

This is the accepted manuscript made available via CHORUS. The article has been published as:

Entropy of a vacuum: What does the covariant entropy count?

Yasunori Nomura and Sean J. Weinberg

Phys. Rev. D **90**, 104003 — Published 4 November 2014

DOI: [10.1103/PhysRevD.90.104003](https://doi.org/10.1103/PhysRevD.90.104003)

The Entropy of a Vacuum: What Does the Covariant Entropy Count?

Yasunori Nomura and Sean J. Weinberg

*Berkeley Center for Theoretical Physics, Department of Physics,
University of California, Berkeley, CA 94720, USA*

Theoretical Physics Group, Lawrence Berkeley National Laboratory, CA 94720, USA

Abstract

We argue that a unitary description of the formation and evaporation of a black hole implies that the Bekenstein-Hawking entropy is the “entropy of a vacuum”: the logarithm of the number of possible independent ways in which quantum field theory on a fixed classical spacetime background can emerge in a full quantum theory of gravity. In many cases, the covariant entropy counts this entropy—the degeneracy of emergent quantum field theories in full quantum gravity—with the entropy of particle excitations in each quantum field theory giving only a tiny perturbation. In the Rindler description of a (black hole) horizon, the relevant vacuum degrees of freedom manifest themselves as an extra hidden quantum number carried by the states representing the second exterior region; this quantum number is invisible in the emergent quantum field theory. In a distant picture, these states arise as exponentially degenerate ground and excited states of the intrinsically quantum gravitational degrees of freedom on the stretched horizon. The formation and evaporation of a black hole involve processes in which the entropy of collapsing matter is transformed into that of a vacuum and then to that of final-state Hawking radiation. In the intermediate stage of this evolution, entanglement between the vacuum and (early) Hawking radiation develops, which is transferred to the entanglement among final-state Hawking quanta through the evaporation process. The horizon is kept smooth throughout the evolution; in particular, no firewall develops. Similar considerations also apply for cosmological horizons, for example for the horizon of a meta-stable de Sitter space.

1 Introduction and Summary

Despite much effort, there remains confusion about how a quantum theory of gravity works, especially in dynamical spacetime. Much of this confusion arises from the lack of clear understanding of the relation between the classical description of gravity, as suggested by general relativity, and the structure/dynamics of the microscopic degrees of freedom from which the classical picture of spacetime is supposed to arise. A major step toward such an understanding was the discovery of the Bekenstein-Hawking entropy [1], which suggests that a black hole—despite its unique nature in general relativity—is somehow associated with $\mathcal{A}/4l_{\text{P}}^2$ quantum degrees of freedom, where \mathcal{A} is the area of the horizon and $l_{\text{P}} \simeq 1.62 \times 10^{-35}$ m the Planck length. A question, however, remains. Where are these degrees of freedom? In other words, what does this entropy count?

A naive interpretation of the Bekenstein-Hawking entropy as the entanglement entropy between the interior and exterior regions within the framework of a quantum field theory on fixed classical spacetime background leads to the fundamental loss of information, which contradicts the basic principles of quantum mechanics [2]. To avoid this problem, a unitary description of the black hole formation and evaporation processes was put forward [3]—when viewed from a distance, a *complete* description of these processes is obtained in terms of the degrees of freedom located on and outside the stretched horizon (a surface located about l_{P} proper distance away from the mathematical horizon), and the interior spacetime arises manifestly only after one adopts a different, though equivalent, infalling description [4]. This “complementarity” picture beautifully addresses some of the possible issues associated with the unitary description, in particular possible cloning of infalling quantum information into the interior and exterior regions [5], and it can form a basis of a consistent quantum mechanical treatment of eternally inflating multiverse cosmology [6, 7]. Recently, however, it was argued that complementarity cannot be a consistent story [8–11]; the argument essentially boiled down to the incompatibility of the uniqueness of the (infalling) vacuum and the distant unitary description [12, 13]. We disagreed with this conclusion [14–16]. We argued that the apparent problem had arisen from an overly simplistic picture on how classical spacetime emerges in a full quantum theory of gravity. In particular, we argued that there are exponentially many black hole vacuum states corresponding to a single semi-classical black hole, and that there can be a semi-classical world built on *each* of them, all of which are described by the same quantum field theory on a fixed spacetime background although they represent different quantum states at the full quantum gravity level.

The aim of this paper is to elaborate further on the picture described just above and to elevate it to general statements about the relation between full quantum gravity and emergent quantum field theories in classical spacetime backgrounds. Our basic points about black hole physics can be summarized as follow.

- The Bekenstein-Hawking entropy is the “entropy of a vacuum”: the logarithm of the number

of possible independent ways in which quantum field theory on classical near-horizon black hole spacetime can emerge in a full quantum theory of gravity. In general, the entropy of a system is given by the logarithm of the number of states N with a specified macroscopic property of the system, which is given by the product of the number of vacuum states, $e^{S_{\text{vac}}}$, and the number of excited states that can be built on each of them, $e^{S_{\text{mat}}}$:

$$N = e^{S_{\text{vac}}} \times e^{S_{\text{mat}}}. \quad (1)$$

For a black hole (more precisely, a near-horizon region of a black hole that can be described by a near-horizon theory), the entropies of the vacuum and matter, which includes massless matter i.e. radiation, are

$$S_{\text{vac}} \approx \frac{\mathcal{A}}{4l_{\text{P}}^2}, \quad S_{\text{mat}} \approx O\left(\frac{\mathcal{A}^n}{l_{\text{P}}^{2n}}\right), \quad (2)$$

where $n < 1$, presumably $n \simeq 3/4$ [17]. We therefore find $S_{\text{vac}} \gg S_{\text{mat}}$, and the black hole entropy is given by

$$S = S_{\text{vac}} + S_{\text{mat}} \approx S_{\text{vac}}. \quad (3)$$

We emphasize that the entropy here is the *fine-grained* entropy, i.e. the logarithm of the number of possible independent quantum states in which the *entire* system, including the horizon degrees of freedom, can be. This is the origin of the Bekenstein-Hawking entropy.

- When the black hole is described in a distant reference frame, the exponential degeneracy of the field theory vacuum states, indicated by S_{vac} , arises from (intrinsically quantum gravitational) degrees of freedom on the stretched horizon. In particular, we postulate that
 - (a) the stretched horizon degrees of freedom can take exponentially many different configurations, labeled by $k = 1, \dots, e^{\approx \mathcal{A}/4l_{\text{P}}^2}$, which are (approximately) degenerate in energy, and all of which can comprise a field theory vacuum;
 - (b) there are infinitely many internally excited states *for each of* these configurations, and these excited states as well as the ground state (for each k) are entangled with the near exterior states $|i\rangle$ in a specific manner, determined by Boltzmann factors, for a black hole to be in a vacuum state.

By labeling internal excitations by the index \tilde{i} , the black hole vacuum states can then be written as¹

$$|\psi_k\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\frac{\beta}{2} E_i} |i\rangle |\tilde{i}; k\rangle; \quad Z = \sum_i e^{-\beta E_i}, \quad (4)$$

where E_i is the energy of the state $|i\rangle$ measured in the asymptotic region, and β the inverse Hawking temperature.

¹Here and below, the sum over i implies the corresponding sum over \tilde{i} as well; for example, $\sum_{i=1}^{\infty} |i\rangle |\tilde{i}; k\rangle = |1\rangle |\tilde{1}; k\rangle + |2\rangle |\tilde{2}; k\rangle + \dots$

- The structure described above implies that the stretched horizon degrees of freedom provide the states necessary to compose (the $e^{\approx \mathcal{A}/4l_P^2}$ copies of) the second exterior region of the Rindler space, and hence the near-horizon eternal black hole geometry. In fact, the quantum mechanical structure of a collapse-formed black hole after the horizon is stabilized to a generic state is, at each instant of time, well approximated by that of an eternal black hole at the microscopic level. In particular, the form of the states in Eq. (4) allows us to define the mode operators acting on $|\tilde{i}; k\rangle$ (which can be interpreted to have arisen as collective excitations of the stretched horizon degrees of freedom) and hence the infalling mode operators *for each* k , following the standard Unruh-Israel prescription [18]. These infalling mode operators satisfy the algebra

$$[a_\sigma^{(k)}, a_{\sigma'}^{(k')\dagger}] = \delta_{\sigma\sigma'} \delta_{kk'}, \quad [a_\sigma^{(k)}, a_{\sigma'}^{(k')}] = [a_\sigma^{(k)\dagger}, a_{\sigma'}^{(k')\dagger}] = 0, \quad (5)$$

where σ represents a set of spacetime and internal quantum numbers, and $a_\sigma^{(k)}$ annihilates all the vacuum states, i.e.

$$a_\sigma^{(k)} |\psi_{k'}\rangle = 0, \quad (6)$$

for all σ, k, k' . A state in which matter exists in the interior of the black hole can be constructed by acting (a finite number of) $a_\sigma^{(k)\dagger}$'s on $|\psi_k\rangle$.

- The internal dynamics of the stretched horizon is such that the time evolution operator describing physics of an infalling object can be organized in a way that makes it manifest that the object smoothly passes through the horizon. In particular, the Hamiltonian describing the infalling object can be written as

$$H = \sum_{k=1}^{e^{\approx \mathcal{A}/4l_P^2}} H_{\text{UI}}(a_\sigma^{(k)}, a_\sigma^{(k)\dagger}; c_{\mathbf{p}}, c_{\mathbf{p}}^\dagger) P_k, \quad (7)$$

where $H_{\text{UI}}(a_\sigma, a_\sigma^\dagger; c_{\mathbf{p}}, c_{\mathbf{p}}^\dagger)$ is the Hamiltonian in the standard Unruh-Israel description, with a_σ and $c_{\mathbf{p}}$ being infalling mode operators and operators for far exterior modes, respectively, and P_k is the projection operator defined by $P_k |\tilde{i}; k'\rangle = \delta_{kk'} |\tilde{i}; k\rangle$. This implies that an infalling observer finds that the horizon is smooth with a probability of 1. When the observer interacts with the black hole state, which involves both the stretched horizon states and the near exterior states with which the stretched horizon states are strongly entangled, he/she “measures” the black hole in the basis $\{|\psi_k\rangle\}$, all of which lead to the same semi-classical physics predicted by general relativity.

- When viewed from a distance, unitarity of the black hole evolution is preserved in such a way that, at an intermediate stage of the evolution, the information about the initial collapsing matter is encoded in how the field theory vacuum on the black hole background is realized at the fundamental quantum gravity level, which will later be transformed into the state of

final-state Hawking radiation. The flow of information is thus schematically

$$\text{collapsing matter} \longrightarrow \begin{array}{c} \text{field theory vacuum} \\ (\text{index } k) \end{array} \longrightarrow \text{Hawking radiation.} \quad (8)$$

This exchange of information between matter/radiation and a vacuum is a characteristic feature of the black hole formation and evaporation processes. Note that physics outside the stretched horizon can still be completely local in the conventional sense throughout this process.²

The features of black hole physics summarized above lead to the following picture on the emergence of quantum field theories (built on classical spacetime backgrounds) in the full theory of quantum gravity. Starting from the theory of the most fundamental quantum degrees of freedom, we can make a “classical approximation” only on certain degrees of freedom, corresponding to spacetime, keeping the full quantum nature for the rest of the degrees of freedom—this is what quantum field theory is. This classical approximation, as usual in such approximations, involves coarse-graining huge degrees of freedom. As in many other systems, however, the entropy associated with these degrees of freedom—the entropy of a vacuum S_{vac} —is still visible in the coarse-grained theory, i.e. quantum field theory, through thermodynamic considerations. A special feature of the black hole evaporation process in this respect is that the information contained in fine-grained degrees of freedom (i.e. constituents of spacetime) can get back to that in coarse-grained degrees of freedom (i.e. Hawking radiation), which does not happen in many systems. Since our macroscopic world ultimately appears after many of the coarse-grained degrees of freedom, i.e. matter, are also classicalized, we may say that quantum field theories represent “intermediate approximations” in which a (major) part of the classicalization needed to go from the most fundamental theory to our classical world is taken into account explicitly.

Armed with the lessons we learned in our study of black hole physics, we also extend our considerations to more general cases. We first consider a relatively straightforward application of the dynamics of the stretched horizon described above to a de Sitter horizon. We argue that, as in the black hole case, the stretched horizon degrees of freedom are organized into states labeled as $|\tilde{i}; k\rangle$, and that the de Sitter vacuum states take the form in Eq. (4), where $|i\rangle$ represents states in the interior of the stretched de Sitter horizon. The index k runs over $1, \dots, e^{\approx \mathcal{A}/4l_{\text{P}}^2}$; here, $\mathcal{A} = 4\pi/H^2$ is the area of the (stretched) de Sitter horizon, where H is the Hubble parameter. The

²**Note added:** More recently, this process has been analyzed in detail in Ref. [19]. In order for the process to be local, e.g. to respect causality in spacetime, a (small) portion of the information about the index k must be regarded as being delocalized into the whole zone region, $r \lesssim 3Mt_{\text{P}}^2$, at the field theory level. In our notation here, this implies that a part of the information about k must be carried by the near exterior state $|i\rangle$ in Eq. (4), instead of $|\tilde{i}; k\rangle$. Since the amount of information that needs to be delocalized to the whole zone region is much smaller than $\mathcal{A}/4l_{\text{P}}^2$, however, our discussions in this paper may persist essentially without changes (except for points related to the issue described here). For more complete and updated discussions on these points, see Ref. [19].

same analysis as in the black hole case implies that a de Sitter horizon is smooth: an object that hits the horizon can be thought of as going to space outside the horizon. The information about the object that goes outside will be stored in the state constructed purely from the interior and the stretched horizon degrees of freedom. Such information may thus be recovered later. This recovery may not necessarily be in the form of Hawking radiation, if the system evolves, for example, into Minkowski space or another de Sitter space with a smaller vacuum energy.

We also discuss implications of our observations for more general spacetimes in quantum gravity. Here we adopt the picture advocated in Refs. [7, 15] that the Hilbert space for quantum gravity can be organized in such a way that the system is viewed from a freely falling (local Lorentz) reference frame. We argue that in general the fine-grained entropy of the system arises from both vacuum and matter/radiation contributions, and conjecture that it saturates the covariant entropy bound [20] if the degrees of freedom on the horizon (which may be located at spatial infinity as in Minkowski space) are included:

$$S = S_{\text{vac}} + S_{\text{mat}} \approx \frac{\mathcal{A}}{4l_{\text{P}}^2}. \quad (9)$$

For the contribution from a horizon at which Planckian physics is important, such as the black hole or de Sitter horizon, the vacuum contribution typically dominates: $S_{\text{vac}} \gg S_{\text{mat}}$.

The framework presented in this paper largely builds on the basic picture presented in Ref. [16] (and particularly emphasized in [21]) that the number of degrees of freedom relevant to describe the black hole interior is a tiny fraction of the total number of degrees of freedom available for a black hole. The explicit realization of the idea, however, is different in this paper. In Ref. [16], it was considered, following [22], that a variety of black hole vacuum states $|\psi_k\rangle$ is allowed because of a freedom in the way the near horizon and stretched horizon modes are entangled; specifically, we considered a freedom in quantum mechanical phase factors appearing in the entangled states. This approach, however, leads to the following issue. When an arbitrary black hole state $|\psi\rangle = \sum_k c_k |\psi_k\rangle$ is considered, it in general does not lead to the thermal density matrix in the exterior region, implying that radiation emitted from the black hole does not have the Hawking spectrum. (The population probability deviates by an $O(1)$ fraction for each energy level.) The spectrum approximately looks like the Hawking form if a sufficient coarse-graining is performed in energy (or we may recover the exact Hawking form if we postulate particular dynamics that enforces a very specific form of entanglement between $|\psi_k\rangle$ and the environment), but it is still uncomfortable that the picture implied by semi-classical analyses receives such a major correction in the regime where we do not expect it. In our framework presented here, the variety of the black hole vacuum states (index k) originates purely from the stretched horizon degrees of freedom. This avoids the above issue of large deviations from the Hawking spectrum—the spectrum is exactly the Hawking form at each moment of emission (although there can be correlations between Hawking quanta emitted at different moments, needed to preserve unitarity)—and it allows us to construct interior quantum

field theory operators explicitly by simply following the standard Unruh-Israel prescription. The framework is also consistent with the criterion for a smooth horizon in Ref. [23]: a smooth horizon requires a near-maximal entanglement in two sets of basis states, which was not the case in Ref. [16].

The organization of the rest of the paper is as follows. In Section 2, we present our framework for describing black hole physics. We motivate each of the hypotheses we introduce about the dynamics of the stretched horizon, and show that they allow us to reconstruct the interior spacetime consistently with unitarity of the evolution of the system. We argue that the Bekenstein-Hawking entropy is the entropy of a vacuum. In Section 3, we extend our considerations to more general cases. We discuss how the same dynamics of the stretched horizon as in the black hole case applies to a de Sitter horizon. We also discuss the fine-grained entropy of more general spacetimes in quantum gravity and its relation to the covariant entropy bound. In the appendix, we present the full Hilbert space needed to describe the evolution of a black hole.

While completing this paper, we received Ref. [24] discussing the black hole interior in AdS/CFT, which seems to employ some similar ideas.

2 Black Hole Entropy as the Entropy of a Vacuum

In this section, we discuss our framework for black hole physics.

2.1 A unitary description of black hole evolution

Suppose we describe the formation and evaporation of a black hole in a distant reference frame. Following Refs. [3, 4], we postulate that there exists a unitary description which involves only the outside and the (stretched) horizon degrees of freedom of the black hole. Suppose some matter that is not entangled with the rest of the system collapses into a black hole of mass M_0 . This process can be described as

$$|M_{\text{init}}\rangle|E_{\text{init}}\rangle \rightarrow |\psi(M_0)\rangle|E\rangle, \quad (10)$$

where $|M_{\text{init}}\rangle$ represents the initial state of matter, $|\psi(M_0)\rangle$ the state of the black hole shortly after the formation, and $|E_{\text{init}}\rangle$ and $|E\rangle$ the states of the rest of the system at the respective moments. (The meaning of the state of the black hole will become clearer later.) Now, consider forming the black hole of the same mass (and angular momentum and charge) by collapsing matter in different initial states $|M_{\text{init},a}\rangle$ ($a = 1, 2, \dots$). Unitarity then implies that the state of the black hole must also carry the index a :

$$|M_{\text{init},a}\rangle|E_{\text{init}}\rangle \rightarrow |\psi_a(M_0)\rangle|E\rangle. \quad (11)$$

Namely, there must be many different black hole quantum states that correspond to the same (semi-)classical black hole.

How many black hole microstates $n(M)$ are there for the black hole with a fixed mass M ? The validity of the generalized second law of thermodynamics suggests that it is given by the exponential of the Bekenstein-Hawking entropy

$$n(M) = e^{\frac{\mathcal{A}}{4l_P^2}} = e^{4\pi M^2 l_P^2}, \quad (12)$$

where $\mathcal{A} = 16\pi M^2 l_P^4$ is the area of the horizon; here, we have assumed for simplicity that the black hole under consideration is (well approximated by) a Schwarzschild black hole in 4-dimensional spacetime.³ Note that while the number of black hole states that can be directly formed by a single collapse, as in Eq. (11), is much smaller than $e^{\mathcal{A}/4l_P^2}$ (presumably of order $e^{c\mathcal{A}^{3/4}/l_P^{3/2}}$ where c is an $O(1)$ coefficient [17]), all the $e^{\mathcal{A}/4l_P^2}$ black hole states are expected to be realized for more complicated histories, for example by producing a larger black hole and then evaporating down.

The existence of exponentially many black hole microstates

$$|\psi_k(M)\rangle; \quad k = 1, \dots, n(M) = e^{4\pi M^2 l_P^2}, \quad (13)$$

allows a unitary description of the black hole formation and evaporation processes as viewed from a distant reference frame. A crucial question is how this picture can be compatible with the implication of the equivalence principle that an infalling object does not feel anything special at the horizon, which seems to suggest that the black hole must be in the *unique* vacuum state from the viewpoint of the infalling object (after the scrambling time of order $t_{\text{sc}} = M_0 l_P^2 \ln(M_0 l_P)$ [5, 25] is passed since the formation, which we assume to be the case). In particular, the Unruh-Israel description of the black hole seems to imply that it must be in a unique state in which the degrees of freedom in the “two exterior regions” are almost maximally entangled [18]. Our first goal then is to figure out what the relation is between the unitary description, in which the black hole state has the index k as in Eq. (13), and the Unruh-Israel-type description which seems to imply the unique state.

2.2 The Unruh-Israel description of a black hole

Recall that the Unruh-Israel description of a horizon is obtained by expanding a quantum field in two different sets of normal modes. (Here we consider only a single bosonic quantum field for simplicity. The extension to the other cases is straightforward.) Let us denote the annihilation operators for the Minkowski (corresponding to the infalling) modes by $a_{\Omega, \xi}$ while those for the two exterior modes of the Rindler (distant) expansion by $b_{\omega, \xi}$ and $\tilde{b}_{\omega, \xi}$, respectively. Here, ξ collectively

³More precisely, $n(M)$ is the number of black hole states with their masses in a range between M and $M + \delta M$. The precise value of δM is unimportant for our purposes because it only leads to a logarithmic correction in the exponent of $n(M)$ (unless δM is chosen exponentially small), which we will ignore. For definiteness, one may take δM to be of order the decay width of a black hole to a lighter black hole and a Hawking quantum, $\delta M \sim 1/M l_P^2$.

represents the quantum numbers associated with the directions parallel to the horizon, e.g. the two angular momentum quantum numbers l and m for a spherical horizon, and Ω and ω the frequencies. The two sets of modes are related by a Bogoliubov transformation

$$a_{\Omega,\xi} = \sum_{\omega} (\alpha_{\Omega,\omega} b_{\omega,\xi} + \gamma_{\Omega,\omega} b_{\omega,\xi}^{\dagger} + \zeta_{\Omega,\omega} \tilde{b}_{\omega,\xi} + \eta_{\Omega,\omega} \tilde{b}_{\omega,\xi}^{\dagger}), \quad (14)$$

where $\alpha_{\Omega,\omega}$, $\gamma_{\Omega,\omega}$, $\zeta_{\Omega,\omega}$, and $\eta_{\Omega,\omega}$ are coefficients.

The Minkowski/infalling vacuum state $|\psi\rangle$ is defined by

$$\forall \Omega, \xi, \quad a_{\Omega,\xi} |\psi\rangle = 0, \quad (15)$$

which is represented in the Rindler/distant frame by

$$|\psi\rangle \propto \prod_{\omega,\xi} \exp\left(e^{-\frac{\beta\omega}{2}} b_{\omega,\xi}^{\dagger} \tilde{b}_{\omega,\xi}^{\dagger}\right) |\emptyset\rangle, \quad (16)$$

where $|\emptyset\rangle$ is the Rindler vacuum and β the inverse temperature. The temperature $1/\beta$ may in general depend on the mode (ω, ξ) , but this dependence is essentially absent when ω is defined in the asymptotic region, since the temperature and frequencies redshift in the same way. (In the true Rindler space, this requires the introduction of an infrared cutoff. In the black hole case, the Unruh-Israel description is valid only for a region close to the horizon, and the local Hawking temperature and mode frequencies at the outer edge of this region are scaled from their respective asymptotic values by the same $O(1)$ factor.) We thus drop the possible dependence of β on (ω, ξ) below, assuming that ω represents the frequency measured in the asymptotic region.

We now decompose $|\emptyset\rangle$ as

$$|\emptyset\rangle = |0\rangle |\tilde{0}\rangle, \quad (17)$$

where $|0\rangle$ and $|\tilde{0}\rangle$ are the vacuum states in the two exterior regions on which $b_{\omega,\xi}^{\dagger}$'s and $\tilde{b}_{\omega,\xi}^{\dagger}$'s act, respectively. By tracing out the exterior states in one side (called Region III), i.e. $|\tilde{0}\rangle$, $\tilde{b}_{\omega,\xi}^{\dagger} |\tilde{0}\rangle$, $\frac{1}{\sqrt{2^{\delta_{\omega\omega'}\delta_{\xi\xi'}}}} \tilde{b}_{\omega,\xi}^{\dagger} \tilde{b}_{\omega',\xi'}^{\dagger} |\tilde{0}\rangle, \dots$, we obtain the density matrix in the other exterior region (Region I):

$$\rho_{\text{I}} = \text{Tr}_{\text{Region III}} |\psi\rangle \langle \psi| = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle \langle i|, \quad (18)$$

where $Z = \sum_i e^{-\beta E_i}$. Here, $|i\rangle$ represents a state in Region I, specified by the number of excitations $n_{\omega,\xi}$ in each mode (ω, ξ) :

$$|i\rangle = \left(\prod_{\omega,\xi} \frac{1}{\sqrt{n_{\omega,\xi}!}} (b_{\omega,\xi}^{\dagger})^{n_{\omega,\xi}} \right) |0\rangle, \quad (19)$$

while $E_i = \sum_{\omega,\xi} n_{\omega,\xi} \omega$ is the energy of the state $|i\rangle$. We identify Region I to represent the side exterior to the Schwarzschild horizon, which leads to $\beta = 1/T_H = 8\pi M l_{\text{P}}^2$, where T_H is the Hawking

temperature. Note that the Unruh-Israel state provides a description only of a spacetime region close to the horizon, e.g. $r \lesssim 3Ml_{\text{P}}^2$, so to describe the entire region outside the Schwarzschild horizon we need to consider quantum states describing the far region, e.g. $r \gtrsim 3Ml_{\text{P}}^2$, in addition to this state. (The validity of this division of the entire system into two subsystems is granted by locality, which we assume to be preserved outside the stretched horizon at length scales larger than the fundamental, or string, length l_* .) On the other hand, the interior region of the Schwarzschild horizon is described (fully) by quantum field theory built on $|\psi\rangle$ by acting $a_{\Omega,\xi}^\dagger$ operators (after an appropriate modification from a planar to the spherical horizon is made).

The von Neumann entropy of the density matrix ρ_{I} in Eq. (18) is given by

$$S_{\text{I}} = -\text{Tr}(\rho_{\text{I}} \ln \rho_{\text{I}}) = \frac{\mathcal{A}}{4l_{\text{P}}^2} \left\{ 1 + O\left(\frac{l_{\text{P}}^{2n}}{\mathcal{A}^n}; n > 0\right) \right\} \approx \frac{\mathcal{A}}{4l_{\text{P}}^2}, \quad (20)$$

where \mathcal{A} is the area of the horizon [26], implying that the number of terms in the last expression of Eq. (18) having unsuppressed coefficients is (effectively) $e^{\approx \mathcal{A}/4l_{\text{P}}^2}$. Here and below, the approximate symbol indicates that the expression is valid at the leading order in expansion in inverse powers of $\mathcal{A}/l_{\text{P}}^2$. Note that to obtain the coefficient of $1/4$ in Eq. (20), one needs to include the effect of the counterterm renormalizing Newton's constant. This implies that the number of $1/4$ is obtained only after we include all the ultraviolet (including trans-Planckian) states in $|i\rangle$ in Eq. (18), which is implicitly done through the counterterm.

How should we interpret the Unruh-Israel result described above? The conventional interpretation is that it describes the *unique* infalling vacuum state in which a black hole must be at late times, specifically after the scrambling time. Our interpretation is different—we consider that a black hole at late times consists, as suggested by unitarity, of exponentially many infalling vacuum states $|\psi_k\rangle$, and that the Unruh-Israel description arises as an emergent effective quantum field theory in *each* of these vacuum states, which is responsible for describing an object falling into the horizon. As we will see in Section 2.5, this structure allows us to avoid the arguments for firewalls and to keep the horizon smooth consistently with the unitarity of the black hole formation and evaporation processes. We will now discuss this picture in more detail.

2.3 The entropy of a vacuum and emergent quantum field theories

Our starting point is to adopt a set of hypotheses that we consider natural from the viewpoint of a distant reference frame:

- (i) The formation and evaporation of a black hole are unitary processes. This implies that there are exponentially many black hole vacuum states $|\psi_k\rangle$.
- (ii) The number of black hole states $|\psi_k\rangle$ for a fixed mass M is $n(M) = e^{\approx \mathcal{A}/4l_{\text{P}}^2}$, where $\mathcal{A} = 16\pi M^2 l_{\text{P}}^4$. This is motivated by the success of the generalized second law of thermodynamics.

- (iii) For any black hole state $|\psi_k\rangle$, the region near and outside the stretched horizon—which we assume to be well described by local quantum field theory—is given by the mixed, thermal state as in Eq. (18):

$$\rho_{\text{ext}} = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle\langle i|. \quad (21)$$

Here, $|i\rangle$ represents the states near and *outside* the stretched horizon; in particular, it does not include the (Planckian) degrees of freedom associated with the stretched horizon.

- (iv) The state in Eq. (21) is purified by the (intrinsically quantum gravitational) degrees of freedom located on the stretched horizon; namely, the stretched horizon degrees of freedom play the role of the second exterior region in the Unruh-Israel description [16]. Item (ii) above then implies that there are $e^{\approx A/4l_{\text{P}}^2}$ different ways in which ρ_{ext} is purified by the stretched horizon states:

$$\rho_{\text{ext}} \rightarrow |\psi_k\rangle, \quad (22)$$

where $k = 1, \dots, e^{\approx A/4l_{\text{P}}^2}$. Since the states $|i\rangle$ that have unsuppressed Boltzmann coefficients in Eq. (21) represent modes localized near but outside the stretched horizon, the black hole state $|\psi_k\rangle$ must be thought of as representing the states of the stretched horizon degrees of freedom *as well as* the exterior modes represented by the $|i\rangle$'s, which are highly entangled with each other. (Note that black hole states $|\psi_k\rangle$ can be further entangled with states representing the rest of the system, in which case the state of the black hole can only be represented by a density matrix in the space spanned by the $|\psi_k\rangle$'s; see Section 2.4 for further discussion.)

We now postulate the following structure for the stretched horizon states. As suggested by the existence of exponentially many black hole states, Eq. (13), we consider that the stretched horizon degrees of freedom can take exponentially many different configurations which are (approximately) degenerate in energy. We label these configurations by the index k , which runs over

$$k = 1, \dots, n(M) = e^{\approx \frac{A}{4l_{\text{P}}^2}}, \quad (23)$$

for a fixed black hole mass M . (More precisely, the stretched horizon degrees of freedom can take $e^{\approx A/4l_{\text{P}}^2}$ configurations in the energy range between M and $M + \delta M$; see footnote 3.) We consider that there are (an infinite number of) internally excited states for each of these configurations, and we label these excited as well as the ground states by \tilde{i} . The stretched horizon states can then be denoted as $|\tilde{i}; k\rangle$, which form an orthonormal set

$$\langle \tilde{i}; k | \tilde{i}'; k' \rangle = \delta_{\tilde{i}\tilde{i}'} \delta_{kk'}. \quad (24)$$

Motivated by the Unruh-Israel description, we consider that the ground and excited states for each k are entangled with the near exterior states $|i\rangle$ in a specific manner, with the coefficients

determined by Boltzmann factors. In particular, we assume that the black hole vacuum states take the specifically entangled form

$$|\psi_k(M)\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\frac{\beta}{2} E_i} |i\rangle |\tilde{i}; k\rangle. \quad (25)$$

This structure satisfies all the requirements in (i) – (iv) above. In particular, upon integrating out the stretched horizon states $|\tilde{i}; k\rangle$, we find that the reduced density matrix for the exterior states takes the form of Eq. (21) for any of the states $|\psi_k(M)\rangle$. Based on a simple field theory estimate, we expect $S_{\text{ext}} = -\text{Tr}(\rho_{\text{ext}} \ln \rho_{\text{ext}}) \approx \gamma \mathcal{A}/l_{\text{P}}^2$, where $\gamma \approx O(1)$. This implies that the number of terms in the right-hand side of Eq. (25) whose coefficients are nonnegligible is $e^{\approx \gamma \mathcal{A}/l_{\text{P}}^2}$.⁴

We now argue that we can build an effective quantum field theory describing the interior and near exterior regions of the horizon on each of the black hole vacuum states $|\psi_k\rangle$, and that all these quantum field theories are isomorphic with each other, taking the form identical to the Unruh-Israel description of the black hole. (In Section 2.5, we will argue that this quantum field theory is the one responsible for describing the fate of an infalling object.) Let us consider states built on a specific black hole vacuum state $|\psi_k\rangle$. Since the states $|i\rangle$ are specified by a set of occupation numbers $n_{\omega, \xi}$ for all the modes (ω, ξ) , we write them as

$$|i\rangle \rightarrow |\{n_{\omega, \xi}\}\rangle. \quad (26)$$

Similarly, we can write the stretched horizon states as

$$|\tilde{i}; k\rangle \rightarrow |\{\tilde{n}_{\omega, \xi}\}; k\rangle, \quad (27)$$

where $\tilde{n}_{\omega, \xi}$'s are the occupation numbers of the (ω, ξ) modes *of* $|i\rangle$, not $|\tilde{i}; k\rangle$, to which $|\tilde{i}; k\rangle$ is coupled *in the vacuum state* $|\psi_k\rangle$ in Eq. (25). (Note that $\{\tilde{n}_{\omega, \xi}\}$ here is simply used to label states $|\tilde{i}; k\rangle$ through Eq. (25), whose meaning is still the occupation numbers for the exterior states $|i\rangle$. We will, however, see later that it can be understood as the occupation numbers for some “quasi-particles” represented by the stretched horizon states $|\tilde{i}; k\rangle$.) A general black hole—*not necessarily vacuum*—state $|\phi_k\rangle$ obtained by exciting $|\psi_k\rangle$ can then be written as

$$|\phi_k\rangle = \frac{1}{\sqrt{Z_\phi}} \sum_{\{n_{\omega, \xi}\}, \{\tilde{n}_{\omega, \xi}\}} f_{\{n_{\omega, \xi}\}, \{\tilde{n}_{\omega, \xi}\}} |\{n_{\omega, \xi}\}\rangle |\{\tilde{n}_{\omega, \xi}\}; k\rangle, \quad (28)$$

⁴In the present picture, the quantity $S_{\text{ext}} (\approx \gamma \mathcal{A}/l_{\text{P}}^2)$ is not related with the number of black hole states $n(M)$ in Eq. (23), so we generally expect $\gamma \neq 1/4$. In fact, γ will be sensitive to the precise division of the degrees of freedom into the stretched horizon and near exterior modes, which is somewhat arbitrary. Note that the quantity S_{ext} here is different from S_{I} in Eq. (20) in which the ultraviolet modes (corresponding to the stretched horizon modes here) are included implicitly in $|i\rangle$. In our picture, S_{I} counts the *total* number of effective degrees of freedom existing in one side of the *mathematical* Schwarzschild horizon (which we view as the only physical degrees of freedom), thus giving $S_{\text{I}} \approx \ln \dim\{|\psi_k(M)\rangle\} \approx \mathcal{A}/4l_{\text{P}}^2$. In other words, calculations such as in Ref. [26]—after including the counterterm—count $\ln n(M)$ in the language here (by considering the fictitious “vacuum state” $\sum_k |\psi_k\rangle |\tilde{\psi}_k\rangle$).

where $Z_\phi = \sum_{\{n_{\omega,\xi}\},\{\tilde{n}_{\omega,\xi}\}} |f_{\{n_{\omega,\xi}\},\{\tilde{n}_{\omega,\xi}\}}|^2$. The vacuum state $|\psi_k\rangle$ is a special case in which

$$f_{\{n_{\omega,\xi}\},\{\tilde{n}_{\omega,\xi}\}} = e^{-\frac{\beta}{2}E_{\{n_{\omega,\xi}\}}} \delta_{\{n_{\omega,\xi}\},\{\tilde{n}_{\omega,\xi}\}}, \quad (29)$$

where $\delta_{\{n_{\omega,\xi}\},\{\tilde{n}_{\omega,\xi}\}} \equiv \prod_{\omega,\xi} \delta_{n_{\omega,\xi},\tilde{n}_{\omega,\xi}}$.

Suppose that we define operators $\tilde{b}_{\omega,\xi}^{(k)}$ and $\tilde{b}_{\omega,\xi}^{(k)\dagger}$ acting on the stretched horizon degrees of freedom by

$$\tilde{b}_{\omega,\xi}^{(k)} |\{\tilde{n}_{\omega',\xi'}\}; k'\rangle = \delta_{kk'} \sqrt{\tilde{n}_{\omega,\xi}} |\{\tilde{n}_{\omega',\xi'} - \delta_{\omega\omega'} \delta_{\xi\xi'}\}; k\rangle, \quad (30)$$

$$\tilde{b}_{\omega,\xi}^{(k)\dagger} |\{\tilde{n}_{\omega',\xi'}\}; k'\rangle = \delta_{kk'} \sqrt{\tilde{n}_{\omega,\xi} + 1} |\{\tilde{n}_{\omega',\xi'} + \delta_{\omega\omega'} \delta_{\xi\xi'}\}; k\rangle. \quad (31)$$

The vacuum state $|\psi_k\rangle$ can then be written as in Eq. (25):

$$|\psi_k\rangle = \frac{1}{\sqrt{Z}} \sum_{i=\{n_{\omega,\xi}\}} e^{-\frac{\beta E_i}{2}} |\tilde{i}; k\rangle, \quad (32)$$

with $|\tilde{i}; k\rangle$'s now given by

$$|\tilde{i}; k\rangle = \left(\prod_{\omega,\xi} \frac{1}{\sqrt{n_{\omega,\xi}!}} (\tilde{b}_{\omega,\xi}^{(k)\dagger})^{n_{\omega,\xi}} \right) |\tilde{0}; k\rangle. \quad (33)$$

Here, $|\tilde{0}; k\rangle$ is the ground state of the k -th configuration of the stretched horizon degrees of freedom, satisfying

$$\forall \omega, \xi, \quad \tilde{b}_{\omega,\xi}^{(k)} |\tilde{0}; k\rangle = 0. \quad (34)$$

The commutation relations between operators $\tilde{b}_{\omega,\xi}^{(k)}$ and $\tilde{b}_{\omega,\xi}^{(k)\dagger}$ are obtained from Eqs. (30, 31) as

$$[\tilde{b}_{\omega,\xi}^{(k)}, \tilde{b}_{\omega',\xi'}^{(k')\dagger}] = \delta_{\omega\omega'} \delta_{\xi\xi'} \delta_{kk'}, \quad [\tilde{b}_{\omega,\xi}^{(k)}, \tilde{b}_{\omega',\xi'}^{(k)}] = [\tilde{b}_{\omega,\xi}^{(k)\dagger}, \tilde{b}_{\omega',\xi'}^{(k')\dagger}] = 0. \quad (35)$$

This implies that we can interpret $\tilde{b}_{\omega,\xi}^{(k)}$ and $\tilde{b}_{\omega,\xi}^{(k)\dagger}$ as the annihilation and creation operators for “quasi-particle” quanta with negative energy $-\omega$, which arise as (collective) excitation modes of the stretched horizon degrees of freedom. As becomes clear below, these are precisely the Hawking partner modes that can be excited in the k -th vacuum state $|\psi_k\rangle$.

For a fixed k , the form of Eqs. (32 – 35) is identical to that in the standard Unruh-Israel description. In view of this, we define the infalling mode operators associated with the k -th vacuum state $|\psi_k\rangle$ by

$$a_{\Omega,\xi}^{(k)} = \sum_{\omega} (\alpha_{\Omega,\omega} b_{\omega,\xi} P_k + \gamma_{\Omega,\omega} b_{\omega,\xi}^\dagger P_k + \zeta_{\Omega,\omega} \tilde{b}_{\omega,\xi}^{(k)} + \eta_{\Omega,\omega} \tilde{b}_{\omega,\xi}^{(k)\dagger}), \quad (36)$$

where P_k is a projection operator acting on the stretched horizon states as $P_k |\{\tilde{n}_{\omega,\xi}\}; k'\rangle = \delta_{kk'} |\{\tilde{n}_{\omega,\xi}\}; k\rangle$, and $\alpha_{\Omega,\omega}$, $\gamma_{\Omega,\omega}$, $\zeta_{\Omega,\omega}$, and $\eta_{\Omega,\omega}$ are the same coefficients as in Eq. (14). Note that

$a_{\Omega,\xi}^{(k)}$ involves projection on k : $a_{\Omega,\xi}^{(k)} = a_{\Omega,\xi}^{(k)} P_k$ because of Eqs. (30, 31). These operators satisfy the algebra for creation/annihilation operators

$$[a_{\Omega,\xi}^{(k)}, a_{\Omega',\xi'}^{(k')\dagger}] = \delta_{\Omega\Omega'} \delta_{\xi\xi'} \delta_{kk'}, \quad [a_{\Omega,\xi}^{(k)}, a_{\Omega',\xi'}^{(k')}] = [a_{\Omega,\xi}^{(k)\dagger}, a_{\Omega',\xi'}^{(k')\dagger}] = 0, \quad (37)$$

and their actions on the vacuum states are given by

$$\forall \Omega, \xi, k, \quad a_{\Omega,\xi}^{(k)} |\psi_k\rangle = 0, \quad (38)$$

(and $a_{\Omega,\xi}^{(k)} |\psi_{k'}\rangle = a_{\Omega,\xi}^{(k)\dagger} |\psi_{k'}\rangle = 0$ for $k \neq k'$). We can therefore construct quantum states in an infalling reference frame by acting $a_{\Omega,\xi}^{(k)\dagger}$ operators on the vacuum state $|\psi_k\rangle$ for each k .

How many quantum states can we build on each $|\psi_k\rangle$? By acting (a finite number of) $a_{\Omega,\xi}^{(k)\dagger}$ s on $|\psi_k\rangle$, one can construct a state in which matter exists in the interior of the black hole of mass M , when viewed from an infalling observer.⁵ We expect that the number of such states is of order $e^{\approx \mathcal{A}^n / l_P^{2n}}$ with $n < 1$, presumably $n \simeq 3/4$ [17]. The number of all the black hole states of mass M is therefore

$$\left[n(M) = e^{\approx \frac{\mathcal{A}}{4l_P^2}} \right] \times e^{\approx c \frac{\mathcal{A}^n}{l_P^{2n}}} = e^{\approx \frac{\mathcal{A}}{4l_P^2}}, \quad (39)$$

which is consistent with the holographic/covariant entropy bound [17,20,27]. In particular, Eq. (39) implies that the black hole (or covariant) entropy is saturated by *the entropy of a vacuum*

$$S_{\text{vac}} = \ln\{n(M)\} \approx \frac{\mathcal{A}}{4l_P^2}, \quad (40)$$

at the leading order in l_P^2/\mathcal{A} , and that the entropy from usual matter (and radiation), $S_{\text{mat}} \approx \mathcal{A}^n / l_P^{2n}$ ($n < 1$), gives only a sub-leading contribution. (This statement applies even if we include all the near-horizon—not necessarily interior—excitations obtained by acting the $a_{\Omega,\xi}^{(k)\dagger}$ s.) As will be seen more explicitly in Section 2.5, $e^{S_{\text{vac}}}$ represents the number of possible independent ways in which quantum field theory on a fixed classical spacetime background, which allows for the description of the black hole interior, can emerge in a full quantum theory of gravity. The black hole entropy mostly counts the logarithm of this number!

2.4 The formation and evaporation of a black hole—the distant view

We now discuss how the black hole formation and evaporation processes are described from the point of view of a distant reference frame. Here we focus on basic aspects of this description. A detailed discussion on the structure of the Hilbert space, relevant to describe the evolution of a black hole, is given in the Appendix.

⁵Strictly speaking, operating $a_{\Omega,\xi}^{(k)\dagger}$ s on $|\psi_k\rangle$ changes the mass of the black hole. This, however, can be compensated by adjusting the mass associated with the singularity at the center. The argument below persists with this adjustment.

Consider collapsing matter represented by state $|M_{\text{init}}\rangle$ that has not been entangled with the rest of the system $|r_{\text{init}}\rangle$. The formation of a black hole (of the initial mass M_0) is then described as the following evolution of the entire system:

$$|\Psi\rangle = |M_{\text{init}}\rangle|r_{\text{init}}\rangle \rightarrow \left(e^{\approx \mathcal{A}(t)/4l_{\text{P}}^2} \sum_{k=1} c_k(t) |\psi_k(M(t))\rangle \right) |r(t)\rangle, \quad (41)$$

where $\mathcal{A}(t) = 16\pi M^2(t)l_{\text{P}}^4$ is the area of the horizon at time t , and $M(t) = M_0$ at the time of the formation.⁶ Strictly speaking, the black hole shortly after the formation is not yet in a vacuum state represented by (a superposition of) $|\psi_k\rangle$'s; instead, it is in a more general state represented by $|\phi_k\rangle$'s in Eq. (28). After the scrambling time of order $t_{\text{sc}} = M_0 l_{\text{P}}^2 \ln(M_0 l_{\text{P}})$, however, the black hole state takes the form shown in the biggest parentheses in the last expression, with c_k 's expected to take generic values

$$|c_k(t)|^2 \sim O\left(e^{\approx -\frac{\mathcal{A}(t)}{4l_{\text{P}}^2}}\right). \quad (42)$$

At this early stage in the evolution of the black hole, the state of the entire system is well approximated by the expression in Eq. (41). In particular, entanglement between the black hole and the rest of the system may still be neglected (for more precise discussion, see below).

As time passes, however, the black hole becomes more and more entangled with the rest of the system in the sense that the ratio of the entanglement entropy between the black hole and the rest, $S_{\text{ent}}(t)$, to the Bekenstein-Hawking entropy, $S_{\text{BH}}(t) = 4\pi M(t)^2 l_{\text{P}}^2$, keeps growing, which saturates the maximum value $S_{\text{ent}}(t)/S_{\text{BH}}(t) = 1$ after the Page time $t_{\text{Page}} \sim M_0^3 l_{\text{P}}^4$ [28]. Therefore, the state of the system at late times must be written more explicitly as [14, 15]

$$|\Psi(t)\rangle = e^{\approx \mathcal{A}(t)/4l_{\text{P}}^2} \sum_{k=1} d_k(t) |\psi_k(M(t))\rangle |r_k(t)\rangle, \quad (43)$$

where $|r_k\rangle$'s represent states of the subsystem complement to the black hole, i.e. those for the region $r \gtrsim 3Ml_{\text{P}}^2$, which include states of the Hawking radiation emitted earlier. In other words, at these late times the logarithm of the dimension of space spanned by the $|r_k\rangle$'s is of order S_{BH} (and equal to S_{BH} after the Page time), while at much earlier times it is negligible compared with S_{BH} . The state at early times, therefore, can be well approximated by the expression in Eq. (41) for the purpose of discussing internal properties of the black hole.

As the black hole evaporation progresses, the information contained in a set of coefficients $\{d_k(t)\}$ is gradually transferred into that in states $|r_{k'}(t')\rangle$ with $t' > t$, specifically in the correlations

⁶To be more precise, states well after the formation involve superpositions of black hole states with different M 's, reflecting the probabilistic nature of Hawking emission. We ignore this effect, as well as a possible spread of the initial black hole mass caused, e.g., by quantum effects associated with the collapse, since they do not affect our argument.

of Hawking quanta emitted at different moments. The dynamics of this process is governed by the interaction Hamiltonian coupling the stretched horizon and exterior degrees of freedom, whose form is determined by the intrinsically quantum gravitational, Planckian physics. We will be agnostic about the precise form of this Hamiltonian.⁷ After the black hole has completely evaporated, the state of the system becomes that of the final-state Hawking quanta (and matter that did not collapse). At some late time, we denote it by $|\Psi\rangle = |r_{\text{fin}}\rangle$.

At each moment in the evolution, the state of the black hole can be given by a density matrix in the space spanned by the $|\psi_k\rangle$'s, obtained by integrating out the rest of the system:

$$\rho_{\text{BH}} = \sum_{k,l} f_{kl} |\psi_k\rangle \langle \psi_l|, \quad (44)$$

where f_{kl} is a positive semi-definite Hermitian matrix with $\sum_k f_{kk} = 1$. (At early stages in which the entanglement between the black hole and the rest is neglected, f_{kl} takes the form $\simeq c_k c_l^*$; in other words, the black hole can be represented by a pure state $|\psi_{\text{BH}}\rangle = \sum_k c_k |\psi_k\rangle$ with $\sum_k |c_k|^2 = 1$.) By integrating out the stretched horizon states $|\tilde{z}; k\rangle$ in ρ_{BH} , we reproduce the exact thermal state, Eq. (21), for the near exterior states:

$$\rho_{\text{ext}} = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle \langle i|,$$

where we have used Eq. (24). The spectrum of Hawking radiation, therefore, is exactly thermal, up to gray-body factors which arise from (calculable) effects of potential barriers on the outgoing quanta.

The evolution of the state described above, which can be summarized by Eqs. (41, 43), implies that the information about the initial collapsing matter is encoded mostly in the coefficients c_k at an early stage of the black hole evolution, and then in the d_k 's and $|r_k\rangle$'s (or in the coefficients d_k and g_{ka} if $|r_k\rangle$'s are expanded in fixed basis states $|e_a\rangle$ describing the far exterior region, $|r_k\rangle = \sum_a g_{ka} |e_a\rangle$). Finally, after the black hole is evaporated, the information is contained in the state of final-state Hawking radiation. We can write this transfer of the information schematically as⁸

$$|M_{\text{init}}\rangle \rightarrow \{c_k\} \rightarrow \{d_k, |r_k\rangle\} \rightarrow |r_{\text{fin}}\rangle. \quad (45)$$

Since all the information about the initial state is kept, the evolution is unitary. Note that unitarity is preserved here in such a way that the entropy of (or the information about) the initial collapsing

⁷**Note added:** At the level of semi-classical field theory, the information transfer is viewed as occurring through a part of the information about the vacuum state which is delocalized in the whole zone region. For further discussions on this point, see Ref. [19].

⁸Note that this description is only schematic. For example, at an early stage of the evolution, the information about the initial matter may be contained not only in c_k 's but also in the deviation of the black hole state from a vacuum state, i.e. the appearance of general $|\phi_k\rangle$'s instead of $|\psi_k\rangle$'s in Eq. (41). The following description, however, still gives the correct overall picture for the transfer of the entropy throughout the whole process.

matter is transformed into that of a vacuum and then to that of final-state Hawking radiation. In the intermediate stage of this evolution, entanglement between the vacuum and (early) Hawking radiation develops, which is transferred to the entanglement among final-state Hawking quanta after the evaporation. This exchange of information between matter/radiation and a vacuum makes the black hole formation and evaporation processes particularly interesting (and, perhaps, is the main reason why these processes are hard to understand).

2.5 The fate of an infalling object and interior spacetime

What happens if an object falls into the horizon? From the viewpoint of a distant reference frame, the object will interact with the stretched horizon degrees of freedom which have a Planckian temperature—it is absorbed into the stretched horizon, and after time (at least) of order $Ml_P^2 \ln(Ml_P)$, its information will be sent back to the exterior in the form of Hawking radiation. On the other hand, general relativity tells us that the infalling object should not feel anything special at the horizon and must simply fall into the interior. In order to reproduce this picture, the physics of the stretched horizon after the object has fallen (which is strongly interacting when viewed from a distant reference frame) must be able to be organized into a form that allows for the interpretation that the object falls freely through the interior spacetime region.

We now discuss how this picture can come out in our framework. The state of the system with a black hole is given by Eq. (43) (or its special case, Eq. (41)). Suppose we let an object, which was originally located in a far exterior region, fall into the black hole. The state of the entire system at the beginning of the fall can then be written as

$$|\Psi_0\rangle = \left(\sum_{k=1}^{e^{\approx \mathcal{A}/4l_P^2}} d_k |\psi_k\rangle |r_k\rangle \right) |\chi\rangle, \quad (46)$$

where $|\chi\rangle$ is the initial state of the object. The subsequent evolution of the system, in particular the falling of the object into the black hole, can be described by acting the time evolution operator e^{-iHt} on this state $|\Psi_0\rangle$. In accordance with the complementarity hypothesis, we assume that this operator can be organized in a way that makes it manifest that the object passes through the horizon without being disrupted. (In the language of Ref. [7], this corresponds to changing the reference frame from a distant to an infalling one.) What is the form of the Hamiltonian H in such a description?

Let us denote the Hamiltonian in the standard Unruh-Israel description of a black hole written in terms of infalling modes by

$$H_{\text{UI}}(a_{\Omega,\xi}, a_{\Omega,\xi}^\dagger; c_{\mathbf{p}}, c_{\mathbf{p}}^\dagger) \simeq H_{\text{UI, Rind}}(a_{\Omega,\xi}, a_{\Omega,\xi}^\dagger) + H_{\text{UI, far}}(c_{\mathbf{p}}, c_{\mathbf{p}}^\dagger), \quad (47)$$

where $c_{\mathbf{p}}^\dagger/c_{\mathbf{p}}$ are the creation/annihilation operators for the modes in the far exterior region, $r \gtrsim 3Ml_P^2$. In the last expression, we have separated the Hamiltonian into those describing the Rindler

(\approx interior + near exterior) region, $H_{\text{UI, Rind}}$, and the far exterior region, $H_{\text{UI, far}}$, by ignoring the terms coupling $a_{\Omega, \xi}$'s and $c_{\mathbf{p}}$'s. (For simplicity, here we focus on the dynamics of an object falling into a black hole of fixed mass M , which is a good approximation given the timescale of any object to hit the singularity.) We postulate that the dynamics of the infalling object can be well described by the “infalling Hamiltonian”

$$H = \sum_{k=1}^{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} H^{(k)} P_k; \quad H^{(k)} = H_{\text{UI}}(a_{\Omega, \xi}^{(k)}, a_{\Omega, \xi}^{(k)\dagger}; c_{\mathbf{p}}, c_{\mathbf{p}}^\dagger), \quad (48)$$

where $a_{\Omega, \xi}^{(k)}$ and $a_{\Omega, \xi}^{(k)\dagger}$ are given by Eq. (36) and its conjugate, respectively, and P_k is the projection operator defined just below it. The evolution of the state $|\Psi_0\rangle$ is then

$$|\Psi_0\rangle \rightarrow e^{-iHt} \left(\sum_{k=1}^{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} d_k |\psi_k\rangle |r_k\rangle \right) |\chi\rangle = \sum_{k=1}^{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} d_k \left(e^{-iH^{(k)}t} |\psi_k\rangle |r_k\rangle |\chi\rangle \right). \quad (49)$$

We thus find that each “branch” of the state, labeled by k , evolves independently with its own Hamiltonian $H^{(k)}$, all of which, however, have the form identical to that of the standard (infalling) Unruh-Israel Hamiltonian. Moreover, since stretched horizon states with different k values are orthogonal (see Eq. (24)), each branch behaves as an independent world that does not interfere with others. This, therefore, explicitly realizes the idea suggested in Ref. [14] to have a smooth horizon consistently with unitary evolution of a black hole.

We note that the Hamiltonian in Eq. (48) may also be written as

$$H \simeq \left(\sum_{k=1}^{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} H_{\text{UI, Rind}}(a_{\Omega, \xi}^{(k)}, a_{\Omega, \xi}^{(k)\dagger}) \right) + H_{\text{UI, far}}(c_{\mathbf{p}}, c_{\mathbf{p}}^\dagger), \quad (50)$$

where we have ignored the terms involving both $a_{\Omega, \xi}^{(k)}$'s and $c_{\mathbf{p}}$'s, which exist near the boundary between the near and far regions, and have used $a_{\Omega, \xi}^{(k)} = a_{\Omega, \xi}^{(k)} P_k$ and $\sum_k P_k = 1$ in the first and second terms, respectively. This expression makes it clear that the Hamiltonian in the far exterior region has not been modified from the standard form in Eq. (47), as is naturally expected. In fact, if we define the infalling mode operators by

$$a_{\Omega, \xi} \equiv \sum_k a_{\Omega, \xi}^{(k)}, \quad a_{\Omega, \xi}^\dagger \equiv \sum_k a_{\Omega, \xi}^{(k)\dagger}, \quad (51)$$

then we find that all the usual expressions for field theory operators go through because of the projection operator P_k involved in $a_{\Omega, \xi}^{(k)}$. For example, the Hamiltonian is given simply by $H = H_{\text{UI}}(a_{\Omega, \xi}, a_{\Omega, \xi}^\dagger; c_{\mathbf{p}}, c_{\mathbf{p}}^\dagger)$, and the creation/annihilation operator algebra by $[a_{\Omega, \xi}, a_{\Omega', \xi'}^\dagger] = \delta_{\Omega\Omega'} \delta_{\xi\xi'}$, $[a_{\Omega, \xi}, a_{\Omega', \xi'}] = [a_{\Omega, \xi}^\dagger, a_{\Omega', \xi'}^\dagger] = 0$.

The analysis described above indicates that the stretched horizon degrees of freedom provide the states necessary to compose (the $e^{\approx \mathcal{A}/4l_P^2}$ copies of) the second exterior region of the Rindler space, and hence the near-horizon eternal black hole geometry. In particular, the quantum mechanical structure of a collapse-formed black hole (often called a one-sided black hole) after the horizon is stabilized to a generic state is that of an eternal (two-sided) black hole at the microscopic level. We can summarize these points by the following statement:

One-sided black hole with a stretched horizon = Two-sided black hole

$$= \text{A "superposition" of } e^{\approx \frac{\mathcal{A}}{4l_P^2}} \text{ Unruh-Israel near-horizon quantum field theories.} \quad (52)$$

Here, the quotation marks around the word “superposition” indicate that the states of the (near-horizon) Unruh-Israel theories may be entangled with far exterior states, as in Eq. (43). We emphasize that the correspondence between the collapse-formed and eternal black holes discussed here applies at each instant of time (or in a sufficiently short time period compared with the timescale for the evolution of the black hole). In particular, the mass of the corresponding (hypothetical) eternal black holes must be taken as that of the evolving black hole at each moment $M(t)$, not the initial mass M_0 .

We now argue that the dynamics of the stretched horizon postulated in Eq. (48) implies that an infalling observer finds that the horizon is smooth (no drama) with a probability of 1. Suppose the initial state before the infall was given by $|\Psi_0\rangle$ in Eq. (46), and that the observer does not interact strongly with the system entangled with the black hole (early Hawking radiation) throughout the falling. Now, the state $|r_k\rangle$ is in general a superposition of decohered classical states $|r_m^{\text{cl}}\rangle$, $|r_k\rangle = \sum_m U_{km} |r_m^{\text{cl}}\rangle$ where U_{km} is a unitary matrix, and the observer finds himself/herself to live in one of these worlds with the (collapsed) state given by

$$|\Psi_0^{(m)}\rangle = \frac{1}{\sqrt{\sum_{k'=1}^{e^{\approx \mathcal{A}/4l_P^2}} |d_{k'} U_{k'm}|^2}} \sum_{k=1}^{e^{\approx \mathcal{A}/4l_P^2}} d_k U_{km} |\psi_k\rangle |r_m^{\text{cl}}\rangle |\chi\rangle. \quad (53)$$

According to Eq. (49), when this observer interacts with the black hole state, he/she will find himself/herself to be in a particular vacuum $|\psi_k\rangle$ with probability $|d_k U_{km}|^2 / \sum_{k'} |d_{k'} U_{k'm}|^2$ (i.e. the measurement basis is $|\psi_k\rangle$), but all of these vacua lead to the same semi-classical physics dictated by the Hamiltonian H_{UI} , i.e. general relativity. The fact that the observer finds a smooth horizon with a probability of 1 does not change even if he/she performs an arbitrary measurement on early Hawking radiation before jumping into the black hole. Without loss of generality, we may assume that the outcome of the measurement was $\sum_k V_{nk} |r_k\rangle$, where V_{nk} is an arbitrary unitary matrix. By repeating the same argument as above with U_{km} replaced by V_{kn}^\dagger , we find that the observer does not see anything special at the horizon.⁹

⁹On the other hand, if a falling observer could directly measure states entangled with the horizon and then enter

How does the analysis described above evade arguments for firewalls? As discussed in Ref. [14], the entropy argument based on the strong subadditivity relation [8] is avoided because the entropies relevant for discussing unitarity are different from those for the smoothness of the horizon. Specifically, unitarity requires the von Neumann entropies of subsystems consisting of A , B , and R (the stretched horizon, near horizon region, and far exterior region, respectively) calculated using the entire quantum state $e^{-iHt} |\Psi_0\rangle$ to satisfy $S_{BR} < S_R$. On the other hand, the smoothness of the horizon requires $\tilde{S}_{AB}^{(k)} \approx 0$ (for all k), where $\tilde{S}_X^{(k)}$ is the von Neumann entropy of subsystem X calculated using the “branch state” $e^{-iH^{(k)}t} U_{km} |\psi_k\rangle |r_m^{\text{cl}}\rangle |\chi\rangle / |U_{km}|$ without summation over k . These two relations are *not* incompatible with each other, unlike the case if they were both calculated using the same quantum state.

The typicality argument of Ref. [10] is also avoided because the black hole vacuum states $|\psi_k\rangle$ cannot all be transformed into eigenstates of the number operator for a near exterior mode, $\hat{b}^\dagger \hat{b}$, by performing a unitary rotation in the space spanned by $|\psi_k\rangle$, \mathcal{H}_ψ . (For a similar discussion, see Ref. [16].) Specifically, by expanding near horizon states $|i\rangle$ by the $\hat{b}^\dagger \hat{b}$ eigenstates $|e_j\rangle$ as $|i\rangle = \sum_j c_j^i |e_j\rangle$, the black hole vacuum states become

$$|\psi_k\rangle = \frac{1}{\sqrt{Z}} \sum_j |e_j\rangle \left(\sum_i e^{-\frac{\beta}{2} E_i} c_j^i |\tilde{i}; k\rangle \right). \quad (54)$$

We find that because of the index k in the stretched horizon states, we cannot find a basis change in \mathcal{H}_ψ that makes all the $|\psi_k\rangle$ ’s $\hat{b}^\dagger \hat{b}$ eigenstates, and hence there is no reason to expect that typical black hole states have firewalls. Indeed, by calculating the average number of high energy quanta in an infalling frame (i.e. quanta with $\Omega \gg 1/Ml_{\text{P}}^2$), we obtain

$$\bar{N} \equiv \frac{\text{Tr}_{\mathcal{H}_\psi} \hat{N}_a}{\text{Tr}_{\mathcal{H}_\psi} \mathbf{1}} = \frac{\sum_{k=1}^{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} \langle \psi_k | \hat{N}_a | \psi_k \rangle}{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} \approx 0, \quad (55)$$

because of Eq. (38), where \hat{N}_a is the number operator for the quanta with the indices Ω and ξ

$$\hat{N}_a = \sum_{k=1}^{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} a_{\Omega, \xi}^{(k)\dagger} a_{\Omega, \xi}^{(k)}. \quad (56)$$

We find that typical black hole states do *not* have firewalls. (In fact, no black hole state has a firewall.)

to it right after (i.e. before the state of the black hole changes), then he/she may see a firewall. (It is not clear if such a measurement can indeed be performed; it is possible that there is some dynamical, or perhaps computational [29], obstacle to it.) This, however, does not violate the equivalence principle, since the same argument applies to any surface in a low curvature region, i.e. there is nothing special about the black hole horizon.

2.6 Complementarity as a unitary reference frame change

We have seen that the dynamics of the stretched horizon is such that it can produce $e^{\approx \mathcal{A}/4l_{\text{P}}^2}$ copies of quantum field theories describing physics in the interior region of a black hole. Is it then possible to organize the description of the entire system in a way that makes manifest the local nature of the interior spacetime region while keeping unitarity at the full quantum level? Here, following Ref. [7], we consider that such an “infalling description” is obtained by performing a unitary complementarity transformation on a distant description, which corresponds to changing the (local Lorentz) reference frame from a distant one to an infalling one. What does this description look like?

If the complementarity transformation is indeed unitary, the $e^{\approx \mathcal{A}/4l_{\text{P}}^2}$ states $|\psi_k\rangle$ must be transformed into $e^{\approx \mathcal{A}/4l_{\text{P}}^2}$ different states which must all look locally like Minkowski vacuum states. In particular, this implies that in the limit that the black hole is large $\mathcal{A} \rightarrow \infty$, i.e. in the limit that the horizon under consideration is a Rindler horizon, there are infinitely many Minkowski vacuum states labeled by $k = 1, \dots, e^{\approx \mathcal{A}/4l_{\text{P}}^2} = \infty$. This seems to contradict our experience that we can do physics without knowing which of the Minkowski vacua we live in. Isn’t the Minkowski vacuum unique, e.g., in QED? Otherwise, we do not seem to be able to do any physics without having the (infinite amount of) information on the Minkowski vacua.

The answers to these questions arise by noticing that when described in an infalling reference frame, the black hole spacetime is Minkowski vacuum-like only locally, and the nonzero curvature effect can lead to a “horizon” (as viewed from the infalling frame, not the original one as viewed from a distant frame) at a finite spatial distance, a distance of order the Schwarzschild radius away from the origin of the reference frame (see Ref. [15] for a related discussion). This makes it possible that the $e^{\approx \mathcal{A}/4l_{\text{P}}^2}$ different states in the infalling description correspond to different configurations of the degrees of freedom on this “horizon,” and that physics in the interior region is given by the same local Hamiltonian in all these $e^{\approx \mathcal{A}/4l_{\text{P}}^2}$ states, consistent with the postulate in Eq. (48). In order for an observer to know which $|\psi_k\rangle$ vacuum he/she is in, he/she needs to probe (intrinsically quantum gravitational) degrees of freedom on his/her “horizon”; the effect he/she can probe locally away from the “horizon” is expected to be suppressed exponentially in $\mathcal{A}/l_{\text{P}}^2$. In the true Minkowski space limit of $\mathcal{A} \rightarrow \infty$, this “horizon” is located only at spatial infinity, which no physical observer can access. The uniqueness of the Minkowski vacuum (for the purpose of describing local physics) is recovered in this way.

Finally, we comment on unitary equivalence relations between subsystems in distant and infalling reference frames. When viewed from a distant reference frame, the information about an object absorbed into the stretched horizon stays there for, at least, a time of order $Ml_{\text{P}}^2 \ln(Ml_{\text{P}})$, which will later be sent back to the exterior in Hawking radiation. On the other hand, when we describe the same object in an infalling reference frame (whose origin closely follows the trajectory

of the object), the description of the object being in the interior spacetime is available only until the object or the origin of the reference frame hits the singularity a time of order Ml_{P}^2 after the fall; after this time the system is described only by “singularity states” [7]: intrinsically quantum gravitational states that do not allow for a spacetime interpretation. (The infalling reference frame is obtained by performing a boost transformation on a distant reference frame at some time; see Ref. [30] for more detailed discussions on this prescription.) This implies that when the complementary transformation is defined as the relation between the two descriptions at the same proper time of the origin of their respective reference frames, then the interior spacetime can correspond only to the stretched horizon degrees of freedom while Hawking radiation only to the singularity degrees of freedom:

$$\begin{array}{ccc}
\text{infalling} & & \text{distant} \\
\hline
\text{interior} & \subset & \text{stretched horizon} \\
\text{singularity} & \supset & \text{Hawking radiation.}
\end{array} \tag{57}$$

In particular, this seems to prevent the possibility of “gravity/gravity” duality in black hole physics, in which information in Hawking radiation is directly mapped to that in the interior spacetime at the same proper time of the two (distant and infalling) reference frames.

We may even go further. As we have seen, and is particularly clear in Eq. (51), an object traveling in the interior spacetime is described by isomorphic excitations of the $e^{\mathcal{A}/4l_{\text{P}}^2}$ vacuum states $|\psi_k\rangle$, without their coefficients d_k in the state of the entire system changing throughout the falling process. Namely, when viewed from a distance, the information about such an object is contained in the deviations of the black hole states from the vacuum states, $|\phi_k\rangle \neq |\psi_k\rangle$, and not in d_k . In fact, by the time the information about the fallen object becomes distributed into the coefficients d_k in the distant picture, the object in the corresponding interior picture has already been absorbed into the “horizon” surrounding it. Note that the distance to the “horizon” from the origin of the infalling reference frame becomes smaller as the singularity is approached (and becomes less than l_* at some point near the singularity) because of the increase of the curvature effect there. This implies that there is no complementary description of the information scrambling, or evaporation, process in the interior picture in the regime where the semi-classical spacetime description is applicable.

3 More General Spacetimes

In this section, we discuss how the picture developed so far in black hole physics can be extended to more general spacetimes. The structure of the section is such that the discussion becomes more conjectural as it progresses. We first consider a relatively straightforward application of the dynamics of the stretched horizon we have learned in black hole physics to de Sitter space. We then discuss possible implications of our picture for the general structure of Hilbert space in quantum

gravity.

3.1 de Sitter Space

Consider de Sitter space with the Hubble parameter H . The de Sitter horizon is located at $r = 1/H$, where r is the radial coordinate of the static coordinate system (t, r, θ, ϕ) . The stretched horizon is located where the local Gibbons-Hawking temperature [31]

$$T(r) = \frac{H/2\pi}{\sqrt{1 - H^2 r^2}}, \quad (58)$$

becomes of order the fundamental, or string, scale $1/l_*$: $T(r_*) = 1/2\pi l_*$ (where the factor of 2π is chosen so that the local proper acceleration of a fixed spatial coordinate point at the stretched horizon is $1/l_*$), i.e.

$$r_* = \frac{1}{H} - \frac{1}{2} H l_*^2. \quad (59)$$

Following the discussion in the black hole case, we consider that the stretched horizon degrees of freedom are organized into the states labeled by \tilde{i} and k :

$$|\tilde{i}; k\rangle \quad \text{with} \quad \langle \tilde{i}; k | \tilde{i}'; k' \rangle = \delta_{\tilde{i}\tilde{i}'} \delta_{kk'}, \quad (60)$$

and that the \tilde{i} index is entangled with the states in the interior of the stretched de Sitter horizon $|i\rangle$ (corresponding to the near-horizon states outside the stretched Schwarzschild horizon) as

$$|\psi_k\rangle = \frac{1}{\sqrt{\sum_{i'} e^{-\beta E_{i'}}}} \sum_i e^{-\frac{\beta}{2} E_i} |i\rangle |\tilde{i}; k\rangle, \quad (61)$$

in the de Sitter vacuum states. Here, E_i is the energy of $|i\rangle$ measured at the origin $r = 0$, and $\beta = 2\pi/H$ the inverse Gibbons-Hawking temperature; the index k runs over

$$k = 1, \dots, e^{\approx \frac{\mathcal{A}}{4l_P^2}}, \quad (62)$$

where $\mathcal{A} = 4\pi/H^2$ is the area of the (stretched) de Sitter horizon.

We assume, given the absence of evidence otherwise, that the analysis of the stretched horizon in the black hole case can be straightforwardly adapted to de Sitter space (with the obvious interchange of the “interior” and “exterior”: the region outside the de Sitter horizon corresponds to the region inside the Schwarzschild horizon). When viewed from a reference frame associated with the static coordinates, the entropy of the system is saturated at the leading order in l_P^2/\mathcal{A} by the logarithm of the possible number of vacuum states, $|\psi_k\rangle$:

$$S_{\text{ds}} \approx \ln \dim \mathcal{H}_\psi \approx \frac{\mathcal{A}}{4l_P^2}, \quad (63)$$

where \mathcal{H}_ψ is the Hilbert space spanned by $|\psi_k\rangle$. The number of possible states the interior region can be in (without forming a black hole, which would change the horizon structure) as well as the number of possible states for the near exterior region are expected to contribute only negligibly, by an amount smaller powers in \mathcal{A}/l_p^2 . (The states in which there is matter in the near exterior region can be obtained by acting creation operators $a_{\Omega,\xi}^{(k)\dagger}$ on $|\psi_k\rangle$, constructed analogously to the black hole case.) When a reference frame change corresponding to a shift of the origin of the reference frame is performed, a spacetime region outside the original horizon can be reconstructed, with the excitations in the near exterior region corresponding to perturbations of $|\psi_k\rangle$ by $a_{\Omega,\xi}^{(k)\dagger}$'s while those in the far exterior region involving the index k . With a succession of such reference frame changes, the (approximate) picture of global de Sitter space may be obtained, which, however, grossly overcounts the number of degrees of freedom if the horizon degrees of freedom (e.g. in each Hubble volume) are also included in the description [6, 7].

Following the same analysis as in the black hole case, we find that the de Sitter horizon is smooth: an object that hits the horizon can be thought of as going to space outside the horizon. The information about the object that goes outside will be stored in the general state of the form

$$|\phi_k\rangle = \frac{1}{\sum_{i',\tilde{j}'} |f_{i,\tilde{j}}|^2} \sum_{i,\tilde{j}} f_{i,\tilde{j}} |i\rangle |\tilde{j}; k\rangle, \quad (64)$$

constructed purely from the interior and the stretched horizon degrees of freedom. Such information may thus be recovered later. This information recovery may not necessarily be in the form of Hawking radiation if the system evolves, for example, into Minkowski space or another de Sitter space with a smaller vacuum energy. Indeed, this is believed to have happened to density fluctuations generated in the early inflationary phase in our universe [32].

In the limit $H \rightarrow 0$ in which the de Sitter space approaches Minkowski space, the number of vacuum states becomes infinity

$$\dim \mathcal{H}_\psi \rightarrow \infty. \quad (65)$$

Since the horizon is located at spatial infinity in this limit, probing the structure of (infinitely many) Minkowski vacua will require access to the horizon at infinity. This is the same picture as the one arrived at in Section 2.6 by taking the large mass limit of a black hole.

3.2 Hilbert space for quantum gravity and the entropy bound

What does the picture developed so far imply for more general spacetimes in quantum gravity? Following Refs. [7, 15], here we consider that the Hilbert space for quantum gravity can be organized such that the system is viewed from a freely falling (local Lorentz) reference frame. (This corresponds to partially fixing large gauge redundancies in full quantum gravity.) We will be agnostic about its detailed implementation, e.g., whether a null or spacelike quantization is used inside horizons as viewed from the reference frame.

The evolution of a system is described by giving a quantum state at each time τ , taken as the proper time measured at the spatial origin $p(\tau)$ of the reference frame. These states are in general superpositions of component states that represent configurations on well-defined semi-classical (“equal-time”) spacetime hypersurfaces. (More precisely, the state of the system may also contain “singularity states” that do not allow for a spacetime interpretation; these states are relevant when $p(\tau)$ hits a singularity.) Now, we can group these component states, which span a Hilbert space \mathcal{H} , into classes $\mathcal{H}_{\partial\mathcal{M}}$ that represent all possible physical configurations in all possible spacetime hypersurfaces that share the same boundary $\partial\mathcal{M}$:

$$\mathcal{H} = \bigoplus_{\partial\mathcal{M}} \mathcal{H}_{\partial\mathcal{M}}. \quad (66)$$

Our interest is in how many independent quantum states there are in $\mathcal{H}_{\partial\mathcal{M}}$ with a fixed $\partial\mathcal{M}$, and what the entropy associated with them (i.e. the logarithm of that number) corresponds to. Note that quantum states we discuss here are those for the entire system; in particular, they include states for the boundary degrees of freedom.

Let us consider a fixed spacetime, corresponding roughly to some fixed $\partial\mathcal{M}$. The origin of the reference frame $p(\tau)$ is then typically surrounded by horizons where Planckian physics becomes important, which we take as our boundaries. (Note that the horizons may be located at spatial infinity as in the case of Minkowski space.) For example, if $p(\tau)$ is located outside the Schwarzschild horizon in a de Sitter space with a black hole, then $p(\tau)$ will “see” the black hole horizon in some directions and the de Sitter horizon in the others. Specifically, suppose that the quantization surface at time τ is parameterized by coordinates (λ, θ, ϕ) which, for infinitesimal λ , are spherical coordinates for the locally inertial reference frame of $p(\tau)$. Then, a radial axis with constant (θ, ϕ) will hit either the (stretched) black hole or de Sitter horizon, depending on the values of (θ, ϕ) . Now, the area of these horizons are

$$\mathcal{A}_{\text{BH}} \sim 16\pi M^2 l_{\text{P}}^4, \quad \mathcal{A}_{\text{dS}} \sim \frac{4\pi}{H^2}, \quad (67)$$

where M and H are the mass of the black hole and the Hubble parameter, respectively, and we have assumed $M l_{\text{P}}^2 \ll 1/H$. What is the entropy of this system, i.e. the logarithm of the number of possible independent quantum states that have the same horizon/boundary structure?

The covariant entropy bound [20] suggests that the (fine-grained) entropy of this spacetime, more precisely the logarithm of the number of independent quantum states consistent with the boundary structure described above, is given by

$$S \approx \frac{1}{4l_{\text{P}}^2} (\mathcal{A}_{\text{BH}} + \mathcal{A}_{\text{dS}}), \quad (68)$$

where we have assumed that the bound is indeed saturated. What does this entropy count? Performing an analysis similar to the one in Ref. [17], we expect that the entropy arising from

the “interior” configurations, i.e. configurations between the Schwarzschild and de Sitter horizons, is negligible. (Note that we should not count the thermal entropy of Hawking radiation for this purpose; it could arise even if the entire state, which includes the horizon degrees of freedom, were unique, i.e. $S = 0$, as long as the interior and horizon degrees of freedom are entangled.) The leading contributions then must come from the horizons as we discussed in Sections 2 and 3.1. These entropies are the entropies of a vacuum—when a reference frame change is performed, the relevant degrees of freedom are mapped into those in the horizons of the new reference frame, which dictate how many interior quantum field theories (which all look identical) can be realized at the full quantum gravity level. For example, if $p(\tau)$ is transformed to be in the interior of the black hole, then S is realized mostly as the logarithm of possible configurations of the degrees of freedom on the “horizon” discussed in Section 2.6. When $p(\tau)$ is transformed to be in the exterior of the de Sitter horizon, S mostly counts the logarithm of possible configurations of the degrees of freedom on the new de Sitter horizon surrounding the new, transformed $p(\tau)$.

More generally, the fine-grained entropy of the system—the logarithm of the number of independent states in $\mathcal{H}_{\partial\mathcal{M}}$ —is given by the area of the Planckian boundaries surrounding $p(\tau)$ (which may exist at spatial infinity):¹⁰

$$S \approx \frac{1}{4l_{\text{P}}^2} \int_{(\theta, \phi)} d\mathcal{A}, \quad (69)$$

where we have assumed that the spacelike projection theorem of Ref. [20] applies if a spacelike quantization is employed. There is, however, one important caveat. Suppose we follow a radial axis, parameterized by λ , in the direction of constant (θ, ϕ) and find that it hits an apparent horizon before hitting a Planckian boundary such as the stretched black hole or de Sitter horizon. Here, the apparent horizon is defined locally as a surface on which the expansion of the past-directed outgoing light rays, emitted from (a portion of) the constant λ surface, first crosses zero. (An example is given by a surface at $r = (1/a(t)l_{\text{P}})\sqrt{3/8\pi\rho(t)}$ in the Friedmann-Robertson-Walker universe, where we have assumed that $p(\tau)$ is comoving at $r = 0$, and $a(t)$ and $\rho(t)$ are the scale factor and energy density, respectively.) If this happens, then the surface used to bound the entropy must be replaced with the apparent horizon. Specifically, the expression in Eq. (69) must be modified to [15]

$$S \approx \frac{1}{4l_{\text{P}}^2} \int_{(\theta, \phi)} d\mathcal{A}(\lambda_{\text{H}}(\theta, \phi)); \quad \lambda_{\text{H}}(\theta, \phi) = \min \{ \lambda_{\text{P}}(\theta, \phi), \lambda_{\text{app}}(\theta, \phi) \}, \quad (70)$$

where $d\mathcal{A}(\lambda_{\text{H}}(\theta, \phi))$ is an area element for a surface $\lambda = \lambda_{\text{H}}(\theta, \phi)$, and $\lambda_{\text{P}}(\theta, \phi)$ and $\lambda_{\text{app}}(\theta, \phi)$ are the radial coordinate distances from $p(\tau)$ to the nearest Planckian and apparent horizons, respectively.

¹⁰The general definition of the structures of two boundaries being the same is not obvious; one possible definition is to require that they explicitly have the same induced metric in terms of the coordinates used in defining the theory, e.g. (λ, θ, ϕ) used above; see Ref. [15] for a discussion on this point.

We may decompose the expression for S in Eq. (70) into two pieces as

$$S \approx S_{\text{P}} + S_{\text{app}}, \quad (71)$$

where

$$S_{\text{P}} \approx \frac{1}{4l_{\text{P}}^2} \int_{(\theta, \phi)_{\text{P}}} d\mathcal{A}(\lambda_{\text{P}}(\theta, \phi)), \quad S_{\text{app}} \approx \frac{1}{4l_{\text{P}}^2} \int_{(\theta, \phi)_{\text{app}}} d\mathcal{A}(\lambda_{\text{app}}(\theta, \phi)), \quad (72)$$

with $(\theta, \phi)_{\text{P}}$ and $(\theta, \phi)_{\text{app}}$ indicating the values of (θ, ϕ) in which $\lambda_{\text{P}}(\theta, \phi)$ is smaller and larger than $\lambda_{\text{app}}(\theta, \phi)$, respectively. We have seen that, when a reference frame change is performed, S_{P} can be viewed as the entropy of a vacuum. On the other hand, as seen in Ref. [33], S_{app} can be saturated by the contribution from quantum field theory degrees of freedom, i.e. the matter/radiation contribution from the region outside the apparent horizon. We therefore find that typically S_{P} comes dominantly from the entropy of a vacuum while S_{app} can come either from a vacuum or matter/radiation contribution:

$$S_{\text{P}} \approx S_{\text{P, vac}} \gg S_{\text{P, mat}}, \quad S_{\text{app}} \approx S_{\text{app, vac}} \text{ or } S_{\text{app, mat}}, \quad (73)$$

where $S_{\text{X, vac}}$ and $S_{\text{X, mat}}$ represent the vacuum and matter/radiation contributions to S_{X} . In any event, the fine-grained entropy arises in general from both vacuum and matter/radiation contributions:

$$S = S_{\text{vac}} + S_{\text{mat}}, \quad (74)$$

where $S_{\text{vac}} = S_{\text{P, vac}} + S_{\text{app, vac}}$ and $S_{\text{mat}} = S_{\text{P, mat}} + S_{\text{app, mat}}$, and only the S_{mat} contribution is explicitly included as the dynamical degrees of freedom in quantum field theories. With the contribution from a vacuum included, we conjecture that the covariant entropy bound is indeed saturated for all $\mathcal{H}_{\partial\mathcal{M}}$

$$S = S_{\text{vac}} + S_{\text{mat}} \approx \frac{1}{4l_{\text{P}}^2} \int_{(\theta, \phi)} d\mathcal{A}(\lambda_{\text{H}}(\theta, \phi)), \quad (75)$$

which indeed seems to be the case in all the systems we have considered.

Acknowledgments

This work was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, of the US Department of Energy under Contract DE-AC02-05CH11231, and in part by the National Science Foundation under grant PHY-1214644.

A Hilbert Space for the Evolution of a Black Hole

Here we present the Hilbert space relevant for describing the formation and evaporation of a black hole (Schwarzschild black hole in 4-dimensional spacetime) from a distant reference frame.¹¹ We first consider a system with a black hole *of fixed mass* M and decompose it into three subsystems:

A : the degrees of freedom associated with the stretched horizon;

C : the degrees of freedom associated with the spacetime region close to, but outside, the stretched horizon, e.g. $r \lesssim 3Ml_{\text{P}}^2$;

R : the rest of the system (which may contain Hawking radiation emitted earlier).

Among all the possible quantum states for the C degrees of freedom, some are strongly entangled with the states representing A . We call the set of these quantum states B :

B : the quantum states representing the states for the C degrees of freedom that are strongly entangled with the degrees of freedom described by A .

(The identification of B depends on the possible existence of matter in the region $2Ml_{\text{P}}^2 < r \lesssim 3Ml_{\text{P}}^2$. If there is extra matter beyond the black hole in $r \lesssim 3Ml_{\text{P}}^2$, it changes the identification of the B states in the Hilbert space for the C degrees of freedom.) Below we will ignore the center-of-mass drift and spontaneous spin-up of a black hole [30,34], which give only minor effects on the dynamics. Including these effects, however, is straightforward—we simply have to add indices for the center-of-mass location and angular momentum of the black hole to the states.

We have now divided the system with a black hole of fixed mass M into three subsystems A , C , and R . Since the black hole mass varies with time, however, the Hilbert space in which the state of the entire system evolves unitarily must take the form

$$\mathcal{H} = \bigoplus_M (\mathcal{H}_{A(M)} \otimes \{\mathcal{H}_{B(M)} \oplus \mathcal{H}_{C(M)-B(M)}\} \otimes \mathcal{H}_{R(M)}) \equiv \bigoplus_M \mathcal{H}_M, \quad (76)$$

where we have explicitly shown the M dependence of A , B , C , and R . Here, $\mathcal{H}_{C(M)-B(M)}$ is the Hilbert space spanned by the states for the C degrees of freedom orthogonal to B (i.e. not entangled with A), and we define \mathcal{H}_0 to be the Hilbert space for the system without a black hole. As the black hole evolves, the state of the system moves between different \mathcal{H}_M 's; for example, a state that is an element of \mathcal{H}_{M_1} with some M_1 will later become an element of \mathcal{H}_{M_2} with $M_2 < M_1$ (more precisely, a superposition of elements in various \mathcal{H}_{M_2} 's). In this language, $|\psi_k(M)\rangle$ in Eqs. (41, 43)

¹¹The Hilbert space described here is similar to that in Ref. [16]. A major difference is the structure of the stretched horizon degrees of freedom, denoted by A here and by \tilde{B} in Ref. [16]. In particular, the dimension of the Hilbert space factor $\mathcal{H}_{A(M)}$ here is much larger than $e^{\approx A/4l_{\text{P}}^2}$ (specified by both the indices \tilde{i} and k ; see Section 2), while that of $\mathcal{H}_{\tilde{B}(M)}$ in Ref. [16] is $e^{\approx A/4l_{\text{P}}^2}$.

are elements of $\mathcal{H}_{A(M)} \otimes \mathcal{H}_{B(M)}$, while $|r\rangle$ and $|r_k\rangle$ are those of $\mathcal{H}_{R(M)}$; $|r_{\text{fin}}\rangle$, representing final-state Hawking radiation, is an element of \mathcal{H}_0 .

In our present picture, the Hilbert space factor for the stretched horizon degrees of freedom can be decomposed as

$$\mathcal{H}_{A(M)} = \bigoplus_{k=1}^{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} \mathcal{H}_{A(M)}^{(k)}, \quad (77)$$

where $\mathcal{A} = 16\pi M^2 l_{\text{P}}^4$, and $\mathcal{H}_{A(M)}^{(k)}$ is the Hilbert space factor of which the stretched horizon states $|\tilde{z}; k\rangle$ are the elements. Plugging this expression into Eq. (76), we find that the Hilbert space for the system with a black hole of mass M takes the form

$$\mathcal{H}_M = \bigoplus_{k=1}^{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} \left(\mathcal{H}_{A(M)}^{(k)} \otimes \{ \mathcal{H}_{B(M)} \oplus \mathcal{H}_{C(M)-B(M)} \} \otimes \mathcal{H}_{R(M)} \right) \equiv \bigoplus_{k=1}^{e^{\approx \mathcal{A}/4l_{\text{P}}^2}} \mathcal{H}_M^{(k)}. \quad (78)$$

We find that \mathcal{H}_M consists of $e^{\approx \mathcal{A}/4l_{\text{P}}^2}$ Hilbert subspaces $\mathcal{H}_M^{(k)}$, each of which takes the form identical to the Hilbert space of quantum field theory on a fixed classical spacetime background with a black hole of mass M .

References

- [1] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975) [Erratum-ibid. **46**, 206 (1976)].
- [2] S. W. Hawking, Phys. Rev. D **14**, 2460 (1976).
- [3] G. 't Hooft, Nucl. Phys. B **335**, 138 (1990); C. R. Stephens, G. 't Hooft and B. F. Whiting, Class. Quant. Grav. **11**, 621 (1994) [arXiv:gr-qc/9310006].
- [4] L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. D **48**, 3743 (1993) [arXiv:hep-th/9306069]; L. Susskind and L. Thorlacius, Phys. Rev. D **49**, 966 (1994) [hep-th/9308100].
- [5] P. Hayden and J. Preskill, JHEP **09**, 120 (2007) [arXiv:0708.4025 [hep-th]].
- [6] Y. Nomura, JHEP **11**, 063 (2011) [arXiv:1104.2324 [hep-th]]; Phys. Rev. D **86**, 083505 (2012) [arXiv:1205.5550 [hep-th]].
- [7] Y. Nomura, Found. Phys. **43**, 978 (2013) [arXiv:1110.4630 [hep-th]].
- [8] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, JHEP **02**, 062 (2013) [arXiv:1207.3123 [hep-th]].
- [9] A. Almheiri, D. Marolf, J. Polchinski, D. Stanford and J. Sully, JHEP **09**, 018 (2013) [arXiv:1304.6483 [hep-th]].
- [10] D. Marolf and J. Polchinski, Phys. Rev. Lett. **111**, 171301 (2013) arXiv:1307.4706 [hep-th].
- [11] See also, S. L. Braunstein, arXiv:0907.1190v1 [quant-ph]; S. D. Mathur, Class. Quant. Grav. **26**, 224001 (2009) [arXiv:0909.1038 [hep-th]]; S. B. Giddings, Class. Quant. Grav. **28**, 025002 (2011) [arXiv:0911.3395 [hep-th]].
- [12] R. Bousso, Phys. Rev. D **88**, 084035 (2013) [arXiv:1308.2665 [hep-th]].
- [13] See also, S. G. Avery, B. D. Chowdhury and A. Puhm, JHEP **09**, 012 (2013) [arXiv:1210.6996 [hep-th]]; B. D. Chowdhury, JHEP **10**, 034 (2013) [arXiv:1307.5915 [hep-th]].
- [14] Y. Nomura and J. Varela, JHEP **07**, 124 (2013) [arXiv:1211.7033 [hep-th]].
- [15] Y. Nomura, J. Varela and S. J. Weinberg, Phys. Lett. B **733**, 126 (2014) [arXiv:1304.0448 [hep-th]].
- [16] Y. Nomura, J. Varela and S. J. Weinberg, Phys. Rev. D **88**, 084052 (2013) [arXiv:1308.4121 [hep-th]].
- [17] G. 't Hooft, gr-qc/9310026.
- [18] W. G. Unruh, Phys. Rev. D **14**, 870 (1976); W. Israel, Phys. Lett. A **57**, 107 (1976).
- [19] Y. Nomura and S. J. Weinberg, arXiv:1406.1505 [hep-th].

- [20] R. Bousso, JHEP **07**, 004 (1999) [arXiv:hep-th/9905177]; Rev. Mod. Phys. **74**, 825 (2002) [hep-th/0203101].
- [21] Y. Nomura, talk at KITP rapid response workshop “Black Holes: Complementarity, Fuzz, or Fire?,” KITP Santa Barbara, August 2013.
- [22] E. Verlinde and H. Verlinde, arXiv:1306.0515 [hep-th]; arXiv:1306.0516 [hep-th].
- [23] M. Van Raamsdonk, Gen. Rel. Grav. **42**, 2323 (2010) [Int. J. Mod. Phys. D **19**, 2429 (2010)] [arXiv:1005.3035 [hep-th]].
- [24] K. Papadodimas and S. Raju, arXiv:1310.6334 [hep-th]; arXiv:1310.6335 [hep-th].
- [25] Y. Sekino and L. Susskind, JHEP **10**, 065 (2008) [arXiv:0808.2096 [hep-th]].
- [26] R. D. Sorkin, in 10th International Conference on General Relativity and Gravitation, Contributed Papers, vol. II, 1167, eds. B. Bertotti, F. de Felice, and A. Pascolini (Roma, Consiglio Nazionale Delle Ricerche, 1983); L. Susskind and J. Uglum, Phys. Rev. D **50**, 2700 (1994) [hep-th/9401070]; J. H. Cooperman and M. A. Luty, arXiv:1302.1878 [hep-th].
- [27] L. Susskind, J. Math. Phys. **36**, 6377 (1995) [arXiv:hep-th/9409089].
- [28] D. N. Page, Phys. Rev. Lett. **71**, 3743 (1993) [hep-th/9306083].
- [29] D. Harlow and P. Hayden, JHEP **06**, 085 (2013) [arXiv:1301.4504 [hep-th]].
- [30] Y. Nomura, J. Varela and S. J. Weinberg, Phys. Rev. D **87**, 084050 (2013) [arXiv:1210.6348 [hep-th]].
- [31] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
- [32] S. W. Hawking, Phys. Lett. B **115**, 295 (1982); A. A. Starobinsky, Phys. Lett. B **117**, 175 (1982); A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); see also, V. F. Mukhanov and G. V. Chibisov, JETP Lett. **33**, 532 (1981) [Pisma Zh. Eksp. Teor. Fiz. **33**, 549 (1981)].
- [33] R. Bousso, B. Freivogel and S. Leichenauer, Phys. Rev. D **82**, 084024 (2010) [arXiv:1003.3012 [hep-th]].
- [34] D. N. Page, Phys. Rev. Lett. **44**, 301 (1980).