Self-accelerating massive gravity: Covariant perturbation theory

Pavel Motloch and Wayne Hu

Phys. Rev. D **90**, 104027 — Published 19 November 2014

DOI: 10.1103/PhysRevD.90.104027

Self-accelerating Massive Gravity: Covariant Perturbation Theory

Pavel Motloch[1] and Wayne Hu[2]

[1]*Kavli Institute for Cosmological Physics, Department of Physics, University of Chicago, Chicago, Illinois 60637, U.S.A*

[2]*Kavli Institute for Cosmological Physics, Department of Astronomy & Astrophysics, Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637, U.S.A*

We undertake a complete and covariant treatment for the quadratic Lagrangian of all of the degrees of freedom of massive gravity with a fixed flat fiducial metric for arbitrary massive gravity parameters around any isotropic self-accelerating background solution. Generically, 3 out of 4 Stückelberg degrees of freedom propagate in addition to the usual 2 tensor degrees of freedom of general relativity. The complete kinetic structure typically is only revealed at an order in the graviton mass that is equivalently to retaining curvature terms in a locally flat expansion. These results resolve several apparent discrepancies in the literature where zero degrees of freedom propagate in either special cases or approximate treatments as well as decoupling limit analyses which attempt to count longitudinal degrees of freedom.

I. INTRODUCTION

The theory of massive gravity with a second static flat fiducial metric [1–4] possesses solutions that accelerate the cosmological expansion in the absence of a true cosmological constant [5–15]. Because the second metric is non-dynamical, this theory of massive gravity breaks diffeomorphism invariance. While covariance can be restored with the Stückelberg trick, for a homogeneous and isotropic spacetime background, the Stückelberg fields must be inhomogeneous to accommodate the two metrics. Moreover, except for a special class of open universe solutions [11], there is no coordinate system where both the spacetime and fiducial metrics can be made simultaneously diagonal, homogeneous and isotropic [10] even though the spacetime metric itself can accommodate self-accelerating solutions with Friedman-Robertson-Walker backgrounds for any desired matter content or curvature [12].

Inhomogeneity in the Stückelberg fields or equivalently the relationship between the spacetime and Minkowski metrics causes both technical and theoretical challenges for understanding the self-accelerating solutions. In the special open universe case where the metrics themselves are simultaneously homogeneous and isotropic in the same coordinates, standard analyses apply. There the massive gravity sector is strongly coupled and propagates no degrees of freedom in the quadratic Lagrangian and possesses an instability at higher order [16]. Furthermore the solutions can evolve to a coordinate singularity which cannot be resolved by charts with overlapping domains of validity [17].

For the more generic case, technical difficulties of incorporating an inhomogeneous fiducial metric background, which breaks translation invariance, has hitherto prevented a full analysis. Results for the longitudinal or isotropic modes with the exact background obtained in Ref. [18] showed one propagating degree of freedom in the quadratic Lagrangian, as might be expected from the 4 Stückelberg fields, the Boulware-Deser ghost free construction and isotropy indicating that the behavior of the open universe solution is not generic. However this degree of freedom obeys unusual but stable first order dynamics with no quadratic coupling to other fields [18] and an unbounded Hamiltonian [19]. Furthermore these results seem to contradict analyses that used the decoupling limit [20] or locally flat approximations [21] which found that generically two and zero isotropic modes propagate respectively in the quadratic Lagrangian. Indeed in the locally flat approximation, no anisotropic modes propagated either.

It is the aim of this paper to resolve these issues and present a complete covariant treatment of the quadratic Lagrangian for the massive gravity degrees of freedom around any isotropic self-accelerating background. We begin in §II with a brief review of the massive gravity theory to establish notation. In §III we reanalyze the isotropic modes for the class of solutions considered in the existing literature and show that inconsistencies in the counting of degrees of freedom are resolved by a complete analysis at the level of first order curvature corrections to the locally flat approximation and a consistent treatment of gauge degrees of freedom. Since the full dynamics of the modes involves spacetime curvature, in §IV we provide a general, covariant treatment of *all* massive gravity degrees of freedom for *any* isotropic background solution on the self-accelerating branch for the *entire* class of massive gravity parameters. We discuss these results in §V.

II. MASSIVE GRAVITY

The Boulware-Deser ghost free theory of massive gravity adds a mass term to the Einstein-Hilbert Lagrangian density [4]

$$\mathcal{L}^{(\mathrm{MG})} = -\frac{m^2 M_{\rm pl}^2}{2}\sqrt{-g}\sum_{k=0}^{4}\frac{\beta_k}{k!}F_k\left(\boldsymbol{\gamma}\right), \qquad (1)$$

where $M_{\rm pl} = (8\pi G)^{-1}$ is the reduced Planck mass and the F_k terms are functions of the matrix $\boldsymbol{\gamma}$

$$\begin{aligned}
&F_0(\boldsymbol{\gamma}) = 1, \\
&F_1(\boldsymbol{\gamma}) = [\boldsymbol{\gamma}], \\
&F_2(\boldsymbol{\gamma}) = [\boldsymbol{\gamma}]^2 - [\boldsymbol{\gamma}^2], \\
&F_3(\boldsymbol{\gamma}) = [\boldsymbol{\gamma}]^3 - 3[\boldsymbol{\gamma}][\boldsymbol{\gamma}^2] + 2[\boldsymbol{\gamma}^3], \\
&F_4(\boldsymbol{\gamma}) = [\boldsymbol{\gamma}]^4 - 6[\boldsymbol{\gamma}]^2[\boldsymbol{\gamma}^2] + 3[\boldsymbol{\gamma}^2]^2 + 8[\boldsymbol{\gamma}][\boldsymbol{\gamma}^3] - 6[\boldsymbol{\gamma}^4],
\end{aligned} \tag{2}$$

where [] denotes the trace of the enclosed matrix. The parameters of the theory are m, the graviton mass, and β_k. Not all of the latter parameters are independent since

$$\begin{aligned}
\beta_0 &= -12(1 + 2\alpha_3 + 2\alpha_4), \\
\beta_1 &= 6(1 + 3\alpha_3 + 4\alpha_4), \\
\beta_2 &= -2(1 + 6\alpha_3 + 12\alpha_4), \\
\beta_3 &= 6(\alpha_3 + 4\alpha_4), \\
\beta_4 &= -24\alpha_4,
\end{aligned} \tag{3}$$

leaving two remaining independent parameters $\{\alpha_3, \alpha_4\}$.

The presence of the matrix $\boldsymbol{\gamma}$ breaks diffeomorphism invariance since it is constructed from the square root of the product of the inverse spacetime metric $g^{\mu\nu}$ and a flat fiducial metric $\Sigma_{\mu\nu}$

$$\gamma^{\mu}{}_{\alpha}\gamma^{\alpha}{}_{\nu} = g^{\mu\alpha}\Sigma_{\alpha\nu}, \tag{4}$$

singling out a specific coordinate choice, called unitary gauge where $\Sigma_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski metric. Nonetheless, diffeomorphism invariance can be restored by the Stückelberg trick of using the coordinates of unitary gauge as auxiliary fields ϕ^a to express the fiducial metric in an arbitrary coordinate system

$$\Sigma_{\mu\nu} = \partial_\mu\phi^a\partial_\nu\phi^b\eta_{ab}. \tag{5}$$

It is important to note that the Stückelberg fields ϕ^a transform as spacetime scalars and form a Lorentz vector only in the internal Minkowski space. Beyond the leading order, locally flat approximation to the spacetime metric, Stückelberg indices should not be conflated with spacetime indices [22]. We shall see in §IV how to construct spacetime vectors out of Stückelberg components. Throughout, Greek indices denote the spacetime and are lowered and raised with $g_{\mu\nu}$ and its inverse; Latin indices likewise by the Minkowski metric η_{ab}.

III. ISOTROPIC MODES

In this section, we resolve discrepancies in the literature for the dynamics of spherically symmetric Stückelberg perturbations around certain self-accelerating vacuum solutions [7], first analyzed in the decoupling limit [20], then in a locally flat limit [21], and finally in the exact background [18, 19] with contradictory results. In §III A we treat the quadratic Lagrangian consistently to leading order in curvature corrections, $\mathcal{O}(m^2)$, and show that kinetic terms for a single longitudinal or isotropic mode only arise at this order, which is then consistent with exact results. In §III B we discuss the problem of equating a locally flat expansion with the decoupling limit around a Minkowski background. In §III C, we show that miscounting of degrees of freedom can also arise from gauge fixing in the Lagrangian.

A. Kinetic Terms and Curvature

We focus here only on the specific case of certain solutions for the $\alpha_3 = \alpha_4 = 0$ model [7] as these suffice to show our main points and have been the most analyzed in the literature. We consider the general case in §IV A. With this choice, the unitary gauge solution is described by the metric

$$\begin{aligned}
g_{\mu\nu}{\rm d}x^\mu{\rm d}x^\nu = &-\mathcal{C}(R){\rm d}T^2 + 2\mathcal{D}(R){\rm d}T{\rm d}R + \mathcal{A}(R){\rm d}R^2 \\
&+\mathcal{B}(R)\left({\rm d}\theta^2 + \sin\theta^2{\rm d}\phi^2\right),
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
\mathcal{A}(R) &= \frac{4C^2}{9}\left(1 + v^2 + \frac{m^2R^2}{9}\right), \\
\mathcal{B}(R) &= \frac{4}{9}R^2, \\
\mathcal{C}(R) &= \frac{4C^2}{9}\left(1 - \frac{m^2R^2}{9}\right), \\
\mathcal{D}(R) &= \frac{4C^2}{9}\frac{mR}{3}\sqrt{v^2 + \frac{m^2R^2}{9}},
\end{aligned} \tag{7}$$

with [23]

$$v^2 = \frac{1}{C^2} - 1. \tag{8}$$

Here $0 < C \leq 1$ is an integration constant in the solutions. From this exact expression we would like to focus on a locally flat patch where $mR \to 0$. There is a subtlety in taking this limit associated with the parameter v. The metric component

$$\mathcal{D}(R) = \frac{4C^2}{9}\frac{mR}{3}v\left[1 + \frac{m^2R^2}{18v^2} + \mathcal{O}\left(\frac{m^4R^4}{v^4}\right)\right]. \tag{9}$$

Hence the validity of the expansion is confined to a radius where $mR \ll v$ rather than $mR \ll 1$. For the special case of $v = 0$ ($C = 1$), this domain of validity shrinks to $R = 0$ and instead

$$\mathcal{D}(R) = \frac{4}{9}\left(\frac{mR}{3}\right)^2, \qquad (v = 0). \tag{10}$$

The result is an apparent discontinuity in the limit $v \to 0$ of the expansion. We shall see that this limit is the case where the background solution has no Stückelberg vector component.

Next when considering fluctuations around this background solution it is more convenient to choose a different gauge where the background metric is described near the origin $r = t = 0$ to order $\mathcal{O}(m^2)$ by the conformal form [20]

$$\bar{g}_{\mu\nu} = \left[1 - \frac{m^2(r^2 - t^2)}{8}\right]\eta_{\mu\nu} + \mathcal{O}(m^3), \qquad (11)$$

where $\eta_{\mu\nu}$ is the Minkowski metric in spherical coordinates

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin\theta^2 \mathrm{d}\phi^2\right). \quad (12)$$

The unitary gauge coordinates (T, R) can be expressed in conformal coordinates (t, r) as

$$\begin{aligned} T &= \frac{3t}{2C}\left[1 + \frac{m^2(t^2+3r^2)}{48}\right] + \frac{mr^2}{8}(3v + \delta^{\mathrm{K}}_{v0} mr) \\ &\quad + \mathcal{O}(m^3), \\ R &= \frac{3r}{2}\left[1 - \frac{m^2(r^2-t^2)}{16}\right] + \mathcal{O}(m^3). \end{aligned} \qquad (13)$$

Here, the Kronecker delta

$$\delta^{\mathrm{K}}_{v0} = \begin{cases} 1 & v = 0 \\ 0 & v \neq 0 \end{cases} \qquad (14)$$

accounts for the discontinuity at $v = 0$ in the expansion of $\mathcal{D}(R)$ above. Note that the discontinuity appears only at $\mathcal{O}(m^2)$ and hence was omitted in Ref. [20]. For $v \neq 0$, unitary time T contains terms that are $\mathcal{O}(m)$.

The background Stückelberg fields are then the unitary gauge coordinates (T, R) of the solution [7] expressed in conformal coordinates (t, r)

$$\begin{aligned} \bar{\phi}^0 &= T, \\ \bar{\phi}^i &= R\frac{x^i}{r}. \end{aligned} \qquad (15)$$

Note that in the special case that $v = 0$, the difference in the unitary and conformal coordinates can be derived from a Lorentz scalar quantity ignoring curvature corrections

$$\bar{\phi}_\mu - x_\mu = \frac{1}{4}\partial_\mu(r^2 - t^2) + \mathcal{O}(m^2). \qquad (16)$$

This $v = 0$ case is said to have no vector Stückelberg field in the background. Note that the Stückelberg index, which is always raised an lowered by the Minkowski metric, cannot be treated as a spacetime index at $\mathcal{O}(m^2)$. In Ref. [20], the background Stückelberg were truncated already before $\mathcal{O}(m^2)$. However we shall see that since the dynamics of perturbations enter at $O(m^2)$, this is not sufficient.

To see this consider spherically symmetric fluctuations in the Stückelberg fields

$$\begin{aligned} \delta\phi^0 &\equiv -a_t(t, r), \\ \delta\phi^i &\equiv a_r(t, r)\frac{x^i}{r}, \end{aligned} \qquad (17)$$

where we keep the definition that Stückelberg indices are raised and lowered by the Minkowski metric at the expense of a_μ not forming a spacetime vector at $O(m^2)$. We correct this notational abuse in §IV. Expanding the Lagrangian (1) to quadratic order in the Stückelberg fluctuations, we obtain

$$\begin{aligned} \mathcal{L}_2 = M_{\mathrm{pl}}^2 m^2 \frac{(mr)^2\sin\theta}{4(1+C)}\Big[&\Big(\frac{4Cv}{mr} + 2\delta^{\mathrm{K}}_{v0}\Big) r a_r a_t' + 3Ca_r^2 \\ &+ 2ra_t\dot{a}_r + \mathcal{O}(m)\Big]. \end{aligned} \qquad (18)$$

Here and in the following, we equate Lagrangians which are equal up to total derivative terms. The primes denote derivatives with respect to the radial coordinate r, dots with respect to t.

Note that we do not consider mixing with metric perturbations here. Unlike in the Minkowski background, the longitudinal or isotopic mode gains a kinetic term from curvature corrections rather than demixing with the metric fluctuations [24] as can explicitly be shown given that the self-accelerating solution is exact for isotropic metrics, perturbed or not [18]. We shall return to this topic in §IV where anisotropic modes and their mixing are considered.

Aside from the overall factor of $M_{\mathrm{pl}}^2 m^2$ from Eq. (1), the leading order terms in Eq. (18) come in at $\mathcal{O}(m^2)$ if there is no vector background and $\mathcal{O}(m)$ with a vector background. Furthermore even in the latter case the $\mathcal{O}(m)$ terms carry no time derivatives and thus propagate no degrees of freedom. This explains the result of Ref. [21], where all terms of order $\mathcal{O}(m^2)$ were omitted. The dynamical term from $\dot{a}_r$ only enters in at $\mathcal{O}(m^2)$ whereas a_t is non-dynamical. The first order structure of this Lagrangian implies that a_r obeys a first order equation of motion that is independent of a_t whereas a_t obeys a first order equation that depends on a_r. This result is in accordance with the exact background [18].

This set of equations of motion do not combine into the usual wave equation for one degree of freedom. Nonetheless a Hamiltonian analysis shows that (a_r, a_t) form a single degree of freedom due to the presence of constraints. In particular the field momenta

$$\begin{aligned} p_{a_t} &= \frac{\partial\mathcal{L}_2}{\partial\dot{a}_t} = 0, \\ p_{a_r} &= \frac{\partial\mathcal{L}_2}{\partial\dot{a}_r} = \frac{M_{\mathrm{pl}}^2 m^4 r^3\sin\theta}{2(1+C)}a_t, \end{aligned} \qquad (19)$$

cannot be inverted to express velocities $\dot{a}_t, \dot{a}_r$ in terms of momenta, which indicates the presence of two primary constraints

$$\begin{aligned} \phi_1 &= p_{a_t}, \\ \phi_2 &= p_{a_r} - \frac{M_{\mathrm{pl}}^2 m^4 r^3\sin\theta}{2(1+C)}a_t. \end{aligned} \qquad (20)$$

To determine whether (ϕ_1, ϕ_2) exhausts all of the constraints we define the total Hamiltonian

$$\mathcal{H}_T = -\frac{M_{\rm pl}^2 m^4 r^2 \sin\theta}{4(1+C)}\Big[\Big(\frac{4Cv}{mr}+2\delta^{\rm K}_{v0}\Big) r a_r a_t' + 3C a_r^2\Big] \\ +u_1(t,r)\phi_1 + u_2(t,r)\phi_2, \tag{21}$$

where u_i are (at the moment) auxiliary variables. Using Poisson brackets and the Hamilton equation, we see that

$$\dot{\phi}_i = \{\phi_i, \mathcal{H}_T\} = 0 \tag{22}$$

uniquely determine u_i and we thus conclude there are no additional constraints in the system.

Because $\{\phi_1, \phi_2\}$ is nonzero, both primary constraints are second class and together remove one of the two degrees of freedom from the problem. This leaves us with a single propagating degree of freedom.

This result is in full agreement with that of the exact theory [18, 19], which also found one propagating degree of freedom for all values of C. On the constrained surface $\phi_i = 0$ we can utilize the constraint (20) and rewrite the Hamiltonian entirely in terms of a_r, p_{a_r}:

$$\mathcal{H} = \frac{2Cv}{mr}\Big(\frac{2}{r}a_r + a_r'\Big)p_{a_r} + \delta^{\rm K}_{v0}\Big(\frac{3}{r}a_r + a_r'\Big)p_{a_r} \\ -\frac{3CM_{\rm pl}^2 m^4 r^2 \sin\theta}{4(1+C)} a_r^2. \tag{23}$$

The Hamiltonian is linear in the now unbounded p_{a_r}, which means the Hamiltonian is unbounded from below. This is also in agreement with the results of the full theory. We concluded that an expansion to the level of curvature corrections in the locally flat limit is sufficient to recover the Stückelberg dynamics.

B. Scaling vs. Decoupling

The analysis in Ref. [20] is based on a scaling limit with m motivated by the decoupling of scalars, vectors and tensors or helicity states of the graviton around a Minkowski background. We show here why this limit is misleading for the self-accelerating background given a lack of decoupling. Naive use of this scaling limit would erroneously imply zero degrees of freedom rather than one.

Around a Minkowski background, scaling the Stückelberg fluctuations according to

$$a_\mu = \frac{mA_\mu + \partial_\mu \pi}{M_{\rm pl} m^2} \tag{24}$$

leads to the so-called decoupling limit where $m \to 0$ at fixed $M_{\rm pl}m^2$. In this limit A_μ is a free vector field with a canonical Maxwell Lagrangian and π is a scalar which gets a canonical kinetic term once demixed from the tensor metric fluctuation [25]. In the spherically symmetric configuration studied in this section, this would lead to one scalar or helicity-0 mode since the Maxwell Lagrangian propagates only transverse degrees of freedom.

In this sense π is an additional Stückelberg field that restores $U(1)$ symmetry to the vector by separating out its longitudinal component. We shall return to this point in the next section.

Rewriting the Stückelberg fluctuations in the form Eq. (24) of course cannot change the dynamics or the number of propagating degrees of freedom. The problem is that the motivation for this scaling disappears around the self-accelerating background where kinetic terms only come in at $\mathcal{O}(m^2)$. We shall therefore refer to the decomposition of Eq. (24) as the Minkowski scaling limit rather than the decoupling limit.

With this scaling we can write the $v \neq 0$ Lagrangian of Eq. (18) as an $\mathcal{O}(m^0)$ term

$$\mathcal{L}_2^{(0)} = \frac{r^2 \sin\theta}{1+C}\left[Cv\pi'(A_t' - \dot{A}_r) + \frac{3}{4}(C\pi'^2 - \dot{\pi}^2)\right] \tag{25}$$

plus terms that appear to be higher order

$$\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \frac{r^2\sin\theta}{4\,(1+C)}\Big\{m\Big[2r\dot{A}_r\dot{\pi} + 4CvA_rA_t' \\ +6CA_r\pi' + 2rA_t\dot{\pi}' + 2\delta^{\rm K}_{v0} r\pi'(A_t' - \dot{A}_r)\Big] \\ +m^2(3CA_r^2 + 2rA_t\dot{A}_r + 2\delta^{\rm K}_{v0} rA_rA_t')\Big\}. \tag{26}$$

Note that unlike the decoupling limit, A_μ does not possess a Maxwell term nor does it decouple from π. Ref. [20] kept only the $\mathcal{L}_2^{(0)}$ term [see their Eqs. (5.8), (5.9) and (5.26)] based on the assumption that taking the $m \to 0$ with the Minkowski scaling was self-consistent. However, we know that (A_t, A_r, π) together form a single degree of freedom. Dropping any interaction between these fields by simply assuming that A_μ should scale differently with m is thus dangerous.

Indeed the Hamiltonian analysis shows $\mathcal{L}_2^{(0)}$ propagates no degrees of freedom for $v \neq 0$. The primary constraints are

$$\phi_1 = p_{A_t}, \\ \phi_2 = p_{A_r} + \frac{Cvr^2\sin\theta}{1+C}\pi', \tag{27}$$

while we are able to express the velocity $\dot{\pi}$ in terms of the momentum p_π.

Time evolution of these two constraints by calculating their Poisson brackets with the total Hamiltonian provides two secondary constraints

$$\phi_3 \sim \left(r^2\pi'\right)', \\ \phi_4 \sim r^2\left(\frac{p_\pi}{r^2}\right)'. \tag{28}$$

Their time evolution does not provide any more constraints on the dynamics of $\mathcal{L}_2^{(0)}$. Overall, there are four constraints and the structure of the Poisson brackets between them reads

$$\{\phi_i(r), \phi_j(r')\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d(r,r') \\ 0 & 0 & 0 & -d(r,r') \\ 0 & -d(r,r') & d(r,r') & 0 \end{pmatrix}, \tag{29}$$

where d is a nonzero distribution. Constraint ϕ_1 is clearly a first-class constraint, while the matrix shows that a linear combination $\phi_2 + \phi_3$ is also a first-class constraint. The remaining two independent constraints ϕ_3, ϕ_4 are then second-class, which means the constraints in total remove three physical degrees of freedom. There were only three degrees of freedom in our problem described by $\mathcal{L}_2^{(0)}$, which means none of them is a physical degree of freedom. This can also be seen directly from the $\mathcal{L}_2^{(0)}$ Lagrangian itself. Variation with respect to A_r and A_t produce constraints on π rather than an equation of motion, in particular $\dot{\pi}' = 0$ or $\phi_4 = 0$.

In the special case of no vector background $v = 0$, the decomposition of Eq. (24) in fact leads to the same $\mathcal{L}_2^{(0)}$ as Eq. (25) since the additional term from $\delta^{\rm K}_{v0}$ enters as a total time derivative to $\mathcal{O}(m^0)$. Thus, since $C = 1$

$$\mathcal{L}_2^{(0)} = \frac{3}{8} r^2 \sin\theta \left(\pi'^2 - \dot{\pi}^2\right), \tag{30}$$

which appears to be a normal kinetic term for π that is a ghost in this $\alpha_3 = \alpha_4 = 0$ theory and potentially healthy in other cases studied by Ref. [7]. The vector component from (A_t, A_r) appears to be a strongly coupled degree of freedom with no kinetic term or coupling to the scalar at quadratic level. Although we are left with the correct answer of one degree of freedom, it does not have the same dynamics as the correct expansion in m since the equation of motion for π admits wavelike solutions. Furthermore, the number of degrees of freedom would appear to be discontinuous as $v \to 0$, unlike in the correct analysis.

Thus for no v does the Minkowski scaling limit provide the correct answer. The scaling with m implied by canonical normalization of the degrees of freedom there does not carry over to the self-accelerating solution where the kinetic structure begins at first order in the curvature correction to the Minkowski limit.

C. Gauge Fixing

Ref. [20] in fact came to the conclusion that in the Minkowski scaling limit, the $v \neq 0$ case with a vector background propagates two degrees of freedom rather than the (also erroneous) zero degrees of freedom shown in the previous section. We shall now show that that conclusion arises from fixing a gauge condition directly in the Lagrangian rather than at the level of the equations of motion.

In the Minkowski scaling limit the introduction of the additional Stückelberg field π in Eq. (24) restores $U(1)$ gauge symmetry to the vector A_μ. In this limit, we can take advantage of the gauge freedom to eliminate the non-dynamical field A_t. In particular Ref. [20] chose a Lorenz gauge condition to demand that A_μ be divergence-free

$$\dot{A}_t - \frac{1}{r^2}\left(r^2 A_r\right)' = 0. \tag{31}$$

(We again stress that A_t, A_r as defined are not components of a spacetime vector beyond the Minkowski limit.) This condition can always be satisfied, because the Lagrangian (25) is invariant under a $U(1)$ symmetry

$$A_\mu \to A_\mu + \partial_\mu \varphi, \tag{32}$$

where φ is an arbitrary scalar function.

If we use the divergence-free condition at the Lagrangian level, we can rewrite one of the terms as

$$\int dt\, dr\, r^2 \pi' \dot{A}_r \to \int dt\, dr\, r^2 \dot{\pi} \dot{A}_t, \tag{33}$$

where we omitted unimportant numerical factors. Together with the term $\sim \dot{\pi}^2$ we then have two kinetic terms, which can be diagonalized to give two propagating degrees of freedom. This is in contradiction with the result of the previous section.

The operation which upset the counting of degrees of freedom is the use of the divergence-free condition (33) at the Lagrangian level. This is not allowed, because the condition (31) is a mere fixing of the gauge redundancy brought about by the introduction of π in Eq. (24).

Given the $U(1)$ gauge symmetry, it is perhaps useful to illustrate the problem in the more familiar setting of classical electromagnetism. The Maxwell Lagrangian

$$\mathcal{L}^{(\rm EM)} = -\frac{1}{4}\sqrt{-g} F_{\mu\nu} F^{\mu\nu} \tag{34}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ propagates two degrees of freedom due to presence of two first-class constraints. Since it possesses the same $U(1)$ symmetry of Eq. (32), we can choose

$$A_t = 0, \tag{35}$$

which defines the so-called temporal gauge. If we impose this condition on the Lagrangian level and drop all terms with A_t in the Minkowski limit of Eq. (34), we lose the constraint that it imposes. The result is a Lagrangian with three degrees of freedom rather than the correct two. This is the same problem that occurs by gauge fixing the Minkowski scaling limit Lagrangian (25) except that the spherical symmetry assumption eliminates the two correct degrees of freedom.

IV. COVARIANT PERTURBATIONS

In §IV A, we construct a manifestly covariant form for the quadratic Lagrangian for all Stückelberg and metric perturbations and all parameters of the massive gravity model extending the techniques of Ref. [24]. This form involves tensors constructed from the self-accelerating background solution, obtained in exact form for any isotropic solution, including inhomogeneous ones in §IV B. We use these relations to study the kinetic structure of the quadratic Lagrangian in the exact background and the locally flat expansion in §IV C. Finally in §IV D we apply these general results to the specific case studied in §III.

A. Covariant Quadratic Lagrangian

Given that the kinetic terms of the Stückelberg fields only appear at $\mathcal{O}(m^2)$ or equivalently as a curvature correction to the Minkowski limit, ϕ^a cannot be viewed as a spacetime vector even for fluctuations around a locally flat patch. Instead they transform as a vector in the fiducial or internal space and as 4 scalars in the spacetime. Nonetheless by aligning the tetrad of the spacetime metric with the internal space by a choice of vierbein, we can construct objects from the Stückelberg scalars that transform as vectors in the background spacetime [24]. From these objects we can construct a manifestly covariant quadratic Lagrangian for the Stückelberg and metric perturbations.

In our case where we know the solution to the Stückelberg fields in the background, this construction is particularly simple [17, 26, 27]. Given that

$$\bar{\gamma}^{\mu}{}_{\alpha}\bar{\gamma}^{\alpha}{}_{\nu} = \bar{g}^{\mu\alpha}\bar{\Sigma}_{\alpha\nu}, \tag{36}$$

and that $\partial_\mu \phi^a$ is an inverse vierbein of the fiducial metric $\Sigma_{\mu\nu}$, it is easy to show that the quantity $e^{\mu}{}_{a}$ constructed as the matrix manipulation of

$$\bar{\gamma}^{\mu}{}_{\nu} = e^{\mu}{}_{a}\partial_\nu \bar{\phi}^a \tag{37}$$

is a vierbein of the background spacetime metric

$$\bar{g}_{\mu\nu} e^{\mu}{}_{a} e^{\nu}{}_{b} = \eta_{ab}. \tag{38}$$

Thus

$$\Sigma^{\mu}{}_{\nu} = g^{\mu\alpha}(e^{\rho}{}_{a}\partial_\alpha\phi^a)(e^{\sigma}{}_{b}\partial_\nu\phi^b)\bar{g}_{\rho\sigma} \tag{39}$$

is constructed out of an object that now transforms as a tensor in the background spacetime

$$\begin{aligned}\bar{g}_{\mu\sigma} e^{\sigma}{}_{a}\partial_\nu\phi^a &= \bar{g}_{\mu\sigma} e^{\sigma}{}_{a}\partial_\nu(\bar{\phi}^a + \delta\phi^a) \\ &= \bar{\gamma}_{\mu\nu} + \bar{g}_{\mu\sigma} e^{\sigma}{}_{a}\partial_\nu(\delta\phi^a).\end{aligned} \tag{40}$$

Note that we raise and lower spacetime indices with the background metric to leading order. Although this quantity transforms as a spacetime tensor, its relation to the spacetime vector built out of the Stückelberg fields

$$V^\mu = e^{\mu}{}_{a}\delta\phi^a \tag{41}$$

requires the introduction of connection coefficients [24]

$$\begin{aligned}\bar{g}_{\nu\sigma} e^{\sigma}{}_{a}\partial_\mu(\delta\phi^a) &= \partial_\mu V_\nu - C^{\sigma}_{\mu\nu}V_\sigma \\ &= V_{\nu;\mu} - [C^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\mu\nu}]V_\sigma,\end{aligned} \tag{42}$$

Here Γ is the usual Christoffel symbol formed from $\bar{g}_{\mu\nu}$ and defines covariant derivatives or parallel transport of vectors in the spacetime. C is the connection coefficient associated with the space-time dependence of the alignment of the internal space and tetrad encapsulated by the change in the vierbein

$$C^{\sigma}_{\mu\nu} = \partial_\mu(\bar{g}_{\nu\lambda} e^{\lambda}{}_{a})\bar{g}^{\rho\sigma}[e^{-1}]^{a}{}_{\rho}. \tag{43}$$

Note that

$$C'^{\sigma}_{\mu\nu} = C^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\mu\nu} \tag{44}$$

is the difference of two connections and thus transforms as a tensor even though connection coefficients do not.

We can now characterize the quadratic Lagrangian of the gravitational sector in terms of these variables. For notational simplicity we divide the Lagrangian into terms involving the Stückelberg fields, metric perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{45}$$

and cross terms. For convenience we factor out common terms following the spherically symmetric results of Ref. [18] and break the terms into component pieces

$$\begin{aligned}\mathcal{L}_2 = P_1'(x_0)m^2 M_{\rm pl}^2\sqrt{-\bar{g}}\,(L_{SS} + L_{Sh} + L_{hh}) \\ + \mathcal{L}_{hh}^{(\rm EH)} + \mathcal{L}_{hh}^{(\Lambda)},\end{aligned} \tag{46}$$

where $P_1'(x_0)$ is a model parameter dependent constant whose definition we will give in Eq. (60).

The quadratic Lagrangian for the pure Stückelberg terms should then take the form

$$L_{SS} = B^{\mu\nu\alpha\beta}[V_{\nu;\mu} - C'^{\rho}_{\mu\nu}V_\rho][V_{\beta;\alpha} - C'^{\sigma}_{\alpha\beta}V_\sigma], \tag{47}$$

where B is a tensor formed from background quantities $\bar{g}_{\mu\nu}, \bar{\gamma}_{\mu\nu}$. Unlike Ref. [24] we factor out $\sqrt{-\bar{g}}$ so that B transforms as a tensor.

We can similarly determine the functional form of the coupling of the Stückelberg fields to the metric perturbation

$$L_{Sh} = D^{\mu\nu\alpha\beta}h_{\mu\nu}\left(V_{\beta;\alpha} - C'^{\sigma}{}_{\alpha\beta}V_\sigma\right), \tag{48}$$

where D is a tensor constructed out of the background quantities.

Finally, the Lagrangian quadratic in the metric perturbations can be split into a part coming from the Einstein-Hilbert action and a part coming from massive gravity. The Einstein-Hilbert piece takes the same form as in general relativity (e.g. [28, 29])

$$\begin{aligned}\frac{\mathcal{L}_{hh}^{(\rm EH)}}{\sqrt{-\bar{g}}M_{\rm pl}^2} &= \left(\frac{1}{2}h^{\mu\alpha}h_{\alpha}{}^{\nu} - \frac{1}{4}hh^{\mu\nu}\right)\bar{R}_{\mu\nu} \\ &+ \left(\frac{1}{16}h^2 - \frac{1}{8}h_{\mu\nu}h^{\mu\nu}\right)\bar{R} - \frac{1}{8}h^{\mu\nu;\alpha}h_{\mu\nu;\alpha} \\ &+ \frac{1}{4}h^{\mu\nu;\alpha}h_{\nu\alpha;\mu} + \frac{1}{8}h_{;\alpha}h^{;\alpha} - \frac{1}{4}h^{\mu\nu}{}_{;\nu}h_{;\mu},\end{aligned} \tag{49}$$

where $h = h^{\alpha}{}_{\alpha}$. For the massive gravity metric-metric terms, first we have the term that depends on the effective cosmological constant of the self-accelerating background which represents a non-dynamical change in the measure. To see this note that a true cosmological constant has a contribution to the action of

$$\mathcal{L}^{(\Lambda)} = -M_{\rm pl}^2\sqrt{-g}\Lambda, \tag{50}$$

and its non-dynamical quadratic metric terms are given by the expansion

$$\sqrt{-g} \approx \sqrt{-\bar g}\left[1 + \frac{1}{2}h + \frac{1}{2}\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu}\right)\right] \tag{51}$$

as

$$\frac{\mathcal{L}^{(\Lambda)}_{hh}}{\sqrt{-\bar g}M_{\rm pl}^2} = \left(\frac{1}{4}h_{\mu\nu}h^{\mu\nu} - \frac{1}{8}h^2\right)\Lambda. \tag{52}$$

This piece will cancel terms in the Einstein-Hilbert Lagrangian by virtue of the Einstein equations in the background. We shall see this feature explicitly in the construction of the perturbed stress energy tensor below. The remaining massive gravity terms can be parameterized as

$$L_{hh} = E^{\mu\nu\alpha\beta}h_{\mu\nu}h_{\alpha\beta}, \tag{53}$$

where $E^{\mu\nu\alpha\beta}$ is a tensor that depends on the background quantities.

This completes the general description of the structural form for the covariant quadratic Lagrangian derived of the gravitational sector. We now turn to the construction of the background tensors B, D, E.

B. Fluctuations around Isotropic Backgrounds

For all isotropic background solutions on the self-accelerating branch [12], there is a single universal form for the relationship between B, D, E and the background tensors $\bar g_{\mu\nu}$, $\bar\gamma_{\mu\nu}$. This includes the vacuum self-accelerating solutions of Ref. [7] as well as its approximation in conformal coordinates, Eq. (11) that was considered in §III. It also includes the special cases of the open self-accelerating solution [11] which is known to propagate no extra degrees of freedom from the mass term at quadratic level.

We therefore utilize the general construction of Ref. [12]. As some aspects of this construction will be useful for extracting the background tensors, we review its salient features here. Any spherically symmetric metric can be written in isotropic coordinates as

$$\bar g_{\mu\nu}{\rm d}x^\mu {\rm d}x^\nu = -b^2(t,r){\rm d}t^2 \;+ a^2(t,r)\big({\rm d}r^2 + r^2{\rm d}\theta^2 + r^2\sin^2\theta{\rm d}\phi^2\big), \tag{54}$$

whereas the background Stückelberg fields can again be given by the isotropic form of Eq. (15). The spacetime metric is diagonal in these coordinates and $\bar g^{\mu\alpha}\bar\Sigma_{\alpha\nu} = \bar\gamma^{\mu}{}_{\alpha}\bar\gamma^{\alpha}{}_{\nu}$ has off diagonal entries only in the (t,r) cases. It is convenient to use matrix notation here and so we define the (t,r) block as

$$\bar{\boldsymbol\gamma}_2 \equiv \begin{pmatrix} \bar\gamma^t{}_t & \bar\gamma^t{}_r \\ \bar\gamma^r{}_t & \bar\gamma^r{}_r \end{pmatrix}. \tag{55}$$

Note that although $\gamma_{\mu\nu}$ is symmetric $\gamma^{\mu}{}_{\nu}$ is not. Its square is related to the background Stückelberg fields as

$$\bar{\boldsymbol\gamma}_2\bar{\boldsymbol\gamma}_2 = \begin{pmatrix} \dfrac{\dot T^2 - \dot R^2}{b^2} & \dfrac{\dot T T' - \dot R R'}{b^2} \\ \dfrac{\dot R R' - \dot T T'}{a^2} & \dfrac{R'^2 - T'^2}{a^2} \end{pmatrix}. \tag{56}$$

The general solution to the matrix square root is given by the Cayley-Hamilton theorem

$$[\bar{\boldsymbol\gamma}_2]\bar{\boldsymbol\gamma}_2 = \bar{\boldsymbol\gamma}_2\bar{\boldsymbol\gamma}_2 + ({\rm det}\bar{\boldsymbol\gamma}_2)\mathbf{I}_2, \tag{57}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix. The determinant can be written in terms of the determinant of the square of the matrix and hence in terms of the Stückelberg background

$${\rm det}\bar{\boldsymbol\gamma}_2 = \frac{\dot T R' - \dot R T'}{ab}, \tag{58}$$

and the trace similarly by taking the trace of Eq. (57). Using this solution in the Lagrangian, we obtain the equations of motion for the background Stückelberg fields and find that on the self-accelerating branch

$$\begin{aligned} R(t,r) &= x_0 r a(t,r), \\ [\bar{\boldsymbol\gamma}_2] &= \frac{1}{x_0}{\rm det}\bar{\boldsymbol\gamma}_2 + x_0, \end{aligned} \tag{59}$$

where x_0 is a constant that solves $P_1(x_0) = 0$ with

$$P_1(x) = 2(3-2x)+6(x-1)(x-3)\alpha_3+24(x-1)^2\alpha_4. \tag{60}$$

The second equation may be rewritten as an equation of motion for $T(t,r)$

$$\begin{aligned} b^2T'^2 + 2ar(a'\dot T^2 - \dot a\dot T T') + r^2(a'\dot T - \dot a T')^2 \\ = x_0^2\left(a'^2b^2r^2 + 2a'ab^2r - \dot a^2a^2r^2\right). \end{aligned} \tag{61}$$

These solutions then require

$$\bar\gamma^\theta{}_\theta = \bar\gamma^\phi{}_\phi = x_0. \tag{62}$$

Note that in terms of the massive gravity parameter dependence $T, R, \bar\gamma \propto x_0$.

These solutions imply an effective cosmological constant in the background stress energy tensor

$$\bar T_{\mu\nu} = -\Lambda M_{\rm pl}^2\bar g_{\mu\nu} = -\frac{1}{2}P_0(x_0)m^2M_{\rm pl}^2\bar g_{\mu\nu}, \tag{63}$$

where

$$\begin{aligned} P_0(x) = -12 - 2x(x-6) - 12(x-1)(x-2)\alpha_3 \\ - 24(x-1)^2\alpha_4. \end{aligned} \tag{64}$$

Knowing $\bar{\boldsymbol{\gamma}}$ and the background Stückelberg fields, we can construct the vierbein $e^{\mu}_{\ a}$ by solving Eq. (37).

One useful consequence of Eqs. (57) and (59) is that a certain combination of $\bar{g}_{\mu\nu}$ and $\bar{\gamma}_{\mu\nu}$

$$\bar{\chi}_{\mu\nu} = \frac{1}{x_0}\bar{\gamma}_{\mu\nu} - \bar{g}_{\mu\nu} \tag{65}$$

obeys special properties. First note that it is independent of the model parameter choices for m, α_3, α_4. It is only non-zero for $\bar{\chi}_{tt}, \bar{\chi}_{tr}$ and $\bar{\chi}_{rr}$. Defining again this 2×2 block with upper and lower indices as $\boldsymbol{\chi}_2$ these components satisfy

$$[\bar{\boldsymbol{\chi}}_2]\bar{\boldsymbol{\chi}}_2 = \bar{\boldsymbol{\chi}}_2\bar{\boldsymbol{\chi}}_2, \tag{66}$$

or equivalently $\det(\bar{\boldsymbol{\chi}}_2) = 0$. More explicitly, after lowering indices

$$\bar{\chi}_{tt}\bar{\chi}_{rr} = \bar{\chi}_{tr}^2. \tag{67}$$

Likewise we can write Eq. (66) in 4×4 index notation as

$$\bar{\chi}_{\mu\nu}\bar{g}^{\nu\alpha}\bar{\chi}_{\alpha\beta} = [\bar{\boldsymbol{\chi}}]\bar{\chi}_{\mu\beta}. \tag{68}$$

We can now construct the tensors B, E and D from these background tensors, specifically $\bar{g}_{\mu\nu}$ and $\bar{\chi}_{\mu\nu}$. Beginning with B, we need to determine the perturbation to the $\gamma^{\mu}_{\ \nu}$ solution given a Stückelberg perturbation in its square $\Sigma^{\mu}_{\ \nu}$. To determine values of components $B^{\mu\nu\alpha\beta}$ it is sufficient to keep track of the coefficient of $V_{\nu;\mu}V_{\beta;\alpha}$ in the expansion of the Lagrangian Eq. (1) to second order in the perturbations V. This in turns means we need expansion of $\gamma^{\mu}_{\ \nu}$ to second order in V.

We start with the defining relation

$$\Sigma^{\mu}_{\ \nu} = \gamma^{\mu}_{\ \alpha}\gamma^{\alpha}_{\ \nu} \tag{69}$$

and expand the tensor Σ order by order in V,

$$\Sigma^{\mu}_{\ \nu} = \bar{\Sigma}^{\mu}_{\ \nu} + {\Sigma^{(1)}}^{\mu}_{\ \nu} + {\Sigma^{(2)}}^{\mu}_{\ \nu} + \ldots, \tag{70}$$

and similarly for γ. The various ${\Sigma^{(i)}}^{\mu}_{\ \nu}$ can be directly obtained in terms of background quantities $\bar{\gamma}, \bar{g}$ and V from Eqs. (39), (40) and (42).

Using the zeroth order solution

$$\bar{\Sigma}^{\mu}_{\ \nu} = \bar{\gamma}^{\mu}_{\ \alpha}\bar{\gamma}^{\alpha}_{\ \nu}, \tag{71}$$

we match orders as

$$\begin{aligned} {\Sigma^{(1)}}^{\mu}_{\ \nu} &= \bar{\gamma}^{\mu}_{\ \alpha}{\gamma^{(1)}}^{\alpha}_{\ \nu} + {\gamma^{(1)}}^{\mu}_{\ \alpha}\bar{\gamma}^{\alpha}_{\ \nu}, \\ {\Sigma^{(2)}}^{\mu}_{\ \nu} &= \bar{\gamma}^{\mu}_{\ \alpha}{\gamma^{(2)}}^{\alpha}_{\ \nu} + {\gamma^{(1)}}^{\mu}_{\ \alpha}{\gamma^{(1)}}^{\alpha}_{\ \nu} + {\gamma^{(2)}}^{\mu}_{\ \alpha}\bar{\gamma}^{\alpha}_{\ \nu}. \end{aligned} \tag{72}$$

Each order represents 16 linear equations for components of $\gamma^{(i)}$ and can be readily solved iteratively.

With the explicit form for γ up to second order in Stückelberg perturbations, we can perturb the Lagrangian density $\mathcal{L}^{(\rm MG)}$ and read off L_{SS}. The coefficients of the various terms form the B-tensor. For a general spherically symmetric background solution, it is possible to express these components in terms of the background tensors $\bar{\chi}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. Because of the relation (68) and definition (65), all tensor structures involving more than two gamma matrices contracted with an inverse metric such as

$$\bar{\gamma}_{\alpha\beta}\bar{g}^{\beta\kappa}\bar{\gamma}_{\kappa\rho}\bar{g}^{\rho\sigma}\bar{\gamma}_{\sigma\delta} \tag{73}$$

can be written as a linear combination of $\bar{\chi}_{\alpha\delta}$, $\bar{g}_{\alpha\delta}$ with coefficients which are spacetime scalars built out of traces $[\bar{\boldsymbol{\chi}}]$. This relation greatly reduces the number of terms we have to take into account as only 12 of them are in principle independent, such as $\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}$ and $\bar{\chi}_{\mu\beta}\bar{g}_{\nu\alpha}$. Coefficients in front of these terms must be spacetime scalars, which must be functions of the trace $[\bar{\boldsymbol{\chi}}]$. In principle these scalars would depend also on $\bar{\gamma}^{\alpha}_{\ \beta}\bar{\gamma}^{\beta}_{\ \alpha}, \bar{\gamma}^{\alpha}_{\ \beta}\bar{\gamma}^{\beta}_{\ \rho}\bar{\gamma}^{\rho}_{\ \alpha}, \ldots$ but in the present case these can be expressed as functions of $[\bar{\boldsymbol{\chi}}]$ by a (repeated) use of Eq. (68). Taking into account the relation (67) for the off-diagonal elements $\bar{\chi}_{tr}$, it is possible to reduce the coefficients in front of the tensorial structures into simple forms.

The Stückelberg-Stückelberg Lagrangian obtained in this manner can be written as

$$L_{SS} = (\tilde{B}^{\mu\nu\alpha\beta} - \Delta B^{\mu\nu\alpha\beta})[V_{\nu;\mu} - C'^{\rho}_{\mu\nu}V_{\rho}][V_{\beta;\alpha} - C'^{\sigma}_{\alpha\beta}]. \tag{74}$$

Here we have separated out a term that is a total derivative and hence may be dropped from the Lagrangian

$$\begin{aligned}\Delta B^{\mu\nu\alpha\beta} = \frac{x_0 P_2'(x_0)}{8P_1'(x_0)}\Big[&(1+[\bar{\boldsymbol{\chi}}])(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\alpha\nu}) \\ &+(\chi^{\mu\beta}g^{\nu\alpha} + \chi^{\nu\alpha}g^{\mu\beta} - \chi^{\mu\nu}g^{\beta\alpha} - \chi^{\alpha\beta}g^{\nu\mu})\Big]\end{aligned} \tag{75}$$

with

$$P_2(x) = -2 + 12\alpha_3(x-1) - 24\alpha_4(x-1)^2 \tag{76}$$

from the dynamical piece which itself can be broken up into terms that are symmetric and antisymmetric in permutation of indices

$$\tilde{B}^{\mu\nu\alpha\beta} = \tilde{B}^{(\mu\nu)(\alpha\beta)} + \tilde{B}^{[\mu\nu][\alpha\beta]}, \tag{77}$$

where

$$\begin{aligned}\tilde{B}^{(\mu\nu)(\alpha\beta)} &= -\frac{1}{8}\left(\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} - \frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} - \frac{1}{2}\bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right), \\ \tilde{B}^{[\mu\nu][\alpha\beta]} &= \frac{1}{16}\left(\bar{\chi}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{\chi}^{\nu\beta}\bar{g}^{\mu\alpha} - \bar{\chi}^{\mu\beta}\bar{g}^{\nu\alpha} - \bar{\chi}^{\nu\alpha}\bar{g}^{\mu\beta}\right) \\ &\quad - \frac{1+[\bar{\boldsymbol{\chi}}]}{16}\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} - \bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right).\end{aligned} \tag{78}$$

We can form an alternate representation of the tensor B by removing any combination of the total derivative term. In particular the form

$$B^{\mu\nu\alpha\beta} = \tilde{B}^{\mu\nu\alpha\beta} + \frac{P_1'(x_0)}{x_0 P_2'(x_0)}\Delta B^{\mu\nu\alpha\beta}, \tag{79}$$

or explicitly

$$B^{\mu\nu\alpha\beta} = \frac{[\bar{\chi}]}{8}\left(\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} - \frac{1}{2}\bar{g}^{\mu\beta}\bar{g}^{\nu\alpha} - \frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\right) + \frac{1}{16}\left(\bar{g}^{\mu\alpha}\bar{\chi}^{\nu\beta} + \bar{g}^{\nu\beta}\bar{\chi}^{\mu\alpha} + \bar{g}^{\mu\beta}\bar{\chi}^{\nu\alpha} + \bar{g}^{\nu\alpha}\bar{\chi}^{\mu\beta}\right) - \frac{1}{8}\left(\bar{g}^{\mu\nu}\bar{\chi}^{\alpha\beta} + \bar{g}^{\alpha\beta}\bar{\chi}^{\mu\nu}\right), \tag{80}$$

is useful as we shall see below. To keep these representations distinct we reserve the B tensor symbol for this form. Note that it is symmetric under the exchange of the first or last two indices.

Similarly we can determine expressions for $D^{\mu\nu\alpha\beta}$ for the Stückelberg-metric terms of Eq. (48) and $E^{\mu\nu\alpha\beta}$ for the metric-metric terms of Eq. (53)

$$D^{\mu\nu\alpha\beta} = -x_0 B^{\mu\nu\alpha\beta}, \qquad E^{\mu\nu\alpha\beta} = \frac{x_0^2}{4}B^{\mu\nu\alpha\beta}. \tag{81}$$

Note that these expressions contain contributions from varying both $\sqrt{-g}$ and Σ with respect to the metric.

Thus the whole quadratic Lagrangian can be written very compactly as

$$\mathcal{L}_2 = \mathcal{L}^{(\mathrm{EH})}_{hh} + \mathcal{L}^{(\Lambda)}_{hh} + P_1'(x_0)m^2 M_{\rm pl}^2\sqrt{-\bar{g}}B^{\mu\nu\alpha\beta}W_{\mu\nu}W_{\alpha\beta}, \tag{82}$$

where

$$W_{\mu\nu} = V_{\nu;\mu} - C'^{\rho}_{\mu\nu}V_\rho - \frac{x_0 h_{\mu\nu}}{2}. \tag{83}$$

This result represents the full quadratic Lagrangian of the gravitational sector in any isotropic self-accelerating branch solution of the theory. It trivially allows the addition of minimally coupled matter but does not necessarily hold beyond the isotropic assumption. Interestingly this includes the case where the background spacetime metric is exactly Minkowski due to the canceling impact of a bare cosmological constant. This case is still not the same as the Minkowski decoupling limit, since self-accelerating branch solutions always have non-trivial Stückelberg backgrounds given by Eq. (59). As we shall see, this generalizes the result of the previous section, that the locally flat expansion of a self-accelerating solution is not the same as the Minkowski decoupling limit.

With the explicit formulae for D and E and expansions (48), (53) we can also construct the linear fluctuations in the stress energy tensor away from the self-accelerating background of Eq. (63),

$$\delta T_\mu^{\ \nu} = -\frac{2}{\sqrt{-g}}g^{\nu\alpha}\frac{\delta\mathcal{L}_{\rm MG}}{\delta g^{\mu\alpha}} - \bar{T}_\mu^{\ \nu} \approx -2P_1'(x_0)m^2M_{\rm pl}^2 x_0 B_\mu^{\ \nu\alpha\beta}W_{\alpha\beta}. \tag{84}$$

It is now clear why we grouped terms in Eq. (53). Since the stress energy fluctuation is the source of $h_{\mu\nu}$ through the Einstein equations, these are the only terms with dynamical impact on the metric. The stress tensor constructed in this way through D and E agrees with the expansion of the exact result [12] and serves as a check on their derivation. Note that the equations of motion derived from the quadratic Lagrangian satisfy covariant conservation of the massive gravity stress-energy tensor $\nabla^\mu T_{\mu\nu} = 0$ regardless of the matter content.

C. Kinetic Structure

Although the quadratic Lagrangian of Eq. (82) with the explicit form for the background B tensor of Eq. (80) is complete, its implication for the dynamics of the Stückelberg fields V^μ is not yet explicit. Is is therefore useful to further isolate the pieces associated with Maxwell type terms involving the antisymmetric field strength tensor

$$f_{\mu\nu} = V_{\nu;\mu} - V_{\mu;\nu} \tag{85}$$

and reorganize the terms in L_{SS} by the number of appearances of the field strength tensor

$$L_{SS} = L_{ff} + L_{fV} + L_{VV}. \tag{86}$$

Reducing the Lagrangian to this form is simpler in the $\tilde{B}$ representation of Eq. (78). First note that we can add total derivatives to rewrite

$$\begin{aligned}
&\tilde{B}^{(\mu\nu)(\alpha\beta)}V_{\nu;\mu}V_{\beta;\alpha} + \frac{1}{8}\left(V^\mu V^\nu_{\ ;\nu}\right)_{;\mu} - \frac{1}{8}\left(V^\mu V^\nu_{\ ;\mu}\right)_{;\nu} \\
&= \frac{1}{16}V_{\mu;\nu}V^{\mu;\nu} - \frac{1}{16}V_{\mu;\nu}V^{\nu;\mu} - \frac{1}{8}V^\mu\left(V^\nu_{\ ;\mu\nu} - V^\nu_{\ ;\nu\mu}\right) \\
&= \frac{1}{32}f_{\mu\nu}f^{\mu\nu} - \frac{1}{8}V^\mu V^\sigma \bar{R}_{\sigma\mu}.
\end{aligned} \tag{87}$$

$\bar{R}_{\sigma\mu}$ denotes the usual Ricci tensor built out of the background metric $\bar{g}$.

After similar integrations by parts, we arrive at the result

$$\begin{aligned}
L_{ff} &= -\frac{1}{32}[\bar{\chi}]f_{\mu\nu}f^{\mu\nu} + \frac{1}{16}\bar{\chi}^\nu_{\ \beta}f_{\mu\nu}\bar{f}^{\mu\beta}, \\
L_{fV} &= C'^{\alpha}_{\mu\nu}(2\tilde{B}^{(\mu\nu)(\rho\sigma)}V_\sigma f_{\rho\alpha} - \tilde{B}^{[\mu\nu][\rho\sigma]}V_\alpha f_{\rho\sigma}), \\
L_{VV} &= \Big[2\tilde{B}^{(\mu\nu)(\alpha\sigma)}C'^{\rho}_{\mu\nu;\alpha} - \tilde{B}^{(\mu\nu)(\rho\sigma)}C'^{\alpha}_{\mu\nu;\alpha} \\
&\quad + \tilde{B}^{\mu\nu\alpha\beta}C'^{\rho}_{\mu\nu}C'^{\sigma}_{\alpha\beta} - \frac{1}{8}\bar{R}^{\rho\sigma}\Big]V_\sigma V_\rho.
\end{aligned} \tag{88}$$

In simplifying the expressions we have integrated by parts and used the fact that $\tilde{B}^{(\mu\nu)(\alpha\beta)}_{\quad\quad\quad ;\rho=0}$ as it is constructed from products of the metric in Eq. (78).

The only place that time derivatives appear in the SS terms are in the ff and fV pieces. Given the antisymmetry of $f_{\mu\nu}$ it is clear that the field V_t is nondynamical reflecting the absence of the Boulware-Deser ghost. This structure of the Stückelberg Lagrangian is expected based on general theoretical arguments [24].

Now consider the *ff* terms that would usually provide quadratic kinetic terms and hence second order equations of motion. Inspection of Eq. (88) shows that the f_{tr}^2 term always vanishes identically. This in turn means that around spherically symmetric solutions, there is no Maxwell term for spherically symmetric perturbations, which is in full agreement with the investigations of previous sections and with the full theory [18, 19].

The terms $f_{t\theta}^2, f_{t\phi}^2$ have coefficients that are proportional to

$$\bar{\chi}_{rr} \propto \frac{R'^2 - T'^2}{x_0^2} - a^2, \tag{89}$$

and give Maxwell-like kinetic terms to the transverse modes when non-vanishing. Note that for the special open universe solution of Ref. [11], the fiducial metric is diagonal in isotropic coordinates and this quantity vanishes. Thus the strongly-coupled anisotropic modes of that model is an artifact of this special symmetry that is imposed.

The remaining kinetic terms are first order. From the *ff* term, we have the mixed terms $f_{t\theta}f_{r\theta}$, $f_{t\phi}f_{r\phi}$ which appear with coefficients proportional to

$$\bar{\chi}_{tr} = \sqrt{\bar{\chi}_{rr}\bar{\chi}_{tt}}. \tag{90}$$

Compared with Eq. (89), this means that the mixed terms will scale differently from the pure kinetic Maxwell terms due by a factor of $\sqrt{\bar{\chi}_{tt}/\bar{\chi}_{rr}}$.

The fV terms have a general structure

$$\begin{aligned} L_{fV} = & K_1\left(V_\theta f_{r\theta}\sin^2\theta + V_\phi f_{r\phi}\right) \\ & + K_2\left(V_\theta f_{t\theta}\sin^2\theta + V_\phi f_{t\phi}\right) \\ & + K_3 V_t f_{tr} + K_4 V_r f_{tr}. \end{aligned} \tag{91}$$

The last two coefficients can be rewritten in a succinct form

$$\begin{aligned} K_3 &= \frac{bR'\mathrm{det}\bar{\gamma}_2 - x_0^2 a\dot{T}}{2x_0[\bar{\gamma}_2]a^2b^3R}, \\ K_4 &= \frac{x_0^2 bT' - a\dot{R}\mathrm{det}\bar{\gamma}_2}{2x_0[\bar{\gamma}_2]a^3b^2R}, \end{aligned} \tag{92}$$

while the expressions for K_1, K_2 are more involved and will not be given here. Note that the K_1 term is nondynamical as is K_2 and K_4 since, e.g.

$$V_\phi f_{t\phi} = \frac{1}{2}\frac{\partial V_\phi^2}{\partial t} - V_\phi V_{t,\phi} \tag{93}$$

so that the time derivative can be moved onto the background by integration by parts. For the special case of the open universe solution [11], $K_3 = 0$ and combined with the angular terms this means that all 3 Stückelberg fields are non-dynamical.

In the general case Stückelberg dynamics are supplied by the terms $f_{t\theta}^2$, $f_{t\phi}^2$, $f_{t\theta}f_{r\theta}$, $f_{t\phi}f_{r\phi}$ and $V_t f_{tr}$. It is interesting to generalize the considerations of §III A for fluctuations around a locally flat patch to see at what order in curvature corrections that each contributes. Any isotropic metric can be considered locally as Minkowski plus curvature corrections and hence

$$a, b = 1 + \mathcal{O}(m^2). \tag{94}$$

For notational simplicity we have here assumed vacuum self-acceleration cases here; more generally we would replace $\mathcal{O}(m^2)$ with $\mathcal{O}(\bar{R})$. Thus given Eq. (59) for the exact solution, we may approximate

$$R = x_0 r + \mathcal{O}(m^2). \tag{95}$$

The other Stückelberg equation of motion (61) then implies

$$T'^2 = \mathcal{O}(m^2) \tag{96}$$

which means the unitary gauge time T does not depend on the spatial coordinate in the leading order, $T = T(t) + \mathcal{O}(m)$. With this solution, we can write down the components of the background tensor χ

$$\chi_{\alpha\beta} = \begin{cases} 1 - \dot{T}/x_0 + \mathcal{O}(m) & \text{if } \alpha = t, \beta = t \\ \mathcal{O}(m) & \text{if } \alpha = t, \beta = r \\ \mathcal{O}(m^2) & \text{if } \alpha = r, \beta = r \\ 0 & \text{otherwise} \end{cases}. \tag{97}$$

From Eq. (89) it follows that the kinetic Maxwell terms $f_{t\theta}^2$ and $f_{t\phi}^2$ are at most $\mathcal{O}(m^2)$. The leading order kinetic *ff* terms are $f_{t\theta}f_{r\theta}$, $f_{t\phi}f_{r\phi}$ which appear already at order $\mathcal{O}(m)$ due to the square root in Eq. (90) and $\mathcal{O}(m^2)$ suppression of χ_{rr}.

From Eq. (58)

$$\mathrm{det}\bar{\gamma}_2 = x_0\dot{T} + \mathcal{O}(m^2), \tag{98}$$

and so $K_3 V_t f_{tr}$ also starts at most at $\mathcal{O}(m^2)$.

On the other hand, the spatial derivative terms in the Lagrangian do not necessarily begin at suppressed orders. We find that the space-space Maxwell terms can have contributions at $\mathcal{O}(m^0)$

$$\begin{aligned} L_{ff} = & \frac{1}{16}\left(1 - \frac{\dot{T}}{x_0}\right)\left(f_{r\theta}f^{r\theta} + f_{r\phi}f^{r\phi} + f_{\phi\theta}f^{\phi\theta}\right) \\ & + \mathcal{O}(m). \end{aligned} \tag{99}$$

For the case with $\dot{T} = x_0 + \mathcal{O}(m)$, the terms $2ara'\dot{T}^2$, $2x_0^2 a'ab^2 r$ in the equation of motion cancel in the leading order and we are left with

$$T'^2 = \mathcal{O}(m^4). \tag{100}$$

This means that in fact $T = x_0 t + \mathcal{O}(m^2)$ and

$$\chi_{\alpha\beta} = \mathcal{O}(m^2). \tag{101}$$

In this case, which corresponds to $v = 0$ in the example of §III, all *ff* terms in the Lagrangian are suppressed and start at linear order in curvature $\mathcal{O}(m^2)$. This result is consistent with the vanishing of the Maxwell term for $v = 0$ in the decoupling limit uncovered in Ref. [5].

The fV terms follow a similar pattern. For $\dot{T} \neq x_0 + \mathcal{O}(m)$, the coefficients K_2, K_4 start in the linear order in m, while the other two coefficients K_1, K_3 are suppressed by an additional power of m and start at $\mathcal{O}(m^2)$. If $\dot{T} = x_0 + \mathcal{O}(m)$ then all these coefficients start at the order $\mathcal{O}(m^2)$ and this is thus also order at which we recover the dynamics of the Stückelberg perturbations.

There are also time derivative terms from the Stückelberg-metric contributions. In fact there are two terms with time derivatives on V_t, $h_{\theta\theta}V_{t;t}$ and $h_{\phi\phi}V_{t;t}$ which might seem problematic for the non-dynamical nature of V_t. However, as argued in Ref. [24], these do not change the dynamics and hence the reappearance of the Boulware-Deser ghost because the derivatives can be moved to $h_{\theta\theta}, h_{\phi\phi}$ by integration by parts. This integration by parts leaves V_t manifestly nondynamical, while not disturbing the non-dynamical nature of $h_{0\mu}$ for imposing constraints.

It turns out that in the flat patch approximation Vh coupling gives kinetic mixing terms to the spatial Stückelberg V_r, V_θ, V_ϕ at $\mathcal{O}(m)$ for the case $\dot{T} \neq x_0 + \mathcal{O}(m)$, while in the case without the vector in the background $\dot{T} = x_0 + \mathcal{O}(m^2)$ these kinetic terms start at $\mathcal{O}(m^2)$. The metric-metric Lagrangian has kinetic terms from only the usual Einstein-Hilbert Lagrangian. We thus conclude that as expected the full Lagrangian generally has kinetic terms for the 3 spatial Stückelberg fields and the usual 2 tensor modes for a total of 5 modes. In no case are there Stückelberg kinetic terms at $\mathcal{O}(m^0)$ consistent with §III and Ref. [21]. For special cases they may begin at $\mathcal{O}(m^2)$ or be absent entirely.

D. Example

To make these considerations concrete, we return here to the specific solutions considered in §III. Recall that these solutions are for the $\alpha_3 = \alpha_4 = 0$ case where $P_1'(x_0) = -4$.

For these background solutions we have

$$\begin{aligned}
\bar{\chi}_{tt} &= \left(1 - \frac{1}{C}\right) - \frac{m^2\left(r^2(1+C^2) + t^2v^2C^2\right)}{8C(1+C)}, \\
\bar{\chi}_{tr} &= -\frac{mrv}{2(1+C)} - \frac{m^2 r(tv^2C + 2r\delta^{\rm K}_{v0})}{8(1+C)}, \\
\bar{\chi}_{rr} &= -\frac{m^2r^2}{4C(1+C)},
\end{aligned} \tag{102}$$

plus terms which are higher order in graviton mass. The remaining components are given by the general formulae as described in the previous section. Note that this explicit form is consistent with the general considerations of Eq. (97) and (101) for the $v \neq 0$ and $v = 0$ cases respectively.

Using the results of the previous section, we can then write down the Stückelberg-Stückelberg quadratic Lagrangian as

$$\begin{aligned}
\frac{\sqrt{-\bar{g}}}{\sin\theta} L_{SS} = &-\frac{(4v^2C^2 + m^2r^2)}{64C(1+C)}\left(f_{r\phi}^2 \csc^2\theta + f_{r\theta}^2\right) \\
&-\frac{v^2C\csc^2\theta}{16(1+C)r^2} f_{\theta\phi}^2 - \frac{m^2r^2}{64C(1+C)}\left(f_{t\theta}^2 + f_{t\phi}^2\csc^2\theta\right) \\
&+mr\frac{4v + m(tCv^2 + 2r\delta^{\rm K}_{v0})}{64(1+C)}\left(f_{r\theta}f_{t\theta} + f_{r\phi}f_{t\phi}\csc^2\theta\right) \\
&-\frac{m^2r(2+C)}{16(1+C)}\left(V_\theta f_{r\theta} + V_\phi f_{r\phi}\csc^2\theta\right) \\
&-m\frac{4Cv + 3Cmr\delta^{\rm K}_{v0}}{16(1+C)}\left(V_\theta f_{\theta t} + V_\phi f_{\phi t}\csc^2\theta\right) \\
&-mr^2\frac{mrV_t - (2Cv + Cmr\delta^{\rm K}_{v0})V_r}{8(1+C)} f_{tr} \\
&-\frac{m^2r^2}{16(1+C)}[(1+2C)V_r^2 - 3CV_t^2] \\
&-\frac{m^2(1+2C)}{16(1+C)}\left(V_\theta^2 + V_\phi^2\csc^2\theta\right).
\end{aligned} \tag{103}$$

Even if we ignore kinetic mixing with the metric, a Hamiltonian analysis shows that the Stückelberg-Stückelberg Lagrangian itself propagates both transverse modes V_θ, V_ϕ and the longitudinal mode V_r, giving three dynamical degrees of freedom. This Hamiltonian is unbounded with respect to the spherically symmetric perturbations $V_\theta = V_\phi = 0$. This is related to the unboundedness of a_r, a_t from §III since

$$\begin{aligned}
V_t &= a_t(t,r) + mr\frac{vC}{2(1+C)}a_r(t,r) + \mathcal{O}(m^2), \\
V_r &= a_r(t,r) + mr\frac{vC}{2(1+C)}a_t(t,r) + \mathcal{O}(m^2).
\end{aligned} \tag{104}$$

Note that the $v = 0$ case is also special in that terms from the tetrad alignment do not appear until $\mathcal{O}(m^2)$.

The Stückelberg-Stückelberg Maxwell terms follow the general behavior pointed out in the previous section. For the case of no vector in the background $v = 0$ and $C = 1$, all terms start at most at $\mathcal{O}(m^2)$, while in the other cases spatial derivative terms $f_{r\theta}^2, f_{r\phi}^2$ and $f_{\theta\phi}^2$ start at $\mathcal{O}(m^0)$ and $f_{r\theta}f_{t\theta}, f_{r\phi}f_{t\phi}, V_r f_{tr}, V_\theta f_{\theta t}, V_\phi f_{\phi t}$ start at $\mathcal{O}(m)$.

Focusing on the terms which appear before $\mathcal{O}(m^2)$, only $f_{r\theta}f_{t\theta}, f_{r\phi}f_{t\phi}$ can provide any dynamics in $\mathcal{L}_{SS}$ as the remaining time derivatives can be integrated out. However, as the Hamiltonian analysis shows, the $\mathcal{O}(m)$ Stückelberg-Stückelberg Lagrangian does not propagate all three modes and we have to go to $\mathcal{O}(m^2)$ if we want to capture the correct dynamics with $\mathcal{L}_{SS}$ only. This once more stresses the importance of retaining all $\mathcal{O}(m^2)$ terms in the Lagrangian to correctly describe the dynamics of the system.

V. DISCUSSION

We have provided a complete and covariant treatment for the quadratic Lagrangian of all of the degrees of freedom of massive gravity with a fixed flat fiducial metric around any isotropic self-accelerating background for any set of massive gravity parameters. We find that for generic cases 3 out of 4 Stückelberg degrees of freedom propagate in addition to the usual 2 tensor degrees of freedom of general relativity. The complete kinetic structure typically is only revealed at $\mathcal{O}(m^2)$ or equivalently curvature terms in a locally flat expansion.

These results resolve a number of apparent discrepancies in the literature. The kinetic terms for all additional degrees of freedom vanish in the leading order, Minkowski term in the locally flat approximation and are only fully established at the order of curvature corrections, omitted in the analysis of Ref. [21]. This result differs from the usual Minkowski decoupling limit because on the self-accelerating branch of solutions there is always a non-trivial background Stückelberg field. Because the Minkowski scaling limit is not justified around self-accelerated solutions, analyses that are based on it can be misinterpreted. It is important to distinguish between an imposed scaling of parameters with the graviton mass and a true decoupling limit where degrees of freedom are both preserved and decoupled. It is also important to note that Stückelberg fields restore gauge invariance and the redundancy that exists because of their introduction should be fixed as a gauge freedom. Together they explain the results of Ref. [20]. Finally the case of open universe solutions where the spacetime and fiducial metrics are simultaneously diagonal, homogeneous and isotropic is extremely special and propagate no degrees of freedom about the exact solution [11].

The covariant quadratic Lagrangian exhibits several notable and potentially problematic features. Spatial derivatives of the degrees of freedom can appear at a lower order than temporal derivatives. Relatedly, as shown in Ref. [18], anisotropic stresses can dominate the stress energy tensor of fluctuations. We leave a full stability analysis of the joint Stückelberg, metric, and matter system for a future work.

Acknowledgments.– We thank P. Adshead, G. D'Amico, P. Gratia, A. Joyce, K. Koyama, M. Mirbabayi, and L.T. Wang for useful discussions. This work was supported by U.S. Dept. of Energy contract DE-FG02-13ER41958 and the Kavli Institute for Cosmological Physics at the University of Chicago through grants NSF PHY-0114422 and NSF PHY-0551142.

[1] G. Gabadadze, Phys.Lett. **B681**, 89 (2009), arXiv:0908.1112 [hep-th].
[2] C. de Rham, Phys.Lett. **B688**, 137 (2010), arXiv:0910.5474 [hep-th].
[3] C. de Rham and G. Gabadadze, Phys.Rev. **D82**, 044020 (2010), arXiv:1007.0443 [hep-th].
[4] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys.Rev.Lett. **106**, 231101 (2011), arXiv:1011.1232 [hep-th].
[5] C. de Rham, G. Gabadadze, L. Heisenberg, and D. Pirtskhalava, Phys.Rev. **D83**, 103516 (2011), arXiv:1010.1780 [hep-th].
[6] K. Koyama, G. Niz, and G. Tasinato, Phys.Rev.Lett. **107**, 131101 (2011), arXiv:1103.4708 [hep-th].
[7] K. Koyama, G. Niz, and G. Tasinato, Phys.Rev. **D84**, 064033 (2011), arXiv:1104.2143 [hep-th].
[8] T. M. Nieuwenhuizen, Phys.Rev. **D84**, 024038 (2011), arXiv:1103.5912 [gr-qc].
[9] L. Berezhiani, G. Chkareuli, C. de Rham, G. Gabadadze, and A. J. Tolley, Phys.Rev. **D85**, 044024 (2012), arXiv:1111.3613 [hep-th].
[10] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, *et al.*, Phys.Rev. **D84**, 124046 (2011), arXiv:1108.5231 [hep-th].
[11] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, JCAP **1111**, 030 (2011), arXiv:1109.3845 [hep-th].
[12] P. Gratia, W. Hu, and M. Wyman, Phys.Rev. **D86**, 061504 (2012), arXiv:1205.4241 [hep-th].
[13] T. Kobayashi, M. Siino, M. Yamaguchi, and D. Yoshida, Phys.Rev. **D86**, 061505 (2012), arXiv:1205.4938 [hep-th].
[14] M. S. Volkov, Phys.Rev. **D86**, 061502 (2012), arXiv:1205.5713 [hep-th].
[15] M. S. Volkov, Phys.Rev. **D86**, 104022 (2012), arXiv:1207.3723 [hep-th].
[16] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, JCAP **1203**, 006 (2012), arXiv:1111.4107 [hep-th].
[17] P. Gratia, W. Hu, and M. Wyman, (2013), arXiv:1305.2916 [hep-th].
[18] M. Wyman, W. Hu, and P. Gratia, Phys. Rev. D **87**, 084046 (2013), arXiv:1211.4576 [hep-th].
[19] N. Khosravi, G. Niz, K. Koyama, and G. Tasinato, JCAP **1308**, 044 (2013), arXiv:1305.4950 [hep-th].
[20] K. Koyama, G. Niz, and G. Tasinato, JHEP **1112**, 065 (2011), arXiv:1110.2618 [hep-th].
[21] G. D'Amico, Phys.Rev. **D86**, 124019 (2012), arXiv:1206.3617 [hep-th].
[22] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys.Lett. **B711**, 190 (2012), arXiv:1107.3820 [hep-th].
[23] Our parameters v, C are related to those used in [7] as $Q_0 = 3v/4$, $\Delta_0 = 16C^2/81$, $\gamma = v^2$.
[24] M. Mirbabayi, Phys.Rev. **D86**, 084006 (2012), arXiv:1112.1435 [hep-th].
[25] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Annals Phys. **305**, 96 (2003), arXiv:hep-th/0210184 [hep-th].
[26] S. Nibbelink Groot, M. Peloso, and M. Sexton, Eur.Phys.J. **C51**, 741 (2007), arXiv:hep-th/0610169 [hep-th].
[27] C. Deffayet, J. Mourad, and G. Zahariade, JHEP **1303**, 086 (2013), arXiv:1208.4493 [gr-qc].
[28] M. Natsuume and M. Ohta, Prog.Theor.Phys. **124**, 931 (2010), arXiv:1008.4142 [hep-th].
[29] I. Gullu, T. C. Sisman, and B. Tekin, Phys.Rev. **D82**, 124023 (2010), arXiv:1010.2411 [hep-th].