Low-energy behavior of gluons and gravitons from gauge invariance

Zvi Bern, Scott Davies, Paolo Di Vecchia, and Josh Nohle

Phys. Rev. D **90**, 084035 — Published 20 October 2014

DOI: 10.1103/PhysRevD.90.084035

UCLA/14/TEP/104, NORDITA-2014-78

Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance

Zvi Bern[a], Scott Davies[a], Paolo Di Vecchia[bc] and Josh Nohle[a]

[a] *Department of Physics and Astronomy, UCLA, Los Angeles, CA 90095-1547, USA*

[b] *The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK 2100 Copenhagen, Denmark*

[c] *Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-10691 Stockholm, Sweden*

Abstract

We show that at tree level, on-shell gauge invariance can be used to fully determine the first subleading soft-gluon behavior and the first two subleading soft-graviton behaviors. Our proofs of the behaviors for n-gluon and n-graviton tree amplitudes are valid in D dimensions and are similar to Low's proof of universality of the first subleading behavior of photons. In contrast to photons coupling to massive particles, in four dimensions the soft behaviors of gluons and gravitons are corrected by loop effects. We comment on how such corrections arise from this perspective. We also show that loop corrections in graviton amplitudes arising from scalar loops appear only at the second soft subleading order. This case is particularly transparent because it is not entangled with graviton infrared singularities. Our result suggests that if we set aside the issue of infrared singularities, soft-graviton Ward identities of extended BMS symmetry are not anomalous through the first subleading order.

I. INTRODUCTION

Interest in the soft behavior of gravitons and gluons has recently been renewed by a proposal from Strominger and collaborators [1, 2] showing that soft-graviton behavior follows from Ward identities of extended Bondi, van der Burg, Metzner and Sachs (BMS) symmetry [3, 4]. This has stimulated a variety of studies of the subleading soft behavior of gravitons and gluons. In four spacetime dimensions, Cachazo and Strominger [2] showed that tree-level graviton amplitudes have a universal behavior through second subleading order in the soft-graviton momentum. In Ref. [5] an analogous description of tree-level soft behavior for gluons at first subleading order was given. Interestingly, these universal behaviors hold in D dimensions as well [6]. In four dimensions, there is an interesting connection between the subleading soft behavior in gauge theory and conformal invariance [7, 8]. There are also recent constructions of twistor-related theories with the desired soft properties [9]. Soft behavior in string theory and for higher-dimension operators has also been discussed [8, 10].

Soft theorems have a long history and were recognized in the 1950s and 1960s to be an important consequence of local on-shell gauge invariance [11–14]. (For a discussion of the low-energy theorem for photons see Chapter 3 of Ref. [15].) For photons, Low's theorem [12] determines the amplitudes with a soft photon from the corresponding amplitudes without a photon, through $\mathcal{O}(q^0)$, where q is the soft-photon momentum. The theorem links the first subleading soft behavior to the universal leading behavior via gauge invariance.

The universal leading soft-graviton behavior was first discussed by Weinberg [13]. The leading behavior is uncorrected to all loop orders [16]. Using dispersion relations, Gross and Jackiw analyzed the particular example of Compton scattering of gravitons on massive scalar particles [17]. They showed that, for fixed angle, the Born contributions have no corrections up to, but not including, fourth order in the soft momentum. Jackiw then applied gauge-invariance arguments similar to those of Low to reanalyze this case [18]. However, for our purposes this case is too special because the degenerate kinematics of $2 \to 2$ scattering leads to extra suppression not only at tree level, but at loop level as well. In particular, the soft limits are finite at fixed angle. This may be contrasted with the behavior for larger numbers of legs, where the amplitudes at all loop orders diverge as a graviton becomes soft, matching the tree behavior. Thus, the results of Refs. [17, 18] cannot be directly applied to our discussion of n-point behavior. A more recent discussion of the generic subleading behavior of soft gluons and gravitons is given in Refs. [19, 20].

Soft-gluon and graviton behaviors are, in general, modified by loop effects [21, 22]. This is not surprising given that loop corrections arising from infrared singularities occur in QCD, starting with the leading behavior [23, 24]. We note that Ref. [25] proposed that by keeping the dimensional-regularization parameter $\epsilon = (4-D)/2 < 0$ finite as one takes the soft limit, loop corrections can be avoided, as explicitly shown in five-point $\mathcal{N} = 4$ super-Yang-Mills examples. However, this prescription is not physically sensible because it does not get soft physics correct and, in particular, ruins the cancellation of leading infrared divergences in QCD. One can instead view this as a prescription on integrands prior to loop integration; in this way, the five-point $\mathcal{N} = 4$ super-Yang-Mills results in Ref. [25] were extended to all numbers of loops and legs for planar amplitudes [8].

Extended BMS symmetry gives us a remarkable new understanding for the behavior of soft gravitons in four spacetime dimensions [1]. However, given that universal soft behavior holds also in D dimensions as well as for gluons, we expect that there is a more general explanation not tied to four dimensions. In this paper, we show that, just as for photons [12], on-shell gauge invariance can be used to fully determine subleading behavior. We show that in nonabelian gauge theory, on-shell gauge invariance dictates that at tree level the first subleading term is universal and controlled by the amplitude with the soft gluon removed. Similarly, in gravity the first two subleading terms at tree level are universal. Our proof is valid in D dimensions because it uses only on-shell gauge invariance together with D-dimensional three-point vertices.

We shall also explain how loop corrections arise in this context. In nonabelian gauge theory and gravity, there are "factorizing" loop corrections to the three-vertex controlling the soft behavior. However, in gravity, generically the dimensionful nature of the coupling implies that there are no loop corrections to the leading behavior [16], no corrections beyond one loop to the first subleading behavior, and no corrections beyond two loops to the second subleading behavior [21].

As shown long ago, the factorizing contributions are suppressed in gauge theory: They vanish at leading order in the soft limit [23, 24] but are nontrivial at the first subleading order [21, 22]. Similarly, in gravity we prove that for the case of a scalar circulating in the loop, the loop corrections to the soft-graviton behavior vanish not only for the leading order but for the first subleading order as well. This case is particularly transparent because there are no infrared singularities [26] or contributions to the soft operators arising from them. We expect that for all other particles circulating in the loop, only contributions associated with infrared singularities will appear at the first subleading soft order. Indeed, this suppression has been observed in the explicit examples of infrared-finite amplitudes studied in Refs. [21, 22]. These results suggest that, up to issues associated with infrared singularites, the soft Ward identities of BMS symmetry [1] are not anomalous. We note that while there are loop corrections to the first subleading soft-graviton behavior linked with infrared singularities, they come from a well-understood source and therefore should not be too disruptive when studying the connection to BMS symmetry.

This paper is organized as follows. In Sect. II, we review Low's theorem for the case of a soft photon coupled to n scalars, showing how gauge invariance determines the first subleading behavior. In Sect. III, we repeat the analysis for a soft graviton. Next, in Sect. IV, we study the case of a soft gluon where all external particles are gluons and discuss spin contributions in some detail. The analysis for a soft graviton is extended to the case where all external particles are gravitons in Sect. V. In Sect. VI, we explain how loop corrections to the soft operators arise from the perspective of on-shell gauge invariance and show that there are no corrections to the first subleading soft-graviton behavior for scalars in the loop. We give our conclusions in Sect. VII.

Added note

While this manuscript was being finalized, a paper appeared constraining soft behavior using Poincaré and gauge invariance, as well as from a condition arising from the distributional nature of scattering amplitudes [27]. In this way, the authors determine the form of the subleading soft differential operators up to a numerical constant for every leg.

II. PHOTON SOFT LIMIT WITH n SCALAR PARTICLES

In this section, we review the classic theorem due to Low [12] on the subleading soft behavior of photons, for simplicity focusing on the case of a single photon coupled to n scalars. As explained by Low in 1958, gauge invariance enforces the universality of the first subleading behavior, allowing us to fully determine it in terms of the amplitude without the

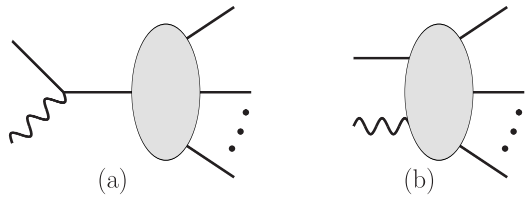

FIG. 1: Diagrams of the form (a) give universal leading soft behavior. The subleading behavior comes from both diagrams types (a) and (b).

soft photon. In subsequent sections, we will apply a similar analysis to cases with gravitons and gluons.

As illustrated in Fig. 1, the scattering amplitude of a single photon and n scalar particles arises from (a) contributions with a pole in the soft momentum q and (b) contributions with no pole:

$$A_n^\mu(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} e_i \frac{k_i^\mu}{k_i \cdot q} T_n(k_1, \ldots, k_i + q, \ldots, k_n) + N_n^\mu(q; k_1, \ldots, k_n)\,. \tag{2.1}$$

For our purposes, it is convenient to not include the polarization vectors until the end of the discussion. The full amplitude is obtained by contracting A_n^μ with the physical photon polarization $\varepsilon_{q\mu}$. The first term in Eq. (2.1) corresponds to the emission of the photon from one of the scalar external lines as illustrated in Fig. 1(a) and is divergent in the soft-photon limit, while the second term, illustrated in Fig. 1(b), is finite in the soft-photon limit. The electric charge of particle i is e_i.

On-shell gauge invariance implies

$$\begin{aligned} 0 &= q_\mu A_n^\mu(q; k_1, \ldots, k_n) \\ &= \sum_{i=1}^{n} e_i T_n(k_1, \ldots, k_i + q, \ldots, k_n) + q_\mu N_n^\mu(q; k_1, \ldots, k_n)\,, \end{aligned} \tag{2.2}$$

valid for any value of q. Expanding around $q = 0$, we have

$$\begin{aligned} 0 = \sum_{i=1}^{n} e_i \left[T_n(k_1, \ldots, k_i, \ldots, k_n) + q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_i, \ldots, k_n) \right] \\ + q_\mu N_n^\mu(q = 0; k_1, \ldots, k_n) + \mathcal{O}(q^2)\,. \end{aligned} \tag{2.3}$$

At leading order, this equation is

$$\sum_{i=1}^{n} e_i = 0\,, \tag{2.4}$$

which is simply a statement of charge conservation [13]. At the next order, we have

$$q_\mu N_n^\mu(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n)\,. \tag{2.5}$$

This equation tells us that $N_n^\mu(0; k_1, \ldots, k_n)$ is entirely determined up to potential pieces that are separately gauge invariant. However, it is easy to see that the only expressions local in q that vanish under the gauge-invariance condition $q_\mu E^\mu = 0$ are of the form,

$$E^\mu = (B_1 \cdot q) B_2^\mu - (B_2 \cdot q) B_1^\mu\,, \tag{2.6}$$

where B_1^μ and B_2^μ are arbitrary vectors that are local in q and constructed with the momenta of the scalar particles. The explicit factor of the soft momentum q in each term means that they are suppressed in the soft limit and do not contribute to $N_n^\mu(0; k_1, \ldots, k_n)$. We can therefore remove the q_μ from Eq. (2.5), leaving

$$N_n^\mu(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n)\,, \tag{2.7}$$

thereby determining $N_n^\mu(0; k_1, \ldots, k_n)$ as a function of the amplitude without the photon. Inserting this into Eq. (2.1) yields

$$A_n^\mu(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[k_i^\mu - i q_\nu J_i^{\mu\nu}\right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q)\,, \tag{2.8}$$

where

$$J_i^{\mu\nu} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right)\,, \tag{2.9}$$

is the orbital angular-momentum operator and $T_n(k_1, \ldots, k_n)$ is the scattering amplitude involving n scalar particles. Eq. (2.8) is Low's theorem for the case of one photon and n scalars.

Low's theorem is unchanged at loop level for the simple reason that even at loop level, all diagrams containing a pole in the soft momentum are of the form shown in Fig. 1(a), with loops appearing only in the blob and not correcting the external vertex. If the scalars are massive, the integrals will not have infrared discontinuities that could lead to loop corrections of the type described in Ref. [21].

It is also interesting to see if there is any further information at higher orders in the soft expansion. If we go one order further in the expansion, we find the extra condition,

$$\frac{1}{2} \sum_{i=1}^{n} e_i q_\mu q_\nu \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \ldots, k_n) + q_\mu q_\nu \frac{\partial N_n^\mu}{\partial q_\nu}(0; k_1, \ldots, k_n) = 0\,. \tag{2.10}$$

This implies

$$\sum_{i=1}^{n} e_i \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \ldots, k_n) + \left[\frac{\partial N_n^\mu}{\partial q_\nu} + \frac{\partial N_n^\nu}{\partial q_\mu}\right](0; k_1, \ldots, k_n) = 0 \,, \tag{2.11}$$

where the final set of arguments belongs to both terms in the bracket. Gauge invariance determines only the symmetric part of the quantity $\frac{\partial N_n^\nu}{\partial q_\mu}(0; k_1, \ldots, k_n)$. The antisymmetric part is not fixed by gauge invariance; indeed, this corresponds exactly to terms of the type in Eq. (2.6). Then, up to this order, we have

$$\begin{aligned} A_n^\mu(q; k_1, \ldots, k_n) = &\sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[k_i^\mu - i q_\nu J_i^{\mu\nu} \left(1 + \frac{1}{2} q_\rho \frac{\partial}{\partial k_{i\rho}}\right)\right] T_n(k_1, \ldots, k_n) \\ &+ \frac{1}{2} q_\nu \left[\frac{\partial N_n^\mu}{\partial q_\nu} - \frac{\partial N_n^\nu}{\partial q_\mu}\right](0; k_1, \ldots, k_n) + O(q^2) \,. \end{aligned} \tag{2.12}$$

It is straightforward to see that one gets zero by saturating the previous expression with q_μ.

In order to write our universal expression in terms of the amplitude, we contract $A_n^\mu(q; k_1, \ldots, k_n)$ with the photon polarization $\varepsilon_{q\mu}$. From Eq. (2.8), we have the soft-photon limit of the single-photon, n-scalar amplitude:

$$A_n(q; k_1, \ldots, k_n) \to \left[S^{(0)} + S^{(1)}\right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q) \,, \tag{2.13}$$

where

$$\begin{aligned} S^{(0)} &\equiv \sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q} \,, \\ S^{(1)} &\equiv -i \sum_{i=1}^{n} e_i \frac{\varepsilon_{q\mu} q_\nu J_i^{\mu\nu}}{k_i \cdot q} \,, \end{aligned} \tag{2.14}$$

and $J_i^{\mu\nu}$ is given in Eq. (2.9).

III. GRAVITON SOFT LIMIT WITH n SCALAR PARTICLES

We now turn to the case of gravitons coupled to n scalars. We shall see that in the graviton case, gauge invariance can be used to fully determine the first two subleading orders in the soft-graviton momentum q. Together with the subsequent sections, this shows that the tree behavior through second subleading soft order uncovered in Ref. [2] can be understood as a consequence of on-shell gauge invariance.

In the case of a graviton scattering on n scalar particles, Eq. (2.1) becomes

$$M_n^{\mu\nu}(q;k_1,\ldots,k_n) = \sum_{i=1}^{n} \frac{k_i^\mu k_i^\nu}{k_i\cdot q} T_n(k_1,\ldots,k_i+q,\ldots,k_n) + N_n^{\mu\nu}(q;k_1,\ldots,k_n)\,, \tag{3.1}$$

where $N_n^{\mu\nu}(q;k_1,\ldots,k_n)$ is symmetric under the exchange of μ and ν. For simplicity, we have set the gravitational coupling constant to unity. Similar to the gauge-theory case, we contract with the graviton polarization tensor $\varepsilon_{q\mu\nu}$ at the end. On-shell gauge invariance of the soft leg requires that the amplitude be invariant under the shift in the polarization tensor,

$$\varepsilon_{q\mu\nu} \to \varepsilon_{q\mu\nu} + q_\mu \varepsilon_{q\nu} f(q,k_i)\,, \tag{3.2}$$

where $\varepsilon_{q\nu}$ satisfies $\varepsilon_{q\nu}\cdot q = 0$ and $f(q,k_i)$ is an arbitrary function of the momenta. This implies that

$$\begin{aligned} 0 &= q_\mu M_n^{\mu\nu}(q;k_1,\ldots,k_n) \\ &= \sum_{i=1}^{n} k_i^\nu T_n(k_1,\ldots,k_i+q,\ldots,k_n) + q_\mu N_n^{\mu\nu}(q;k_1,\ldots,k_n)\,. \end{aligned} \tag{3.3}$$

Strictly speaking, Eq. (3.3) is true only after contracting the ν index with either $\varepsilon_{q\nu}$ or a conserved current. Since we contract with polarizations at the end, we can use Eq. (3.3). At leading order in q, we then have

$$\sum_{i=1}^{n} k_i^\mu = 0\,, \tag{3.4}$$

which is satisfied due to momentum conservation. (As noted by Weinberg [13], had there been different couplings to the different particles, it would have prevented this from vanishing in general; this shows that gravitons have universal coupling.)

At first order in q, Eq. (3.3) implies

$$\sum_{i=1}^{n} k_i^\nu \frac{\partial}{\partial k_{i\mu}} T_n(k_1,\ldots,k_n) + N_n^{\mu\nu}(0;k_1,\ldots,k_n) = 0\,, \tag{3.5}$$

while at second order in q, it gives

$$\sum_{i=1}^{n} k_i^\nu \frac{\partial^2}{\partial k_{i\mu}\partial k_{i\rho}} T_n(k_1,\ldots,k_n) + \left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho} + \frac{\partial N_n^{\rho\nu}}{\partial q_\mu}\right](0;k_1,\ldots,k_n) = 0\,. \tag{3.6}$$

As in the case of the photon, this is true up to gauge-invariant contributions to $N_n^{\mu\nu}$. However, the requirement of locality prevents us from writing any expression that is local in q

yet not sufficiently suppressed in q. In fact, the most general local expression that obeys the gauge-invariance condition $q_\mu E^{\mu\nu} = q_\nu E^{\mu\nu} = 0$ is of the form,

$$E^{\mu\nu} = \Big((B_1 \cdot q)B_2^\mu - (B_2 \cdot q)B_1^\mu\Big)\Big((B_3 \cdot q)B_4^\nu - (B_4 \cdot q)B_3^\nu\Big)\,, \tag{3.7}$$

where the B_i^μ are local in q and constructed in terms of the momenta of the scalar particles. In the amplitude, $E^{\mu\nu}$ will be contracted against the symmetric traceless graviton-polarization tensor $\varepsilon_{q\mu\nu}$, so there is no need to include potential terms proportional to q^μ, q^ν or $\eta^{\mu\nu}$. Terms of the form in Eq. (3.7) have two powers of q and therefore will not contribute to the soft expansion at the orders in which we are interested.

Using Eqs. (3.5) and (3.6) in Eq. (3.1), we write the expression for a soft graviton as

$$\begin{aligned} M_n^{\mu\nu}(q; k_1 \dots k_n) = &\sum_{i=1}^{n} \frac{k_i^\nu}{k_i \cdot q}\left[k_i^\mu - i q_\rho J_i^{\mu\rho}\left(1 + \frac{1}{2} q_\sigma \frac{\partial}{\partial k_{i\sigma}}\right)\right] T_n(k_1, \dots, k_n) \\ &+ \frac{1}{2} q_\rho \left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho} - \frac{\partial N_n^{\rho\nu}}{\partial q_\mu}\right](0; k_1, \dots, k_n) + \mathcal{O}(q^2)\,. \end{aligned} \tag{3.8}$$

This is essentially the same as Eq. (2.12) for the photon except that there is a second Lorentz index in the graviton case. We will show that, unlike the case of the photon, the antisymmetric quantity in the second line of the previous equation can also be determined from the amplitude $T_n(k_1, \dots, k_n)$ without the graviton.

But, before we proceed further, let us check gauge invariance. Saturating the previous expression with q_μ, we see that the first term is vanishing because of momentum conservation, while all other terms are vanishing because $q_\mu q_\rho$ is saturated with terms that are antisymmetric in μ and ρ. If, instead, we saturate the amplitude with q_ν, the first term is vanishing as before due to momentum conservation, while the first term depending on angular momentum is vanishing because of angular-momentum conservation. The remaining terms are

$$\begin{aligned} q_\nu M_n^{\mu\nu}(q; k_1, \dots, k_n) &= \frac{1}{2} q_\rho q_\sigma \Bigg\{ \sum_{i=1}^{n} \left(k_i^\mu \frac{\partial}{\partial k_{i\rho}} - k_i^\rho \frac{\partial}{\partial k_{i\mu}}\right) \frac{\partial}{\partial k_{i\sigma}} T_n(k_1, \dots, k_n) \\ &\qquad + \left[\frac{\partial N_n^{\mu\sigma}}{\partial q_\rho} - \frac{\partial N_n^{\rho\sigma}}{\partial q_\mu}\right](0; k_1, \dots, k_n) \Bigg\} \\ &= 0\,, \end{aligned} \tag{3.9}$$

where the vanishing follows from Eq. (3.6), remembering that $N_n^{\mu\nu}$ is a symmetric matrix. Therefore the amplitude in Eq. (3.8) is gauge invariant. Actually, Eq. (3.6) allows us to write the relation ,

$$-i\sum_{i=1}^{n} J_i^{\mu\rho}\frac{\partial}{\partial k_{i\sigma}}T_n(k_1,\ldots,k_n) = \left[\frac{\partial N_n^{\rho\sigma}}{\partial q_\mu} - \frac{\partial N_n^{\mu\sigma}}{\partial q_\rho}\right](0;k_1,\ldots,k_n)\,, \tag{3.10}$$

which fixes the antisymmetric part of the derivative of $N_n^{\mu\nu}$ in terms of the amplitude $T_n(k_1,\ldots,k_n)$ without the graviton. Inserting this into Eq. (3.8), we can then rewrite the terms of $\mathcal{O}(q)$ as follows:

$$\begin{aligned} M_n^{\mu\nu}(q;k_1,\ldots,k_n)\big|_{\mathcal{O}(q)} &= -\frac{i}{2}\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{k_i\cdot q}\left[k_i^\nu J_i^{\mu\rho}\frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma J_i^{\mu\rho}\frac{\partial}{\partial k_{i\nu}}\right]T_n(k_1,\ldots,k_n) \\ &= -\frac{i}{2}\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{k_i\cdot q}\left[J_i^{\mu\rho}k_i^\nu\frac{\partial}{\partial k_{i\sigma}} - \left(J_i^{\mu\rho}k_{i\nu}\right)\frac{\partial}{\partial k_{i\sigma}}\right. \\ &\qquad\qquad \left. - J_i^{\mu\rho}k_i^\sigma\frac{\partial}{\partial k_{i\nu}} + \left(J_i^{\mu\rho}k_i^\sigma\right)\frac{\partial}{\partial k_{i\nu}}\right]T_n(k_1,\ldots,k_n) \\ &= \frac{1}{2}\sum_{i=1}^{n}\frac{1}{k_i\cdot q}\left[\Big((k_i\cdot q)(\eta^{\mu\nu}q^\sigma - q^\mu\eta^{\nu\sigma}) - k_i^\mu q^\nu q^\sigma\Big)\frac{\partial}{\partial k_i^\sigma}\right. \\ &\qquad\qquad \left. - q_\rho J_i^{\mu\rho}q_\sigma J_i^{\nu\sigma}\right]T_n(k_1,\ldots,k_n)\,. \end{aligned} \tag{3.11}$$

Finally, we wish to write our soft-limit expression in terms of the amplitude, so we contract with the physical polarization tensor of the soft graviton, $\varepsilon_{q\mu\nu}$. We see that the physical-state conditions set to zero the terms in Eq. (3.11) that are proportional to $\eta^{\mu\nu}$, q^μ and q^ν. We are then left with the following expression for the graviton soft limit of a single-graviton, n-scalar amplitude:

$$M_n(q;k_1,\ldots,k_n) \to \left[S^{(0)} + S^{(1)} + S^{(2)}\right]T_n(k_1,\ldots,k_n) + \mathcal{O}(q^2)\,, \tag{3.12}$$

where

$$\begin{aligned} S^{(0)} &\equiv \sum_{i=1}^{n}\frac{\varepsilon_{\mu\nu}k_i^\mu k_i^\nu}{k_i\cdot q}\,, \\ S^{(1)} &\equiv -i\sum_{i=1}^{n}\frac{\varepsilon_{\mu\nu}k_i^\mu q_\rho J_i^{\nu\rho}}{k_i\cdot q}\,, \\ S^{(2)} &\equiv -\frac{1}{2}\sum_{i=1}^{n}\frac{\varepsilon_{\mu\nu}q_\rho J_i^{\mu\rho}q_\sigma J_i^{\nu\sigma}}{k_i\cdot q}\,. \end{aligned} \tag{3.13}$$

These soft factors follow from gauge invariance and agree with those computed in Ref. [2].

We have also looked at higher-order terms and found that gauge invariance does not fully determine them in terms of derivatives acting on $T_n(k_1,\ldots,k_n)$.

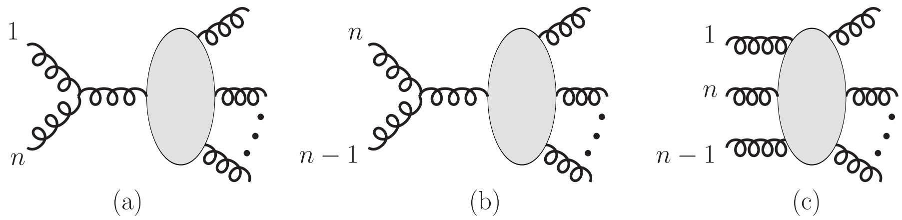

FIG. 2: Diagrams (a) and (b) give leading universal soft-gluon behavior. The first subleading behavior of the amplitude contained in the non-pole diagram (c) can be determined via on-shell gauge invariance.

IV. SOFT LIMIT OF n-GLUON AMPLITUDES

A. Behavior of gluon tree amplitudes

In this section, we generalize the procedure of Sect. II to the case of n-gluon tree amplitudes prior to turning to the case of n gravitons in the next section. As we shall discuss in Sect. VI, the soft-gluon behavior has loop corrections.

We consider a tree-level color-ordered amplitude (see e.g. Ref. [28]) where gluon n becomes soft, where we define $q \equiv k_n$. As before, we find it convenient to contract the expression with polarization vectors only at the end to obtain the full amplitude. In the case of n gluons, we have two pole terms: one where the soft gluon is attached to leg 1 (see Fig. 2(a)) and the other where the soft gluon is attached to leg $n-1$ (see Fig. 2(b)). In addition to the contributions containing a pole in the soft momentum, we have the usual term $N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1})$ that is regular in the soft limit (see Fig. 2(c)). Together, the contributions in Fig. 2 give

$$
\begin{aligned}
&A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1}) \\
&\quad = \frac{\delta^{\mu_1}_\rho k_1^\mu + \eta^{\mu\mu_1} q_\rho - \delta^\mu_\rho q^{\mu_1}}{\sqrt{2}(k_1\cdot q)}\, A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1+q,k_2,\ldots,k_{n-1}) \\
&\qquad - \frac{\delta^{\mu_{n-1}}_\rho k_{n-1}^\mu + \eta^{\mu_{n-1}\mu} q_\rho - \delta^\mu_\rho q^{\mu_{n-1}}}{\sqrt{2}(k_{n-1}\cdot q)}\, A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-2},k_{n-1}+q) \\
&\qquad + N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1})\,.
\end{aligned} \tag{4.1}
$$

We have dropped terms from the three-gluon vertex that vanish when saturated with the external-gluon polarization vectors in addition to using the current-conservation conditions,

$$
\begin{aligned}
&(k_1+q)_\rho \, A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1+q, k_2, \ldots, k_{n-1}) = 0\,, \\
&(k_{n-1}+q)_\rho \, A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \ldots, k_{n-2}, k_{n-1}+q) = 0\,,
\end{aligned} \tag{4.2}
$$

which are valid once we contract with the polarization vectors carrying the μ_j indices. By introducing the spin-one angular-momentum operator,

$$
(\Sigma_i^{\mu\sigma})^{\mu_i\rho} \equiv i\left(\eta^{\mu\mu_i}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\mu_i\sigma}\right)\,, \tag{4.3}
$$

we can write Eq. (4.1) as

$$
\begin{aligned}
&A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1, \ldots, k_{n-1}) \\
&\quad = \frac{\delta_\rho^{\mu_1} k_1^\mu - i q_\sigma (\Sigma_1^{\mu\sigma})^{\mu_1}{}_\rho}{\sqrt{2}(k_1\cdot q)}\, A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1+q, k_2, \ldots, k_{n-1}) \\
&\qquad - \frac{\delta_\rho^{\mu_{n-1}} k_{n-1}^\mu - i q_\sigma (\Sigma_{n-1}^{\mu\sigma})^{\mu_{n-1}}{}_\rho}{\sqrt{2}(k_{n-1}\cdot q)}\, A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \ldots, k_{n-2}, k_{n-1}+q) \\
&\qquad + N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1, \ldots, k_{n-1})\,.
\end{aligned} \tag{4.4}
$$

Notice that the spin-one terms independently vanish when contracted with q_μ.

The on-shell gauge invariance of Eq. (4.4) requires

$$
\begin{aligned}
0 &= q_\mu A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1, \ldots, k_{n-1}) \\
&= \frac{1}{\sqrt{2}} A_{n-1}^{\mu_1\mu_2\cdots\mu_{n-1}}(k_1+q, k_2, \ldots, k_{n-1}) - \frac{1}{\sqrt{2}} A_{n-1}^{\mu_1\cdots\mu_{n-2}\mu_{n-1}}(k_1, \ldots, k_{n-2}, k_{n-1}+q) \\
&\quad + q_\mu N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1, \ldots, k_{n-1})\,.
\end{aligned} \tag{4.5}
$$

For $q=0$, this is automatically satisfied. At the next order in q, we obtain

$$
-\frac{1}{\sqrt{2}}\left[\frac{\partial}{\partial k_{1\mu}} - \frac{\partial}{\partial k_{n-1\mu}}\right] A_{n-1}^{\mu_1\cdots\mu_{n-1}}(k_1, k_2 \ldots k_{n-1}) = N_n^{\mu;\mu_1\cdots\mu_{n-1}}(0; k_1, \ldots, k_{n-1})\,. \tag{4.6}
$$

Similar to the photon case, we ignore local gauge-invariant terms in $N_n^{\mu;\mu_1\cdots\mu_{n-1}}$ because they are necessarily of a higher order in q. Thus, $N_n^{\mu;\mu_1\cdots\mu_{n-1}}(0; k_1, \ldots, k_{n-1})$ is determined in terms of an expression without the soft gluon. With this, the total expression in Eq. (4.4) becomes

$$
\begin{aligned}
A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1 \ldots k_{n-1}) = &\left(\frac{k_1^\mu}{\sqrt{2}(k_1\cdot q)} - \frac{k_{n-1}^\mu}{\sqrt{2}(k_{n-1}\cdot q)}\right) A_{n-1}^{\mu_1\cdots\mu_{n-1}}(k_1, \ldots, k_{n-1}) \\
&- i\frac{q_\sigma (J_1^{\mu\sigma})^{\mu_1}{}_\rho}{\sqrt{2}(k_1\cdot q)}\, A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1, \ldots, k_{n-1}) \\
&+ i\frac{q_\sigma (J_{n-1}^{\mu\sigma})^{\mu_{n-1}}{}_\rho}{\sqrt{2}(k_{n-1}\cdot q)}\, A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \ldots, k_{n-1}) + \mathcal{O}(q)\,,
\end{aligned} \tag{4.7}
$$

where

$$(J_i^{\mu\sigma})^{\mu_i\rho} \equiv L_i^{\mu\sigma}\eta^{\mu_i\rho} + (\Sigma_i^{\mu\sigma})^{\mu_i\rho}, \tag{4.8}$$

the spin-one angular-momentum operator is given in Eq. (4.3), and the orbital angular-momentum operator is

$$L_i^{\mu\sigma} \equiv i\left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}}\right). \tag{4.9}$$

Both angular-momentum operators satisfy the same commutation relations,

$$\begin{aligned}
[L_i^{\mu\nu}, L_i^{\rho\sigma}] &= i\left(\eta^{\nu\rho}L_i^{\mu\sigma} + \eta^{\mu\rho}L_i^{\sigma\nu} + \eta^{\mu\sigma}L_i^{\nu\rho} + \eta^{\nu\sigma}L_i^{\rho\mu}\right), \\
[\Sigma_i^{\mu\nu}, \Sigma_i^{\rho\sigma}] &= i\left(\eta^{\nu\rho}\Sigma_i^{\mu\sigma} + \eta^{\mu\rho}\Sigma_i^{\sigma\nu} + \eta^{\mu\sigma}\Sigma_i^{\nu\rho} + \eta^{\nu\sigma}\Sigma_i^{\rho\mu}\right),
\end{aligned} \tag{4.10}$$

where the suppressed indices on $\Sigma_i^{\mu\nu}$ should be treated as matrix indices.

In order to write the final result in terms of full amplitudes, we contract with external polarization vectors. On the right-hand side of Eq. (4.7), we must pass polarization vectors $\varepsilon_{1\mu_1}$ and $\varepsilon_{n-1\mu_{n-1}}$ through the spin-one angular-momentum operator such that they will contract with the ρ index of, respectively, $A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1,\ldots,k_{n-1})$ and $A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-1})$. It is convenient write the spin angular-momentum operator as

$$\varepsilon_{i\mu_i}(\Sigma_i^{\mu\sigma})^{\mu_i}{}_\rho A^\rho = i\left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}}\right)\varepsilon_{i\rho}A^\rho. \tag{4.11}$$

We may therefore write

$$A_n(q;k_1,\ldots,k_{n-1}) \to \left[S_n^{(0)} + S_n^{(1)}\right] A_{n-1}(k_1,\ldots,k_{n-1}) + \mathcal{O}(q), \tag{4.12}$$

where

$$\begin{aligned}
S_n^{(0)} &\equiv \frac{k_1\cdot\varepsilon_n}{\sqrt{2}\,(k_1\cdot q)} - \frac{k_{n-1}\cdot\varepsilon_n}{\sqrt{2}\,(k_{n-1}\cdot q)}, \\
S_n^{(1)} &\equiv -i\varepsilon_{n\mu}q_\sigma\left(\frac{J_1^{\mu\sigma}}{\sqrt{2}\,(k_1\cdot q)} - \frac{J_{n-1}^{\mu\sigma}}{\sqrt{2}\,(k_{n-1}\cdot q)}\right).
\end{aligned} \tag{4.13}$$

Here

$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + \Sigma_i^{\mu\sigma}, \tag{4.14}$$

where

$$\Sigma_i^{\mu\sigma} \equiv i\left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}}\right). \tag{4.15}$$

In using Eq. (4.12), one must interpret $L_i^{\mu\sigma}$ as not acting on explicit polarization vectors, i.e., $L_i^{\mu\sigma}\varepsilon_i = 0$. If one instead interprets polarization vectors as functions of momenta (see e.g. Sect. 5.9 of Ref. [29]) and returns a nonzero value for $L_i^{\mu\sigma}\varepsilon_i$, then one should not include the spin term (4.15). To be concrete, we define the action of the total angular-momentum operator on momenta and polarizations by

$$\begin{aligned} J_i^{\mu\sigma} k_i^\rho &= i\left(\eta^{\sigma\rho} k_i^\mu - \eta^{\mu\rho} k_i^\sigma\right) \,, \\ J_i^{\mu\sigma} \varepsilon_i^\rho &= i\left(\eta^{\sigma\rho} \varepsilon_i^\mu - \eta^{\mu\rho} \varepsilon_i^\sigma\right) \,. \end{aligned} \tag{4.16}$$

We comment more on the action of the operator on polarization vectors in Sect. IV B.

In conclusion, the first two leading terms in the soft-gluon expansion of an n-gluon amplitude are given directly in terms of the amplitude without the soft gluon. This derivation is valid in D dimensions. We have explicitly checked the soft-gluon formula (4.12) using explicit four-, five- and six-gluon tree amplitudes of gauge theory in terms of formal polarization vectors.

B. Connection to spinor helicity

To connect with the spinor-helicity formalism used in e.g. Refs. [2, 21, 22], we show that, up to a gauge transformation, the action of the above subleading soft operators on polarization vectors expressed in terms of spinor helicity is identical to the ones defined as differential operators acting on spinors. In the spinor-helicity formalism, polarization vectors are expressed directly in terms of spinors depending on the momenta:

$$\varepsilon_i^{+\,\rho}(k_i, k_r) = \frac{\langle r|\,\gamma^\rho\,|i]}{\sqrt{2}\,\langle r\,i\rangle} \,, \qquad\qquad \varepsilon_i^{-\,\rho}(k_i, k_r) = -\frac{\langle i|\,\gamma^\rho\,|r]}{\sqrt{2}\,[r\,i]} \,, \tag{4.17}$$

where k_i is the momentum of gluon i and k_r is a null reference momentum. Henceforth, we will leave the k_i argument implicit and only display the reference momentum. The spinors are standard Weyl spinors. We follow the conventions of Ref. [28] aside from our use of angle and square brackets instead of the $\pm$ angle-bracket convention. In our convention, we have

$$\langle i| = \langle i^-| \,, \qquad [i| = \langle i^+| \,, \qquad |i\rangle = |i^+\rangle \,, \qquad |i] = |i^-\rangle \,. \tag{4.18}$$

In terms of spinors, the subleading soft factor for a tree-level gauge-theory amplitude is [5]

$$S_n^{(1)\lambda} = \frac{1}{\langle (n-1)\, n\rangle} \tilde\lambda_n^{\dot\alpha} \frac{\partial}{\partial \tilde\lambda_{n-1}^{\dot\alpha}} - \frac{1}{\langle 1\, n\rangle} \tilde\lambda_n^{\dot\alpha} \frac{\partial}{\partial \tilde\lambda_1^{\dot\alpha}} \,, \tag{4.19}$$

where $\lambda^\alpha \equiv |i^+\rangle^\alpha$ and $\tilde\lambda^{\dot\alpha} \equiv |i^-\rangle^{\dot\alpha}$. We consider the explicit action of $S_n^{(1)\lambda}$ in Eq. (4.19) and $S_n^{(1)}$ in Eq. (4.13) on $\varepsilon_1^{\pm\,\rho}(k_{r_1})$ to show equivalence after contraction with the polarization-stripped amplitude. The action on $\varepsilon_{n-1}^{\pm\,\rho}(k_{r_{n-1}})$ follows similarly. We act with Eq. (4.19) on the vectors in Eq. (4.17)—with $i \to 1$ and $k_r \to k_{r_1}$—in turn:

$$S_n^{(1)\lambda}\,\varepsilon_1^{+\,\rho}(k_{r_1}) = -\frac{1}{\langle 1\,n\rangle}\frac{\langle r_1|\,\gamma^\rho\,|n]}{\sqrt{2}\,\langle r_1\,1\rangle} = -\frac{\langle r_1\,n\rangle}{\langle r_1\,1\rangle\,\langle 1\,n\rangle}\varepsilon_n^{+\,\rho}(k_{r_1})\,, \tag{4.20}$$

and

$$\begin{aligned} S_n^{(1)\lambda}\,\varepsilon_1^{-\,\rho}(k_{r_1}) &= -\frac{1}{\langle 1\,n\rangle}\left(-\frac{\langle 1|\,\gamma^\rho\,|r_1]}{\sqrt{2}}\right)\left(-\frac{[r_1\,n]}{[r_1\,1]^2}\right) \\ &= \frac{[r_1\,n]}{[r_1\,1]\,\langle 1\,n\rangle}\varepsilon_1^{-\,\rho}(k_{r_1}) \\ &= \frac{[r_1\,n]}{[r_1\,1]\,\langle 1\,n\rangle}\left[\varepsilon_1^{-\,\rho}(k_n) + \frac{\sqrt{2}\,[r_1\,n]}{[r_1\,1]\,[n\,1]}\,k_1^\rho\right] \\ &= \frac{[r_1\,n]}{[r_1\,1]\,[1\,n]}\left[\varepsilon_n^{+\,\rho}(k_1) - \frac{\sqrt{2}\,[r_1\,n]}{[r_1\,1]\,\langle 1\,n\rangle}\,k_1^\rho\right], \end{aligned} \tag{4.21}$$

where we used

$$\varepsilon_i^{-\rho}(k_r) = \varepsilon_i^{-\rho}(k_{\tilde r}) + \frac{\sqrt{2}\,[r\,\tilde r]}{[r\,i]\,[\tilde r\,i]}\,k_i^\rho\,, \tag{4.22}$$

in the second-to-last line. The last line of Eq. (4.21) follows from

$$\varepsilon_j^{+\,\rho}(k_i) = \frac{[i\,j]}{\langle i\,j\rangle}\,\varepsilon_i^{-\,\rho}(k_j)\,. \tag{4.23}$$

We can write Eq. (4.21) more simply as

$$S_n^{(1)\lambda}\,\varepsilon_1^{-\,\rho}(k_{r_1}) \cong \frac{[r_1\,n]}{[r_1\,1]\,[1\,n]}\,\varepsilon_n^{+\,\rho}(k_1)\,, \tag{4.24}$$

where the symbol $\cong$ denotes equivalence up to a term proportional to k_1^ρ. Such terms will vanish when contracted with the polarization-stripped $(n-1)$-point amplitude, so we are free to drop them. Similar spinor-helicity algebra reveals that the action of $S_n^{(1)}$ from Eq. (4.13) on $\varepsilon_1^{\pm\,\rho}(k_{r_1})$ yields

$$\begin{aligned} S_n^{(1)}\,\varepsilon_1^{+\,\rho}(k_{r_1}) &= -i\varepsilon_{n\,\mu}^{+}(k_{r_n})k_{n\,\sigma}\frac{\Sigma_1^{\mu\sigma}}{\sqrt{2}\,(k_1\cdot k_n)}\varepsilon_1^{+\,\rho}(k_{r_1}) \\ &= -\frac{\langle r_1\,n\rangle}{\langle r_1\,1\rangle\,\langle 1\,n\rangle}\,\varepsilon_n^{+\,\rho}(k_{r_1})\,, \end{aligned} \tag{4.25}$$

and

$$S_n^{(1)}\,\varepsilon_1^{-\,\rho}(k_{r_1}) = \frac{[r_1\,n]}{[r_1\,1]\,[1\,n]}\,\varepsilon_n^{+\,\rho}(k_1)\,. \tag{4.26}$$

We can summarize the action of the operators as

$$S_n^{(1)\lambda}\,\varepsilon_1^{\pm\,\rho}(k_{r_1}) \cong S_n^{(1)}\,\varepsilon_1^{\pm\,\rho}(k_{r_1}) = -\left(\frac{\varepsilon_1^{\pm}(k_{r_1})\cdot p_n}{\sqrt{2}(p_1\cdot p_n)}\right)\times\begin{cases}\varepsilon_n^{+\,\rho}(k_{r_1}), & \text{for } +\,,\\ \varepsilon_n^{+\,\rho}(k_1), & \text{for } -\,.\end{cases} \tag{4.27}$$

We see that, up to terms proportional to k_1^ρ, the action of $S_n^{(1)\lambda}$ and $S_n^{(1)}$ on the polarization vectors yield completely equivalent expressions as expected.

V. SOFT LIMIT OF n-GRAVITON AMPLITUDES

In this section, we generalize what has been done for the case of n gluons to the case of n gravitons. As before, we write the amplitude as a sum of two pieces: the first contains terms where the soft graviton is attached to one of the other $n-1$ external gravitons, giving a contribution divergent as $1/q$ for $q\to 0$, while in the second the soft graviton attaches to one of the internal graviton lines and is of $\mathcal{O}(q^0)$ in the same limit. Leaving the expression uncontracted with polarization tensors for now, we write

$$\begin{aligned} &M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1})\\ &\qquad=\sum_{i=1}^{n-1}\frac{1}{k_i\cdot q}\left[k_i^\mu\eta^{\mu_i\alpha}-iq_\rho(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\left[k_i^\nu\eta^{\nu_i\beta}-iq_\sigma(\Sigma_i^{\mu\sigma})^{\nu_i\beta}\right]\\ &\qquad\qquad\times M_{n-1}{}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i+q,\ldots,k_{n-1})\\ &\qquad\quad+N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1})\,, \end{aligned} \tag{5.1}$$

where

$$(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\equiv i\left(\eta^{\mu\mu_i}\eta^{\alpha\rho}-\eta^{\mu\alpha}\eta^{\mu_i\rho}\right)\,. \tag{5.2}$$

The simple form of the three-vertex used in Eq. (5.1) can be obtained from the standard one using current conservation and the tracelessness properties of external polarization tensors and M_{n-1}, as well as assigning terms to N_n where the $i/k_i\cdot q$ propagator cancels. We note that it is important to keep the lowered indices of M_{n-1} in their appropriate slots. On-shell gauge invariance implies

$$
\begin{aligned}
0 &= q_\mu M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1}) \\
&= \sum_{i=1}^{n-1}\left[k_i^\nu\eta^{\nu_i\beta} - iq_\rho(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right] M_{n-1}^{\mu_1\nu_1\cdots\mu_i}{}_{\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i+q,\ldots,k_{n-1}) \\
&\quad + q_\mu N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1})\,,
\end{aligned}
\tag{5.3}
$$

provided that, as usual, we contract all free indices of M_n with polarization tensors at the end. This includes contracting the ν index with a polarization vector ε_n^ν satisfying $\varepsilon_n\cdot q = 0$. Expanding the previous expression for small q, we find that the leading term vanishes because of momentum conservation, while the next-to-leading term gives two conditions by taking the symmetric and antisymmetric parts:

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n-1}\eta^{\mu_i\alpha}\eta^{\nu_i\beta}\left(k_i^\mu\frac{\partial}{\partial k_{i\nu}} + k_i^\nu\frac{\partial}{\partial k_{i\mu}}\right) M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1}) \\
&\qquad\qquad = -N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(0;k_1,\ldots,k_{n-1})\,,
\end{aligned}
\tag{5.4}
$$

and

$$
\sum_{i=1}^{n-1}\left[L_i^{\nu\rho}\eta^{\nu_i\beta} + 2(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right] M_{n-1}^{\mu_1\nu_1\cdots\mu_i}{}_{\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1}) = 0\,. \tag{5.5}
$$

As in the earlier cases, we can ignore potential terms that are local in q and vanish when dotted into q^μ since they will not contribute to the desired order. The first condition determines $N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(0;k_1,\ldots,k_{n-1})$ in terms of the amplitude without the soft graviton, while the second one reflects conservation of total angular momentum. The factor of 2 in front of the spin term in Eq. (5.5) reflects the fact that the graviton has spin 2.

Finally, the terms of order q^2 in Eq. (5.3) imply the following condition:

$$
\begin{aligned}
&\sum_{i=1}^{n-1} q_\rho\left[k_i^\nu\eta^{\nu_i\beta}\frac{\partial^2}{\partial k_{i\rho}\partial k_{i\mu}} - 2i(\Sigma_i^{\nu\rho})^{\nu_i\beta}\frac{\partial}{\partial k_{i\mu}}\right] M_{n-1}^{\mu_1\nu_1\cdots\mu_i}{}_{\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1}) \\
&\qquad = -q_\rho\left[\frac{\partial N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\rho} + \frac{\partial N_n^{\rho\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\mu}\right](0;k_1,\ldots,k_{n-1})\,.
\end{aligned}
\tag{5.6}
$$

Using the previous results, for a soft graviton of momentum q, we have

$$
\begin{aligned}
&M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1})\\
&\quad=\sum_{i=1}^{n-1}\frac{1}{k_i\cdot q}\Big\{k_i^\mu k_i^\nu\eta^{\mu_i\alpha}\eta^{\nu_i\beta}\\
&\qquad-\frac{i}{2}q_\rho\Big[k_i^\mu\eta^{\mu_i\alpha}\left[L_i^{\nu\rho}\eta^{\nu_i\beta}+2(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right]+k_i^\nu\eta^{\nu_i\beta}\left[L_i^{\mu\rho}\eta^{\mu_i\alpha}+2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\Big]\\
&\qquad-\frac{i}{2}q_\rho q_\sigma\Big[k_i^\nu\eta^{\mu_i\alpha}\eta^{\nu_i\beta}L_i^{\mu\rho}\frac{\partial}{\partial k_{i\sigma}}-2i(\Sigma_i^{\mu\rho})^{\mu_i\alpha}(\Sigma_i^{\nu\sigma})^{\nu_i\beta}-2k_i^\sigma\eta^{\nu_i\beta}(\Sigma_i^{\nu\rho})^{\nu_i\beta}\frac{\partial}{\partial k_{i\mu}}\\
&\qquad\qquad+2\left[\eta^{\mu_i\alpha}k_i^\mu(\Sigma_i^{\nu\rho})^{\nu_i\beta}+\eta^{\nu_i\beta}k_i^\nu(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\frac{\partial}{\partial k_{i\sigma}}\Big]\Big\}\\
&\qquad\qquad\times M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1})\\
&\qquad+\frac{1}{2}q_\rho\left[\frac{\partial N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\rho}-\frac{\partial N_n^{\rho\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\mu}\right](0;k_1,\ldots,k_{n-1})\\
&\qquad+\mathcal{O}(q^2)\,.
\end{aligned}
\tag{5.7}
$$

As in the case of gluon scattering, it may seem that we cannot determine the order q contributions in terms of M_{n-1} because the antisymmetric part of the matrix N_n is still present in Eq. (5.7). However, it turns out that there is additional information from on-shell gauge invariance. When we saturate it with q_μ, we get of course zero because this is the way that Eq. (5.7) is constructed. When we saturate it with q_ν, however, we obtain the extra condition:

$$
\begin{aligned}
0&=q_\nu M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1})\\
&=q_\rho q_\sigma\Big\{\sum_{i=1}^{n-1}\left[L_i^{\mu\rho}\eta^{\mu_i\alpha}+2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\frac{\partial}{\partial k_{i\sigma}}M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha}{}^{\nu_i\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1})\\
&\qquad+i\left[\frac{\partial N_n^{\mu\sigma;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\rho}-\frac{\partial N_n^{\rho\sigma;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\mu}\right](0;k_1,\ldots,k_{n-1})\Big\}\,,
\end{aligned}
\tag{5.8}
$$

which implies

$$
\begin{aligned}
&\sum_{i=1}^{n-1}q_\rho\left[L_i^{\mu\rho}\eta^{\mu_i\alpha}+2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\frac{\partial}{\partial k_{i\sigma}}M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha}{}^{\nu_i\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1})\\
&\qquad=-iq_\rho\left[\frac{\partial N_n^{\mu\sigma;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\rho}-\frac{\partial N_n^{\rho\sigma;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\mu}\right](0;k_1,\ldots,k_{n-1})\,.
\end{aligned}
\tag{5.9}
$$

We can now use it in Eq. (5.7) to obtain our final expression giving the soft limit entirely in terms of the $(n-1)$-point amplitude:

$$
\begin{aligned}
&M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1}) \\
&\quad = \sum_{i=1}^{n-1} \frac{1}{k_i\cdot q}\bigg\{ k_i^\mu k_i^\nu \eta^{\mu_i\alpha}\eta^{\nu_i\beta} \\
&\qquad - \frac{i}{2} q_\rho \Big[k_i^\mu \eta^{\mu_i\alpha}\left[L_i^{\nu\rho}\eta^{\nu_i\beta} + 2(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right] + k_i^\nu \eta^{\nu_i\beta}\left[L_i^{\mu\rho}\eta^{\mu_i\alpha} + 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\Big] \\
&\qquad - \frac{1}{2} q_\rho q_\sigma \Big[\left[L_i^{\mu\rho}\eta^{\mu_i\alpha} + 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\left[L_i^{\nu\sigma}\eta^{\nu_i\beta} + 2(\Sigma_i^{\nu\sigma})^{\nu_i\beta}\right] - 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}(\Sigma_i^{\nu\sigma})^{\nu_i\beta}\Big]\bigg\} \\
&\qquad\qquad \times M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1}) + \mathcal{O}(q^2)\,.
\end{aligned}
\tag{5.10}
$$

In order to write our expression in terms of amplitudes, we saturate with graviton polarization tensors using $\varepsilon_{\mu\nu}\to\varepsilon_\mu\varepsilon_\nu$ where ε_μ are spin-one polarization vectors. As we did for the case with gluons, we must pass the polarization vectors through the spin-one operators. We are then left with

$$
M_n(q;k_1,\ldots,k_{n-1}) = \left[S_n^{(0)} + S_n^{(1)} + S_n^{(2)}\right] M_{n-1}(k_1,\ldots,k_{n-1}) + \mathcal{O}(q^2)\,, \tag{5.11}
$$

where

$$
\begin{aligned}
S_n^{(0)} &\equiv \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu}k_i^\mu k_i^\nu}{k_i\cdot q}\,, \\
S_n^{(1)} &\equiv -i\sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu}k_i^\mu q_\rho J_i^{\nu\rho}}{k_i\cdot q}\,, \\
S_n^{(2)} &\equiv -\frac{1}{2}\sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu}q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i\cdot q}\,.
\end{aligned}
\tag{5.12}
$$

Here

$$
J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + \Sigma_i^{\mu\sigma}\,, \tag{5.13}
$$

with

$$
L_i^{\mu\sigma} \equiv i\left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}}\right)\,, \qquad\qquad \Sigma_i^{\mu\sigma} \equiv i\left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}}\right)\,. \tag{5.14}
$$

Since the graviton polarization tensor is quadratic in spin-one polarization vectors ε_i^μ, the differential operator in Eq. (5.14) picks up factors of 2 as required for Eq. (5.11) to be compatible with Eq. (5.10).

In conclusion, in the case of a soft graviton, on-shell gauge invariance completely determines the first two subleading contributions. Using the Kawai-Lewellen-Tye relations [30] we have generated graviton amplitudes with formal polarization tensors up to six points. Using these we analytically confirmed Eq. (5.11) through five points and numerically through six points.

VI. COMMENTS ON LOOP CORRECTIONS

In gauge and gravity theories in four dimensions, the operators describing the soft behavior have nontrivial loop corrections [21, 22]. Indeed, in QCD, loop corrections linked to infrared singularities are present already at leading order in the soft limit [23, 24]. One may wonder how loop corrections to the soft operators arise from the perspective of the constraints imposed by on-shell gauge invariance. In this section we explain this. We first describe the case of gauge theory before turning to gravity.

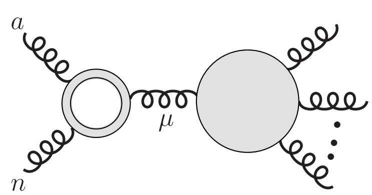

FIG. 3: The potential factorizing contributions to the one-loop corrections to the soft operators. Leg n is the soft leg which carries momentum q.

As explained in Ref. [23], we can separate the contributions into two distinct sources. The first source of potential corrections is the "factorizing" one that arises from loop corrections of the form displayed in Fig. 3 [21–23]. The second source of contributions is the "nonfactorizing" infrared-divergent one that can come from discontinuities in the amplitudes associated with infrared divergences [31]. (Alternatively, these nonfactorizing contributions can be pushed into factorizing contributions that have light-cone denominators coming from a careful application of unitarity [24].)

Here we will focus on the factorizing pieces. In gauge theory, we will explain why they do not enter in the leading soft behavior [23, 24]. In gravity, for the case of scalars in the loops, which is an especially clean case since there are no infrared singularities even for massless scalars, we show that there are no loop corrections at the leading and first subleading orders of the soft-graviton expansion. This suppression was noticed earlier in explicit examples of soft limits of one-loop infrared-finite gravity amplitudes [21, 22].

A. Gauge theory

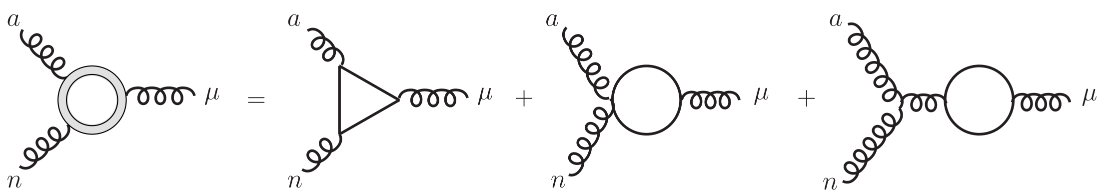

FIG. 4: The diagrams contributing to the factorizing contribution to the one-loop soft function.

As a warm up for the gravity case, we first discuss the well-understood gauge-theory case. The explicit forms of the factorizing one-loop corrections to the soft behavior have been described in some detail in Refs. [23, 24] for QCD at leading order in the soft (and collinear) limits.

For the case of external gluons, the potential factorizing contributions to one-loop modifications of the soft behavior are shown in Fig. 3. We can expand these corrections into triangle and bubble diagrams as shown in Fig. 4. As derived in Ref. [23], these diagrams evaluate to

$$D^{\mu,\mathrm{fact}} = \frac{i}{\sqrt{2}}\frac{1}{3}\frac{1}{(4\pi)^2}\Big(1-\frac{n_f}{N_c}+\frac{n_s}{N_c}\Big)(q-k_a)^\mu\Big[(\varepsilon_n\cdot\varepsilon_a)-\frac{(q\cdot\varepsilon_a)(k_a\cdot\varepsilon_n)}{(k_a\cdot q)}\Big]\,, \tag{6.1}$$

where n_f is the number of fundamental representation fermions, n_s the number of fundamental representation complex scalars (using the normalization conventions of Ref. [23]), and N_c is the number of colors. As usual we take the soft momentum of leg n to be q. After integration this result is both ultraviolet- and infrared-finite, so we have taken $\epsilon = 0$ in the final integrated result. The all orders in ϵ form of Eq. (6.1) is given in Refs. [23, 24].

The result (6.1) has a few surprising features that explain how it evades the link between the leading and first subleading soft contributions via gauge invariance. The first feature is that the correction to the three-vertex is nonlocal because of the pole in $q\cdot k_a$ that arises from the loop integration. Indeed, after we include the intermediate propagator $-i/(k_a+q)^2$, there is a double pole[1] in $q \cdot k_a$. A second curious feature is that the leading contribution is gauge invariant by itself; it vanishes when ε_n^μ is replaced by $q^\mu \equiv k_n^\mu$ for any value of the intermediate off-shell momentum. The nonlocal nature of the result is what allows us to write such a gauge-invariant term with the correct dimensions. A third feature is that, in fact, there is no contribution from Eq. (6.1) to the leading one-loop correction to the soft function, as noted in Refs. [23, 24]. To see this, we sew Eq. (6.1) onto the rest of the amplitude across the factorization channel:

$$D_\mu^{\rm fact} \frac{-i}{2q \cdot k_a} \mathcal{J}^\mu \,, \tag{6.2}$$

as illustrated in Fig. 3. We observe that $\mathcal{J}^\mu$ is a conserved current:

$$(q + k_a)_\mu \mathcal{J}^\mu = 0 \,, \tag{6.3}$$

assuming that all of the remaining legs are contracted with on-shell polarizations. This immediately implies

$$D_\mu^{\rm fact} \frac{-i}{2q \cdot k_a} \mathcal{J}^\mu = \mathcal{O}(q^0) \,, \tag{6.4}$$

because $D_\mu^{\rm fact}$ is proportional to $(q - k_a)_\mu$ which is equivalent to $2q_\mu$ when dotted into a conserved current. This reproduces the fact that there is no leading, $\mathcal{O}(1/q)$, factorizing contribution to the one-loop soft function [23, 24].

Unfortunately, the $\mathcal{O}(q^0)$ terms in Eq. (6.4) are not under control via gauge invariance. Once we allow for an extra $1/(q \cdot k_a)$ nonlocality arising from the loop integration, we lose control over the subleading piece. This cannot happen at tree level because there is no source of a second factor of $1/(q \cdot k_a)$. The $\mathcal{O}(q^0)$ contribution from Eq. (6.4) is not constrained from gauge invariance. These types of contributions have already been described in some detail at one loop on a case-by-case basis in Refs. [32, 33]. Unfortunately, no universal factorization formula is known for these types of corrections, although case-by-case their forms appear to be relatively simple.

Interestingly, these contributions resemble an anomaly that seemingly vanishes if we take the loop integrand strictly in four dimensions. This arises from the integration where a $1/\epsilon$ ultraviolet pole cancels a factor of ϵ from numerator algebra, leaving terms of $\mathcal{O}(1)$. This

[1] While this might seem to violate basic factorization properties of field theories, in fact it does not, because for real momenta the double pole is reduced to a single pole. See Ref. [32] for a detailed discussion of this phenomenon.

is reminiscent of the way the chiral anomaly arises from triangle diagrams in dimensional regularization. Indeed, for the single-minus-helicity case discussed in Refs. [21, 22], not only does this contribution vanish but the entire amplitude would vanish if we were not careful to keep in the integrand in $D = 4 - 2\epsilon$ instead of four dimensions. It is interesting that these types of contributions do not appear in supersymmetric theories.

Besides the loop contributions described above, there is a second type of loop correction to the soft operators (4.13) arising from non-smoothness in the amplitude due to infrared singularities [31]. In QED the integrals are smooth because the electron mass acts as an infrared cutoff, but in QCD or gravity there is no such physical cutoff on gluons or gravitons. It is therefore much more difficult to consistently introduce a mass regulator without breaking gauge symmetry or altering the number of propagating degrees of freedom. As is standard practice, it is far simpler to use dimensional regularization. As discussed in some detail in Refs. [21, 23, 31], as gluons become soft or collinear, the matrix elements develop discontinuities that are absorbed into modifications of the loop splitting or soft operators. Alternatively, by using light-cone gauge or carefully applying unitarity, one introduces light-cone denominators containing a reference momentum, and one can push all contributions into factorizing diagrams [24, 34]. Either way, the conclusion is the same: There are non-trivial contributions due to infrared singularities not accounted for in the naive tree-level soft limit.

B. Gravity

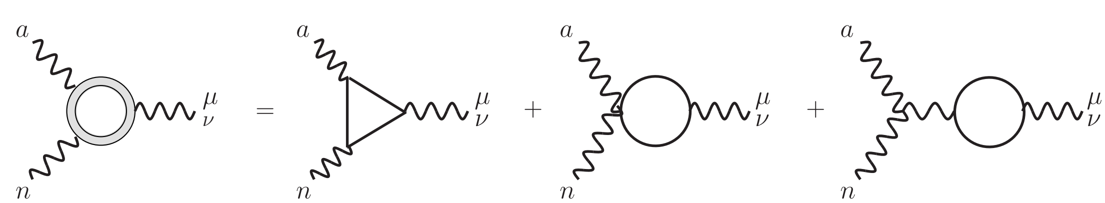

FIG. 5: The diagrams with potential factorizing contributions to the one-loop soft operator in gravity with a scalar in the loop.

We now show that the situation in gravity is similar. Here, the dimensionful coupling ensures that there are no loop corrections at leading order [16], only one-loop corrections at the first subleading order, and only up to two-loop corrections at second subleading order [21]. Thus, we need only to analyze one loop to show that the factorizing contributions do not modify the soft operator at first subleading order.

We focus on the case of a scalar in the loop. This case is particularly transparent because there are no infrared singularities associated with scalars circulating in a loop [26]. This allows us to study the soft behavior without being entangled with the issue of infrared divergences. We can determine the behavior through the first subleading soft order simply by computing the diagrams in Fig. 5.

We have carried out the analogous computation to the one performed in Ref. [23] for gluons, but for gravity with a real scalar in the loop. The result of this computation is

$$\mathcal{D}^{\mu\nu,\text{fact,s}} = -\frac{i}{(4\pi)^2}\left(\frac{\kappa}{2}\right)^3 \frac{1}{30 q\cdot k_a}\Big((\varepsilon_n\cdot\varepsilon_a)(q\cdot k_a) - (q\cdot\varepsilon_a)(k_a\cdot\varepsilon_n)\Big)^2 k_a^\mu k_a^\nu + \mathcal{O}(q^2)\,, \quad (6.5)$$

where we have kept all terms involving no more than one overall power of the soft momentum $q \equiv k_n$. Such terms naively appear to contribute at the first subleading order in the correction to the amplitude. However, as in the gauge-theory case, the diagrams $\mathcal{D}^{\mu\nu,\text{fact,s}}$ contract into a current $\mathcal{J}_{\mu\nu}$ which results in a suppression of an extra factor of the soft momentum q. In the gravity case we find

$$(k_a + q)^\mu \mathcal{J}_{\mu\nu} = f(k_i, \varepsilon_i)(k_a + q)_\nu\,, \quad (6.6)$$

where f is some function of the momenta and polarizations of both the hard and soft legs. With $k_a^\mu k_a^\nu$ contracting with $\mathcal{J}_{\mu\nu}$, we then have

$$\begin{aligned} k_a^\mu k_a^\nu \mathcal{J}_{\mu\nu} &= (k_a + q)^\mu (k_a + q)^\nu \mathcal{J}_{\mu\nu} + \mathcal{O}(q) \\ &= f(k_i, \varepsilon_i)(k_a + q)^2 + \mathcal{O}(q) \\ &= 2 f(k_i, \varepsilon_i) q\cdot k_a + \mathcal{O}(q) \\ &= \mathcal{O}(q)\,. \end{aligned} \quad (6.7)$$

Therefore, as far as the correction to the amplitude is concerned, we can effectively view $\mathcal{D}^{\mu\nu,\text{fact,s}}$ as being of order q^2. We then finally have

$$\mathcal{D}^{\mu\nu,\text{fact,s}} \frac{i}{2q\cdot k_a} \mathcal{J}_{\mu\nu} = \mathcal{O}(q)\,. \quad (6.8)$$

After including the $1/q$ from the intermediate propagator, we find the potential correction to the soft operator is of $\mathcal{O}(q)$ and therefore does not modify the first subleading soft behavior.

Unfortunately, for the second subleading soft behavior we lose control, in much the same way that we did for the first subleading behavior of gauge theory. Indeed, nontrivial contributions are found in explicit examples [21, 22].

As in the QCD case (6.1), we expect the cases with other particles circulating in the loop to be similar and that factorizing contributions not linked to infrared singularities should appear starting only at the second subleading order in the soft expansion. In addition, the explicit gravity examples studied in Refs. [21, 22] are exactly in line with this expectation. We leave a discussion of cases with infrared singularities to future work.

VII. CONCLUSIONS

In this paper, we extended Low's proof of the universality of subleading behavior of photons to nonabelian gauge theory and to gravity. In particular, we showed that in gauge theory, on-shell gauge invariance can be used to fully determine the first subleading soft-gluon behavior at tree level. In gravity, the first two subleading terms in the soft expansion found in Ref. [2] can also be fully determined from on-shell gauge invariance. Our discussion is similar to the ones given by Low [12] for photons and by Jackiw [18] for gravitons coupled to a scalar at four points. We focused mainly on n-gluon and n-graviton amplitudes but also discussed simpler cases with scalars.

A motivation for studying soft-graviton theorems is to understand their relation to the extended BMS symmetry. It will, of course, be very important to understand how BMS symmetry relates to the proof of soft properties in n-graviton amplitudes given here.

Unlike the case of photons, for gluons there are loop corrections to the soft operators starting at leading order. In gauge theory, leading-order corrections are linked to infrared singularities, while subleading-order corrections can also arise from contributions not linked to infrared singularities. Gravity also has loop corrections but not at leading order. In this paper, we proved that for the case of a scalar circulating in the loop, there is no modification to the soft behavior of graviton amplitudes until the second subleading order. We expect this to hold in general for contributions not linked to infrared singularities. On the other hand, graviton loop contributions that are infrared divergent give corrections to the soft operators starting at the first subleading order [21], using the standard definition of dimensional regularization. Since infrared singularities are well-understood, we do not expect this to be too disruptive for studying the consequences of extended BMS symmetry at loop level. We will describe loop level in more detail elsewhere.

Acknowledgments

We thank Yu-tin Huang, Henrik Johansson, Daniele Marmiroli, Raffaele Marotta, Jan Plefka, Radu Roiban and Andrew Strominger for helpful discussions. This work was supported by the US Department of Energy under Award Number DE-SC0009937. We also gratefully acknowledge Mani Bhaumik for his generous support.

---

[1] A. Strominger, arXiv:1312.2229 [hep-th];
T. He, V. Lysov, P. Mitra and A. Strominger, arXiv:1401.7026 [hep-th];
D. Kapec, V. Lysov, S. Pasterski and A. Strominger, arXiv:1406.3312 [hep-th].

[2] F. Cachazo and A. Strominger, arXiv:1404.4091 [hep-th].

[3] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, Proc. Roy. Soc. Lond. A **269**, 21 (1962);
R. K. Sachs, Proc. Roy. Soc. Lond. A **270**, 103 (1962).

[4] G. Barnich and C. Troessaert, Phys. Rev. Lett. **105**, 111103 (2010) [arXiv:0909.2617 [gr-qc]];
G. Barnich and C. Troessaert, JHEP **1112**, 105 (2011) [arXiv:1106.0213 [hep-th]];
G. Barnich and C. Troessaert, JHEP **1311**, 003 (2013) [arXiv:1309.0794 [hep-th]].

[5] E. Casali, arXiv:1404.5551 [hep-th].

[6] B. U. W. Schwab and A. Volovich, arXiv:1404.7749 [hep-th];
N. Afkhami-Jeddi, arXiv:1405.3533 [hep-th].

[7] A. J. Larkoski, arXiv:1405.2346 [hep-th].

[8] M. Bianchi, S. He, Y.-t. Huang and C. Wen, arXiv:1406.5155 [hep-th].

[9] T. Adamo, E. Casali and D. Skinner, arXiv:1405.5122 [hep-th];
Y. Geyer, A. E. Lipstein and L. Mason, arXiv:1406.1462 [hep-th].

[10] B. U. W. Schwab, arXiv:1406.4172 [hep-th].

[11] F. E. Low, Phys. Rev. **96**, 1428 (1954);
M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954);

S. Saito, Phys. Rev. **184**, 1894 (1969).

[12] F. E. Low, Phys. Rev. **110**, 974 (1958).

[13] S. Weinberg, Phys. Rev. **135**, B1049 (1964);
S. Weinberg, Phys. Rev. **140**, B516 (1965).

[14] T. H. Burnett and N. M. Kroll, Phys. Rev. Lett. **20**, 86 (1968);
J. S. Bell and R. Van Royen, Nuovo Cim. A **60**, 62 (1969);
V. Del Duca, Nucl. Phys. B **345**, 369 (1990).

[15] V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti, *"Currents in hadron physics"*, North-Holland, Amsterdam 1974, see Chapter 3.

[16] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B **546**, 423 (1999) [hep-th/9811140].

[17] D. J. Gross and R. Jackiw, Phys. Rev. **166**, 1287 (1968).

[18] R. Jackiw, Phys. Rev. **168**, 1623 (1968).

[19] E. Laenen, G. Stavenga and C. D. White, JHEP **0903**, 054 (2009) [arXiv:0811.2067 [hep-ph]];
E. Laenen, L. Magnea, G. Stavenga and C. D. White, JHEP **1101**, 141 (2011) [arXiv:1010.1860 [hep-ph]].

[20] C. D. White, JHEP **1105**, 060 (2011) [arXiv:1103.2981 [hep-th]].

[21] Z. Bern, S. Davies and J. Nohle, arXiv:1405.1015 [hep-th].

[22] S. He, Y.-t. Huang and C. Wen, arXiv:1405.1410 [hep-th].

[23] Z. Bern, V. Del Duca and C. R. Schmidt, Phys. Lett. B **445**, 168 (1998) [hep-ph/9810409];
Z. Bern, V. Del Duca, W. B. Kilgore and C. R. Schmidt, Phys. Rev. D **60**, 116001 (1999) [hep-ph/9903516].

[24] D. A. Kosower and P. Uwer, Nucl. Phys. B **563**, 477 (1999) [hep-ph/9903515];
D. A. Kosower, Phys. Rev. Lett. **91**, 061602 (2003) [hep-ph/0301069].

[25] F. Cachazo and E. Y. Yuan, arXiv:1405.3413 [hep-th].

[26] S. G. Naculich and H. J. Schnitzer, JHEP **1105**, 087 (2011) [arXiv:1101.1524 [hep-th]];
R. Akhoury, R. Saotome and G. Sterman, Phys. Rev. D **84**, 104040 (2011) [arXiv:1109.0270 [hep-th]].

[27] J. Broedel, M. de Leeuw, J. Plefka, and M. Rosso, arXiv:1406.6574 [hep-th].

[28] L. J. Dixon, in Boulder 1995, QCD and beyond, p. 539-582 [hep-ph/9601359].

[29] S. Weinberg, "The Quantum Theory of Fields. Vol. 1: Foundations," Cambridge Univ. Press

(1995).

[30] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B **269**, 1 (1986).

[31] Z. Bern and G. Chalmers, Nucl. Phys. B **447**, 465 (1995) [hep-ph/9503236].

[32] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **71**, 105013 (2005) [hep-th/0501240];
Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **72**, 125003 (2005) [hep-ph/0505055];
Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **73**, 065013 (2006) [hep-ph/0507005].

[33] D. Vaman and Y. -P. Yao, JHEP **1011**, 028 (2010) [arXiv:1007.3475 [hep-th]];
S. D. Alston, D. C. Dunbar and W. B. Perkins, Phys. Rev. D **86**, 085022 (2012) [arXiv:1208.0190 [hep-th]].

[34] J. C. Collins and G. F. Sterman, Nucl. Phys. B **185**, 172 (1981);
J. C. Collins, D. E. Soper and G. F. Sterman, Nucl. Phys. B **308**, 833 (1988).