On loop corrections to subleading soft behavior of gluons and gravitons

Zvi Bern, Scott Davies, and Josh Nohle

Phys. Rev. D **90**, 085015 — Published 20 October 2014

DOI: 10.1103/PhysRevD.90.085015

UCLA/14/TEP/102

On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons

Zvi Bern, Scott Davies and Josh Nohle

Department of Physics and Astronomy

University of California at Los Angeles

Los Angeles, CA 90095-1547, USA

Abstract

Cachazo and Strominger recently proposed an extension of the soft-graviton theorem found by Weinberg. In addition, they proved the validity of their extension at tree level. This was motivated by a Virasoro symmetry of the gravity S-matrix related to BMS symmetry. As shown long ago by Weinberg, the leading behavior is not corrected by loops. In contrast, we show that with the standard definition of soft limits in dimensional regularization, the subleading behavior is anomalous and modified by loop effects. We argue that there are no new types of corrections to the first subleading behavior beyond one loop and to the second subleading behavior beyond two loops. To facilitate our investigation, we introduce a new momentum-conservation prescription for defining the subleading terms of the soft limit. We discuss the loop-level subleading soft behavior of gauge-theory amplitudes before turning to gravity amplitudes.

PACS numbers: 04.65.+e, 11.15.Bt, 11.25.Db, 12.60.Jv

I. INTRODUCTION

Recent years have seen enormous advances in our ability to calculate scattering amplitudes in gauge and gravity theories. These advances allow us to address various fundamental issues in such theories. Some time ago Weinberg presented a theorem for the universal factorization of scattering amplitudes when gravitons become soft [1]. Recently Weinberg's soft-graviton theorem was shown to be a Ward identity [2] of the Bondi, van der Burg, Metzner and Sachs (BMS) [3] symmetry. Along these lines, Strominger conjectured that an extension of Weinberg's theorem [4] for the first subleading terms in the soft limit follows from BMS symmetry. Supporting evidence has been presented recently by Cachazo and Strominger [5], proving that it holds at tree level. Interestingly, Cachazo and Strominger also showed that the second-order subleading correction to the tree behavior is also universal. These results are similar to the universal subleading soft-photon behavior proven long ago by Low [6]. The first subleading soft-graviton behavior was first discussed by White using eikonal methods [7]. Very recently, the subleading soft behavior at tree level has also been shown to be universal outside of four dimensions [8].

One might hope that at least the first subleading soft behavior is a theorem valid to all loop orders, as suggested by its link to BMS symmetry [5]. However, symmetries at loop level are delicate because of the need to regularize ultraviolet and infrared divergences. The required regularization can modify Ward identities derived from symmetries. In this paper, we demonstrate in a simple way that graviton infrared singularities imply that there are loop corrections to the subleading behavior of scattering amplitudes as external gravitons become soft, when we use the standard definition of such limits. These corrections are effectively a quantum breaking of the symmetry responsible for the tree-level behavior.

In order to understand the loop-level behavior of soft gravitons, it is useful to first look at the well-studied case of loop corrections to soft gluons in quantum chromodynamics (QCD) [9, 10]. The subleading soft-gluon behavior was already discussed using the eikonal approach [11]. A simple proof of the universal subleading soft behavior of gluons at tree level was recently given [12], following the corresponding proof for gravitons [5]. The connection between the two theories is not surprising. Gravity scattering amplitudes are closely related to gauge-theory ones and can even be constructed directly from them [13–17].

At one loop, the modifications to the leading soft-gluon behavior are directly tied to the

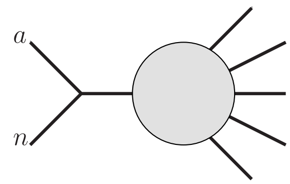

FIG. 1: The diagrams where leading and subleading contributions to the tree soft factor arise. Leg n is the soft leg.

infrared singularities, and can be used to deduce the complete correction including finite parts [9]. When a gluon becomes soft, there is a mismatch between the infrared singularities at n points and at $n-1$ points, so loop corrections to the soft function are required to absorb this mismatch. Following the gauge-theory case, we use the infrared singularities of gravity loop amplitudes [1, 18] to deduce the existence of loop corrections to the subleading soft-graviton behavior. As in QCD, discontinuities in the infrared singularities arise as one goes from n points to $n-1$ points by taking a soft limit in the standard way. In gravity, the leading soft-graviton behavior is smooth because the dimensionful coupling ensures that any discontinuity is suppressed by at least one additional factor of the soft momentum [16]. However, since there is less suppression in subleading soft pieces, loop corrections survive. This allows us to demonstrate in a simple way that the subleading behavior of gravitons indeed has loop corrections similar to the loop corrections that appear in QCD. As the loop order increases, the suppression increases. Hence, the first subleading behavior is protected against corrections starting at two loops and the second subleading behavior is protected against corrections starting at three loops.

This paper is organized as follows. In Sect. II, we give preliminaries on the tree-level behavior of soft gluons and gravitons. In Sect. III, we turn to the main subject of this paper: the behavior of the subleading contributions at loop level, showing that there are nontrivial one-loop corrections to subleading soft-graviton behavior. In Sect. IV, we discuss the all-loop behavior. We give our conclusions in Sect. V.

II. PRELIMINARIES

In this section, we summarize the soft behavior of gravitons and gluons at tree level, including their subleading behavior.

A. Soft gravitons

At tree level, consider the soft scaling of momentum k_n of an n-point amplitude,

$$k_n^{\alpha\dot\alpha} \to \delta k_n^{\alpha\dot\alpha}\,, \qquad \lambda_n^{\alpha} \to \sqrt{\delta}\lambda_n^{\alpha}\,, \qquad \tilde\lambda_n^{\dot\alpha} \to \sqrt{\delta}\tilde\lambda_n^{\dot\alpha}\,, \tag{2.1}$$

where $k_n^{\alpha\dot\alpha} = \lambda_n^{\alpha}\tilde\lambda_n^{\dot\alpha}$ is the standard decomposition of a massless momentum in terms of spinors. (See e.g. Ref. [19] for the spinor-helicity formalism used for scattering amplitudes.) In the limit (2.1), an n-point graviton tree amplitude behaves as [5]

$$M_n^{\rm tree} \to \Big(\frac{1}{\delta}\, S_n^{(0)} + S_n^{(1)} + \delta\, S_n^{(2)}\Big) M_{n-1}^{\rm tree} + \mathcal{O}(\delta^2)\,, \tag{2.2}$$

where δ is taken to be a small parameter. The soft operators are

$$\begin{aligned}
S_n^{(0)} &= \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^{\mu} k_i^{\nu}}{k_n\cdot k_i}\,,\\
S_n^{(1)} &= -i\sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^{\mu} k_{n\rho} J_i^{\nu\rho}}{k_n\cdot k_i}\,,\\
S_n^{(2)} &= -\frac{1}{2}\sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_{n\rho} J_i^{\mu\rho} k_{n\sigma} J_i^{\nu\sigma}}{k_n\cdot k_i}\,,
\end{aligned}\tag{2.3}$$

where $\varepsilon_{\mu\nu}$ is the graviton polarization tensor of the soft leg n and $J_i^{\mu\nu}$ is the angular momentum operator for particle i. $S_n^{(0)}$ is the leading term found long ago by Weinberg [1]. For simplicity, we suppress powers of the gravitational coupling $\kappa/2$ here and in the remaining part of the paper. In a helicity basis with a plus-helicity soft graviton, the explicit forms of the operators are

$$\begin{aligned}
S_n^{(0)} &= -\sum_{i=1}^{n-1} \frac{[n\,i]\,\langle x\,i\rangle\,\langle y\,i\rangle}{\langle n\,i\rangle\,\langle x\,n\rangle\,\langle y\,n\rangle}\,,\\
S_n^{(1)} &= -\frac{1}{2}\sum_{i=1}^{n-1} \frac{[n\,i]}{\langle n\,i\rangle}\left(\frac{\langle x\,i\rangle}{\langle x\,n\rangle} + \frac{\langle y\,i\rangle}{\langle y\,n\rangle}\right)\tilde\lambda_n^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\,,\\
S_n^{(2)} &= -\frac{1}{2}\sum_{i=1}^{n-1} \frac{[n\,i]}{\langle n\,i\rangle}\tilde\lambda_n^{\dot\alpha}\tilde\lambda_n^{\dot\beta}\frac{\partial^2}{\partial\tilde\lambda_i^{\dot\alpha}\partial\tilde\lambda_i^{\dot\beta}}\,,
\end{aligned}\tag{2.4}$$

where λ_x and λ_y are arbitrary massless reference spinors, which reflect gauge invariance. We follow the standard conventions of $s_{ab} = \langle a\, b\rangle\, [b\, a]$. The case of a minus-helicity soft graviton follows from parity conjugation. The first subleading behavior was discussed first in Ref. [7].

It is convenient to present the subleading behavior in terms of a holomorphic scaling of the spinors [5]. An advantage is that it makes the factorization channels clearer because the universal subleading behavior appears as poles in the scattering amplitudes. Taking leg n of an n-point amplitude to be a soft plus-helicity graviton, we scale the spinors as

$$k_n^\mu \to \delta k_n^\mu \,, \qquad \lambda_n^\alpha \to \delta \lambda_n^\alpha \,, \qquad \tilde\lambda_n^{\dot\alpha} \to \tilde\lambda_n^{\dot\alpha} \,. \tag{2.5}$$

Under this rescaling, tree-level graviton amplitudes behave as [5]

$$M_n^{\text{tree}} \to \Big(\frac{1}{\delta^3} S_n^{(0)} + \frac{1}{\delta^2} S_n^{(1)} + \frac{1}{\delta} S_n^{(2)}\Big) M_{n-1}^{\text{tree}} + \mathcal{O}(\delta^0) \,, \tag{2.6}$$

where M_n^{tree} is the n-point amplitude and M_{n-1}^{tree} is the $(n-1)$-point amplitude obtained by removing the soft leg n. The connection of the two scalings is through little-group scaling. The proof of universality [5] of the subleading soft behavior (2.3) relies on all contributions arising from factorizations on $1/(k_a + k_n)^2$ propagators in the soft kinematics (2.5), as illustrated in Fig. 1.

Some care is needed to interpret the soft behavior in Eq. (2.6) because the n-point kinematics of the amplitude on the left-hand side of the equation is not the same as the $(n-1)$-point kinematics normally used to define the amplitude on the right-hand side of the equation. This becomes an issue for the subleading soft terms because of feed down from leading terms to subleading ones, depending on the precise prescription. The prescription chosen by Cachazo and Strominger is to explicitly impose n-point momentum conservation on the amplitude on the left-hand side and $(n-1)$-point momentum conservation on the amplitude on the right-hand side. This constraint is conveniently implemented via

$$\tilde\lambda_1 = -\sum_{i=3}^{m} \frac{\langle 2\, i\rangle}{\langle 2\, 1\rangle} \tilde\lambda_i \,, \qquad \tilde\lambda_2 = -\sum_{i=3}^{m} \frac{\langle 1\, i\rangle}{\langle 1\, 2\rangle} \tilde\lambda_i \,, \tag{2.7}$$

so that $\sum_{i=1}^{m} \lambda_i \tilde\lambda_i = 0$. This constraint is imposed on the amplitudes on the left-hand side of Eq. (2.6) with $m = n$ and on the right-hand side with $m = n - 1$.

For our loop-level study, we use a different prescription. We interpret the expressions on both sides of Eq. (2.6) as carrying the *same* n-point kinematics, without needing to apply any additional constraints on the kinematics. The advantage is that this prevents complicated terms from feeding down from higher- to lower-order terms in the soft expansion, which would obscure the structure at loop level. This change in prescription effectively shifts contributions between different orders in the expansion.[1]

B. Soft gluons

Following the same derivation as for gravitons, tree-level Yang-Mills amplitudes also have a universal subleading soft behavior [12]. If we scale $\lambda_n \to \delta\lambda_n$, the color-ordered amplitude behaves as

$$A_n^{\text{tree}} \to \Big(\frac{1}{\delta^2} S^{(0)}_{n\,\text{YM}} + \frac{1}{\delta} S^{(1)}_{n\,\text{YM}}\Big) A^{\text{tree}}_{n-1}\,, \tag{2.8}$$

where the leading soft factor is

$$S^{(0)}_{n\,\text{YM}} = \frac{k_1\cdot\varepsilon_n}{\sqrt{2}\,k_1\cdot k_n} - \frac{k_{n-1}\cdot\varepsilon_n}{\sqrt{2}\,k_{n-1}\cdot k_n}\,. \tag{2.9}$$

The subleading one is

$$S^{(1)}_{n\,\text{YM}} = -i\varepsilon_{n\mu}k_{n\nu}\left(\frac{J_1^{\mu\nu}}{\sqrt{2}\,k_1\cdot k_n} - \frac{J_{n-1}^{\mu\nu}}{\sqrt{2}\,k_{n-1}\cdot k_n}\right)\,. \tag{2.10}$$

Again we have suppressed the coupling constants. Using spinor-helicity, the plus-helicity gluon leading soft factor is

$$S^{(0)}_{n\,\text{YM}} = \frac{\langle (n-1)\,1\rangle}{\langle (n-1)\,n\rangle\langle n\,1\rangle}\,, \tag{2.11}$$

while the subleading operator is

$$S^{(1)}_{n\,\text{YM}} = \frac{1}{\langle (n-1)\,n\rangle}\tilde\lambda_n^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_{n-1}^{\dot\alpha}} - \frac{1}{\langle 1\,n\rangle}\tilde\lambda_n^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_1^{\dot\alpha}}\,. \tag{2.12}$$

An earlier description was given in Ref. [11].

III. ONE-LOOP CORRECTIONS TO SUBLEADING SOFT BEHAVIOR

As shown by Weinberg [1], the leading soft-graviton behavior has no higher-loop corrections. In Ref. [5], Cachazo and Strominger demonstrated that their proposed theorem for subleading soft-graviton behavior holds at tree level.

[1] We numerically confirmed in many examples that the two prescriptions give identical results through $\mathcal{O}(\delta)$ in Eq. (2.2).

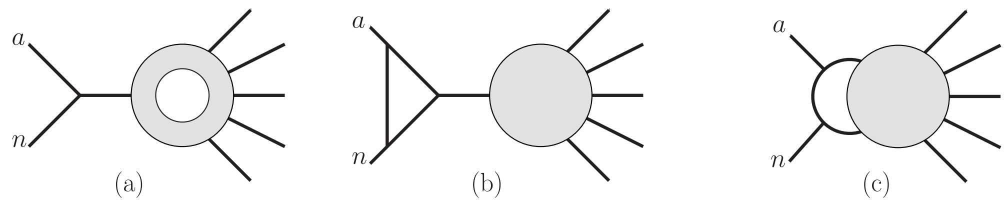

FIG. 2: At one loop, the simple tree-level soft behavior (a) is corrected by factorizing (b) and nonfactorizing (c) contributions [9]. In gravity, the corrections are suppressed by factors of the soft momentum k_n, but they affect the subleading behavior.

Here, we demonstrate that there are nontrivial loop corrections for the subleading soft-graviton behavior analogous to the ones that appear in QCD for the leading soft terms, using the standard definition of soft limits in dimensional regularization. As in QCD, loop corrections linked to infrared divergences necessarily appear because of mismatches in the logarithms of the infrared singularities at n and $n-1$ points. Divergences require a regulator which can break symmetries at the quantum level. In this sense, we can think of the loop corrections as due to an anomaly in the underlying symmetry. Its origin is similar to the twistor-space holomorphic anomaly [20], where extra contributions arise in regions of loop integration that are singular.

In general, the structure of the loop corrections to soft behavior is entangled with the infrared divergences. This phenomenon is familiar in QCD [9, 21], so we discuss this case first before turning to gravity. Besides corrections that arise from infrared singularities, we will find that there are other loop corrections due to nontrivial factorization properties [22–24], even for infrared-finite one-loop amplitudes.

A. One-loop corrections to soft-gluon behavior

In general, loop-level factorization properties of gauge theories are surprisingly nontrivial, in part, because of their entanglement with infrared singularities [21]. This causes naive notions of factorization in soft and other kinematic limits to break down; in massless gauge theories, one can obtain kinematic poles also from the loop integration. However, because the infrared singularities have a universal behavior, they offer a simple means for studying soft limits of loop amplitudes with an arbitrary number of external legs.

Fig. 2 shows the types of contributions to the one-loop soft behavior when the amplitude is represented in terms of the standard covariant basis of integrals. These consist of "factorizing" contributions, illustrated in Fig. 2(b), and "nonfactorizing" contributions, illustrated in Fig. 2(c).[2] The nonfactorizing contributions arise from poles in the S-matrix coming from loop integration and not directly from propagators, as illustrated in Fig. 2(c).

As a simple example, consider the single-external-mass box integral, displayed in Fig. 3. This is one of the basis integrals for one-loop amplitudes. The infrared-divergent terms of this integral are [26]

$$I_4^{1\mathrm{m}} = \frac{2i\, c_\Gamma}{s_{n1}s_{12}}\left[\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{-s_{n1}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{12}}\right)^\epsilon - \left(\frac{\mu^2}{-s_{n12}}\right)^\epsilon\right) + \text{finite}\right], \tag{3.1}$$

where the labels correspond to those in Fig. 3. We also have

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,, \qquad s_{i_1 i_2\cdots i_j} = (k_{i_1}+k_{i_2}+\cdots+k_{i_j})^2\,. \tag{3.2}$$

When leg n goes soft, the integral has a $1/s_{n1}$ kinematic pole from the prefactor. While one might expect such poles to cancel out of amplitudes, they, in fact, remain due to their entanglement with infrared singularities. However, this link ensures that they have a regular pattern. In general, these nonfactorizing contributions need to be accounted for in loop-level soft behavior and other factorization limits in gauge theories. The same holds for the subleading soft behavior of gravity amplitudes.

A one-loop n-gluon amplitude in QCD has ultraviolet and infrared singularities given by [21, 27]

$$A_n^{\text{1-loop}}(1,2,\cdots,n)\Big|_{\text{div.}} = -\frac{1}{\epsilon^2}A_n^{\text{tree}}(1,2,\cdots,n)\sigma_n^{\text{YM}}\,, \tag{3.3}$$

where

$$\sigma_n^{\text{YM}} = c_\Gamma\left[\sum_{j=1}^{n}\left(\frac{\mu^2}{-s_{j,j+1}}\right)^\epsilon + 2\epsilon\left(\frac{11}{6} - \frac{1}{3}\frac{n_f}{N_c} - \frac{1}{6}\frac{n_s}{N_c}\right)\right]. \tag{3.4}$$

In this expression, n_f is the number of quark flavors, n_s is the number of scalar flavors (zero in QCD) and N_c is the number of colors. Here, $\epsilon = (4-D)/2$ is the dimensional-regularization parameter, and μ^2 is the usual dimensional-regularization scale. It turns out that it is best to work with unrenormalized amplitudes containing also ultraviolet divergences because the mismatch in the number of coupling constants at n and $n-1$ points causes an additional

[2] In light-cone gauge or the unitarity approach, by introducing light-cone denominators containing a reference momentum, one can push all contributions into factorizing diagrams [10, 25].

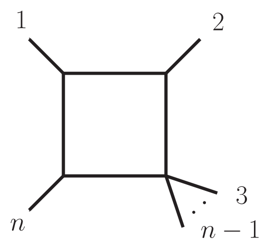

FIG. 3: An example of an integral that has a "nonfactorizing" kinematic pole that contributes to the soft behavior.

(trivial) discontinuity in the soft behavior. By working with unrenormalized amplitudes, we avoid this. A key property of Eq. (3.4) is that the terms depending on the number of quark and scalar flavors is independent of the number of external gluons. The terms in the summation arise from soft-gluon singularities in the loop integration. In general, the expression in Eq. (3.4) should be interpreted as being series expanded in ϵ, since terms beyond $\mathcal{O}(\epsilon^0)$ that are usually not computed can mix nontrivially with these.

Consider the soft limit of the singular parts of the gauge-theory amplitude (3.3). The tree prefactor obeys the simple soft behavior given in Eq. (2.8). The infrared singularities, however, have a mismatch between n points and $n-1$ points:

$$\sigma_n^{\rm YM} = \sigma_{n-1}^{\rm YM} + \sigma_n^{\prime {\rm YM}} + \mathcal{O}(\epsilon^2)\,, \tag{3.5}$$

where

$$\sigma_n^{\prime {\rm YM}} = c_\Gamma\left(1 + \epsilon \log\left(\frac{-\mu^2 s_{(n-1)1}}{s_{(n-1)n} s_{n1}}\right)\right). \tag{3.6}$$

It turns out that this mismatch can be used to deduce the complete one-loop corrections to the leading soft factor by matching the infrared discontinuities in the basis integrals to the infrared discontinuities in the amplitude [9].

The leading soft behavior of an n-gluon amplitude with any matter content for $\lambda_n \to \delta\lambda_n$ is then [9, 10]

$$A_n^{\text{1-loop}} \to S^{(0)}_{n\,{\rm YM}} A_{n-1}^{\text{1-loop}} + S^{(0)\text{1-loop}}_{n\,{\rm YM}} A_{n-1}^{\rm tree}\,, \tag{3.7}$$

where the leading one-loop soft correction function is

$$
\begin{aligned}
S^{(0)\text{1-loop}}_{n\,\text{YM}} &= -S^{(0)}_{n\,\text{YM}}\frac{c_\Gamma}{\epsilon^2}\left(\frac{-\mu^2 s_{(n-1)1}}{s_{(n-1)n}s_{n1}}\right)^{\epsilon}\frac{\pi\epsilon}{\sin(\pi\epsilon)} \\
&= -S^{(0)}_{n\,\text{YM}}c_\Gamma\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\log\left(\frac{-\mu^2 s_{(n-1)1}}{\delta^2 s_{(n-1)n}s_{n1}}\right)+\frac{1}{2}\log^2\left(\frac{-\mu^2 s_{(n-1)1}}{\delta^2 s_{(n-1)n}s_{n1}}\right)+\frac{\pi^2}{6}\right) \\
&\qquad + \mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{3.8}
$$

The form on the first line is valid to all orders in ϵ. In applying this equation, it is important to first expand in ϵ prior taking the soft limit.

Now consider the subleading soft terms. Taking the divergent part of the one-loop amplitude to have a soft limit of the form,

$$
A_n^{\text{1-loop}}\Big|_{\text{div.}} \to \left(\frac{1}{\delta^2}S^{(0)}_{n\,\text{YM}}+\frac{1}{\delta}S^{(1)}_{n\,\text{YM}}\right)A_{n-1}^{\text{1-loop}}\Big|_{\text{div.}} + \left(\frac{1}{\delta^2}S^{(0)\text{1-loop}}_{n\,\text{YM}}+\frac{1}{\delta}S^{(1)\text{1-loop}}_{n\,\text{YM}}\right)A_{n-1}^{\text{tree}}\Big|_{\text{div.}}\,, \tag{3.9}
$$

we then solve for the divergent parts of the one-loop corrections to the soft operators, denoted by $S^{(i)\,\text{1-loop}}_{n\,\text{YM}}$. We do so by comparing the soft expansion of the left-hand side of Eq. (3.9) to the terms on the right-hand side. Applying $S^{(1)}_{n\,\text{YM}}$ to the infrared singularity of the $(n-1)$-point amplitude gives

$$
\begin{aligned}
S^{(1)}_{n\,\text{YM}}\sigma^{\text{YM}}_{n-1} &= -c_\Gamma\epsilon\Big(\frac{[1\,n]}{[1\,(n-1)]\,\langle(n-1)\,n\rangle}-\frac{[(n-1)\,n]}{[(n-1)\,1]\,\langle 1\,n\rangle} \\
&\qquad + \frac{[(n-2)\,n]}{[(n-2)\,(n-1)]\,\langle(n-1)\,n\rangle}-\frac{[2\,n]}{[2\,1]\,\langle 1\,n\rangle}\Big)\,,
\end{aligned}
\tag{3.10}
$$

where we use the form of σ^{YM}_{n-1} exactly as it appears in Eq. (3.4) without any additional momentum-conservation relations imposed. Taking the one-loop correction to the subleading soft function to be

$$
S^{(1)\text{1-loop}}_{n\,\text{YM}} = -\frac{1}{\epsilon^2}\left[\sigma'^{\text{YM}}_n S^{(1)}_{n\,\text{YM}} - \left(S^{(1)}_{n\,\text{YM}}\sigma^{\text{YM}}_{n-1}\right)\right] + \mathcal{O}(\epsilon^0)\,, \tag{3.11}
$$

we find that Eq. (3.9) holds. The simple form of the correction relies on using the specific form for $S^{(1)}\sigma_{n-1}$ in Eq. (3.10). We also interpret both sides of Eq. (3.9) as having the same n-point kinematics.

It would be important to understand the infrared-finite terms as well. These also have nontrivial corrections. For the case of the infrared-finite identical-helicity one-loop amplitudes [28], numerical analysis through 30 points shows that the amplitudes behave exactly as tree-level amplitudes with no nontrivial corrections. However, the one-loop amplitudes with a single minus helicity [22] have nontrivial subleading soft behavior. As an example, consider the one-loop five-gluon amplitude [22, 29],

$$A_5^{\text{1-loop}}(1^-,2^+,3^+,4^+,5^+) = \frac{i}{48\pi^2}\frac{1}{\langle 3\,4\rangle^2}\left[-\frac{[2\,5]^3}{[1\,2]\,[5\,1]} + \frac{\langle 1\,4\rangle^3\,[4\,5]\,\langle 3\,5\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle^2} - \frac{\langle 1\,3\rangle^3\,[3\,2]\,\langle 4\,2\rangle}{\langle 1\,5\rangle\,\langle 5\,4\rangle\,\langle 3\,2\rangle^2}\right], \tag{3.12}$$

as the momentum of leg 5 becomes soft. The four-point one-loop single-minus-helicity amplitude is [30]

$$A_4^{\text{1-loop}}(1^-,2^+,3^+,4^+) = \frac{i}{48\pi^2}\frac{\langle 2\,4\rangle\,[2\,4]^3}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}\,. \tag{3.13}$$

Applying the tree-level operators to the four-point amplitude, as in Eq. (2.8), yields

$$\begin{aligned}\Big(\frac{1}{\delta^2}S^{(0)}_{n\,\text{YM}} + \frac{1}{\delta}S^{(1)}_{n\,\text{YM}}\Big)&A_4^{\text{1-loop}}(1^-,2^+,3^+,4^+)\\ &= \frac{i}{48\pi^2}\frac{\langle 1\,3\rangle^3\,\langle 2\,4\rangle\,[1\,2]}{\langle 2\,3\rangle^2\,\langle 3\,4\rangle^3}\left(\frac{1}{\delta^2}\frac{\langle 4\,1\rangle}{\langle 4\,5\rangle\,\langle 5\,1\rangle} + \frac{1}{\delta}\frac{[5\,2]}{\langle 5\,1\rangle\,[1\,2]}\right)\,.\end{aligned} \tag{3.14}$$

After applying the operators, we applied five-point momentum conservation to remove the anti-holomorphic spinors $\tilde\lambda_3$, $\tilde\lambda_4$.[3] This facilitates comparison with the soft limit of the five-point amplitude (3.12). With the same constraints applied, this is given by

$$\begin{aligned}A_5^{\text{1-loop}}(1^-,2^+,3^+,4^+,5^+) \to \frac{i}{48\pi^2}\Bigg[&\frac{\langle 1\,3\rangle^3\,\langle 2\,4\rangle\,[1\,2]}{\langle 2\,3\rangle^2\,\langle 3\,4\rangle^3}\left(\frac{1}{\delta^2}\frac{\langle 4\,1\rangle}{\langle 4\,5\rangle\,\langle 5\,1\rangle} + \frac{1}{\delta}\frac{[5\,2]}{\langle 5\,1\rangle\,[1\,2]}\right)\\ &+ \frac{1}{\delta}\frac{\langle 1\,4\rangle^3\,\langle 3\,5\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle^3\,\langle 4\,5\rangle^2}(\langle 1\,3\rangle\,[1\,5] + \langle 2\,3\rangle\,[2\,5])\Bigg]\,.\end{aligned} \tag{3.15}$$

While the leading order pieces are identical, the subleading pieces differ in Eqs. (3.14) and (3.15).

The nontrivial behavior of the single-minus-helicity amplitudes is not surprising given that they contain nontrivial complex poles that cannot be interpreted as a straightforward factorization. In general, nonsupersymmetric gauge-theory loop amplitudes contain such nontrivial poles. This phenomenon complicates the construction of gauge and gravity loop amplitudes from their poles and has been described in some detail in Refs. [23, 24]. We leave the discussion of such infrared-finite contributions to the future.

[3] We note that the momentum-conservation prescription of Ref. [5] gives the same conclusion.

B. One-loop corrections to soft-graviton behavior

Applying a similar analysis, it is straightforward to see that one-loop corrections to the subleading soft-graviton behavior do not vanish because of mismatched logarithms in the infrared singularities. At one loop, the n-graviton amplitude contains the dimensionally-regularized infrared-singular terms [18, 31],

$$M_n^{\text{1-loop}}\Big|_{\text{div.}} = \frac{\sigma_n}{\epsilon} M_n^{\text{tree}}\,, \tag{3.16}$$

where M_n^{tree} is the n-graviton tree amplitude, and

$$\sigma_n = -c_\Gamma \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s_{ij} \log\Big(\frac{\mu^2}{-s_{ij}}\Big)\,, \tag{3.17}$$

where c_Γ is defined in Eq. (3.2). As in QCD, the logarithms that appear at n points are not identical to the ones appearing at $(n-1)$ points. The logarithms in the infrared singularity that differ between an n- and $(n-1)$-graviton amplitude are

$$\sigma'_n = -c_\Gamma \sum_{i=1}^{n-1} s_{in} \log\Big(\frac{\mu^2}{-s_{in}}\Big)\,. \tag{3.18}$$

While this mismatch does not affect the leading soft behavior because of the suppression from the s_{in} factors, it does affect subleading terms.

By absorbing the mismatches into corrections to the subleading soft operator, we find that in the soft limit $\lambda_n \to \delta\lambda_n$, the infrared singular terms behave as

$$M_n^{\text{1-loop}}\Big|_{\text{div.}} \to \bigg(\frac{S_n^{(0)}}{\delta^3} + \frac{S_n^{(1)}}{\delta^2} + \frac{S_n^{(2)}}{\delta}\bigg) M_{n-1}^{\text{1-loop}}\Big|_{\text{div.}} + \bigg(\frac{S_n^{(1)\,\text{1-loop}}}{\delta^2} + \frac{S_n^{(2)\,\text{1-loop}}}{\delta}\bigg) M_{n-1}^{\text{tree}}\Big|_{\text{div.}}\,, \tag{3.19}$$

where

$$\begin{aligned}
S_n^{(0)\,\text{1-loop}}\Big|_{\text{div.}} &= 0\,,\\
S_n^{(1)\,\text{1-loop}}\Big|_{\text{div.}} &= \frac{1}{\epsilon}\bigg[\sigma'_n S_n^{(0)} - \Big(S_n^{(1)}\sigma_{n-1}\Big)\bigg]\,,\\
S_n^{(2)\,\text{1-loop}}\Big|_{\text{div.}} &= \frac{1}{\epsilon}\bigg[\sigma'_n S_n^{(1)} - \Big(S_n^{(2)}\sigma_{n-1}\Big) + \sum_{i=1}^{n-1} \frac{[n\,i]}{\langle n\,i\rangle}\bigg(\tilde\lambda_n^{\dot\alpha}\frac{\partial\sigma_{n-1}}{\partial\tilde\lambda_i^{\dot\alpha}}\bigg)\tilde\lambda_n^{\dot\beta}\frac{\partial}{\partial\tilde\lambda_i^{\dot\beta}}\bigg]\,.
\end{aligned} \tag{3.20}$$

Similar to the gauge-theory case, the simple form of these corrections to the subleading soft operators relies on using the form of σ_{n-1} obtained from Eq. (3.17) with no additional momentum-conservation relations imposed. We again also interpret both sides of Eq. (3.19)

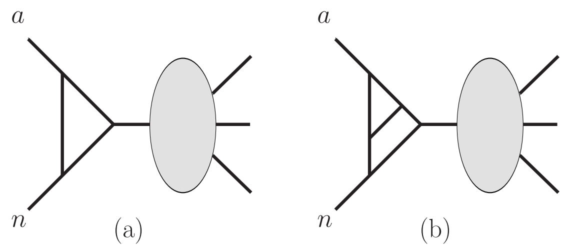

FIG. 4: Sample factorizing (a) one- and (b) two-loop contributions to the soft behavior.

as having the same n-point kinematics. As in QCD, it is important to follow the standard procedure of first series expanding the amplitude in ϵ prior to taking soft limits.

We have checked numerically through 10 points that the infrared-finite identical-helicity graviton amplitudes [32] satisfy the same subleading soft behavior as the tree amplitudes. However, more generally we expect a more complicated behavior due to the nontrivial factorization properties of loop amplitudes [22, 23]. Such nontrivial factorization properties have been discussed for gravity theories in Refs. [24, 33]. Indeed, by numerically analyzing the infrared-finite one-loop five-graviton amplitude with a single minus helicity from Ref. [33] and the one-loop four-graviton amplitude with a single minus helicity from Ref. [34], we find that the second subleading soft behavior has nontrivial corrections. We leave a discussion of the infrared-finite corrections to the graviton soft behavior to the future.

IV. ALL LOOP ORDER BEHAVIOR OF SOFT GRAVITONS

As we demonstrated in the previous section, the subleading soft behavior has loop corrections. In this section, we argue that the first subleading soft behavior has no corrections beyond one loop and that the second subleading behavior has no corrections beyond two loops.

A. General considerations

The all-loop leading soft-graviton behavior has been discussed in some detail in Section 5.2 of Ref. [16]. Here we follow this discussion for the subleading behavior. As already noted for gauge theory, potential contributions to the soft behavior can be divided into "factorizing"

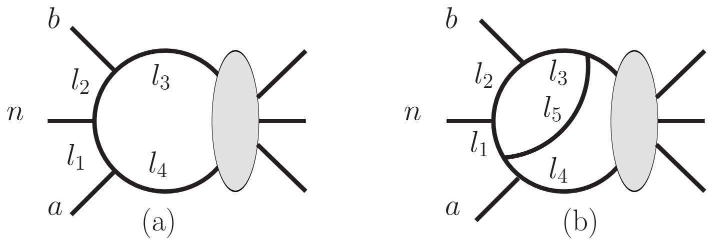

FIG. 5: Sample nonfactorizing (a) one- and (b) two-loop contributions to the soft behavior.

and "nonfactorizing" contributions [21] when the amplitude is expressed in terms of covariant Feynman integrals. We consider these types of contributions in turn.

The factorizing contributions of the type displayed in Fig. 4 depend on the soft momentum k_n and one additional momentum k_a. After the Lorentz indices of polarization tensors are contracted, no other Lorentz invariants are present other than s_{na}. By dimensional analysis, the L-loop correction contains an additional factor κ^{2L} of the gravitational coupling relative to the tree-level contribution in Fig. 1, and therefore must contain relative factors of s_{na}^L. This gives a suppression of one soft momentum k_n for each additional loop.

The nonfactorizing contributions displayed in Fig. 5 have a similar suppression. The nonfactorizing contributions arise in regions where loop momenta become soft in addition to the external soft leg. For example, in the one-loop case displayed in Fig. 5(a), as $k_n \to 0$, we must also have the loop momentum go as $l_1 \to 0$ in order to obtain a nonfactorizing contribution to the soft behavior; otherwise, there would be no large contribution for $k_n \to 0$, or equivalently for $\lambda_n \to 0$. In this region, $l_2 = l_1 - k_n$, $l_3 = l_1 - k_n - k_b$ and $l_4 = l_1 + k_a$ also all become small. After integration, this leads to potential kinematic poles in s_{an} or s_{bn}, or equivalently in λ_n. However, because gravity has an extra power of soft momentum, either k_n or l_1 in the vertex attaching leg n to the loop will suppress the pole. Similarly, at two loops, illustrated in Fig. 5(b), potential contributions arise when additional loop momenta become soft, in this case l_5. Once again, the dimensionful coupling ensures that there will be additional factors of soft momenta in the numerator. More generally, after integration, we get an additional L factors of s_{jn} compared to the gauge-theory case, where j can be any momentum in the amplitude.

The net effect is that there are no loop corrections to the leading soft behavior, no corrections beyond one loop for the first subleading soft behavior, and no corrections beyond two loops for the second subleading soft behavior. We therefore expect the general form of the L-loop behavior for a plus-helicity graviton with $\lambda_n \to \delta\lambda_n$ to have no loop corrections beyond two loops.

B. All loop behavior of leading infrared singularities

Since there should be no corrections beyond two loops, we expect that the L-loop leading infrared-divergent terms should behave in the soft limit as

$$\begin{aligned} M_n^{L\text{-loop}}\Big|_{\text{lead. div.}} \to &\left(\frac{S_n^{(0)}}{\delta^3}+\frac{S_n^{(1)}}{\delta^2}+\frac{S_n^{(2)}}{\delta}\right)M_{n-1}^{L\text{-loop}}\Big|_{\text{lead. div.}} \\ &+\left(\frac{S_n^{(1)\,1\text{-loop}}}{\delta^2}+\frac{S_n^{(2)\,1\text{-loop}}}{\delta}\right)M_{n-1}^{(L-1)\text{-loop}}\Big|_{\text{lead. div.}} \\ &+\frac{S_n^{(2)\,2\text{-loop}}}{\delta}M_{n-1}^{(L-2)\text{-loop}}\Big|_{\text{lead. div.}} . \end{aligned} \tag{4.1}$$

We check this using the known all-loop-order form of infrared singularities in gravity theories [1, 18]. The infrared singularities of gravity amplitudes are given by

$$M_n = \mathcal{S}_n \mathcal{H}_n \,, \tag{4.2}$$

where M_n is a gravity amplitude valid to all loop orders and $\mathcal{H}_n$ is the infrared-finite hard function. The all-loop infrared singularity function is a simple exponentiation of the one-loop function (3.16):

$$\mathcal{S}_n = \exp\left(\frac{\sigma_n}{\epsilon}\right). \tag{4.3}$$

From this equation, we see that the leading infrared singularity at L loops is simply given in terms of the tree amplitude:

$$M_n^{L\text{-loop}}\Big|_{\text{lead. div.}} = \frac{1}{L!}\left(\frac{\sigma_n}{\epsilon}\right)^L M_n^{\text{tree}} . \tag{4.4}$$

This gives us a simple means for testing Eq. (4.1) and also for finding the leading infrared-singular part of the two-loop operator, $S_n^{(2)\,2\text{-loop}}$. We do so by taking the difference of the soft expansion on both sides of Eq. (4.1) and using the previously determined operators in Eq. (3.20). We need the soft expansion of the leading infrared-singular part of $M_n^{L\text{-loop}}$, given by

$$\frac{\sigma_n^L}{L!}M_n^{\text{tree}} \to \frac{(\sigma_{n-1}+\delta\sigma'_n)^L}{L!}\left(\frac{S_n^{(0)}}{\delta^3}+\frac{S_n^{(1)}}{\delta^2}+\frac{S_n^{(2)}}{\delta}\right)M_{n-1}^{\text{tree}} , \tag{4.5}$$

where σ'_n is defined in Eq. (3.18). We also need the results of acting on $(\sigma^L_{n-1}/L!)M^{\rm tree}_{n-1}$ with the tree-level soft operators,

$$\left(\frac{S^{(0)}_n}{\delta^3}+\frac{S^{(1)}_n}{\delta^2}+\frac{S^{(2)}_n}{\delta}\right)\frac{\sigma^L_{n-1}}{L!}M^{\rm tree}_{n-1}\,. \tag{4.6}$$

Evaluating these, we deduce the leading infrared-divergent contribution to the two-loop soft operator to be

$$S^{(2)\,2\text{-loop}}_n\Big|_{\text{lead. div.}}=\frac{1}{\epsilon^2}\left[\frac{1}{2}\left(\sigma'_n\right)^2S^{(0)}_n-\sigma'_n\left(S^{(1)}_n\sigma_{n-1}\right)-\left(\frac{1}{2}\sum_{i=1}^{n-1}\frac{[n\,i]}{\langle n\,i\rangle}\left(\tilde\lambda^{\dot\alpha}_n\frac{\partial\sigma_{n-1}}{\partial\tilde\lambda^{\dot\alpha}_i}\right)^2\right)\right]. \tag{4.7}$$

The lack of higher-loop corrections to the soft operators is a consequence of the fact that they are suppressed by additional powers of the soft momentum. As before, the form of σ_{n-1} in the correction must be specifically as given in Eq. (3.17).

V. CONCLUSIONS

Recently a generalization of Weinberg's soft-graviton theorem for the subleading behavior was proposed [4, 5]. (See also previous work from White [7].) Here we showed that, unlike the leading soft-graviton behavior, the subleading soft behavior requires loop corrections. In QCD, loop corrections to the leading soft functions make up for mismatches in the infrared singularities of n-point and $(n-1)$-point amplitudes. Applying this observation to gravity, we obtained the leading infrared-singular loop contributions to the subleading soft-graviton operators valid to all loop orders. This proves in a simple way that there necessarily are nonvanishing loop corrections to soft-graviton behavior. In addition, in the simple example of a five-graviton amplitude with a single minus helicity, we found additional corrections to the second subleading behavior, not linked to infrared singularities. These come from the nontrivial complex factorization properties of generic loop amplitudes [21–24, 33].

Following the discussion for the leading soft-graviton behavior [1, 16], we argued that there are no loop corrections to the first subleading soft behavior beyond one loop and no new corrections to the second subleading behavior beyond two loops. This is connected to the dimensionful coupling of gravity. In the regions contributing to the soft limit, an extra power of the soft momentum is obtained for each additional loop, suppressing the contributions. By the third loop order, there are a sufficient number of powers of the soft momentum to suppress further corrections to the soft operators.

We also discussed the form of subleading corrections to the soft behavior in gauge theory as a warm-up for the gravity case. It is interesting to note that the subleading soft behavior in QCD might be useful for improved soft-gluon approximations.

An important remaining task is to determine the loop corrections to the general subleading soft behavior of the infrared-finite terms in both gauge and gravity theories. While this is simple in special cases, such as for identical-helicity amplitudes [28, 32], in general, the task is complicated by the nontrivial complex factorization properties of loop amplitudes [21–24, 33], on top of well-understood feed downs from infrared singularities. We leave studies of the soft behavior of infrared-finite terms in gauge and gravity amplitudes to future work.

Added Note

In this paper we have used the standard definition of dimensionally-regularized soft limits where one first series expands in the dimensional-regularization parameter before taking the soft limit. We do so because it matches the one needed for scattering amplitudes and associated physical processes as they are normally computed. After the appearance of the first version of this paper, a new paper appeared [35] showing that in some simple supersymmetric examples, loop corrections to the soft operators can be removed by altering the long-standing standard definition of soft limits. This alteration involves keeping the dimensional-regularization parameter finite before taking the soft limit.

The lack of loop corrections found in the examples of Ref. [35] is not surprising and is a simple consequence of the lack of discontinuities [9, 21] with the reordered limits. This is connected to the well-known fact that with a finite dimensional-regularization parameter $\epsilon < 0$, or equivalently $D > 4$, there are no infrared singularities. One can also view the prescription as equivalent to taking soft limits on integrands instead of the integrated expressions because one can push limits through the integral when they are smooth. (One can apply soft limits directly at the integrand level, but that is a distinct problem from the one for integrated amplitudes.) As an example, we immediately see from the first line of Eq. (3.8) that one-loop corrections to the leading soft function in QCD vanish for $k_n \to 0$ if we hold $\epsilon < 0$ fixed.

However, there are a number of reasons why it is important to use the standard dimensional-regularization procedure of series expanding in ϵ prior to taking soft [9, 10] or other limits. To be useful for obtaining cross sections, soft limits must be compatible with cancellations of infrared singularities between real-emission and virtual contributions. One might imagine keeping ϵ finite in both contributions in an attempt to treat them on an equal footing. However, the use of four-dimensional helicity states on external legs makes this problematic. Even in the well-understood standard definition of soft limits, one must be careful not to violate unitarity because of the incompatible treatment of real-emission and virtual contributions. (See for example Ref. [36].) Moreover, in QCD the modified prescription disrupts the cancellation of leading infrared singularities when $\epsilon \to 0$ because it alters the real-emission sigularities without changing corresponding virtual ones.

Even if there were a way to avoid difficulties with real-emission contributions, keeping ϵ finite in virtual contributions would lead to serious complications as well. In general, loop amplitudes are computed only through a fixed order in ϵ because the higher order contributions are rather complicated, except in simple supersymmetric cases, and do not carry useful physical information for the problem at hand. (For an example of the typical forms that loop amplitudes take, see Ref. [37].)

The single-minus helicity infrared-finite amplitudes are a good example of why it is best to series expand in ϵ. As noted in Sections III A and III B, these amplitudes have another type of loop correction to soft behavior coming from nontrivial complex factorization channels and not from infrared discontinuities. (Since the first version of our paper appeared, He, Huang and Wen thoroughly investigated the single-minus helicity amplitudes [38], among other topics, confirming our finding of nontrivial loop corrections.) In general, such amplitudes are known only for $\epsilon = 0$ [22, 33]. It would be highly nontrivial to obtain the higher order in ϵ contributions for the purpose of attempting to prevent renormalization of the soft operators. Furthermore, we note that loop corrections to soft behavior are, in fact, quite useful for understanding the analytic structure of amplitudes and their associated physical properties. More generally, experience shows that it is overwhelmingly simpler to absorb complications associated with dimensional regularization into loop corrections of soft limits rather than to deal with higher order in ϵ terms in amplitudes.

Consequently, while it may be tempting to change the standard definitions of dimensional regularization and soft limits in order to remove loop corrections to soft operators associated with infrared singularities, we greatly prefer the standard definitions because of their well-understood consistency, simplicity and applicability to problems of physical and theoretical interest.

Acknowledgments

We thank Guillaume Bossard, John Joseph Carrasco, Yu-tin Huang, Henrik Johansson and Radu Roiban for helpful discussions. We also thank Paolo Di Vecchia, Duff Neill and Chris White for pointing out the earlier work on subleading soft limits, as well as for helpful discussions. This work was supported by the US Department of Energy under Award Number DE-SC0009937. We also gratefully acknowledge Mani Bhaumik for his generous support.

[1] S. Weinberg, Phys. Rev. **140**, B516 (1965).

[2] A. Strominger, arXiv:1312.2229 [hep-th];
T. He, V. Lysov, P. Mitra and A. Strominger, arXiv:1401.7026 [hep-th].

[3] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, Proc. Roy. Soc. Lond. A **269**, 21 (1962);
R. K. Sachs, Proc. Roy. Soc. Lond. A **270**, 103 (1962).

[4] A. Strominger, unpublished, 2013.

[5] F. Cachazo and A. Strominger, arXiv:1404.4091 [hep-th].

[6] F. E. Low, Phys. Rev. **110**, 974 (1958);
T. H. Burnett and N. M. Kroll, Phys. Rev. Lett. **20**, 86 (1968);
J. S. Bell and R. Van Royen, Nuovo Cim. A **60**, 62 (1969);
V. Del Duca, Nucl. Phys. B **345**, 369 (1990).

[7] C. D. White, JHEP **1105**, 060 (2011) [arXiv:1103.2981 [hep-th]].

[8] B. U. W. Schwab and A. Volovich, arXiv:1404.7749 [hep-th].

[9] Z. Bern, V. Del Duca and C. R. Schmidt, Phys. Lett. B **445**, 168 (1998) [hep-ph/9810409];
Z. Bern, V. Del Duca, W. B. Kilgore and C. R. Schmidt, Phys. Rev. D **60**, 116001 (1999) [hep-ph/9903516].

[10] D. A. Kosower and P. Uwer, Nucl. Phys. B **563**, 477 (1999) [hep-ph/9903515];
D. A. Kosower, Phys. Rev. Lett. **91**, 061602 (2003) [hep-ph/0301069].

[11] E. Laenen, G. Stavenga and C. D. White, JHEP **0903**, 054 (2009) [arXiv:0811.2067 [hep-ph]]; E. Laenen, L. Magnea, G. Stavenga and C. D. White, JHEP **1101**, 141 (2011) [arXiv:1010.1860 [hep-ph]].

[12] E. Casali, arXiv:1404.5551 [hep-th].

[13] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B **269**, 1 (1986).

[14] F. A. Berends, W. T. Giele and H. Kuijf, Phys. Lett. B **211**, 91 (1988).

[15] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B **530**, 401 (1998) [hep-th/9802162].

[16] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B **546**, 423 (1999) [hep-th/9811140].

[17] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D **78**, 085011 (2008) [arXiv:0805.3993 [hep-ph]]; Phys. Rev. Lett. **105**, 061602 (2010) [arXiv:1004.0476 [hep-th]].

[18] S. G. Naculich and H. J. Schnitzer, JHEP **1105**, 087 (2011) [arXiv:1101.1524 [hep-th]]; R. Akhoury, R. Saotome and G. Sterman, Phys. Rev. D **84**, 104040 (2011) [arXiv:1109.0270 [hep-th]].

[19] L. J. Dixon, in Boulder 1995, QCD and beyond, p. 539-582 [hep-ph/9601359].

[20] F. Cachazo, P. Svrcek and E. Witten, JHEP **0410**, 077 (2004) [hep-th/0409245].

[21] Z. Bern and G. Chalmers, Nucl. Phys. B **447**, 465 (1995) [hep-ph/9503236].

[22] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **71**, 105013 (2005) [hep-th/0501240].

[23] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **72**, 125003 (2005) [hep-ph/0505055]. Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **73**, 065013 (2006) [hep-ph/0507005].

[24] S. D. Alston, D. C. Dunbar and W. B. Perkins, Phys. Rev. D **86**, 085022 (2012) [arXiv:1208.0190 [hep-th]].

[25] J. C. Collins and G. F. Sterman, Nucl. Phys. B **185**, 172 (1981); J. C. Collins, D. E. Soper and G. F. Sterman, Nucl. Phys. B **308**, 833 (1988).

[26] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302**, 299 (1993) [Erratum-ibid. B **318**, 649 (1993)] [hep-ph/9212308].

[27] Z. Kunszt, A. Signer and Z. Trocsanyi, Nucl. Phys. B **420**, 550 (1994) [hep-ph/9401294].

[28] Z. Bern, G. Chalmers, L. J. Dixon and D. A. Kosower, Phys. Rev. Lett. **72**, 2134 (1994) [hep-ph/9312333].

[29] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. Lett. **70**, 2677 (1993) [hep-ph/9302280].

[30] Z. Bern and D. A. Kosower, Nucl. Phys. B **379**, 451 (1992);
Z. Bern and A. G. Morgan, Nucl. Phys. B **467**, 479 (1996) [hep-ph/9511336].

[31] D. C. Dunbar and P. S. Norridge, Nucl. Phys. B **433**, 181 (1995) [hep-th/9408014].

[32] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, Phys. Lett. B **444**, 273 (1998) [hep-th/9809160].

[33] D. C. Dunbar, J. H. Ettle and W. B. Perkins, JHEP **1006**, 027 (2010) [arXiv:1003.3398 [hep-th]].

[34] Z. Bern, D. C. Dunbar and T. Shimada, Phys. Lett. B **312**, 277 (1993) [hep-th/9307001].

[35] F. Cachazo and E. Y. Yuan, arXiv:1405.3413 [hep-th].

[36] Z. Kunszt, A. Signer and Z. Trocsanyi, Nucl. Phys. B **411**, 397 (1994) [hep-ph/9305239].

[37] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. Lett. **70**, 2677 (1993) [hep-ph/9302280].

[38] S. He, Y.-t. Huang and C. Wen, arXiv:1405.1410 [hep-th].