}{\epsilon^3} + \left(\frac{4535}{6144} - \frac{265D_s}{768} + \frac{D_s^2}{32}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{1779}{128} + \frac{783D_s}{128} - \frac{9D_s^2}{16}\right) \text{S2} + \left(\frac{39}{256} + \frac{3D_s}{256}\right) \zeta_2 + \left(-\frac{2263}{2304} + \frac{11D_s}{576}\right) \zeta_3 + \frac{311953}{110592} - \frac{7691D_s}{6912} + \frac{5D_s^2}{64}\right] \frac{1}{\epsilon}$
(j)	$\left(-\frac{3}{32} - \frac{3D_s}{32}\right) \frac{1}{\epsilon^3} + \left(-\frac{41}{32} + \frac{35D_s}{64}\right) \frac{1}{\epsilon^2} + \left[\left(\frac{927}{32} - \frac{333D_s}{32}\right) \text{S2} + \left(-\frac{9}{64} - \frac{9D_s}{64}\right) \zeta_2 + \left(-\frac{11}{4} + \frac{67D_s}{24}\right) \zeta_3 - \frac{1297}{576} + \frac{151D_s}{384}\right] \frac{1}{\epsilon}$
(k)	$\left(\frac{1}{64} + \frac{D_s}{64}\right) \frac{1}{\epsilon^3} + \left(\frac{443}{576} - \frac{347D_s}{1152}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{985}{64} + \frac{365D_s}{64}\right) \text{S2} + \left(\frac{3}{128} + \frac{3D_s}{128}\right) \zeta_2 + \left(\frac{247}{144} - \frac{13D_s}{12}\right) \zeta_3 + \frac{9167}{6912} - \frac{865D_s}{2304}\right] \frac{1}{\epsilon}$
(l)	$\left(\frac{5}{64} + \frac{5D_s}{64}\right) \frac{1}{\epsilon^3} + \left(\frac{295}{576} - \frac{283D_s}{1152}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{869}{64} + \frac{301D_s}{64}\right) \text{S2} + \left(\frac{15}{128} + \frac{15D_s}{128}\right) \zeta_2 + \left(\frac{149}{144} - \frac{41D_s}{24}\right) \zeta_3 + \frac{6397}{6912} - \frac{41D_s}{2304}\right] \frac{1}{\epsilon}$
sum	0

TABLE II: The divergences for the four-graviton amplitude in $\mathcal{N} = 4$ supergravity corresponding to each graph in Fig. 5. To simplify the diagrams we chose the helicities $(1^-2^-3^+4^+)$ on the pure Yang-Mills side of the double-copy decomposition, leaving the states on the super-Yang-Mills side arbitrary. On the pure Yang-Mills side, we use spinor-helicity with reference momenta $q_1 = q_2 = k_3$ and $q_3 = q_4 = k_1$. Each expression includes a permutation sum over external legs and the symmetry factor appropriate to the graph. D_s is the state-counting parameter and $\epsilon = (4 - D)/2$ is the usual dimensional regularization parameter. The transcendental constant S2 is defined in Eq. (4.1). The sum over all contributions in the table vanishes, illustrating the phenomenon of enhanced cancellations. These results do not include subdivergence subtractions, whose sum also vanishes.

diagrams correspond to the nonvanishing ones of $\mathcal{N} = 4$ super-Yang-Mills theory in the BCJ representation [13]. Table II contains all contributions that do not depend on the parameter n_f , and Table III contains all the pieces that do depend on n_f . In the calculation, we take the external states to be gluons on the $\mathcal{N} = 0$ or $\mathcal{N} = 1$ side of the double copy, keeping the polarizations vectors formal. However, to simplify the tables we apply four-dimensional

graph	(divergence) $(4\pi)^6/(\langle 12 \rangle^2 [34]^2 st A^{\text{tree}}(\frac{k}{2})^8 n_f)$
(a)–(d)	0
(e)	$\left(\frac{43}{192} + \frac{D_s}{32}\right) \frac{1}{\epsilon^3} + \left(\frac{821}{432} - \frac{391D_s}{576}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{4627}{192} + \frac{281D_s}{32}\right) \text{S2} + \left(\frac{43}{128} + \frac{3D_s}{64}\right) \zeta_2 + \left(-\frac{271}{36} - \frac{17D_s}{12}\right) \zeta_3 + \frac{59723}{6912} - \frac{3113D_s}{3456} - \frac{5D_s^2}{16} \right] \frac{1}{\epsilon}$
(f)	$\left(\frac{109}{576} - \frac{D_s}{16}\right) \frac{1}{\epsilon^3} + \left(\frac{9}{128} + \frac{D_s}{36}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{689}{64} + \frac{17D_s}{8}\right) \text{S2} + \left(\frac{109}{384} - \frac{3D_s}{32}\right) \zeta_2 + \left(\frac{425}{144} + \frac{35D_s}{144}\right) \zeta_3 - \frac{7649}{5184} - \frac{55D_s}{1728} + \frac{D_s^2}{48} \right] \frac{1}{\epsilon}$
(g)	$\left(-\frac{4}{9} + \frac{D_s}{32}\right) \frac{1}{\epsilon^3} + \left(-\frac{7549}{3456} + \frac{43D_s}{64}\right) \frac{1}{\epsilon^2} + \left[\left(\frac{1849}{48} - \frac{361D_s}{32}\right) \text{S2} + \left(-\frac{2}{3} + \frac{3D_s}{64}\right) \zeta_2 + \left(\frac{749}{144} + \frac{163D_s}{144}\right) \zeta_3 - \frac{160627}{20736} + \frac{3331D_s}{3456} + \frac{7D_s^2}{24} \right] \frac{1}{\epsilon}$
(h)	$\left(-\frac{15}{128} + \frac{D_s}{24}\right) \frac{1}{\epsilon^2} + \left[\left(\frac{9}{8} - \frac{3D_s}{4}\right) \text{S2} + \left(\frac{1}{24} + \frac{95D_s}{144}\right) \zeta_3 + \frac{3481}{6912} - \frac{599D_s}{1728} + \frac{D_s^2}{16} \right] \frac{1}{\epsilon}$
(i)	$\frac{1}{32} \frac{1}{\epsilon^3} + \left(\frac{127}{384} - \frac{D_s}{16}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{153}{32} + \frac{9D_s}{8}\right) \text{S2} + \frac{3}{64} \zeta_2 + \left(-\frac{2}{3} - \frac{89D_s}{144}\right) \zeta_3 + \frac{179}{2304} + \frac{545D_s}{1728} - \frac{D_s^2}{16} \right] \frac{1}{\epsilon}$
(j)	$-\frac{3}{8} \frac{1}{\epsilon^3} + \left(-\frac{97}{48} + \frac{17D_s}{24}\right) \frac{1}{\epsilon^2} + \left[\left(\frac{255}{8} - \frac{21D_s}{2}\right) \text{S2} - \frac{9}{16} \zeta_2 + \left(\frac{43}{6} + \frac{5D_s}{3}\right) \zeta_3 - \frac{757}{96} + \frac{151D_s}{144} + \frac{D_s^2}{3} \right] \frac{1}{\epsilon}$
(k)	$\frac{1}{16} \frac{1}{\epsilon^3} + \left(\frac{337}{288} - \frac{3D_s}{8}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{265}{16} + \frac{45D_s}{8}\right) \text{S2} + \frac{3}{32} \zeta_2 + \left(-\frac{13}{6} - \frac{5D_s}{6}\right) \zeta_3 + \frac{707}{192} - \frac{23D_s}{48} - \frac{D_s^2}{6} \right] \frac{1}{\epsilon}$
(l)	$\frac{5}{16} \frac{1}{\epsilon^3} + \left(\frac{245}{288} - \frac{D_s}{3}\right) \frac{1}{\epsilon^2} + \left[\left(-\frac{245}{16} + \frac{39D_s}{8}\right) \text{S2} + \frac{15}{32} \zeta_2 + \left(-5 - \frac{5D_s}{6}\right) \zeta_3 + \frac{269}{64} - \frac{41D_s}{72} - \frac{D_s^2}{6} \right] \frac{1}{\epsilon}$
sum	0

TABLE III: Additional diagrammatic contributions appearing in the four-graviton amplitude of $\mathcal{N} = 5$ supergravity. This contribution contains also the n_f state-counting parameter. The total $\mathcal{N} = 5$ divergence is given by the sum over these contributions and those in Table II. The vanishing of the sum over the entries in each table individually is a reflection of enhanced cancellations. Subdivergences automatically cancel amongst themselves and are not included. The choice of external helicity states and reference momenta are as in Table II.

spinor-helicity (see Ref. [48] for a recent review) on the polarization vectors and specify the external-gluon states to be $--++$. We also make convenient choices of reference momenta: $q_1 = q_2 = k_3$ and $q_3 = q_4 = k_1$. This choice makes the divergences in diagrams (a)–(d) vanish for both tables. It also results in terms containing a factor of n_f^2 , from two fermion loops, to vanish individually in all diagrams in Table III (instead of in the sum over diagrams). We have not included subdivergence subtractions in the tables, but we have explicitly confirmed that, with the use of the uniform mass regulator, the subdivergences cancel as expected, given that there are no lower-loop divergences. In the tables, the ζ_i are the standard Riemann zeta constants. The transcendental constant S2 appearing in the tables is

$$\text{S2} = \frac{4}{9\sqrt{3}} \text{Cl}_2\left(\frac{\pi}{3}\right), \quad (4.1)$$

where $\text{Cl}_2(x) = \text{Im}(\text{Li}_2(e^{ix}))$ is the Clausen function. As the tables illustrate, when one sums over all diagrams, the result is finite for any choice of the state-counting parameters.

Although we made special helicity choices for the tables, our calculation is based on using

formal polarization states and is therefore valid for any external state that is a direct product of a gluon state and an $\mathcal{N} = 4$ super-Yang-Mills state. This corresponds to a subset of the $\mathcal{N} = 5$ supergravity states. Nevertheless, the result also extends to *any* $\mathcal{N} = 5$ state because the $\mathcal{N} = 1$ super-Yang-Mills supersymmetry identities [49] are powerful enough to relate all four-point amplitudes to the gluonic ones. (A discussion of these identities at two loops is given in Ref. [50].) It is interesting that for *any* values of the state-counting parameters, the divergences vanish.

Summarizing, not only is $\mathcal{N} = 4$ supergravity ultraviolet finite at three loops, but the extra pieces needed to obtain $\mathcal{N} = 5$ supergravity using the decomposition (2.6) are finite by themselves:

$$\begin{aligned} \mathcal{M}_4^{3\text{-loop}} \Big|_{\mathcal{N}=4, \text{div.}} &= 0, \\ \mathcal{M}_4^{3\text{-loop}} \Big|_{(\mathcal{N}=5 - \mathcal{N}=4), \text{div.}} &= 0. \end{aligned} \tag{4.2}$$

The independent vanishings in Eq. (4.2) show that the extra supersymmetry of the $\mathcal{N} = 5$ theory compared to $\mathcal{N} = 4$ theory is not needed to make $\mathcal{N} = 5$ supergravity finite. While this is no surprise given the three-loop finiteness of $\mathcal{N} = 4$ supergravity, it does explicitly demonstrate that ultraviolet cancellations exist in subpieces for which there is no power-counting argument. Thus, the $\mathcal{N} = 5$ case is another explicit example of enhanced ultraviolet cancellations that go beyond the ones that have been understood by any standard-symmetry considerations.

V. FOUR LOOPS

We now consider four loops. We first summarize the calculation of the $\mathcal{N} = 4$ supergravity four-point divergence presented in Ref. [3], giving a few additional intermediate results. We then turn to the corresponding calculation in $\mathcal{N} = 5$ supergravity, showing that the divergence vanishes.

A. Review of $\mathcal{N} = 4$ supergravity

The calculation of the $\mathcal{N} = 4$ supergravity divergence starts from pure Yang-Mills Feynman diagrams, keeping only those diagrams with color factors that match the 82 $\mathcal{N} = 4$ super-Yang-Mills diagrams displayed in Figs. 6 and 7. The color factors are then replaced the BCJ forms of the $\mathcal{N} = 4$ super-Yang-Mills numerators given in Ref. [39]. There are three additional diagrams, displayed in Fig. 8. In pure Yang-Mills they contain ultraviolet divergences (canceled by infrared divergences), but in $\mathcal{N} \geq 4$ supergravity an extra power of zero in the form of an on-shell massless external momentum squared in the $\mathcal{N} = 4$ super-Yang-Mills numerators sets such potential ultraviolet contributions to zero.

After feeding the integrand so constructed through the integration procedure summarized in Sect. II, we find that pure $\mathcal{N} = 4$ supergravity in $D = 4$ is divergent [3]:

$$\mathcal{M}_4^{4\text{-loop}} \Big|_{\mathcal{N}=4, \text{div.}} = \frac{1}{(4\pi)^8} \frac{1}{\epsilon} \left(\frac{\kappa}{2}\right)^{10} \frac{1}{144} (1 - 264 \zeta_3) \mathcal{T}, \tag{5.1}$$

where $\epsilon = (4 - D)/2$ is the dimensional-regularization parameter, and

$$\mathcal{T} = st A_{\mathcal{N}=4}^{\text{tree}} (\mathcal{O}_1 - 28\mathcal{O}_2 - 6\mathcal{O}_3), \tag{5.2}$$

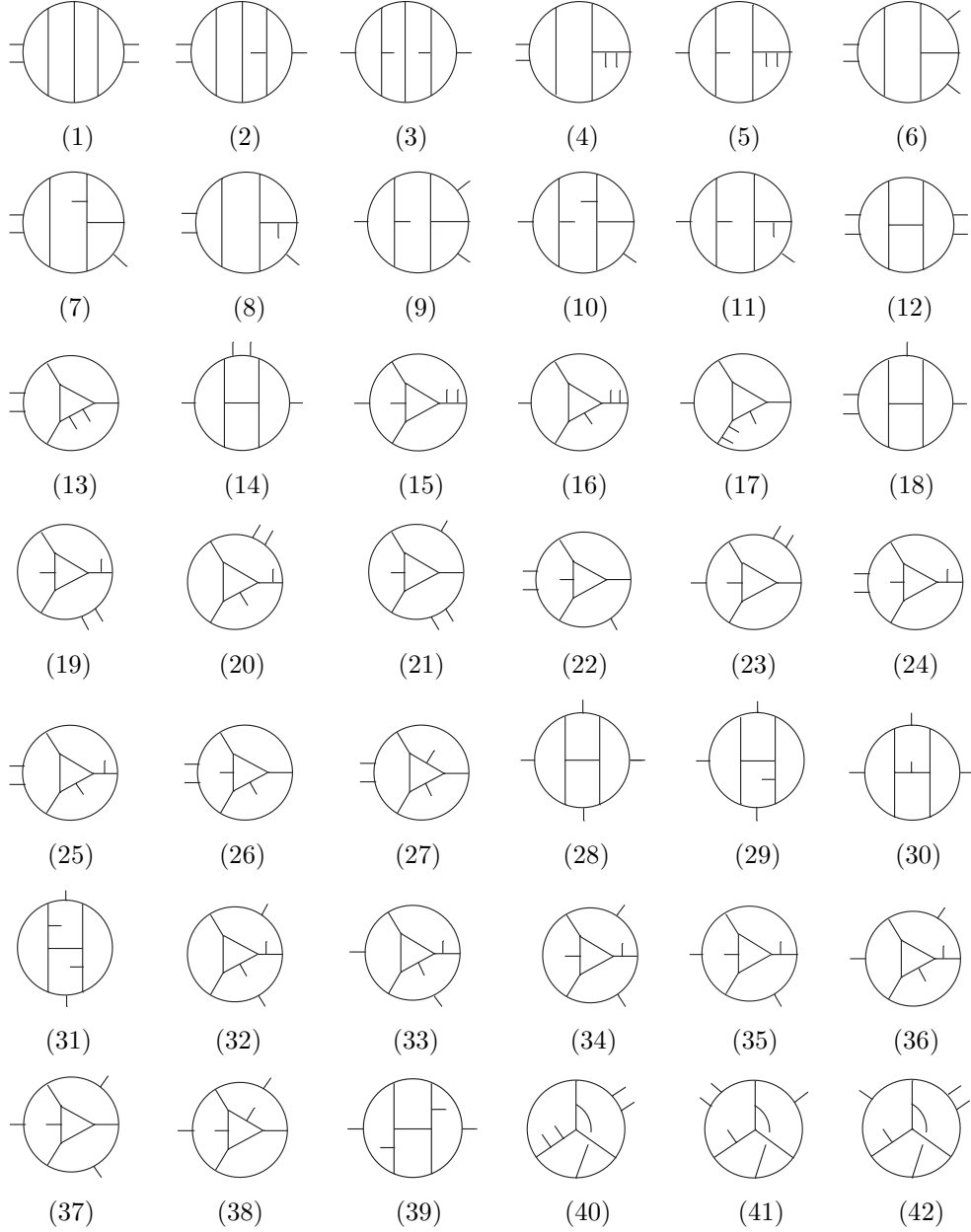


FIG. 6: The first 42 diagrams for the four-loop four-point amplitudes of $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravity. These correspond to the $\mathcal{N} = 4$ super-Yang-Mills diagrams of Ref. [39].

with

$$\begin{aligned}
\mathcal{O}_1 &= \sum_{S_4} (D_\alpha F_{1\mu\nu}) (D^\alpha F_2^{\mu\nu}) F_{3\rho\sigma} F_4^{\rho\sigma}, \\
\mathcal{O}_2 &= \sum_{S_4} (D_\alpha F_{1\mu\nu}) (D^\alpha F_2^{\nu\sigma}) F_{3\rho\sigma} F_4^{\rho\mu}, \\
\mathcal{O}_3 &= \sum_{S_4} (D_\alpha F_{1\mu\nu}) (D_\beta F_2^{\mu\nu}) F_{3\sigma}{}^\alpha F_4^{\sigma\beta}.
\end{aligned} \tag{5.3}$$

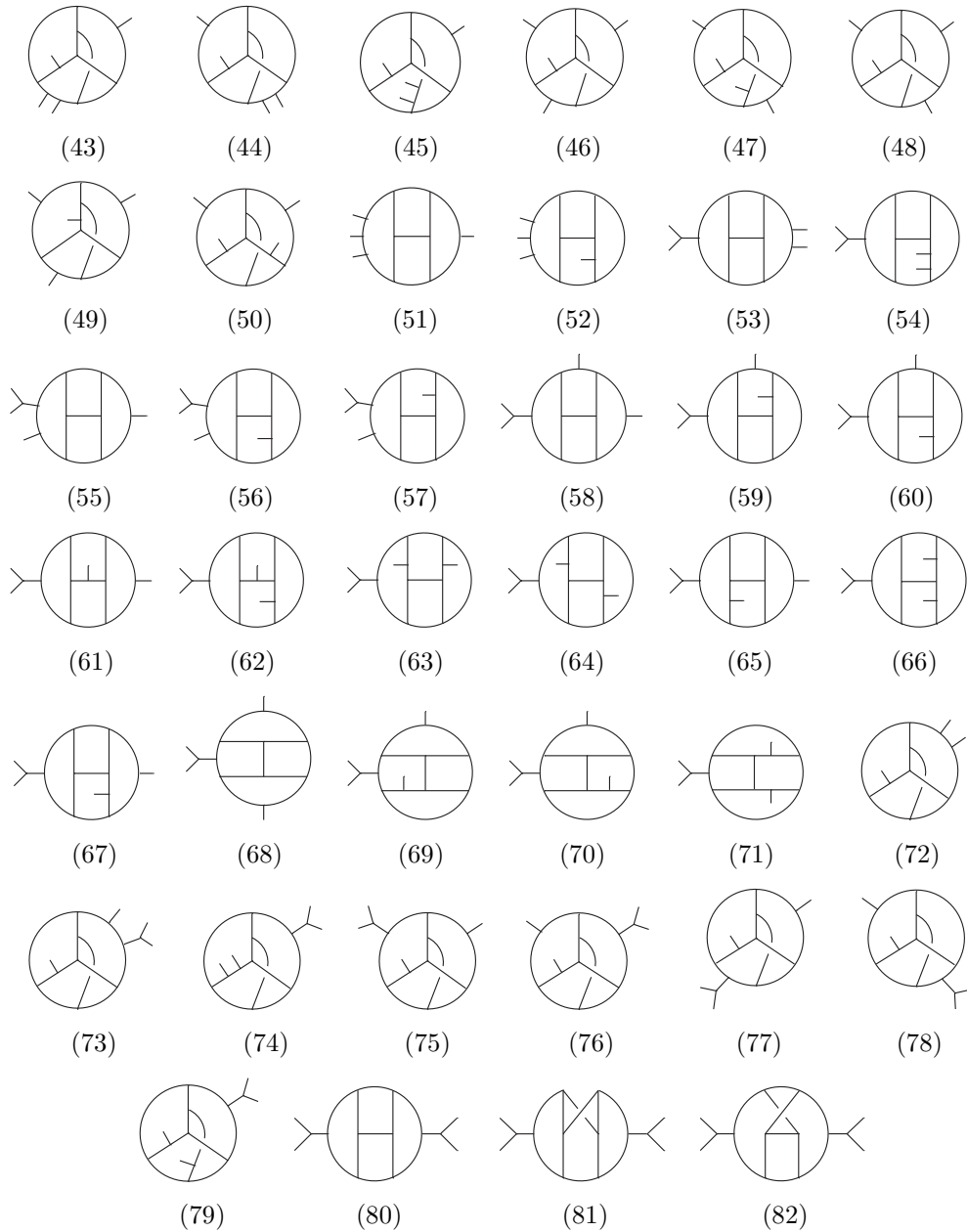


FIG. 7: Diagrams 43–82 for the four-loop four-point amplitudes of $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravity.

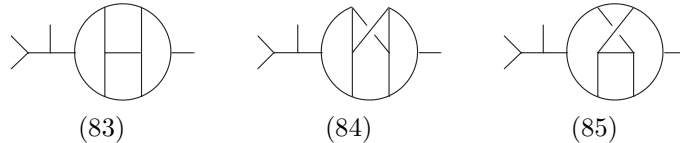


FIG. 8: The bubble-on-external-leg diagrams of $\mathcal{N} = 4$ super-Yang-Mills theory. These do not contribute to $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravity.

The sum runs over all 24 permutations of the external legs. $F_j^{\mu\nu}$ is the linearized field-

graphs	(divergence) $\times u(4\pi)^8 / (-\langle 12 \rangle^2 [34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^{10}$
1–30	$ \begin{aligned} & -\frac{1}{\epsilon^4} \left[\frac{297863}{3981312} s^2 + \frac{7115179}{7962624} st + \frac{1230523}{2654208} t^2 \right] + \frac{1}{\epsilon^3} \left[\frac{183507269}{318504960} s^2 - \frac{121097629}{106168320} st - \frac{125340203}{159252480} t^2 \right] \\ & + \frac{1}{\epsilon^2} \left[\zeta_3 \left(-\frac{54780317}{3686400} s^2 - \frac{364821169}{22118400} st - \frac{19297919}{7372800} t^2 \right) - \zeta_2 \left(\frac{297863}{1990656} s^2 + \frac{7115179}{3981312} st + \frac{1230523}{1327104} t^2 \right) \right. \\ & - \text{S2} \left(\frac{1602535}{73728} s^2 + \frac{10330175}{442368} st - \frac{14079343}{442368} t^2 \right) - \frac{80222068879}{28665446400} s^2 - \frac{949461174731}{57330892800} st - \frac{17877740021}{19110297600} t^2 \\ & \left. + \frac{1}{\epsilon} \left[\zeta_5 \left(\frac{42165713}{92160} s^2 + \frac{12876011}{9216} st + \frac{10040753}{46080} t^2 \right) + \zeta_4 \left(\frac{1162609}{7372800} s^2 + \frac{183267071}{14745600} st + \frac{110749763}{14745600} t^2 \right) \right. \right. \\ & - \zeta_3 \left(\frac{10506518408983}{71663616000} s^2 + \frac{30289233413171}{71663616000} st - \frac{2013863213191}{35831808000} t^2 \right) - \zeta_2 \left(\frac{970317931}{159252480} s^2 + \frac{59367181}{5898240} st \right. \\ & \left. \left. - \frac{719420377}{79626240} t^2 \right) - \text{T1ep} \left(\frac{1602535}{995328} s^2 + \frac{10330175}{5971968} st - \frac{14079343}{5971968} t^2 \right) - \text{S2} \left(\frac{33354691993}{53084160} s^2 \right. \right. \\ & \left. \left. + \frac{19386147397}{10616832} st + \frac{9723954001}{8847360} t^2 \right) - \text{D6} \left(\frac{4137589}{552960} s^2 + \frac{2283701}{184320} st + \frac{527011}{138240} t^2 \right) \right. \\ & \left. - \frac{20252328329611}{143327232000} s^2 - \frac{534679988685821}{1146617856000} st - \frac{8363829769903}{1146617856000} t^2 \right] \end{aligned} $
31–60	$ \begin{aligned} & \frac{1}{\epsilon^4} \left[\frac{1788617}{3981312} s^2 + \frac{20728021}{7962624} st + \frac{2452169}{2654208} t^2 \right] + \frac{1}{\epsilon^3} \left[\frac{527762531}{318504960} s^2 + \frac{1120727089}{106168320} st + \frac{122147731}{53084160} t^2 \right] \\ & + \frac{1}{\epsilon^2} \left[\zeta_3 \left(\frac{6081287}{345600} s^2 + \frac{13983243}{819200} st + \frac{98182043}{22118400} t^2 \right) + \zeta_2 \left(\frac{1788617}{1990656} s^2 + \frac{20728021}{3981312} st + \frac{2452169}{1327104} t^2 \right) \right. \\ & + \text{S2} \left(\frac{3516907}{73728} s^2 + \frac{31188941}{442368} st - \frac{15998365}{442368} t^2 \right) + \frac{545203990507}{28665446400} s^2 + \frac{4109230335503}{57330892800} st + \frac{142686680113}{19110297600} t^2 \\ & \left. + \frac{1}{\epsilon} \left[\zeta_5 \left(-\frac{160438583}{245760} s^2 - \frac{311758955}{147456} st - \frac{119748949}{368640} t^2 \right) - \zeta_4 \left(\frac{5925797}{921600} s^2 + \frac{460780679}{14745600} st + \frac{126445477}{14745600} t^2 \right) \right. \right. \\ & + \zeta_3 \left(\frac{11662905491459}{53747712000} s^2 + \frac{54035183618969}{71663616000} st + \frac{8467395805631}{214990848000} t^2 \right) + \zeta_2 \left(\frac{3059935571}{159252480} s^2 + \frac{789428243}{17694720} st \right. \\ & \left. \left. - \frac{197819569}{26542080} t^2 \right) + \text{T1ep} \left(\frac{3516907}{995328} s^2 + \frac{31188941}{5971968} st - \frac{15998365}{5971968} t^2 \right) + \text{S2} \left(\frac{2658637313}{53084160} s^2 \right. \right. \\ & \left. \left. + \frac{2611873009}{10616832} st + \frac{23301734753}{26542080} t^2 \right) + \text{D6} \left(\frac{6050189}{552960} s^2 + \frac{10479103}{552960} st + \frac{233987}{46080} t^2 \right) \right. \\ & \left. + \frac{455464156513}{1911029760} s^2 + \frac{173334911330293}{229323571200} st + \frac{673760034799}{25480396800} t^2 \right] \end{aligned} $
61–82	$ \begin{aligned} & -\frac{1}{\epsilon^4} \left[\frac{248459}{663552} s^2 + \frac{756269}{442368} st + \frac{610823}{1327104} t^2 \right] + \frac{1}{\epsilon^3} \left[-\frac{17781745}{7962624} s^2 - \frac{5553497}{589824} st - \frac{24110299}{15925248} t^2 \right] \\ & + \frac{1}{\epsilon^2} \left[\zeta_3 \left(-\frac{30260233}{11059200} s^2 - \frac{1590799}{2764800} st - \frac{20144143}{11059200} t^2 \right) - \zeta_2 \left(\frac{248459}{331776} s^2 + \frac{756269}{221184} st + \frac{610823}{663552} t^2 \right) \right. \\ & - \text{S2} \left(\frac{53177}{2048} s^2 + \frac{3476461}{73728} st - \frac{319837}{73728} t^2 \right) - \frac{38748493469}{2388787200} s^2 - \frac{9752373953}{176947200} st - \frac{31202235023}{4777574400} t^2 \\ & \left. + \frac{1}{\epsilon} \left[\zeta_5 \left(\frac{28798009}{147456} s^2 + \frac{35247593}{49152} st + \frac{876065}{8192} t^2 \right) + \zeta_4 \left(\frac{15414589}{2457600} s^2 + \frac{11563067}{614400} st + \frac{7847857}{7372800} t^2 \right) \right. \right. \\ & - \zeta_3 \left(\frac{15920366514887}{214990848000} s^2 + \frac{4001452799633}{11943936000} st + \frac{20550575084777}{214990848000} t^2 \right) - \zeta_2 \left(\frac{52240441}{3981312} s^2 + \frac{30566335}{884736} st \right. \\ & \left. \left. + \frac{12596167}{7962624} t^2 \right) - \text{T1ep} \left(\frac{53177}{27648} s^2 + \frac{3476461}{995328} st - \frac{319837}{995328} t^2 \right) + \text{S2} \left(\frac{767401367}{1327104} s^2 \right. \right. \\ & \left. \left. + \frac{1397856199}{884736} st + \frac{587012725}{2654208} t^2 \right) - \text{D6} \left(\frac{47815}{13824} s^2 + \frac{22675}{3456} st + \frac{17495}{13824} t^2 \right) \right. \\ & \left. - \frac{434546648527}{4478976000} s^2 - \frac{9221628964379}{31850496000} st - \frac{5488842949013}{286654464000} t^2 \right] \end{aligned} $
sum	$\frac{1}{\epsilon} su \frac{1}{72} (264 \zeta_3 - 1)$

TABLE IV: The divergence in the four-graviton amplitude of pure $\mathcal{N} = 4$ supergravity. The first three entries correspond to the sum over diagrams 1–30, 31–60 and 61–82 listed in Figs. 6 and 7, while the final row gives the sum over all diagrams. Subdivergences automatically cancel amongst themselves and are not included. Our choice of external helicity states and reference momenta are as in Table II.

strength tensor given in terms of polarization vectors for leg j as

$$\begin{aligned} F_j^{\mu\nu} &\equiv i(k_j^\mu \varepsilon_j^\nu - k_j^\nu \varepsilon_j^\mu), \\ D^\alpha F_j^{\mu\nu} &\equiv -k_j^\alpha (k_j^\mu \varepsilon_j^\nu - k_j^\nu \varepsilon_j^\mu). \end{aligned} \quad (5.4)$$

This form makes explicit the fact that each state of pure $\mathcal{N} = 4$ supergravity corresponds to a direct product of a color-stripped state of $\mathcal{N} = 4$ super-Yang-Mills theory and of pure nonsupersymmetric Yang-Mills theory.

By taking linear combinations, the divergences can be separated into distinct helicity classifications:

$$\begin{aligned} \mathcal{O}^{--++} &= \mathcal{O}_1 - 4\mathcal{O}_2, & \mathcal{O}^{-+++} &= \mathcal{O}_1 - 4\mathcal{O}_3, \\ \mathcal{O}^{++++} &= \mathcal{O}_2. \end{aligned} \quad (5.5)$$

Each of the obtained operators are nonvanishing only for the indicated helicity configurations and their parity conjugates and relabelings. The helicity labels refer to those of the polarization vectors used in Eq. (5.4) on the pure Yang-Mills side and not the supergravity states, which are obtained by a direct product of these states with those of $\mathcal{N} = 4$ super-Yang-Mills theory. For explicit helicity states in $D = 4$, we have

$$\begin{aligned} \mathcal{O}^{--++} &= 4s^2 t \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}, \\ \mathcal{O}^{-+++} &= -12s^2 t^2 \frac{[24]^2}{[12] \langle 23 \rangle \langle 34 \rangle [41]}, \\ \mathcal{O}^{++++} &= 3st(s+t) \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle}, \end{aligned} \quad (5.6)$$

using four-dimensional spinor-helicity notation.

As explained in Ref. [3], the appearance of the divergences in the three independent helicity configurations in Eq. (5.5) is unexpected and points to the source of the divergence being the Marcus $U(1)$ duality-symmetry anomaly [28]. Without the anomaly, the $--++$ and $++++$ helicity sectors would vanish. Ref. [29] explains how the anomaly leads to poor ultraviolet behavior even in the $--++$ sector.

In Table IV we have collected together groups of diagrams in order to display the non-trivial cancellations between diagrams. The first three entries correspond to the sums over diagrams 1–30, 31–60 and 61–82, while the final one gives the sum over all diagrams. The final sum displays an enormous cancellation between the diagrams to yield a remarkably simple result. We do not include subdivergences which automatically cancel amongst themselves. In the table, the ζ_i are the standard Riemann zeta constants. The value of S2 is already defined in Eq. (4.1), while the other constants are [37]

$$\begin{aligned} \text{T1ep} &= -\frac{45}{2} - \frac{\pi\sqrt{3}\log^2 3}{8} - \frac{35\pi^3\sqrt{3}}{216} - \frac{9}{2}\zeta_2 + \zeta_3 + 6\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right) \\ &\quad - 6\sqrt{3}\text{Im}\left(\text{Li}_3\left(\frac{e^{-i\pi/6}}{\sqrt{3}}\right)\right), \\ \text{D6} &= 6\zeta_3 - 17\zeta_4 - 4\zeta_2\log^2 2 + \frac{2}{3}\log^4 2 + 16\text{Li}_4\left(\frac{1}{2}\right) - 4\left(\text{Cl}_2\left(\frac{\pi}{3}\right)\right)^2. \end{aligned} \quad (5.7)$$

As noted earlier, $\text{Cl}_2(x) = \text{Im}(\text{Li}_2(e^{ix}))$ is the Clausen function. These transcendental constants arise from our use of an infrared mass regulator and, as expected, cancel from the final ultraviolet divergence [44, 45].

B. $\mathcal{N} = 5$ supergravity

Next we turn to $\mathcal{N} = 5$ supergravity in $D = 4$. As discussed in Sect. IIB, we obtain $\mathcal{N} = 5$ supergravity from $\mathcal{N} = 4$ supergravity by adding in the contributions from the BCJ construction based on the direct product of $\mathcal{N} = 4$ super-Yang-Mills with additional contributions from adding a single Majorana fermion to the pure gluon theory. The calculation is somewhat more complicated than the pure $\mathcal{N} = 4$ supergravity calculation because of the long fermion traces that appear at four loops.

Because $\mathcal{N} = 5$ supergravity has no duality-symmetry anomaly, we expect it to be ultraviolet finite at four loops. Indeed, additional $\mathcal{N} = 1$ supersymmetry identities are sufficient to show amplitudes and any associated potential ultraviolet divergences vanish in the $-+++$ and $++++$ helicity sectors in Eq. (5.6). Only the $--++$ sector gives nonvanishing amplitudes and therefore needs checking, although we have calculated the other two sectors as well. We have only computed the case where the external $\mathcal{N} = 5$ supergravity states are those obtained from a direct product of $\mathcal{N} = 4$ super-Yang-Mills and pure Yang-Mills states (i.e. the subset of states that are also in the pure $\mathcal{N} = 4$ supergravity spectrum). However, as mentioned in Sect. IV, $\mathcal{N} = 1$ and $\mathcal{N} = 4$ supersymmetry identities in the direct product [49, 50] allow us to express any of the four-point amplitudes in terms of one of them, so ruling out divergences in this sector rules out *all* four-point divergences.

We find that the four-loop four-point amplitudes of $\mathcal{N} = 5$ supergravity are finite. This may be unsurprising given that the additional supersymmetry compared to the $\mathcal{N} = 4$ supergravity case should improve the ultraviolet properties. However, the fact remains that, at present, there is no standard-symmetry explanation for the finiteness. In addition, as we explain in Sect. III, no covariant diagrammatic formalism can display the cancellations manifestly, so the vanishing of the divergence is another example of enhanced cancellations.

In Table V we give the extra contributions to the potential divergence coming from the additional states that are present in $\mathcal{N} = 5$ supergravity compared to $\mathcal{N} = 4$ supergravity. As can be seen from the final entry in the table, the contribution is equal and opposite to the contribution that comes solely from the $\mathcal{N} = 4$ supergravity states given in Table IV. Therefore the total divergence vanishes:

$$\mathcal{M}_4^{4\text{-loop}} \Big|_{\mathcal{N}=5, \text{div.}} = 0. \quad (5.8)$$

The nontrivial way the cancellations occur in the sum of the entries in Tables IV and V suggests that there should be a better way to see them. While it may be simple to state the obvious, as already explained in Sect. III, finding a formalism that makes these cancellations manifest is nontrivial, given that no covariant diagrammatic representation exists that does so.

Besides the information given in Tables IV and V, in accompanying Mathematica attachments [51] we give the divergences for each diagram for pure $\mathcal{N} = 4$ supergravity as well as for the additional contributions needed for $\mathcal{N} = 5$ supergravity. As in the tables, we do not include subdivergences in these files since they automatically cancel amongst themselves in our calculation.

graphs	(divergence) $\times u(4\pi)^8 / (-12)^2 [34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^{10}$
1-30	$\begin{aligned} & \frac{1}{\epsilon^4} \left[\frac{607}{1990656} s^2 - \frac{1323773}{1990656} st - \frac{14255}{41472} t^2 \right] + \frac{1}{\epsilon^3} \left[\frac{4865671}{19906560} s^2 + \frac{149977}{3317760} st - \frac{20170049}{19906560} t^2 \right] \\ & + \frac{1}{\epsilon^2} \left[\zeta_3 \left(-\frac{3733153}{230400} s^2 - \frac{5900609}{276480} st + \frac{3883097}{691200} t^2 \right) + \zeta_2 \left(\frac{607}{995328} s^2 - \frac{1323773}{995328} st - \frac{14255}{20736} t^2 \right) \right. \\ & - \text{S2} \left(\frac{625357}{36864} s^2 + \frac{5161189}{110592} st - \frac{1428583}{55296} t^2 \right) - \frac{7648139167}{3583180800} s^2 - \frac{22568882383}{3583180800} st - \frac{55681241}{59719680} t^2 \left. \right] \\ & + \frac{1}{\epsilon} \left[\zeta_5 \left(-\frac{225641}{1024} s^2 - \frac{12931021}{18432} st - \frac{2378855}{18432} t^2 \right) - \zeta_4 \left(\frac{4044329}{460800} s^2 + \frac{3646153}{921600} st - \frac{2056603}{153600} t^2 \right) \right. \\ & + \zeta_3 \left(\frac{6076575618157}{17915904000} s^2 + \frac{3396579085657}{3583180800} st + \frac{2089036585637}{8957952000} t^2 \right) - \zeta_2 \left(\frac{51416459}{9953280} s^2 + \frac{801749}{51840} st \right. \\ & - \frac{65544931}{9953280} t^2 \left. \right) - \text{T1ep} \left(\frac{625357}{497664} s^2 + \frac{5161189}{1492992} st - \frac{1428583}{746496} t^2 \right) - \text{S2} \left(\frac{8055438013}{16588800} s^2 \right. \\ & + \frac{555755309}{414720} st + \frac{555207793}{614400} t^2 \left. \right) - \text{D6} \left(\frac{715513}{138240} s^2 + \frac{718247}{76800} st + \frac{285839}{172800} t^2 \right) \\ & \left. + \frac{1916368326173}{71663616000} s^2 + \frac{7258817218703}{71663616000} st + \frac{3175133834231}{35831808000} t^2 \right] \end{aligned}$
31-60	$\begin{aligned} & \frac{1}{\epsilon^4} \left[\frac{509381}{1990656} s^2 + \frac{3991391}{1990656} st + \frac{242555}{331776} t^2 \right] + \frac{1}{\epsilon^3} \left[\frac{50554927}{19906560} s^2 + \frac{13023425}{1327104} st + \frac{8356667}{3317760} t^2 \right] \\ & + \frac{1}{\epsilon^2} \left[\zeta_2 \left(\frac{509381}{995328} s^2 + \frac{3991391}{995328} st + \frac{242555}{165888} t^2 \right) + \zeta_3 \left(\frac{990949}{57600} s^2 + \frac{570691}{1382400} st - \frac{10906963}{691200} t^2 \right) \right. \\ & + \text{S2} \left(\frac{1380997}{36864} s^2 + \frac{9202651}{110592} st - \frac{821453}{27648} t^2 \right) + \frac{65553264229}{3583180800} s^2 + \frac{27992599379}{447897600} st + \frac{12366245939}{1194393600} t^2 \left. \right] \\ & + \frac{1}{\epsilon} \left[\zeta_5 \left(\frac{10240481}{23040} s^2 + \frac{96847583}{92160} st + \frac{3535453}{30720} t^2 \right) + \zeta_4 \left(\frac{816643}{76800} s^2 - \frac{6008467}{307200} st - \frac{51227}{2048} t^2 \right) \right. \\ & - \zeta_3 \left(\frac{8235182625383}{13436928000} s^2 + \frac{25298224196579}{17915904000} st + \frac{11561841643253}{53747712000} t^2 \right) + \zeta_2 \left(\frac{174844657}{9953280} s^2 + \frac{31428727}{663552} st \right. \\ & - \frac{8072393}{1658880} t^2 \left. \right) + \text{T1ep} \left(\frac{1380997}{497664} s^2 + \frac{9202651}{1492992} st - \frac{821453}{373248} t^2 \right) + \text{S2} \left(\frac{2385329963}{16588800} s^2 \right. \\ & + \frac{1077896293}{3317760} st + \frac{1501624967}{2073600} t^2 \left. \right) + \text{D6} \left(\frac{233051}{46080} s^2 + \frac{4649023}{691200} st + \frac{77389}{172800} t^2 \right) \\ & \left. - \frac{273686499733}{2654208000} s^2 - \frac{10212410685517}{35831808000} st - \frac{501121685203}{477574400} t^2 \right] \end{aligned}$
61-82	$\begin{aligned} & -\frac{1}{\epsilon^4} \left[\frac{42499}{165888} s^2 + \frac{148201}{110592} st + \frac{128515}{331776} t^2 \right] + \frac{1}{\epsilon^3} \left[-\frac{27710299}{9953280} s^2 - \frac{21805693}{2211840} st - \frac{29969953}{19906560} t^2 \right] \\ & + \frac{1}{\epsilon^2} \left[\zeta_3 \left(-\frac{25627}{25600} s^2 + \frac{1607353}{76800} st + \frac{3511933}{345600} t^2 \right) + \zeta_2 \left(-\frac{42499}{82944} s^2 - \frac{148201}{55296} st - \frac{128515}{165888} t^2 \right) \right. \\ & + \text{S2} \left(-\frac{10495}{512} s^2 - \frac{673577}{18432} st + \frac{71441}{18432} t^2 \right) - \frac{9650854177}{597196800} s^2 - \frac{7458218987}{132710400} st - \frac{11252621119}{1194393600} t^2 \left. \right] \\ & + \frac{1}{\epsilon} \left[\zeta_5 \left(-\frac{10327117}{46080} s^2 - \frac{1788471}{5120} st + \frac{321979}{23040} t^2 \right) + \zeta_4 \left(-\frac{855529}{460800} s^2 + \frac{10835777}{460800} st + \frac{892711}{76800} t^2 \right) \right. \\ & + \zeta_3 \left(\frac{14908078591061}{53747712000} s^2 + \frac{1396836736049}{2985984000} st - \frac{972377870569}{53747712000} t^2 \right) + \zeta_2 \left(-\frac{61714099}{4976640} s^2 - \frac{35277233}{1105920} st \right. \\ & - \frac{17110573}{9953280} t^2 \left. \right) + \text{T1ep} \left(-\frac{10495}{6912} s^2 - \frac{673577}{248832} st + \frac{71441}{248832} t^2 \right) + \text{S2} \left(\frac{113402161}{331776} s^2 \right. \\ & + \frac{1122715393}{1105920} st + \frac{119104427}{663552} t^2 \left. \right) + \text{D6} \left(\frac{409}{3456} s^2 + \frac{2269}{864} st + \frac{4169}{3456} t^2 \right) \\ & \left. + \frac{2736085919309}{35831808000} s^2 + \frac{1462778758259}{7962624000} st + \frac{1166557609583}{71663616000} t^2 \right] \end{aligned}$
sum	$-\frac{1}{\epsilon} su \frac{1}{72} (264 \zeta_3 - 1)$

TABLE V: The additional contributions in $\mathcal{N} = 5$ supergravity. These include internal states that arise from a direct product of the $\mathcal{N} = 4$ sYM states and a Majorana fermion. The sum of these contributions together with the ones in Table IV vanishes, showing that the $\mathcal{N} = 5$ supergravity amplitude is ultraviolet finite. Subdivergences automatically cancel amongst themselves and are not included. Our choice of external helicity states and reference momenta are as in Table II.

We note that while our calculation proves that there are no four-loop four-point diver-

gences in $\mathcal{N} = 5$ supergravity, it does not rule out five-point R^5 -type divergences. It would of course be interesting to study these as well in the future.

VI. CONCLUSIONS

In this paper, we described the phenomenon of *enhanced ultraviolet cancellations* in supergravity theories. By definition, when all local covariant diagrammatic representations of an amplitude contain terms that have a worse power count than the amplitude as a whole, we have enhanced cancellations. To illustrate this phenomenon, we first discussed $\mathcal{N} = 4$ supergravity in four dimensions at three loops. By power counting maximal cuts, we identified terms in the four-point amplitude that are divergent at three loops, in agreement with supersymmetry and duality-symmetry arguments [19], when, in fact, the amplitude is three-loop finite in $D = 4$ [2]. The theory does diverge at four loops [3], but it appears to be due to a rigid $U(1)$ duality-symmetry anomaly [28, 29]. Such anomalies are not present in $\mathcal{N} \geq 5$ supergravity theories, suggesting that these theories cannot have similar divergences. This motivated us to study the four-loop four-point amplitudes of $\mathcal{N} = 5$ supergravity. Again in this case, power counting maximal cuts identifies divergent terms in $D = 4$, consistent with standard-symmetry considerations [19]. However, explicit calculations performed in this paper shows this amplitude is ultraviolet finite, again illustrating enhanced ultraviolet cancellations. If similar enhanced ultraviolet cancellations hold in $\mathcal{N} = 8$ supergravity, then this theory will be finite at seven loops as well, contradicting predictions based on power counting [18–20].

The underlying reason for enhanced ultraviolet cancellations is not fully understood. There are some indications that the duality between color and kinematics [12, 13] is responsible. An explicit study shows that this duality is responsible for improved ultraviolet behavior in the relatively simple two-loop case of half-maximal supergravity in $D = 5$ [24]. An important challenge is to push this understanding to higher loop orders. Another important question is whether there might be an explanation for enhanced cancellations based on supersymmetry or duality symmetry. Such an explanation would have to be novel, given that enhanced ultraviolet cancellations are nonstandard. The potential three-loop R^4 counterterm of $\mathcal{N} = 4$ supergravity and four-loop counterterm $\mathcal{D}^2 R^4$ counterterm of $\mathcal{N} = 5$ supergravity cannot be written as full superspace integrals [19]. An interesting open question is whether this plays any role in the vanishing of the associated divergences.

While we do not yet know if perturbatively ultraviolet-finite unitary field theories of gravity exist, based on the results of this paper it is clearly premature to conclude otherwise. More generally, nontrivial multiloop enhanced cancellations in gravity theories are a new and surprising phenomenon, contrary to expectations based on standard-symmetry considerations which suggest viable counterterms. The existence of these cancellations gives us confidence that further nontrivial surprises await us as we probe supergravity theories to ever higher loop orders using modern tools.

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