Calabi's diastasis as interface entropy

Constantin P. Bachas, Ilka Brunner, Michael R. Douglas, and Leonardo Rastelli

Calabi's diastasis as interface entropy

Constantin P. Bachas,[a,b] **Ilka Brunner,**[b,c] **Michael R. Douglas,**[d,e] **Leonardo Rastelli** [e,f]

[a] *Laboratoire de Physique Théorique de l'École Normale Supérieure,*[1]
24 rue Lhomond, 75231 Paris cedex, France

[b] *Arnold Sommerfeld Center, Ludwig-Maximilians-Universität,*
Theresienstr. 37, München 80333, Germany

[c] *Excellence Cluster Universe, Technische Universität München,*
Boltzmannstr. 2, Garching 85748, Germany

[d] *Simons Center for Geometry and Physics, Stony Brook University,*
Stony Brook, NY 11794, USA

[e] *C.N. Yang Institute for Theoretical Physics, Stony Brook University,*
Stony Brook, NY 11794-3840, USA

[f] *Institute for Advanced Study, Einstein Dr., Princeton, NJ 08540, USA*

E-mail: bachas@lpt.ens.fr, ilka.brunner@physik.uni-muenchen.de,
mdouglas@scgp.stonybrook.edu, leonardo.rastelli@stonybrook.edu

ABSTRACT: We show that the entropy of certain conformal interfaces between $N = (2,2)$ sigma models that belong to the same moduli space, has a natural geometric interpretation in the large volume limit as Calabi's diastasis function. This is an extension of the well-known relation between the quantum Kähler potential and the overlap of canonical Ramond-Ramond ground states in $N = (2,2)$ models.

---

[1]Unité mixte (UMR 8549) du CNRS et de l'ENS, Paris.

1 Introduction

The purpose of the present note is to establish a geometric formula for the entropy of certain superconformal interfaces between $N = (2, 2)$ superconformal sigma models. As is well known, in the large volume limit the target spaces of such sigma models are Calabi-Yau manifolds. The interfaces of interest separate theories with the same Kähler modulus but different complex structure, or vice versa, and they reduce to the trivial interface when the moduli of the two theories coincide. Our main result is that for such interfaces

$$2 \log g = K(t, \bar{t}) + K(t', \bar{t}') - K(t, \bar{t}') - K(t', \bar{t}), \tag{1.1}$$

where g is the universal degeneracy [1] of the interface [1] , t and t' are the moduli of the theories on either side of the interface, and K is the Kähler function on moduli space.

The right-hand side of the above equation is a known quantity in Kähler geometry; it is the so-called Calabi diastatic function [2]. It can be defined on any Kähler manifold (this requires showing that the analytic continuation of $K(z, \bar{z})$ to independent z and $\bar{z}$ makes sense, which is done in [2]). A nice feature of the combination (1.1) is that the Kähler-Weyl dependence of $K(z, \bar{z})$ cancels out. Furthermore, it agrees with the geodesic distance at small separations, but has the property that it is preserved under restriction to a submanifold.

Equation (1.1) gives a world-sheet definition of the diastatic function that can be used away from the geometric, large volume limit. It is a natural extension of the well-known formula that relates the (quantum) Kähler potential to the norm of a canonical Ramond-Ramond (RR) ground state in the sigma model, [3, 4]

$$_{\rm RR}\langle \bar{0}|0 \rangle_{\rm RR} = e^{-K(t, \bar{t})} . \tag{1.2}$$

Recently, [5, 6] the norm of this RR ground state has been related to the partition function of $N = (2, 2)$ gauge theories on the (squashed) two-sphere, which can be computed exactly using the technique of localization [7, 8]. This is a new way to compute world-sheet instanton corrections to the Kähler potential, and to extract Gromov-Witten invariants, without the need to identify and solve a classical geometric mirror problem.

Similarly, through equation (1.1) one may relate quantum corrections to Calabi's diastasis function to the partition function on the (squashed) two-sphere in the presence of certain $N = 2$ supersymmetric domain walls. Localization techniques for the computation of the latter have been developed recently in [9, 10]. They could be used to extract the relevant "open-string" Gromov-Witten invariants, which are notoriously hard to compute by other means.

We were actually led to this formula while studying the following broader question: how to define alternative metric(s) on spaces of conformal field theories [11] ? One promising proposal

---

[1]When the interface is viewed as an operator between the initial and deformed theory, then g is the image of the identity projected to the identity of the other theory, see Section 4 for more details.

[12] is to define the distance between CFT_1 and CFT_2 as

$$d(1,2) = \min_{\mathcal{S}} \sqrt{\log g} \, , \tag{1.3}$$

where $\mathcal{S}$ is an appropriate set of interfaces which separate the two conformal theories. An appealing feature of such a definition is that d reduces to the Zamolodchikov metric whenever CFT_1 can be obtained from CFT_2 by a small variation of continuous moduli. In the special case of $N = (2,2)$ models this follows immediately from (1.1), but the proof is more general [12] and does not require the use of supersymmetry. Another appealing feature of the above definition is that CFT_1 and CFT_2 need not belong to the same moduli space, or even have the same central charge. In particular, equation (1.1) extends the definition of the diastatic function to pairs of sigma models separated by an $N = 2$ interface, even when these sigma models belong to different moduli spaces.

Despite its intuitive appeal, the proposal (1.3) does not automatically obey the axioms for a proper distance. In particular, conformal interfaces may have negative entropy, and Calabi's diastasis need not always obey, as we will show, the triangle inequality. Ideas for bypassing these obstructions, by restricting the set $\mathcal{S}$ of allowed interfaces, will be discussed elsewhere [12]. Here we concentrate on proving formula (1.1), which is interesting in its own right as a new entry in the "worldsheet versus target-space geometry" dictionary.

The paper is organized as follows: in section 2 we prove formula (1.1) in the simplest case of $N = (2,2)$ sigma model whose target space is the two-dimensional torus. We give both an algebraic and a geometric derivation of g for any moduli deformations, and show that it reduces to (1.1) when either the Kähler or the complex structure are held fixed. In section 3 we extend the geometric derivation to arbitrary Calabi-Yau n-folds with $n > 1$. This uses the well-known "folding trick", to map the interfaces to branes in a product Calabi-Yau manifold. Section 4 presents an algebraic derivation of this result, which only relies on the $N = 2$ supersymmetry of the interface. This shows that interface entropy provides a natural extension of Calabi's diastasis in the non-geometric regime, and even when the two worldsheet theories do not belong to the same moduli space. Finally, in section 5 we show that Calabi's diastasis function does not obey the triangle inequality in spaces with positive sectional curvature, and may hence fail one of the key tests for a proper distance. We conclude the section with some remarks.

2   The two-dimensional torus CFT

The simplest Calabi-Yau manifold is the two-dimensional torus, $T^2 = \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$. As a warm up, we shall first derive the formula (1.1) in this special case.

We parametrize the torus by $(x,y) \in (0,1] \times (0,1]$. The Kähler and complex structure moduli, $\tau = \tau_1 + i\tau_2$ and $\rho = \rho_1 + i\rho_2$, are related to the flat metric, G, and antisymmetric

Neveu-Schwarz field, B, as follows:[2]

$$G = \frac{\tau_2}{\rho_2} \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & |\rho|^2 \end{pmatrix} \quad , \qquad B = \begin{pmatrix} 0 & \tau_1 \\ -\tau_1 & 0 \end{pmatrix} . \tag{2.1}$$

In terms of the complex coordinate $z = x + \rho y$ one has

$$ds^2 = \frac{\tau_2}{\rho_2} \, dz d\bar{z} \iff k = \frac{i}{2} \frac{\tau_2}{\rho_2} \, dz \wedge d\bar{z} = \tau_2 \, dx \wedge dy \ , \tag{2.2}$$

where k is the real Kähler form. It is complexified by the addition of the Neveu-Schwarz 2-form, $\omega = B + ik$ with $B = \tau_1 \, dx \wedge dy$. The holomorphic $(1,0)$ form is $\Omega = dz$, up to an irrelevant multiplicative constant.

The moduli space of $N = (2,2)$ superconformal theories with target space T^2 consists of two copies of the symmetric coset $\mathcal{M} = SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$. One copy parametrizes the complex structure modulus, and the other the Kähler modulus. T-duality exchanges τ and ρ, so that the full moduli space is $(\mathcal{M} \times \mathcal{M})/\mathbb{Z}_2$. The metric on this moduli space derives from the Kähler potential

$$K = K_{\mathrm{K}}(\tau, \bar{\tau}) + K_{\mathrm{C}}(\rho, \bar{\rho}) \ , \tag{2.3}$$

where K_{K} and K_{C} are given by

$$K_{\mathrm{K}} = -\log \left(\int_M k \right) = -\log \tau_2 \ , \qquad K_{\mathrm{C}} = -\log \left(\int_M \frac{i}{2} \Omega \wedge \bar{\Omega} \right) = -\log \rho_2 \ . \tag{2.4}$$

The $SL(2,\mathbb{R})$ transformations of τ and ρ act as Kähler-Weyl transformations on the Kähler potential, $K \to K + f + \bar{f}$ where f is a holomorphic function of τ and ρ. Such transformations leave the metric invariant, as expected.

2.1 Algebraic derivation

Consider now two conformal theories with moduli (τ, ρ) and (τ', ρ'). We will be interested in a special conformal interface between these two theories – the "deformed identity" interface introduced and discussed in [13–16]. This is the deformation of the trivial defect (the "no interface") as the moduli of the second CFT vary continuously from (τ, ρ) to (τ', ρ').

In general, such an interface could depend on the specific deformation path, as well as on (open-string) moduli. However, we expect the deformed identity to only depend on the homotopy class of the deformation path, and its g-function to be independent of open-string moduli. In the case at hand this will be manifest from the explicit expression for g. We will argue below that this holds more generally. Thus $\log g$ is a well-defined function of pairs of points on the covering space of moduli space, i.e. on two copies of the upper-half complex plane.

---

[2]Later, we will refer to the complex structure moduli collectively as t, and to the Kähler moduli as u. But for the 2-torus we use the more canonical notation, ρ and τ.

Both the trivial interface, and its deformations, preserve the $U(1)^4$ symmetry of the toroidal theory. Such symmetry-preserving interfaces were analyzed recently in [16], where it was shown that their g-function can be written as

$$\log g = \frac{1}{2}\log\det(\Lambda_{22}) \,, \qquad \text{with} \qquad \Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \tag{2.5}$$

the $SO(2,2)$ matrix that relates the even self-dual Lorentzian charge lattices of the two theories. Written in 2×2-block form, this matrix obeys $\Lambda^t \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \Lambda = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$.

One can give an explicit formula for the matrix Λ corresponding to the deformed identity by using the expression for the charge lattice of toroidal models in terms of the metric G and Kalb-Ramond field B [17]. The answer is[3]

$$\Lambda = V'V^{-1} \qquad \text{with} \qquad V = \begin{pmatrix} \hat{e} & \hat{e}(B+G) \\ \hat{e} & \hat{e}(B-G) \end{pmatrix} \,, \tag{2.6}$$

where $\hat{e}$ is the vielbein that satisfies $2\,\hat{e}^t\,\hat{e} = G^{-1}$, and there is a similar expression for V'. Inserting these formulae and (2.1) in equation (2.5) leads to the following expression for the g-function of the deformed identity:

$$g_{\text{d.i.}} = \left[\frac{(\tau - \bar{\tau}')(\tau' - \bar{\tau})}{(\tau - \bar{\tau})(\tau' - \bar{\tau}')} + \frac{(\rho - \bar{\rho}')(\rho' - \bar{\rho})}{(\rho - \bar{\rho})(\rho' - \bar{\rho}')} - 1 \right]^{1/2} \,. \tag{2.7}$$

As anticipated earlier, this is a well-defined function of τ', ρ' as these range over the (simply-connected) covering space of the CFT moduli space. Note that $g_{\text{d.i.}}$ is invariant under a simultaneous $SL(2,\mathbb{Z})$ transformation of the primed and unprimed moduli, but not under a transformation of only one of the two CFTs.

2.2 Geometric derivation

The expression (2.7) can be derived more directly, in a way that will generalize to any large volume Calabi-Yau n-fold. The starting point is the folding trick, which maps an interface between σ-models with target spaces M and M' to a boundary of the σ-model with target space $M \times M'$ [13, 18, 19]. When the two σ-models are identical there exists a trivial interface across which all the fields are continuous – the "no interface". This is mapped after folding to the diagonally-embedded middle-dimensional D-brane: $M \to M \times M$ given by $x \to (x,x)$. Now as one of the σ-models is deformed, this diagonal brane is also deformed to a new brane, Δ_f, which we describe (at least locally) as the graph of a function f from M to M'. Put differently, Δ_f is given by the embedding $x \to (x, f(x)) \in M \times M'$.

We may determine f by minimizing the g-function of the brane – this is the condition of conformal invariance. In the large volume limit, the g-function is the appropriately normalized

---

[3]More general $U(1)^4$ symmetric interfaces are given by $\Lambda = V'\hat{\Lambda}V^{-1}$ where $\hat{\Lambda}$ is an element of $O(2,2,\mathbb{Q})$. These have $g = \sqrt{l.c.m. \times |\Lambda_{22}|}$ where $l.c.m.$ is the least common multiple of the matrix elements of $\hat{\Lambda}$ [16].

Dirac-Born-Infeld action (see e.g. [20]):

$$g \simeq \frac{\int_M \det^{1/2}(G - B + f^*G' + f^*B')}{\sqrt{2^d \ Vol(M) \ Vol(M')}} \ . \tag{2.8}$$

Here f^* denotes the pullback from M' to M, and we note that folding flips the sign of B thereby complex-conjugating the Kähler form of the folded σ-model. In the toroidal case at hand this amounts to trading (τ, ρ) for $(-\bar\tau, \rho)$. Note also the normalization of the DBI action by the volume factors in (2.8). This can be fixed by requiring that $g = 1$ for the trivial (or identity) interface. [4]

For toroidal theories, the identity defect is a diagonally-embedded planar D2-brane. As one of the two tori is deformed this D2-brane follows suit, i.e. it is still given by the planar diagonal embedding $x = x'$ and $y = y'$ where (x, y) and (x', y') are the canonically-normalized flat coordinates of the two tori. One thus finds

$$g_{\text{d.i.}} = \frac{1}{\sqrt{4\tau_2\tau_2'}} \det^{1/2}(G + G' - B + B') \ , \tag{2.9}$$

which after inserting (2.1) and doing some simple algebra leads to the result (2.7) for the g-function of the deformed identity. Note that the two, geometric and algebraic, derivations of g give the same result, because the DBI approximation (2.8) is in this case exact.

2.3 Supersymmetry and diastasis

Expression (2.7) simplifies considerably if $\rho = \rho'$, i.e. if one keeps the complex structure fixed and only deforms the Kähler modulus of the torus. The folded interface is in this case the holomorphic brane $z = z'$, and

$$2 \log g_{\text{d.i.}}\Big|_{\rho=\rho'} = -\log(\tau - \bar\tau) - \log(\tau' - \bar\tau') + \log(\tau - \bar\tau') + \log(\tau' - \bar\tau) \ , \tag{2.10}$$

which is precisely Calabi's diastasis function for the potential (2.4).

The same conclusion holds if one only deforms the complex structure, ρ, keeping the Kähler modulus, τ, fixed. The g function of such branes is given again by Calabi's diastasis,

$$2 \log g_{\text{d.i.}}\Big|_{\tau=\tau'} = -\log(\rho - \bar\rho) - \log(\rho' - \bar\rho') + \log(\rho - \bar\rho') + \log(\rho' - \bar\rho) \ . \tag{2.11}$$

This is of course expected by mirror symmetry. For later use, it is nevertheless interesting to understand how supersymmetry is preserved in this case.

To this end, we consider the 2-form $\bar\Omega \wedge \Omega' = d\bar z \wedge dz'$. The D-brane corresponding to the folded interface obeys trivially

$$\text{Im}(e^{i\theta} d\bar z \wedge dz')\Big|_{\Delta_f} = \text{Im}[e^{i\theta} (\rho' - \bar\rho)dx \wedge dy]\Big|_{\Delta_f} = 0 \ , \tag{2.12}$$

---

[4]In string theory, the g-function of a D-brane wrapping some dimensions of the compact space is the mass of the corresponding point-particle in the Einstein frame.

where $-\theta$ is the phase of the complex number $(\rho' - \bar{\rho})$. Furthermore, since $\tau = \tau'$, the restriction of the two Kähler forms on the D-brane is the same,

$$(k - k')\Big|_{\Delta_f} = 0 \ . \tag{2.13}$$

Finally the following two top-forms are equal to the volume form of the doubled torus, up to an irrelevant multiplicative constant:

$$(d\bar{z} \wedge dz') \wedge (dz \wedge d\bar{z}') = \mathcal{C}(k - k')^2 \ . \tag{2.14}$$

The set of conditions (2.12)-(2.14) define special lagrangian submanifolds, which preserve $N = 2$ supersymmetries in any Calabi-Yau space.

In conclusion, interfaces between theories which differ only in complex structure, or in Kähler form, preserve half of the bulk supersymmetries, and their entropy is the diastasis function. Note that if one varies both τ and ρ, then (2.7) is not any more related to the diastasis function.[5] Nevertheless, these two functions do coincide for small deformations at quadratic order. This is actually a general fact: $\sqrt{\log g}$ of the deformed-identity can be shown to reduce to the Zamolodchikov distance for *all* infinitesimal marginal deformations of a 2d conformal theory, whether they preserve supersymmetry or not [12].

3   Large volume Calabi-Yau σ-models

It is straightforward to extend the geometric arguments of the previous subsection to any Calabi-Yau sigma model in the large volume limit. The product of two Calabi-Yau n-folds, $M \times M'$, is also a Calabi-Yau manifold of complex dimension $2n$. Its Kähler form is $k + k'$, and its holomorphic $(2n, 0)$ form $\Omega \wedge \Omega'$. Like all Calabi-Yau manifolds, $M \times M'$ has two types of supersymmetric submanifolds [21–24] : the special Lagrangians (A-type), and the holomorphic submanifolds (B-type). As we will see, these correspond to interfaces between theories with the same complex, respectively Kähler, structures.

3.1   Kähler structure deformation

Consider the trivial interface between two identical σ-models, which after folding becomes the diagonal brane $M \to M \times M$ given by $x \to (x, x)$. This is a holomorphic brane since in complex coordinates we can write $z \to (z, z)$. Let us now deform one of the theories from M to M'. If M and M' have the same complex structure, the above brane will stay holomorphic and, in general, there will be no other nearby holomorphic branes.[6] The g-function of this

---

[5]The reader may here object that any planar brane in a four-torus is half-BPS, and this would continue to be true for the direct product of any two tori. The unbroken supersymmetries mix however in this case the fields of the two tensored CFTs, so they are not local symmetries after unfolding.

[6]If the moduli space is not simply connected, this brane will generally depend on the homotopy class of the path from M to M', and so will its g function. We have already seen an example of this phenomenon in the case of the torus, as $\tau \to \tau + 1$ (for one of the arguments) does not leave the diastasis invariant. Nevertheless, the diastasis *is* indeed invariant under small deformations of the path in the same homotopy class. This is because the holomorphic brane comes back to itself when transported around a homotopically trivial path.

deformed identity is proportional, in the large volume limit, to the Dirac-Born-Infeld action

$$g_{\text{hol}} \simeq \frac{2^{-n} | \int_M (-\bar{\omega} + \omega')^n |}{| \int_M k^n |^{1/2} | \int_M (k')^n |^{1/2}} \ , \tag{3.1}$$

where ω and ω' are the complexified Kähler forms of M and M', and we recall that folding transforms $\omega := B + ik$ to $-\bar{\omega}$. As explained in the previous section, the normalization factor can be fixed by requiring that $g_{\text{hol}} = 1$ for the identity, namely when $\omega = \omega'$.

Taking the logarithm of (3.1), and using the fact that $K(u, \bar{u}) \simeq -\log(\int_M k^n)$ gives precisely Calabi's diastasis

$$2 \log (g_{\text{hol}}) \simeq K(u, \bar{u}) + K(u', \bar{u}') - K(u, \bar{u}') - K(u', \bar{u}) \ , \tag{3.2}$$

where u denotes collectively the Kähler moduli, and we have defined the analytic extension of $K(u, \bar{u})$ to independent u and $\bar{u}$ as follows

$$K(u', \bar{u}) \simeq - \log \int_M [-\bar{\omega}(\bar{u}) + \omega(u')]^n + n \log 2 \ . \tag{3.3}$$

We expect this last formula to make sense in any open neighborhood of a generic point on the Kähler cone. That the analytic continuation of $K(u, \bar{u})$ makes sense (and is unique) is a crucial input in the original work of Calabi.

This proves the equation (1.1) for interfaces separating theories with the same complex structure but different Kähler forms, in the large volume limit.

3.2 Complex structure deformation

What is the the mirror statement to (3.2)? If M and M' have the same Kähler form but different complex structures, then the deformed identity interface does not correspond to a holomorphic submanifold. We will now show that it corresponds to a special Lagrangian (sLag) brane calibrated not by the holomorphic $(2n, 0)$ form, but by the appropriately-normalized mixed $2n$-form $\varphi = i^n \bar{\Omega} \wedge \Omega'$.

The existence of this extra calibrating form, in addition to the holomorphic volume form, is due to the fact that $M \times M'$ is "more special" as it has $SU(n)^2 \subset SU(2n)$ holonomy. One may in particular complex conjugate one of the two manifolds in the product, which gives also a new symplectic form $s = i(k - k')$, different from the standard Kähler form $k + k'$. Clearly, both φ and s are closed forms, and they satisfy, up to normalization, the top-form condition

$$2^{-2n} \varphi \wedge \bar{\varphi} = \frac{1}{(2n)!} s^{2n} = d(volume) \ . \tag{3.4}$$

They can thus be used to construct a new class of sLag submanifolds,[7] different from those constructed with the standard Kähler and holomorphic volume forms.

---

[7] This observation has been exploited, for instance, in the context of $N = (2, 0)$ σ-models in [25].

It is easy to see that the brane Δ_{id} corresponding to the trivial interface belongs to this new class. The necessary and sufficient conditions (see e.g. [26]),

$$s\Big|_{\Delta_{id}} = 0 \qquad \text{and} \qquad \text{Im}\varphi\Big|_{\Delta_{id}} = 0 \ , \tag{3.5}$$

are trivialy satisfied in this case. The sLag conditions can still be imposed if we deform the complex structure of M' without changing its Kähler form (so that $k = k'$). The Lagrangian requirement for the submanifold $\Delta_f \subset M \times M'$ now reads

$$k = f^*k \ , \tag{3.6}$$

which says that f preserves the Kähler form – it is a "symplectomorphism". Many such maps indeed exist, and they can be specified locally by a single function $F(\text{Re}z, \text{Re}z')$ which defines a canonical transformation. This function can then be determined by the (volume-minimizing) calibration condition

$$\text{Im}(e^{i\theta}\varphi)\Big|_{\Delta_f} = \text{Im}(e^{i\theta}i^n\bar{\Omega} \wedge f^*\Omega') = 0 \tag{3.7}$$

for some constant phase θ. This gives an equation for the function f, which can be always solved at least in a local patch. In the toroidal theory of the previous section, f is the map $(x', y') = (x, y)$ as the reader can easily verify.

It has been actually shown [27] that for compact manifolds the moduli space of a sLag submanifold has dimension equal to its first Betti number. By continuity this should be in our case the first Betti number of Δ_{id}, which is isomorphic to the Calabi-Yau manifold M. Since $b_1(M)$ vanishes for complex dimension $n > 1$, the sLag submanifold Δ_f is unique in all cases with the exception of the two-torus.[8]

The g-function of this sLag D-brane is given by its volume, which by the sLag condition is the integral of the calibrating form

$$g_{\text{sLag}} = \frac{|\int_M \bar{\Omega} \wedge f^*\Omega'|}{(|\int_M \Omega \wedge \bar{\Omega}| \, |\int_M \Omega' \wedge \bar{\Omega}'|)^{1/2}} \ . \tag{3.8}$$

The normalization was once again fixed so as to ensure that $g = 1$ for the trivial interface. Notice that we do not need here the Dirac-Born-Infeld action, because the B-field on the D-brane is zero. This is because we have assumed that the two σ-models have the same complexified Kähler moduli, i.e. $k = k'$ and $B = B'$. Since folding flips the sign of B in one of the models, the net field on the sLag brane vanishes.

Expression (3.8) is again suggestive of an exponentiated diastasis function for the Kähler potential in complex structure moduli space. We denote complex moduli by t. The analytic extension of $K(t, \bar{t}) = -\log \int_M \Omega(t) \wedge \bar{\Omega}(\bar{t})$ suggested by (3.8) is

$$K(t', \bar{t}) = -\log \int_M f^*\Omega(t') \wedge \bar{\Omega}(\bar{t}) \ , \tag{3.9}$$

---

[8]For the 2-torus the sLag D2-brane has $b_1 = 2$. Its two geometric moduli, which determine the brane's position on the doubled torus, get complexified by the Wilson lines along the two non-contractible cycles.

with f defined by the calibration condition (3.7). Notice that this condition only depends on t' and $\bar{t}$, so the function f does not introduce any implicit dependence on the conjugate variables, $\bar{t}'$ and t, as claimed. [9] With this analytic extension one has

$$2\log\left(g_{\mathrm{sLag}}\right) = K(t,\bar{t}) + K(t',\bar{t}') - K(t,\bar{t}') - K(t',\bar{t}) ,\qquad(3.10)$$

which proves the advertised identity (1.1) for interfaces between theories with the same Kähler structure but different complex structures.

4 Superconformal $N = (2,2)$ σ-models

We turn next to an algebraic derivation of the basic formula (1.1), which only relies on the $N = 2$ superconformal symmetry of the interface. To this end, we view an interface between two theories, CFT and CFT$'$, as a formal operator mapping the states on the circle of CFT to those of CFT$'$. This has been explained for instance in [13, 14]. Folding converts this operator to a boundary state of the tensor-product theory $\overline{\mathrm{CFT}} \otimes \mathrm{CFT}'$, where here the bar denotes the parity-conjugate theory. We use the same symbol, Δ_f, for the operator, for the corresponding brane, and for its boundary state. Our discussion parallels the analysis of $N = 2$ superconformal boundaries by Ooguri et al [23], and we will therefore adopt the conventions of these authors.

Every interface operator contains a term $g\,|0\rangle\langle 0'| + \cdots$, where $|0\rangle$ is the normalized ground state of theory CFT, and $|0'\rangle$ the normalized ground state of theory CFT$'$. The coefficient of this term is, by definition, the g-function of the interface. Since we will be working with non-normalized ground states, we write more generally

$$g = \frac{\langle 0|q^H\,\Delta_f\,q^{H'}|0'\rangle}{\langle 0|q^H|0\rangle^{\frac{1}{2}}\,\langle 0'|q^{H'}|0'\rangle^{\frac{1}{2}}} ,\qquad(4.1)$$

where H and H' are the Hamiltonians in the closed-string channel. This expression does not depend on the evolution time $\log q$, so one can take $q \to 0$ and replace the ground states by any other states with non-vanishing vacuum components.

4.1 Type-A and type-B boundaries

We are interested in interfaces that preserve a $N = 2$ superconformal algebra, and which can be continuously deformed to the identity operator. Since folding converts these operators to boundary states, we first recall some well-known facts about supersymmetry-preserving boundaries in $N = (2,2)$ superconformal theories.

---

[9]One may of course complex-conjugate (3.7) and express the calibrating map f, equivalently, in terms only of the independent variables $\bar{t}'$ and t.

Supersymmetric branes come in two varieties, type-A and type-B. The boundary states of type-A branes obey the following conditions: [10]

$$(G_L^+ - iG_R^-)\,|A\rangle\!\rangle = (G_L^- - iG_R^+)\,|A\rangle\!\rangle = (J_L - J_R)\,|A\rangle\!\rangle = 0 \ ,$$

$$\text{and} \qquad e^{i\alpha\phi}\,|A\rangle\!\rangle = e^{i\alpha\phi_0}\,|A\rangle\!\rangle \ , \tag{4.2}$$

where $G_L^\pm$ and $G_R^\pm$ are the complex left- and right-moving supercurrents, J_L and J_R are the R-symmetry currents, and $\phi = \int(J_L - J_R) = \phi_L + \phi_R$. Likewise, the B-type boundaries obey the conditions

$$(G_L^+ - iG_R^+)\,|B\rangle\!\rangle = (G_L^- - iG_R^-)\,|B\rangle\!\rangle = (J_L + J_R)\,|B\rangle\!\rangle = 0 \ ,$$

$$\text{and} \qquad e^{i\alpha\tilde{\phi}}\,|B\rangle\!\rangle = e^{i\alpha\tilde{\phi}_0}\,|B\rangle\!\rangle \ , \tag{4.3}$$

where $\tilde{\phi} = \int(J_L + J_R) = \phi_L - \phi_R$. The above conditions imply that type-A branes couple only to the (c, c) and (a, a) fields, while type-B branes couple to (c, a) and (a, c) fields. Here c and a denote chiral and antichiral primaries of the $N = 2$ superconformal algebra, (c, c) is a field that is chiral with respect to both the left and the right algebra etc.

Another consequence of the above conditions, which will be important for our purposes here, has to do with spectral flow. The two spectral-flow operators are $e^{ic\phi/6}$ and $e^{ic\tilde{\phi}/6}$, with c the central charge of the CFT. It follows then from (4.2) and (4.3) that

$$\langle\!\langle A|0\rangle = e^{-ic\phi_0/6}\,\langle\!\langle A|0\rangle_{\mathrm{RR}} \qquad \text{and} \qquad \langle\!\langle B|\tilde{0}\rangle = e^{-ic\tilde{\phi}_0/6}\,\langle\!\langle B|\tilde{0}\rangle_{\mathrm{RR}} \ , \tag{4.4}$$

where $|0\rangle_{\mathrm{RR}}$ and $|\widetilde{0}\rangle_{\mathrm{RR}}$ are the canonical Ramond-Ramond ground states, obtained from the Neveu-Schwarz vacuum by spectral flow. [11]

These statements have a nice geometric interpretation in the large volume limit [23]. The boundary states $|A\rangle\!\rangle$ correspond to D-branes wrapping Lagrangian submanifolds, γ, of the Calabi-Yau n-fold, while the states $|B\rangle\!\rangle$ correspond to (p-dimensional) holomorphic submanifolds $\tilde{\gamma}$. The overlaps with the NS vacuum are the g-factors of the corresponding D-branes, whereas the overlaps with the appropriately-normalized, canonical RR ground states give the D-brane charges

$$\langle\!\langle A|0\rangle_{\mathrm{RR}} = \int_\gamma \Omega \qquad \text{and} \qquad \langle\!\langle B|\tilde{0}\rangle_{\mathrm{RR}} = \int_{\tilde{\gamma}} \frac{1}{p!}\,\omega^p \ . \tag{4.5}$$

---

[10]These are the conditions in the closed-string channel, in which the reality conditions for fermions involve the exchange of left and right movers. In the open-string channel one has $J_L = -J_R$ and $G_L^+ = G_R^-$ for the type-A boundaries, $J_L = J_R$ and $G_L^+ = G_R^+$ for the type-B boundaries.

[11]Readers not familiar with $N = 2$ should compare these facts to the analogous statements for boundary states of a free-boson theory. There, the Dirichlet condition couples only to the closed-string momentum modes, and the coupling has equal strength for all states that differ only in momentum. This is necessary in order to produce the localizing δ-function of the Dirichlet brane. Likewise, a Neumann condition couples only to winding modes, and the coupling depends on winding number through a phase factor. In the context of $N = (2, 2)$ theories, type-A and type-B branes are, respectively Dirichlet and Neumann conditions for the field ϕ, and the two spectral-flow operators inject, respectively, momentum and winding.

In the context of string theory compactified on a Calabi-Yau 3-fold, the equations (4.4) are the BPS mass formulae for the corresponding supersymmetric black holes.

The above disk amplitudes have also an interpretation in terms of topological twists. To compute the first amplitude for example, one puts A-type boundary conditions at the end of a semi-infinite cigar, where the curved region at the tip of the cigar is B-twisted. Due to the topological twist, the identity operator sitting at the end point of the cigar becomes a RR-ground state at the end of the cigar. This is the canonical RR ground state, corresponding to the holomorphic 3-form on a Calabi-Yau manifold.

4.2 Diastasis as entropy of A-type interfaces

The discussion of supersymmetric boundaries can be adapted easily to N=2 superconformal interfaces [28]. These are of A-type or of B-type depending on which combination of superconformal generators they intertwine. Explicitly, in terms of the interface operators one has

$$[G_L^+ - iG_R^-,\, \Delta_{f,A}] = [G_L^- - iG_R^+,\, \Delta_{f,A}] = [J_L - J_R,\, \Delta_{f,A}] = 0 \ ,$$

$$[G_L^+ - iG_R^+,\, \Delta_{f,B}] = [G_L^- - iG_R^-,\, \Delta_{f,B}] = [J_L + J_R,\, \Delta_{f,B}] = 0 \ . \tag{4.6}$$

Likewise, the intertwining of spectral-flow operators reads

$$e^{i\alpha\phi}\, \Delta_{f,A}\, e^{-i\alpha\phi} = e^{i\alpha\phi_0}\, \Delta_{f,A} \ , \quad \text{or} \quad e^{i\alpha\tilde\phi}\, \Delta_{f,B}\, e^{-i\alpha\tilde\phi} = e^{i\alpha\tilde\phi_0}\, \Delta_{f,B} \ , \tag{4.7}$$

where ϕ and $\tilde\phi$ are the fields defined earlier, and ϕ_0, $\tilde\phi_0$ are constant phases. These equations are the unfolded versions of equations (4.2) and (4.3) for the tensor-product theory. To be precise, since folding converts the interfaces to boundaries of the product theory $\overline{\text{CFT}} \otimes \text{CFT}'$, the generators that enter in (4.2) and (4.3) are the sums of the generators for the individual theories, but with theory CFT parity-transformed. [12]

Combining equations (4.1) and (4.7) leads to the following alternative expression for the square of the g function of type-A interfaces:

$$g_A^2 \;=\; \frac{{}_{\text{RR}}\langle\bar0|q^H\, \Delta_{f,A}\, q^{H'}|0'\rangle_{\text{RR}} \times\; {}_{\text{RR}}\langle\bar0'|q^{H'}\, \Delta_{f,A}^\dagger\, q^H|0\rangle_{\text{RR}}}{{}_{\text{RR}}\langle\bar0|q^H|0\rangle_{\text{RR}} \times\; {}_{\text{RR}}\langle\bar0'|q^{H'}|0'\rangle_{\text{RR}}} \ , \tag{4.8}$$

where $|0\rangle_{\text{RR}}$ is the canonical Ramond-Ramond ground state obtained by acting with the spectral-flow operator $e^{ic\phi/6}$ on the Neveu-Schwarz vacuum, and $|\bar0\rangle_{\text{RR}}$ is the conjugate ground state. There is a similar expression for type-B interfaces, with $|0\rangle_{\text{RR}}$ replaced by the twisted canonical Ramond ground state, $|\widetilde0\rangle_{\text{RR}}$, obtained with the spectral-flow operator $e^{ic\tilde\phi/6}$. Taking

---

[12]The tensor-product theory has also superconformal branes that do not involve this parity operation. From the geometric point of view, this is because the target manifold is more special than Calabi-Yau. The corresponding interfaces are not continuously-connected to the identity, and will not concern us here.

the logarithm of (4.8) gives an expression reminiscent of the diastasis function of the previous section,

$$2\log(g_A) = K(t,\bar{t}) + K(t',\bar{t}') - \log(\,_{\rm RR}\langle\bar{0}|\Delta_{f,A}|0'\rangle_{\rm RR}) - \log(\,_{\rm RR}\langle\bar{0}'|\Delta_{f,A}^\dagger|0\rangle_{\rm RR})\,. \qquad (4.9)$$

We used here the fact [4] (see also [3]) that the canonical Ramond ground state, which has holomorphic dependence on the complex structure moduli, has norm

$$\log(\,_{\rm RR}\langle\bar{0}|0\rangle_{\rm RR}) = -K(t,\bar{t})\,. \qquad (4.10)$$

There are analogous expressions for type-B interfaces with $K(t,\bar{t})$ replaced by $K(u,\bar{u})$, the Kähler potential on the moduli space of Kähler structures.

To show that (4.9) is Calabi's diastasis we interpret the expression $\log(\,_{\rm RR}\langle\bar{0}|\Delta_{f,A}|0'\rangle_{\rm RR})$ at large volume. For this, we note that $|0\rangle_{\rm RR}$ becomes the holomorphic three-form for the geometry corresponding to the unprimed theory, whereas $_{\rm RR}\langle\bar{0}|$ corresponds to the anti-holomorphic three-form for the primed theory. The expression then becomes, in the folded picture, the RR-charge of an A-type brane with respect to the canonical RR ground state. The relevant A-brane is the deformed diagonal brane described before. Hence, we can employ (4.5) to conclude that at large volume

$$_{\rm RR}\langle\bar{0}|\Delta_{f,A}|0'\rangle_{\rm RR} = \int_M \bar{\Omega}' \wedge f^*\Omega\,. \qquad (4.11)$$

This is precisely the analytic continuation of the Kähler potential appearing in Calabi's diastasis function. Note that relation (4.11) was computed at large volume. However, the relevant disk one-point functions do not depend on Kähler moduli, hence the above relation can be extrapolated to all length scales.

4.3 Quantum diastasis function

We will now prove that for any B-twistable theory the left hand side of (4.11) depends holomorphically on t' and antiholomorphically on t, in some open region of moduli space. The proof does not rely on a geometrical interpretation, and also goes through practically unchanged for B-type interfaces in A-twistable theories. This shows that

$$_{\rm RR}\langle\bar{0}|\Delta_{f,A}|0'\rangle_{\rm RR} \equiv e^{-K(t',\bar{t})}\,, \quad \text{and} \quad _{\rm RR}\langle\bar{\bar{0}}|\Delta_{f,B}|\tilde{0}'\rangle_{\rm RR} \equiv e^{-K(u',\bar{u})} \qquad (4.12)$$

define the analytic extensions of the quantum Kähler potentials to independent holomorphic and anti-holomorphic moduli.

To establish this holomorphicity property, recall that N=2 supersymmetric conformal theories can be marginally perturbed by suitable descendants of fields from the chiral (c,c) sector, or the twisted chiral (a,c) sector. The two are interchanged by mirror symmetry, and we focus here on chiral perturbations, which geometrically correspond to complex structure deformations. On the level of the action, and on a worldsheet with boundaries, the perturbation reads

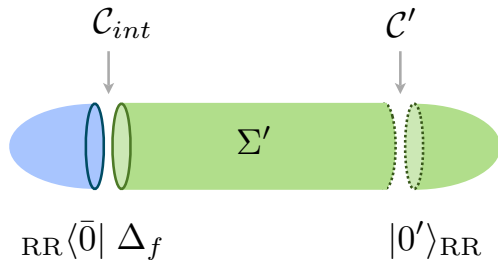$$S \to S + \Delta S(t_i) + \overline{\Delta S}(\bar{t}_i), \qquad (4.13)$$

Figure 1. Schematic representation of the amplitude (4.12) that defines the analytic extension of the Kähler potential to independent holomorphic and antiholomorphic moduli.

where

$$\Delta S = \sum_i t_i \int Q_L^+ Q_R^+ \phi_i + \Delta S_b \,, \qquad \overline{\Delta S} = \sum_i \bar{t}_i \int Q_R^- Q_L^- \bar{\phi}_i + \overline{\Delta S_b} \,, \qquad (4.14)$$

and

$$\Delta S_b + \overline{\Delta S_b} \;=\; -2i \oint_{\mathcal{C}} ds (t_i \phi_i - \bar{t}_i \bar{\phi}_i) \,. \qquad (4.15)$$

The addition of this boundary term is necessary in order to preserve A-type supersymmetry [29], assuming this was the unbroken supersymmetry in the unperturbed theory. The super-charges $Q_L^\pm, Q_R^\pm$ are obtained by contour integration of the supersymmetry currents $G_L^\pm, G_R^\pm$, and they satisfy the standard $N = 2$ algebra

$$\{Q_L^+, Q_L^-\} = 2(H + P), \quad \{Q_R^+, Q_R^-\} = 2(H - P) \,, \qquad (4.16)$$

where H and P are translation operators in Euclidean time and space.

We are interested in perturbations that are restricted to only part of the worldsheet. The resulting interface then connects an unperturbed initial theory to another theory on the same moduli space. If the perturbation is from the (c, c) sector, the interface will be A-type, i.e. it preserves the same supersymmetry as A-type branes. Thus the fusion between (c, c) deformation interfaces and A-type branes is protected by supersymmetry, in agreement with the fact that A-type branes remain supersymmetric under (c, c) perturbations [30]. Viewed as an operator, the A-interface acts naturally on elements of the (c, c) ring, in the same way that A-branes couple to (c, c) fields.

Recall from the discussion in section 3 that complex structure interfaces were related after folding to special Lagrangian branes. The deformation of the original diagonal brane was determined by a map $f : M \to M'$, which ensured that the deformed brane is still special Lagrangian, hence supersymmetric. In the conformal theory, the role of f is played by the boundary perturbation (4.15), which adjusts to the bulk perturbation continuously as long as there are no relevant operators at the interface. After a finite perturbation, on the other hand, the O.P.E. of the perturbing field with the boundary may stop being regular, thereby inducing a renormalization-group flow to some lower-g interface. When a space-time interpretation of

D-branes in terms of BPS states is available, this means that we hit a line of marginal stability.

From now on, we assume that relevant boundary operators do not appear, which should be true in open regions around generic points of moduli space. We would like to show that

$$\frac{\partial}{\partial \bar{t}_i'} \left[{}_{\mathrm{RR}}\langle \bar{0} | \Delta_f | 0' \rangle_{\mathrm{RR}} \right] = 0 , \tag{4.17}$$

so that the amplitude has holomorphic dependence on the primed moduli. Our analysis follows [29]. The amplitude under consideration is drawn in Figure 1. We model the RR ground state $|0'\rangle_{\mathrm{RR}}$ by a semi-infinite cigar in a B-twisted topological theory. Since the operator insertion at the tip of the cigar is the identity, the state appearing at the boundary of the cigar is the canonical RR ground state. Similarly, we create the state ${}_{\mathrm{RR}}\langle \bar{0} |$ by inserting the identity at the tip of a semi-infinite cigar with an anti-topological B-twist. The two cigars are connected by a flat cylinder, on which we locate the interface Δ_f that separates the perturbed (green) region from the unperturbed (blue) region.

We can now prove (4.17) in two steps. First, we use the results of [4] to conclude that the canonical RR ground state has holomorphic dependence on the moduli,

$$\frac{\partial}{\partial \bar{t}_i'} |0'\rangle_{\mathrm{RR}} = 0 . \tag{4.18}$$

Hence, all we need to do is to consider the t_i perturbation on a flat region Σ' between the interface and the boundary of the semi-infinite cigar, see Figure 1. Taking the derivative in (4.17) then amounts to inserting in the amplitude

$$\overline{\Delta S_i} = \int_{\Sigma'} Q_R^- Q_L^- \bar{\phi}_i + 2i \oint_{\mathcal{C}_{int}} \bar{\phi}_i . \tag{4.19}$$

Using the supersymmetry algebra, we can rewrite this insertion as

$$\overline{\Delta S_i} = \int_{\Sigma'} (Q_R^- - iQ_L^+)(Q_L^- + iQ_R^+)\bar{\phi}_i + 2i \oint_{\mathcal{C}'} \bar{\phi}_i , \tag{4.20}$$

where $\mathcal{C}'$ is the boundary of the cigar on the right. The above rewriting follows from the supersymmetry algebra and the fact that $\bar{\phi}_i$ is a anti-chiral field. Together these imply that

$$(Q_R^- - iQ_L^+)(Q_L^- + iQ_R^+)\bar{\phi}_i = (Q_R^- - iQ_L^+)Q_L^- \bar{\phi}_i = (Q_R^- Q_L^- - 2iH)\bar{\phi}_i , \tag{4.21}$$

where H is the generator of translations perpendicular to the interface.

Proving equation (4.17) is therefore equivalent to proving that

$${}_{\mathrm{RR}}\langle \bar{0} | \Delta_f \left[\int_{\Sigma'} (Q_R^- - iQ_L^+)(Q_L^- + iQ_R^+)\bar{\phi}_i + 2i \oint_{\mathcal{C}'} \bar{\phi}_i \right] |0'\rangle_{\mathrm{RR}} = 0 . \tag{4.22}$$

The second piece vanishes, since otherwise the amplitude would violate the selection rules for R-charge. As for the bulk piece, we use that $Q_R^- - iQ_L^+$ intertwines the interface operator Δ_f. By contour deformation, we can thus let it act on the bra state ${}_{\mathrm{RR}}\langle \bar{0} |$, which is annihilated since it is a ground state. Here, we have used that Σ' is flat, hence all contours can be deformed freely. This concludes the proof of (4.17).

5 On the triangle inequality

We conclude this letter with a remark on whether Calabi's diastasis function defines a distance. Although Calabi did not comment on this in [2], the name he chose suggests that he knew it did not. Since his motivation was to study isometric embeddings of Kähler manifolds, this question was not central to his work anyway. If, on the other hand, the proposal (1.3) for a distance between conformal field theories [12] makes sense, it should do so for Calabi-Yau manifolds, for which we have proven equation (1.1). The question, therefore, is whether the square-root of the diastasis function defines a distance in the special case of Calabi-Yau moduli spaces.

Actually, the complex structure moduli space of a Calabi-Yau n-fold can be embedded isometrically in a higher-dimensional projective space. This is because the Kähler potential $K = -\log(\int_M \Omega \wedge \bar\Omega)$ can be interpreted as the Fubini-Study potential

$$K = -\log \eta_{\alpha\beta}\Pi^\alpha \bar\Pi^\beta \tag{5.1}$$

restricted to the embedding

$$t \to \Pi^\alpha \equiv \int_{\Sigma_\alpha} \Omega \ , \tag{5.2}$$

where Σ_α is a basis for $H_n(M,\mathbb{Z})$, and $\eta_{\alpha\beta}$ is the intersection form (antisymmetric for $n = 2k+1$). As a first step, we may thus like to check whether the square root of Calabi's diastatic function on projective space is a distance.

The answer actually depends on the signature of the metric. It is true in the hyperbolic case (signature $-+++\ldots$), which includes the complex structure and Kähler moduli spaces on T^2. To see why, take a coordinate patch $z_0 = 1$ and choose, without loss of generality, the third point at $\vec{z} = 0$. If the other two points are at $\vec{z}_1$ and $\vec{z}_2$ we have

$$d_{i3}^2 = -\log(1 - |\vec{z}_i|^2) \qquad i = 1,2;$$
$$d_{12}^2 = \log \frac{|1 - \vec{z}_1 \cdot \bar{z}_2|^2}{(1 - |\vec{z}_1|^2)(1 - |\vec{z}_2|^2)} = \log|1 - \vec{z}_1 \cdot \bar{z}_2|^2 + d_{13}^2 + d_{23}^2. \tag{5.3}$$

To check the triangle inequality, we need to check that

$$(d_{13} + d_{23})^2 \geq d_{12}^2 \iff 2d_{13}d_{23} \geq 2\log|1 - \vec{z}_1 \cdot \bar{z}_2|. \tag{5.4}$$

The worst case is when $z_1 = x$ and $z_2 = -x$ are antialigned, so it is sufficient to check that

$$|\log(1 - x^2)| \geq \log(1 + x^2). \tag{5.5}$$

This inequality is indeed true for all values of x.

The same analysis for the elliptic case (signature $+++\ldots$) shows that the triangle inequality fails. Take for instance the three points $z_1 = z$, $z_2 = 1$ and $z_3 = 0$ on CP^1 for which one finds

$$d_{13}^2(0,z) = \log(1 + |z|^2) \ , \quad d_{12}^2 = \log\frac{2(1 + |z|^2)}{|1 + z|^2} \ , \quad d_{23}^2 = \log 2 \ . \tag{5.6}$$

Since d_{13} diverges as $z \to \infty$, while d_{12} and d_{23} approach $\sqrt{\log 2}$, the triangle inequality cannot possibly be valid. What about the projective spaces relevant for Calabi-Yau moduli spaces? For general signature (n, m) this contains the elliptic case as a submanifold, so it cannot hold everywhere. Thus the actual embedding is important.

While one could study this, it is simpler to test the claim locally by constructing the expansion for the diastatic function from the intrinsic geometric data of the moduli space. In Kähler normal coordinates, we have

$$K(t, \bar{t}) = \sum_\alpha |t^\alpha|^2 - \frac{1}{4} R_{\alpha\bar{\beta}\gamma\bar{\delta}} t^\alpha \bar{t}^{\bar{\beta}} t^\gamma \bar{t}^{\bar{\delta}} + \dots \tag{5.7}$$

so the Calabi diastasis function is

$$d^2(t_1, t_2) = |t_1 - t_2|^2 - \frac{1}{4} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (t_1^\alpha t_1^\gamma - t_2^\alpha t_2^\gamma)(\bar{t}_1^{\bar{\beta}} \bar{t}_1^{\bar{\delta}} - \bar{t}_2^{\bar{\beta}} \bar{t}_2^{\bar{\delta}}) + \dots \tag{5.8}$$

Setting $t_3 = 0$, the triangle inequality at this order becomes

$$|t_1| + |t_2| - |t_1 - t_2| \geq -\frac{1}{8} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \left[\frac{1}{|t_1 - t_2|} (t_1^\alpha t_1^\gamma - t_2^\alpha t_2^\gamma)(\bar{t}_1^{\bar{\beta}} \bar{t}_1^{\bar{\delta}} - \bar{t}_2^{\bar{\beta}} \bar{t}_2^{\bar{\delta}}) \right.$$
$$\left. - \frac{1}{|t_1|} t_1^\alpha t_1^\gamma \bar{t}_1^{\bar{\beta}} \bar{t}_1^{\bar{\delta}} - \frac{1}{|t_2|} t_2^\alpha t_2^\gamma \bar{t}_2^{\bar{\beta}} \bar{t}_2^{\bar{\delta}} \right]. \tag{5.9}$$

For the special choice $t_1 = x$ and $t_2 = -x$, one finds $R_{x\bar{x}x\bar{x}} < 0$, namely the sectional curvature in the plane $x\bar{x}$ must be negative. This confirms the previous computation, and shows that there is a local condition.

Since Kähler manifolds can have either sign of sectional curvature, we see that in general the diastatic function does not satisfy the triangle inequality. For a Calabi-Yau moduli space, in particular, the sectional curvature is known to be positive near a conifold point [31]. Thus, if the supersymmetric interfaces minimize the entropy, the proposal (1.3) violates the triangle inequality. We hope to return to this problem in the future [12].

Acknowledgments

References

[1] I. Affleck and A. W. W. Ludwig, "Universal noninteger 'ground state degeneracy' in critical quantum systems," Phys. Rev. Lett. **67** (1991) 161.

[2] E. Calabi, "Isometric Imbedding of Complex Manifolds," Ann. Math. **58**, 1 (1953).

[3] V. Periwal and A. Strominger, "Kähler Geometry Of The Space Of N=2 Superconformal Field Theories," Phys. Lett. B **235** (1990) 261.

[4] S. Cecotti and C. Vafa, "Topological antitopological fusion," Nucl. Phys. B **367** (1991) 359.

[5] H. Jockers, V. Kumar, J. M. Lapan, D. R. Morrison and M. Romo, "Two-Sphere Partition Functions and Gromov-Witten Invariants," arXiv:1208.6244 [hep-th].

[6] J. Gomis and S. Lee, "Exact Kahler Potential from Gauge Theory and Mirror Symmetry," JHEP **1304** (2013) 019 [arXiv:1210.6022 [hep-th]].

[7] F. Benini and S. Cremonesi, "Partition functions of N=(2,2) gauge theories on S^2 and vortices," arXiv:1206.2356 [hep-th].

[8] N. Doroud, J. Gomis, B. Le Floch and S. Lee, "Exact Results in D=2 Supersymmetric Gauge Theories," JHEP **1305** (2013) 093 [arXiv:1206.2606 [hep-th]].

[9] K. Hori and M. Romo, "Exact Results In Two-Dimensional (2,2) Supersymmetric Gauge Theories With Boundary," arXiv:1308.2438 [hep-th].

[10] S. Sugishita and S. Terashima, "Exact Results in Supersymmetric Field Theories on Manifolds with Boundaries," arXiv:1308.1973 [hep-th].

[11] M. R. Douglas, "Spaces of Quantum Field Theories," arXiv:1005.2779 [hep-th].

[12] C. P. Bachas, I. Brunner, M. R. Douglas and L. Rastelli, work in progress.

[13] C. Bachas, J. de Boer, R. Dijkgraaf and H. Ooguri, "Permeable conformal walls and holography," JHEP **0206** (2002) 027 [hep-th/0111210].

[14] C. Bachas and I. Brunner , "Fusion of conformal interfaces," JHEP **0802** (2008) 085 [arXiv:0712.0076 [hep-th]].

[15] I. Brunner, H. Jockers and D. Roggenkamp, "Defects and D-Brane Monodromies," Adv. Theor. Math. Phys. **13** (2009) 1077 [arXiv:0806.4734 [hep-th]].

[16] C. Bachas, I. Brunner and D. Roggenkamp , "A worldsheet extension of O(d,d:Z)," JHEP **1210** (2012) 039 [arXiv:1205.4647 [hep-th]].

[17] A. Giveon, M. Porrati, and E. Rabinovici , "Target space duality in string theory," Phys. Rept. **244** (1994) 77 [hep-th/9401139].

[18] C. Bachas and M. Gaberdiel, "Loop operators and the Kondo problem," JHEP **0411** (2004) 065 [hep-th/0411067].

[19] J. Fuchs, C. Schweigert and K. Waldorf, "Bi-branes: Target space geometry for world sheet topological defects," J. Geom. Phys. **58** (2008) 576 [hep-th/0703145 [HEP-TH]].

[20] J. A. Harvey, S. Kachru, G. W. Moore and E. Silverstein, "Tension is dimension," JHEP **0003** (2000) 001 [hep-th/9909072].

[21] K. Becker, M. Becker and A. Strominger, "Five-branes, membranes and nonperturbative string theory," Nucl. Phys. B **456** (1995) 130 [hep-th/9507158].

[22] M. Bershadsky, C. Vafa and V. Sadov, "D-branes and topological field theories," Nucl. Phys. B **463** (1996) 420 [hep-th/9511222].

[23] H. Ooguri, Y. Oz and Z. Yin, "D-branes on Calabi-Yau spaces and their mirrors," Nucl. Phys. B **477** (1996) 407 [hep-th/9606112].

[24] I. Brunner, M. R. Douglas, A. E. Lawrence and C. Römelsberger, "D-branes on the quintic," JHEP **0008** (2000) 015 [hep-th/9906200].

[25] A. Kapustin and K. Setter, "Geometry of Topological Defects of Two-dimensional Sigma Models," arXiv:1009.5999 [hep-th].

[26] D. Joyce, "Lectures on Calabi-Yau and special Lagrangian geometry," math/0108088 [math-dg].

[27] R.C. McLean, "Deformations of calibrated submanifolds," Communications in Analysis and Geometry **6** (1998), 705747.

[28] I. Brunner and D. Roggenkamp, "B-type defects in Landau-Ginzburg models," JHEP **0708** (2007) 093 [arXiv:0707.0922 [hep-th]].

[29] K. Hori, A. Iqbal and C. Vafa, "D-branes and mirror symmetry," hep-th/0005247.

[30] I. Brunner and D. Roggenkamp, "Defects and bulk perturbations of boundary Landau-Ginzburg orbifolds," JHEP **0804** (2008) 001 [arXiv:0712.0188 [hep-th]].

[31] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, "A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory," Nucl. Phys. B **359** (1991) 21.