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Covariant and gauge-invariant linear scalar perturbations in multiple scalar field cosmologies

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Abstract

We derive a set of equations monitoring the evolution of covariant and gauge-invariant linear scalar perturbations of Friedman-Lemaître-Robertson-Walker models with multiple interacting nonlinear scalar fields. We use a dynamical systems' approach in order to perform a stability analysis for some classes of scalar field potentials. In particular, using a recent approximation for the inflationary dynamics of the background solution, we derive conditions under which homogenization occurs for chaotic (quadratic and quartic potentials) and new inflation. We also prove a cosmic no-hair result for power-law inflation and its generalisation for two scalar fields with independent exponential potentials (assisted power-law inflation).

1 Introduction

Nonlinear scalar fields ϕ have been important to model the presently observed accelerated cosmological expansion [1] as well as the inflationary phase of the early universe [2].

The simplest of these models has potentials $\mathcal{V}(\phi)$ with a strictly positive lower bound, which is a straightforward generalisation of the positive cosmological constant [3] and mimics it at late times. In that case, the Bianchi models of types I-VIII were studied in detail by Rendall [4] and their nonlinear stability by Ringström in [5]. In subsequent works, Rendall considered potentials with zero lower bound when ϕ is either infinite [6] or finite [7]. For the former class of solutions, it was shown that if $\frac{d\mathcal{V}}{d\phi}/\mathcal{V}$ satisfies an upper bound which rules out too rapid exponential decay, accelerated expansion is expected to exist indefinitely and has a dynamical behaviour between the power-law [8, 9] and exponential types, commonly termed as intermediate inflation [10, 11, 12, 13]. Later, in [7], Rendall considered potentials which are positive and tend to zero, but which do not experience accelerated expansion indefinitely, as the Klein-Gordon field. These potentials are very useful for studying the early inflationary stage of the universe in models of chaotic inflation (see [2]), since they allow the physical process of reheating [14, 15].

Dynamical systems's techniques can be applied when the Einstein field equations (EFE) reduce to a system of ordinary differential equations (ODEs). For an exponential potential, the flat and isotropic power-law inflationary solution was studied by Halliwell using phase-plane methods [8] (see also [75, 74]) and by Burd and Barrow for Bianchi types I and III, as well as Kantowski-Sachs models [9]. Polynomial type potential were also studied using dynamical systems' techniques by a number of authors: Isotropic models were first studied by Belinskii et al. [16, 17, 18] whereas spatially homogeneous and anisotropic models by [19, 20, 21]. In [22], Rendall revisited some results of [17] by giving a rigorous asymptotic analysis of the inflationary dynamics using center manifold theory.

A procedure which has proved to be very useful when using techniques from dynamical systems' theory applied to Cosmology is the reduction of the original system of equations, by using *Hubble-normalized-variables* [23, 24, 25]. For scalar field cosmologies, such variables were defined by Coley et al. [26] in the context of a single scalar field with an exponential potential. There, the Bianchi type models I-VIII were studied in detail and previous results in the literature [27, 28, 29] were treated in an unified way, in particular, it was possible to test whether a given model inflates and/or isotropizes at late times and thus test the validity of the cosmic no-hair conjecture in those settings. More precisely, it was shown that

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the flat isotropic power-law inflationary solution is an attractor for all ever expanding Bianchi models with an exponential potential [31]. The flat Friedman-Lemaître-Robertson-Walker (FLRW) model with exponential potential coupled to matter was studied in [32], whereas the case of several independent exponential potentials was considered in [33]. For more details see Coley [34] and references therein.

Recently, Hubble-normalized variables have also been used in the study of the Klein-Gordon field by Ureña-López and Reyes-Ibarra in [35, 36], see also [37, 38]. Contrary to the exponential potential situation, in this case the ODEs system does not decouple from the Raychaudhuri equation and it is necessary to introduce new expansion-normalized variables. By treating this new variable as a potential parameter, they found an analytical approximation for the inflationary dynamics and claimed that, for small values of the parameter, it works as a first order correction to the usual slow-roll approximation. Recently, in [73], it was found that the inflationary solution is a one-dimensional unstable center submanifold of a de-Sitter fixed point, and approximations based on series expansion were given and compared with the slow-roll approximation.

The above results concern spatially homogeneous backgrounds. Here, instead, we shall be interested in the evolution of inhomogeneous spacetimes resulting from linear perturbations of FLRW backgrounds and in the application of dynamical systems techniques to these settings.

In this paper, we use the approach to linear perturbation theory developed by Ellis and Bruni [39, 40, 41]. This consists in starting from exact non-linear equations using the *1+3 covariant formalism* which, in view of the fundamental lemma of Stewart and Walker [42], are then linearized about exact FLRW models. The advantage of this approach with respect to other metric formalisms [43, 44], relies on the fact that the perturbations variables are covariant and gauge-invariant by construction, having a clear geometrical and well defined physical interpretation [41], see also [45]. Exact evolution equations for linear perturbations of FLRW with a perfect-fluid as matter source were given in [39, 40] and the extension for an imperfect-fluid can be found in Hwang and Vishniac [46]. The imperfect-fluid case was also applied to describe perturbations in a multi-component fluid by Dunsby et al. [47, 48] using the methods of King and Ellis [49] and, to minimally coupled scalar-fields by Bruni et al. [50] using the results by Madsen [51] and the field-fluid relation. More recently, this has also been applied to charged multifluids [52] and magnetized cosmologies [53, 54].

To study the evolution of inhomogeneities, we shall then employ a dynamical systems' approach following the methods of Woszczyna [55, 56, 57, 58]. These were also used to study stability problems in a universe with dust and radiation [59], magnetized cosmologies [60] and locally rotational symmetric (LRS) Bianchi I models [61].

The plan of the paper is the following: in Section 1 we revise the background dynamics and relevant results for the upcoming sections. We start by introducing the Hubble normalized variables and the resulting reduced dynamical system. This shall be done in a particular way so that case of exponential potentials and the approximations for polynomial potentials can be treated in a unified way, simplifying the analysis of last section. In sections 2 and 3, we construct the system of equations governing the evolution of linear scalar perturbations of a *Friedman-Lemaître-Robertson-Walker-scalar field* (FLRWsf) background with multiple interacting scalar fields generalising the works of [48, 50]. In Section 5, we shall then apply the dynamical systems' approach of Woszczyna and show how can it be generalised to the case of multiple scalar fields. In particular, we consider in detail the examples of one and two scalar fields with exponential potentials as well as the approximations for models with polynomial potentials.

2 The background spacetime

In this section, we revise the background setting as well as some results that will be used in subsequent sections. This will be done by presenting the formalism in an unified way for several classes of scalar field potentials.

We will consider N minimally coupled scalar fields ϕ_A , $A = 1, \dots, N$, with arbitrary self-interaction potentials $\mathcal{V}_A = \mathcal{V}(\phi_A)$ and a general interaction potential between the fields $\mathcal{W} = \mathcal{W}(\phi_1, \dots, \phi_N)$. The action associated to this scenario is given by

$$S[\phi_A, \mathbf{g}] = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\frac{1}{2\chi} R - \frac{1}{2} \sum_{A=1}^N (\nabla_\lambda \phi_A) (\nabla^\lambda \phi_A) - \sum_{A=1}^N \mathcal{V}_A - \mathcal{W} \right], \quad (1)$$

where g is the determinant of the metric \mathbf{g} , R is the Ricci scalar and the Einstein summation convention is understood on the greek indices. We will also use units such that $\chi = \frac{8\pi G}{c^4} = 1$. The energy-momentum

(EM) tensor is then

$$T_{\mu\nu} = \sum_{A=1}^N \left[(\nabla_{\mu}\phi_A)(\nabla_{\nu}\phi_A) - g_{\mu\nu} \left\{ \frac{1}{2} (\nabla_{\lambda}\phi_A)(\nabla^{\lambda}\phi_A) + \mathcal{V}_A \right\} \right] - g_{\mu\nu}\mathcal{W}, \quad (2)$$

and the generalized Euler-Lagrange equations give a system of N evolution equations for the scalar fields

$$\square_{\mathbf{g}}\phi_A - \frac{d\mathcal{V}_A}{d\phi_A} = \frac{\partial\mathcal{W}}{\partial\phi_A} \quad A = 1, \dots, N, \quad (3)$$

where $\square_{\mathbf{g}}$ is the D'Alembertian for the metric \mathbf{g} . On a FLRW background and using comoving coordinates, the line element reads

$$ds^2 = -dt^2 + a^2(t) \left[\frac{1}{1-kr^2} dr^2 + r^2 d\Omega^2 \right], \quad (4)$$

where $a(t)$ is the scale factor, t the proper time, $d\Omega^2$ the spherical 2-metric and $k = 1, 0, -1$ the curvature of spatial hypersurfaces. On such a background, the scalar fields ϕ_A are functions of time t only. Making use of the momentum density variable defined by

$$\psi_A := \frac{\partial\mathcal{L}}{\partial\dot{\phi}_A} = \dot{\phi}_A, \quad (5)$$

where the dot denotes differentiation with respect to proper time, the EFEs together with the scalar field evolution equations give an autonomous system of first order ODEs

$$\dot{\psi}_A = -3H\psi_A - \frac{d\mathcal{V}_A}{d\phi_A} - \frac{\partial\mathcal{W}}{\partial\phi_A} \quad , \quad A = 1, \dots, N \quad (6)$$

$$\dot{H} = -H^2 - \frac{1}{3} \sum_{A=1}^N \psi_A^2 + \frac{1}{3} \sum_{A=1}^N \mathcal{V}_A + \frac{1}{3} \mathcal{W} \quad (7)$$

and the Friedman constraint

$${}^3R = -6H^2 + \sum_{A=1}^N \psi_A^2 + 2 \sum_{A=1}^N \mathcal{V}_A + 2\mathcal{W}, \quad (8)$$

where $H := \dot{a}/a$ is the Hubble function and ${}^3R = 6k/a^2$ the Ricci scalar of the spatial metric. An important quantity in cosmology is the deceleration parameter given by

$$q := -\frac{\ddot{a}a}{\dot{a}^2} = - \left[1 + \frac{\dot{H}}{H^2} \right] \quad (9)$$

and a model has accelerated expansion, $\ddot{a} > 0$, if and only if, $q < 0$. We will use the *Hubble normalized variables* for scalar field cosmologies defined by

$$\Psi_A := \frac{\psi_A}{\sqrt{6}H} \quad , \quad \Phi_A := \left(\frac{\mathcal{V}_A}{3H^2} \right)^{\frac{1}{2n}} \quad , \quad \Theta := \left(\frac{\mathcal{W}}{3H^2} \right)^{\frac{1}{2n}} \quad , \quad K := -\frac{{}^3R}{6H^2}, \quad (10)$$

where $n \in \mathbb{N}$ takes values for specific potentials. For instance, in the case of an exponential potential, $n = 1$, the variables coincide with those of [26] and, for the polynomial type potentials, with those of [35, 36, 37, 38]. We will also make use of the logarithmic time variable τ

$$\frac{d\tau}{dt} = H \quad , \quad H' = -(1+q)H, \quad (11)$$

so that $\tau \rightarrow -\infty$, as $t \rightarrow 0^+$, and denote differentiation with respect to τ by a prime. Using these variables, the system of ODEs governing the background dynamics becomes

$$\begin{aligned} \Psi'_A &= (q-2)\Psi_A - n\sqrt{6} \left[\Phi_A^{2n-1} \frac{\partial\Phi_A}{\partial\phi_A} + \Theta^{2n-1} \frac{\partial\Theta}{\partial\phi_A} \right] \\ \Phi'_A &= \frac{1}{n}(q+1)\Phi_A + \sqrt{6} \frac{\partial\Phi_A}{\partial\phi_A} \Psi_A \\ \Theta' &= \frac{1}{n}(q+1)\Theta + \sqrt{6} \sum_{A=1}^N \frac{\partial\Theta}{\partial\phi_A} \Psi_A \end{aligned} \quad (12)$$

subject to the Friedman constraint

$$K = 1 - \sum_{A=1}^N \Psi_A^2 - \sum_{A=1}^N \Phi_A^{2n} - \Theta^{2n} \quad (13)$$

and with

$$q = 2 \sum_{A=1}^N \Psi_A^2 - \sum_{A=1}^N \Phi_A^{2n} - \Theta^{2n}. \quad (14)$$

These equations will allow us to treat, in a unified way, various families of scalar field potentials given that the system remains autonomous, which is the case for exponential potentials and the polynomial potentials' approximation. In general, if the scalar fields do not interact with each other in flat FLRW models then $(\Psi_1, \dots, \Psi_N, \Phi_1, \dots, \Phi_N) \in [-1, 1]^N \times [-1, 1]^N$, but for the models under consideration we will only need to take the dynamical system state space

$$\left\{ (\Psi_1, \dots, \Psi_N, \Phi_1, \dots, \Phi_N) \in [-1, 0]^N \times [0, 1]^N : \sum_{A=1}^N \Psi_A^2 + \sum_{A=1}^N \Phi_A^{2n} = 1 \right\} \quad (15)$$

with

$$q = 2 - 3 \sum_{A=1}^N \Phi_A^{2n} = 3 \sum_{A=1}^N \Psi_A^2 - 1,$$

and it is straightforward to get:

Lemma 1. *For N non-interacting scalar fields in flat backgrounds with $\Theta = K = 0$, the fixed points \mathcal{P} and \mathcal{Q} of the system (12), when they exist, are given by*

$$\mathcal{P} : \frac{\Phi_A^{2n}}{\sum_{B=1}^N \Phi_B^{2n}} \frac{\partial \Phi_A}{\partial \phi_A} = -\frac{\sqrt{6}}{2n} \Psi_A \Phi_A \quad \text{and} \quad \frac{\Psi_A^2}{\sum_{B=1}^N \Psi_B^2} \frac{\partial \Phi_A}{\partial \phi_A} = -\frac{\sqrt{6}}{2n} \Psi_A \Phi_A$$

and

$$\mathcal{Q} : \Phi_A = \Psi_A = 0$$

with $A = 1, \dots, N$.

The fixed points \mathcal{P} correspond to physical solutions depending on the potential, while the point \mathcal{Q} is unphysical. For a single scalar field, the Friedman constraint (13) reads

$$\Psi^2 + \Phi^{2n} = 1 \quad (16)$$

and the linearised matrix of the system (12), at \mathcal{P} , is

$$\begin{pmatrix} 9\Psi_{\mathcal{P}}^2 - 3 & \frac{\Phi_{\mathcal{P}}^{2n}}{\Psi_{\mathcal{P}}\Phi_{\mathcal{P}}} \left(3(2n-1)\Psi_{\mathcal{P}}^2 + \frac{2n^2}{\Phi_{\mathcal{P}}} \left(\frac{\partial^2 \Phi}{\partial \phi^2} \right)_{\mathcal{P}} \right) \\ \frac{3}{n} \Psi_{\mathcal{P}} \Phi_{\mathcal{P}} & \frac{3}{n} \Psi_{\mathcal{P}}^2 - \frac{2n}{\Phi_{\mathcal{P}}} \left(\frac{\partial^2 \Phi}{\partial \phi^2} \right)_{\mathcal{P}} \end{pmatrix} \quad (17)$$

with characteristic polynomial

$$\omega^2 - \left\{ \left(9 + \frac{3}{n} \right) \Psi_{\mathcal{P}}^2 - 3 - 2n \frac{1}{\Phi_{\mathcal{P}}} \left(\frac{\partial^2 \Phi}{\partial \phi^2} \right)_{\mathcal{P}} \right\} \omega + \frac{18}{n} \Psi_{\mathcal{P}}^2 \left\{ 1 - (n+1) \Phi_{\mathcal{P}}^{2n} - \frac{2n^2}{3\Phi_{\mathcal{P}}} \left(\frac{\partial^2 \Phi}{\partial \phi^2} \right)_{\mathcal{P}} \right\}.$$

The eigenvalues of the matrix (17) are

$$\begin{aligned} \omega_{\mathcal{P}}^{\pm} = & \frac{3}{2} \left[\left(3 + \frac{1}{n} \right) \Psi_{\mathcal{P}}^2 - 1 - \frac{2n}{3\Phi_{\mathcal{P}}} \left(\frac{\partial^2 \Phi}{\partial \phi^2} \right)_{\mathcal{P}} \right] \\ & \pm \frac{3}{2} \sqrt{\left[1 + \frac{2n}{3\Phi_{\mathcal{P}}} \left(\frac{\partial^2 \Phi}{\partial \phi^2} \right)_{\mathcal{P}} \right]^2 + 2\Psi_{\mathcal{P}}^2 \left[\left(1 - \frac{1}{n} \right) - \frac{1}{3} \left(\frac{2}{n} - \frac{1}{n^2} - 1 \right) \Psi_{\mathcal{P}}^2 + \frac{2}{3} (n-1) \frac{2n}{3\Phi_{\mathcal{P}}} \left(\frac{\partial^2 \Phi}{\partial \phi^2} \right)_{\mathcal{P}} \right]}. \end{aligned}$$

If we denote the respective eigenvectors by $(\delta\Psi \ \delta\Phi)_{\pm}^T$, the general solution to the perturbations around \mathcal{P} reads

$$\begin{pmatrix} \delta\Psi \\ \delta\Phi \end{pmatrix} = C_- \begin{pmatrix} \delta\Psi \\ \delta\Phi \end{pmatrix}_- e^{\omega_- \tau} + C_+ \begin{pmatrix} \delta\Psi \\ \delta\Phi \end{pmatrix}_+ e^{\omega_+ \tau}.$$

However, (16) implies, to linear order, that

$$\Psi_{\mathcal{P}}\delta\Psi + n\Phi_{\mathcal{P}}^{2n-1}\delta\Phi = 0 \Leftrightarrow \left(\Psi_{\mathcal{P}} \quad n\Phi_{\mathcal{P}}^{2n-1}\right) \begin{pmatrix} \delta\Psi \\ \delta\Phi \end{pmatrix} = 0$$

and the evolution of linear perturbations around the fixed points \mathcal{P} reduces to a single equation

$$\delta\Phi' = \left(-3\Phi_{\mathcal{P}}^{2n} + \frac{3}{n}\Psi_{\mathcal{P}}^2 - \frac{2n}{\Phi_{\mathcal{P}}}\left(\frac{\partial^2\Phi}{\partial\phi^2}\right)_{\mathcal{P}}\right)\delta\Phi. \quad (18)$$

Then, there is a single eigenvalue solution

$$\omega_{\mathcal{P}}^- = -3\left(1 + \frac{1}{n}\right)\Phi_{\mathcal{P}}^{2n} + \frac{3}{n} - \frac{2n}{\Phi_{\mathcal{P}}}\left(\frac{\partial^2\Phi}{\partial\phi^2}\right)_{\mathcal{P}} \quad (19)$$

which is proportional to the eigenvector $(\delta\Psi \ \delta\Phi)_{\mathcal{P}}^T$.

With the above, it is easy to show that, at \mathcal{P} , the single scalar field solutions are inflationary if and only if

$$\Phi_{\mathcal{P}}^{2n} > \frac{2}{3} \Leftrightarrow \Psi_{\mathcal{P}}^2 < \frac{1}{3}. \quad (20)$$

We note that this result does not involve a specific class of potentials.

We shall now revise the particular examples of exponential and polynomial potentials which will be useful in the stability analysis of Section 5.

2.1 Exponential Potentials

Using the above framework, we now review the flat (assisted) power-law solutions due to exponential potentials

$$\mathcal{V}_A = \Lambda e^{\lambda_A\phi_A} \quad (21)$$

where λ_A and Λ are positive constants. For such potentials, the Hubble normalized variables (10) are defined with $n = 1$, the zero curvature invariant set (15) is a higher dimensional sphere \mathbb{S}^N and

$$\frac{\partial^p\Phi_A}{\partial\phi_A^p} = \left(\frac{\lambda_A}{2}\right)^p \Phi_A, \quad (22)$$

for any $p \in \mathbb{N}$. We will now treat separately the single scalar field and the two scalar fields cases.

2.1.1 Power-law Inflation

If only one scalar field is present, Lemma 1 implies

$$\mathcal{P} : \quad \Phi\left(\Psi + \frac{\lambda}{\sqrt{6}}\right) = 0 \quad (23)$$

satisfying (16). Therefore, there are two fixed points $(\Psi_{\mathcal{P}}, \Phi_{\mathcal{P}}) \in [-1, 0] \times [0, 1]$ in \mathbb{S}^1 such that

$$\mathcal{P}_0 : \quad (\Psi, \Phi) = (-1, 0) \quad (24)$$

$$\mathcal{P}_1 : \quad (\Psi, \Phi) = \left(-\frac{\lambda}{\sqrt{6}}, \frac{\sqrt{6-\lambda^2}}{\sqrt{6}}\right) \quad \text{with} \quad 0 < \lambda < \sqrt{6}. \quad (25)$$

Also, at \mathcal{P} , we have

$$\left(\frac{\partial^2\Phi}{\partial\phi^2}\right)_{\mathcal{P}} = -\frac{\sqrt{6}\lambda}{4}\Psi\Phi,$$

so that, from (19), the eigenvalues are

$$\omega_{\mathcal{P}_0}^- = 3 + \frac{\sqrt{6}}{2}\lambda \quad \text{and} \quad \omega_{\mathcal{P}_1}^- = -\frac{6-\lambda^2}{2}. \quad (26)$$

The point \mathcal{P}_0 corresponds to the well-known massless scalar field solution, which is the early time attractor for the system and is a *source* for all $\lambda \in (0, \sqrt{6})$. The point \mathcal{P}_1 is a *sink*, with the deceleration parameter given by

$$q_{\mathcal{P}_1} = \frac{\lambda^2 - 2}{2}$$

and, by (20), the solution is inflationary if and only if $q_{\mathcal{P}_1} < 0$, i.e. for

$$0 < \lambda < \sqrt{2},$$

which corresponds to the flat homogeneous and isotropic power-law inflationary solution found by Halliwell [8]. This solution is also known to be a future attractor for ever expanding scalar field Bianchi models, see e.g. [34] for details and related references.

2.1.2 Assisted power-law Inflation

For two scalar fields ϕ_1 and ϕ_2 with independent exponential potentials, Lemma 1 implies

$$\mathcal{P} : \quad \Phi_A \Psi_A \left(\frac{\Psi_A \lambda_A}{\Psi^2} \frac{\lambda_A}{2} + \frac{\sqrt{6}}{2} \right) = 0 \quad \text{and} \quad \Phi_A \left(\frac{\Phi_A^2 \lambda_A}{\Phi^2} \frac{\lambda_A}{2} + \frac{\sqrt{6}}{2} \Psi_A \right) = 0, \quad A = 1, 2$$

which leads to the four fixed points

$$\mathcal{P}_0 : \quad (\Psi_1, \Psi_2, \Phi_1, \Phi_2) = \left(-\Psi_0, -\sqrt{1 - \Psi_0^2}, 0, 0 \right), \quad 0 \leq \Psi_0 \leq 1 \quad (27)$$

$$\mathcal{P}_1 : \quad (\Psi_1, \Psi_2, \Phi_1, \Phi_2) = \left(-\frac{\lambda_1}{\sqrt{6}}, 0, \frac{\sqrt{6 - \lambda_1^2}}{\sqrt{6}}, 0 \right), \quad 0 < \lambda_1 < \sqrt{6} \quad (28)$$

$$\mathcal{P}_2 : \quad (\Psi_1, \Psi_2, \Phi_1, \Phi_2) = \left(0, -\frac{\lambda_2}{\sqrt{6}}, 0, \frac{\sqrt{6 - \lambda_2^2}}{\sqrt{6}} \right), \quad 0 < \lambda_2 < \sqrt{6} \quad (29)$$

$$\mathcal{P}_3 : \quad (\Psi_1, \Psi_2, \Phi_1, \Phi_2) = \left(-\frac{\lambda^2}{\sqrt{6}\lambda_1}, -\frac{\lambda^2}{\sqrt{6}\lambda_2}, \frac{\sqrt{\lambda^2(6 - \lambda^2)}}{\sqrt{6}\lambda_1}, \frac{\sqrt{\lambda^2(6 - \lambda^2)}}{\sqrt{6}\lambda_2} \right), \quad \frac{1}{\lambda^2} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \quad (30)$$

and the linearised matrix of the system (12) at \mathcal{P} is

$$\begin{pmatrix} 9\Psi_1^2 + 3(\Psi_2^2 - 1) & 6\Psi_1\Psi_2 & -\sqrt{6}\lambda_1\Phi_1 & 0 \\ 6\Psi_1\Psi_2 & 9\Psi_2^2 + 3(\Psi_1^2 - 1) & 0 & -\sqrt{6}\lambda_2\Phi_2 \\ \left(\frac{\sqrt{6}}{2}\lambda_1 + 6\Psi_1\right)\Phi_1 & 6\Psi_2\Phi_1 & \frac{\sqrt{6}}{2}\lambda_1\Psi_1 + 3(\Psi_1^2 + \Psi_2^2) & 0 \\ 6\Psi_1\Phi_2 & \left(\frac{\sqrt{6}}{2}\lambda_2 + 6\Psi_2\right)\Phi_2 & 0 & \frac{\sqrt{6}}{2}\lambda_2\Psi_2 + 3(\Psi_1^2 + \Psi_2^2) \end{pmatrix}_{\mathcal{P}}$$

with eigenvalues

$$\omega(\mathcal{P}_0) = 6, 0, 3 - \frac{\sqrt{6}}{2}\lambda_1\Psi_0, 3 - \frac{\sqrt{6}}{2}\lambda_2\sqrt{1 - \Psi_0^2} \quad \text{where} \quad 0 \leq \Psi_0 \leq 1 \quad (31)$$

$$\omega(\mathcal{P}_{1,2}) = \frac{\lambda_{1,2}^2}{2}, \lambda_{1,2}^2, -\frac{(6 - \lambda_{1,2}^2)}{2}, -\frac{(6 - \lambda_{1,2}^2)}{2} \quad (32)$$

$$\omega(\mathcal{P}_3) = \lambda^2, \frac{\lambda^2 - 6}{2}, \frac{1}{4} \left\{ (\lambda^2 - 6) \pm \sqrt{(\lambda^2 - 6) + 8\lambda^2(\lambda^2 - 6)} \right\}. \quad (33)$$

Thus, in \mathbb{S}^2 , the point \mathcal{P}_0 is a local source and corresponds to the massless scalar field solution which is the early time attractor. The points \mathcal{P}_1 and \mathcal{P}_2 are saddles which correspond to single power law solutions where either ϕ_1 dominates over ϕ_2 or vice-versa. As before, the Friedman constraint can be used to eliminate the unphysical radial direction, which corresponds to the positive eigenvalue of \mathcal{P}_3 . Therefore, \mathcal{P}_3 , having negative eigenvalues, becomes the stable late-time attractor and corresponds to the assisted power-law solution.

The results of this section will be used in Section 5.2.

2.2 Polynomial Potentials

Recently, the expansion-normalized variables defined in [35, 36, 37, 38] were used to study the dynamical properties of scalar field cosmologies with potentials given by

$$\mathcal{V} = C \frac{(\phi^2 - v^2)^n}{2n}, \quad (34)$$

where $C > 0$ and $v \geq 0$ are constants. Models with $v = 0$ are called *chaotic inflation* and with $v > 0$ *new inflation*, see e.g. [38, 2]. Contrary to the case of exponential potentials, where the reduced dynamical system (12) is autonomous, for the potentials (34) one gets a non-autonomous system, in general. In order to turn the system autonomous, a new Hubble normalised variable must be introduced. An appropriate choice is

$$\mathcal{M} := (6n)^{\frac{n-1}{2n}} n^{\frac{1}{2}} \left(\frac{C}{H^2} \right)^{\frac{1}{2n}}, \quad (35)$$

which, in ever expanding models, is a monotone and growing function obeying the evolution equation

$$\mathcal{M}' = \frac{3}{n} \Psi^2 \mathcal{M}. \quad (36)$$

From (34) and (35) it follows that

$$\frac{\partial \Phi}{\partial \phi} = \frac{\mathcal{M}}{\sqrt{6}n^2} \frac{\sqrt{n^2 \Phi^2 + \mathcal{M}^2 \mathcal{N}^2}}{\Phi} \quad \text{and} \quad \frac{\partial^2 \Phi}{\partial \phi^2} = - \left(\frac{\mathcal{M}}{\sqrt{6}n} \right)^2 \frac{\mathcal{M}^2 \mathcal{N}^2}{n \Phi^3}, \quad (37)$$

where

$$\mathcal{N} := \frac{v}{\sqrt{6}}. \quad (38)$$

To get a better picture of the state space of the new dynamical system, it is useful to make a change of variables and turn the above system into a 2-dimensional system. For $n = 1, 2$ and $v = 0$ this was done in [36], where the new variable Υ was defined as

$$\Psi = \cos(\Upsilon) \quad , \quad \Phi = |\sin(\Upsilon)|^{\frac{1}{n}}, \quad (39)$$

with $(\Upsilon, \mathcal{M}) \in [\frac{\pi}{2}, \pi] \times [0, +\infty)$. The dynamical system (12) coupled to (36) then reads

$$\begin{aligned} \Upsilon' &= \frac{|\sin(\Upsilon)|^{\frac{2(n-1)}{n}}}{\sin(\Upsilon)} \left[3 \cos(\Upsilon) |\sin(\Upsilon)|^{\frac{2}{n}} + \frac{\mathcal{M}}{n} \sqrt{n^2 |\sin(\Upsilon)|^{\frac{2}{n}} + \mathcal{M}^2 \mathcal{N}^2} \right] \\ \mathcal{M}' &= \frac{3}{n} \mathcal{M} \cos^2(\Upsilon). \end{aligned} \quad (40)$$

The fixed points of this system are located at $\Upsilon = \frac{\pi}{2}, \pi$ and $\mathcal{M} = 0$, and are independent of n . The point $(0, \pi)$ corresponds to the massless scalar field solutions $\Psi = -1$ and is unstable, while the point $(0, \frac{\pi}{2})$ is a saddle and corresponds to the potential dominated solutions $\Phi = 1$. There are also heteroclinic curves connecting the unstable point with the saddle point along the stable direction i.e., along the $\mathcal{M} = 0$ axis. Along the unstable direction of the saddle point departs a curve, which for small values of \mathcal{M} , acts as an attractor trajectory in the phase space, see [36] for details when $\mathcal{N} = 0$ (see also Figs. 1, 2, 3, 4). The approach of [35, 38] consists in reducing the 3-dimensional system, obtained by coupling (12) with (36), to a 2-dimensional system with state vector (Ψ, Φ) , by considering \mathcal{M} as a control parameter.

2.2.1 Chaotic Inflation $\mathcal{N} = 0$

For potentials having $\mathcal{N} = 0$, Lemma 1 and (37) give

$$\mathcal{P} : \quad \Psi \Phi = -\frac{\mathcal{M}}{3} \quad \text{with} \quad \frac{\partial^2 \Phi}{\partial \phi^2} = 0 \quad (41)$$

subject to the Friedman constraint (16). Then (19) gives

$$\omega_- = -3 \left(1 + \frac{1}{n} \right) \Phi_{\mathcal{P}}^{2n} + \frac{3}{n} \quad (42)$$

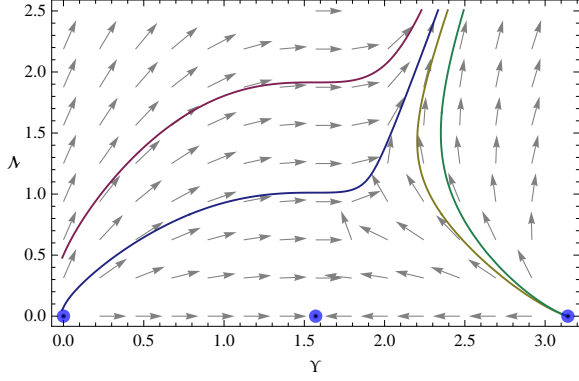


Figure 1: Phase space for a quadratic potential $n = 1$ and $\mathcal{N} = 0$.

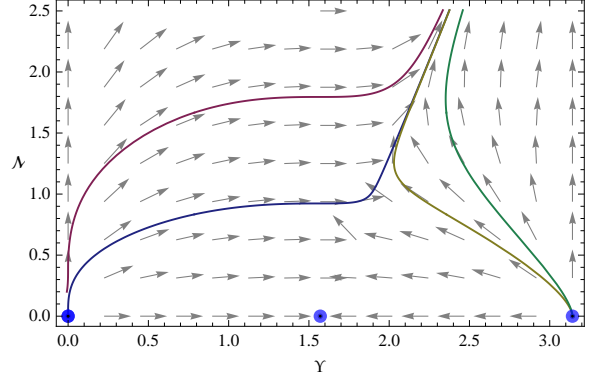


Figure 2: Phase space for a quartic potential $n = 2$ and $\mathcal{N} = 0$.

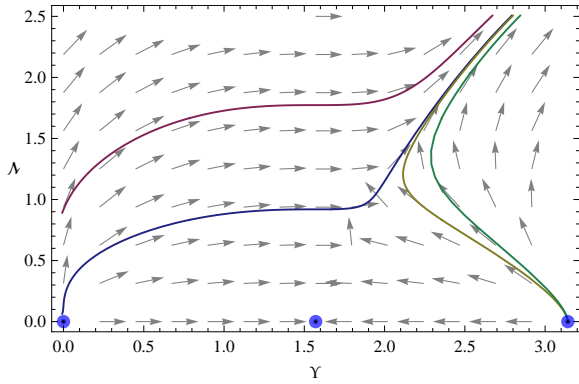


Figure 3: Phase space for new inflation $n = 2$ and $\mathcal{N} = 1$.

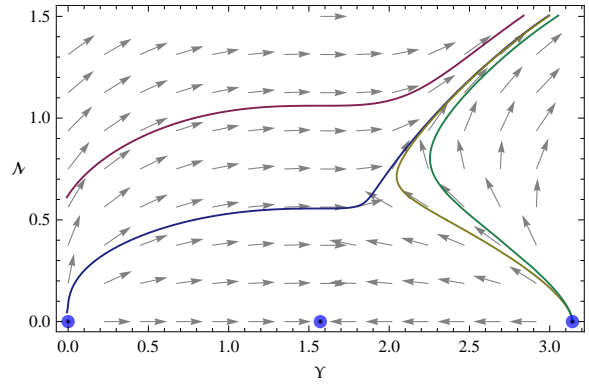


Figure 4: Phase space for new inflation $n = 2$ and $\mathcal{N} = 4$.

and the fixed points \mathcal{P} are stable if $\omega_- < 0$, i.e. if

$$\Phi_{\mathcal{P}}^{2n} > \frac{1}{1+n}, \quad (43)$$

which contains the inflationary solutions (20) for all n .

(i) Quadratic Potential

For a quadratic potential $n = 1$ and (35) reads

$$\mathcal{M} := \frac{m}{H}. \quad (44)$$

where $m = C > 0$. In this case, the fixed points $(\Psi_{\mathcal{P}}, \Phi_{\mathcal{P}}) \in [-1, 0] \times [0, 1]$ in \mathbb{S}^1 are given by condition (41) subject to (16) as

$$\mathcal{P}_0 : (\Psi, \Phi) = \left(-\sqrt{\frac{1}{2} \left(1 + \sqrt{1 - \frac{4}{9}\mathcal{M}^2} \right)}, \sqrt{\frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{9}\mathcal{M}^2} \right)} \right) \quad (45)$$

$$\mathcal{P}_1 : (\Psi, \Phi) = \left(-\sqrt{\frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{9}\mathcal{M}^2} \right)}, \sqrt{\frac{1}{2} \left(1 + \sqrt{1 - \frac{4}{9}\mathcal{M}^2} \right)} \right) \quad (46)$$

with eigenvalues (42)

$$\omega_{\mathcal{P}_0}^- = 3\sqrt{1 - \frac{4}{9}\mathcal{M}^2}, \quad \omega_{\mathcal{P}_1}^- = -3\sqrt{1 - \frac{4}{9}\mathcal{M}^2}. \quad (47)$$

Thus, in this case, the fixed points exist in the unitary circumference for $0 \leq \mathcal{M} \leq \frac{3}{2}$. For $\mathcal{M} < \frac{3}{2}$, \mathcal{P}_0 is the local source which, at $\mathcal{M} = 0$, represents the massless scalar field early attractor and \mathcal{P}_1 is the future attractor. At $(\Psi, \Phi) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, the fixed points have zero eigenvalues. Moreover, from (20), and as was shown in [35], the future attractor \mathcal{P}_1 is inflationary if and only if

$$\mathcal{M} < \sqrt{2}. \quad (48)$$

For $\mathcal{M} = \sqrt{2}$, the attractor point ceases to be inflationary and the value of ϕ at this point (which represents the end of inflation) corresponds to that of the slow-roll approximation.

(ii) Quartic Potential

For a quartic potential one has $n = 2$ and (35) reads

$$\mathcal{M} = \frac{\sqrt{2\sqrt{12}\lambda}}{\sqrt{H}},$$

with $\lambda^4 = C \geq 0$, and the fixed points, given by condition (41) subject to the flat Friedman constraint (16)

$$\Psi^2 + \Phi^4 = 1,$$

are the zeros of the cubic polynomial

$$f(\Psi^2) = \Psi^6 - \Psi^4 + \left(\frac{\mathcal{M}}{3}\right)^4.$$

The discriminant of this polynomial is

$$\Delta = \frac{1}{9^3} \left(\frac{\mathcal{M}^4}{3}\right) \left(\frac{\mathcal{M}^4}{12} - 1\right),$$

so that, for $0 < \mathcal{M}^4 < 12$, it follows that $\Delta < 0$ and there are three distinct real roots. If $\mathcal{M}^4 = 12$, then $\Delta = 0$ and there is a repeated real root, otherwise there are two complex roots. Now, setting

$$\mathcal{M}^4 = 12 \sin^2(\chi) \quad , \quad 0 < \chi < \frac{\pi}{2}, \quad (49)$$

the three distinct roots are explicitly given by

$$\mathcal{P} : \Psi^2 = \frac{1}{3} \left(1 + 2 \cos\left(\frac{2}{3}\chi + \frac{2}{3}\pi l\right)\right), \quad l = 0, \pm 1. \quad (50)$$

The $l = 1$ solution is unphysical since $\Psi^2 < 0$. Denoting the $l = 0$ and $l = -1$ solutions by \mathcal{P}_0 and \mathcal{P}_1 , respectively, then we get from (42)

$$\omega^-(\mathcal{P}_0) = -\frac{3}{2} + 3 \cos\left(\frac{2}{3}\chi\right) \quad , \quad \omega^-(\mathcal{P}_1) = -\frac{3}{2} + 3 \cos\left(\frac{2}{3}(\chi - \pi)\right). \quad (51)$$

For all values of \mathcal{M} for which there are fixed points, \mathcal{P}_0 is a source and at $\mathcal{M} = 0 \Leftrightarrow \chi = 0$ the solution represents that of a massless scalar field with $\Psi_{\mathcal{P}_0}^2 = 1$. In turn, \mathcal{P}_1 is a sink and, at $\mathcal{M} = 0$, represents a potential dominated solution $\Phi_{\mathcal{P}_1}^4 = 1$.

When $\mathcal{M}^4 = 12$, the discriminant Δ of the cubic equation is zero and its roots coincide, having $\Psi^2 = \frac{2}{3}$, which by (51) gives a saddle. Moreover, from equation (20), \mathcal{P}_1 is inflationary whenever

$$\chi < \frac{\pi}{4} \quad \Leftrightarrow \mathcal{M}^4 < 6. \quad (52)$$

2.2.2 New Inflation $\mathcal{N} > 0$

For potentials having $\mathcal{N} > 0$, Lemma 1 and (37) give the fixed points

$$\mathcal{P} : \Psi\Phi = -\frac{\mathcal{M}}{3n} \frac{\sqrt{n^2\Phi^2 + \mathcal{M}^2\mathcal{N}^2}}{\Phi} \quad \text{and} \quad \frac{\partial^2\Phi}{\partial\phi^2} = \frac{\mathcal{M}}{2n^3} \frac{\Psi\mathcal{M}^2\mathcal{N}^2}{\Phi\sqrt{n^2\Phi^2 + \mathcal{M}^2\mathcal{N}^2}} \quad (53)$$

subject to the Friedman constraint (16). Since $\mathcal{N} > 0$, we can write

$$\left(\frac{\partial^2\Phi}{\partial\phi^2}\right)_{\mathcal{P}} = -\frac{3}{2n^2}\Psi^2\Phi + \frac{\mathcal{M}^2}{6n^2\Phi} \quad (54)$$

and then (19) gives

$$\omega_- = -3\left(1 + \frac{2}{n}\right)\Phi_{\mathcal{P}}^{2n} + \frac{6}{n} - \frac{\mathcal{M}^2}{3n\Phi^2} \quad (55)$$

so that the fixed points \mathcal{P} are stable if $\omega_- < 0$ which, as in the case of chaotic inflation, contains the inflationary solutions

$$\omega_- < -2\left(1 + \frac{1}{n}\right) - \frac{\mathcal{M}^2}{3n\Phi^2}. \quad (56)$$

The case $n = 2$, $C = \lambda^{\frac{1}{4}}$, was studied in [38] and the corresponding dynamical system is given by

$$\begin{aligned} \Psi' &= 3\Psi^3 - 3\Psi - \Phi^2\mathcal{M}\sqrt{\Phi^2 + \mathcal{N}^2\mathcal{M}^2} \\ \Phi\Phi' &= \frac{1}{2}\left(3\Psi^2\Phi^2 + \Psi\mathcal{M}\sqrt{\Phi^2 + \mathcal{N}^2\mathcal{M}^2}\right) \end{aligned} \quad (57)$$

subject to the flat Friedman constraint

$$\Psi^2 + \Phi^4 = 1. \quad (58)$$

As in the quadratic case, considering the state vector (Ψ, Φ) with control parameter \mathcal{M} , the fixed points are solutions of

$$\begin{aligned} 3\Psi^2 - 3\Psi - \Phi^2\mathcal{M}\sqrt{\Phi^2 + \mathcal{N}^2\mathcal{M}^2} &= 0 \\ 3\Psi\Phi^2 + \mathcal{M}\sqrt{\sqrt{1 - \Psi^2} + \mathcal{N}^2\mathcal{M}^2} &= 0 \end{aligned} \quad (59)$$

$$3\Psi\sqrt{1 - \Psi^2} + \mathcal{M}\sqrt{\sqrt{1 - \Psi^2} + \mathcal{N}^2\mathcal{M}^2} = 0. \quad (60)$$

It is not possible, in general, to find explicitly the fixed points for this system. We shall then make a numerical stability analysis in Section 5.1.2.

3 Kinematic variables and scalar field sources

The kinematical quantities associated with a timelike congruence in General Relativity were first introduced by Ehlers [62] and Ellis [63]. Given a timelike vector field \mathbf{u} , the unique tensors

$$h_{\alpha\beta} := g_{\alpha\beta} + u_{\alpha}u_{\beta}$$

$$U_{\alpha\beta} = -u_{\alpha}u_{\beta}$$

project, at each point, tensors orthogonal and parallel to \mathbf{u} , respectively. We will use the following notation:

$$\begin{aligned} \dot{f} &= u^{\sigma}\nabla_{\sigma}f \quad , \quad D_{\alpha}f = h_{\alpha}^{\beta}\nabla_{\beta}f \\ T_{\langle\alpha_1\dots\alpha_p\rangle} &= h_{\alpha_1}^{\beta_1}\dots h_{\alpha_p}^{\beta_p}T_{\beta_1\dots\beta_p} \end{aligned}$$

so that the covariant derivative of a scalar field is decomposed into

$$\nabla_{\alpha}f = -u_{\alpha}\dot{f} + D_{\alpha}f.$$

The covariant derivative of \mathbf{u} can also be decomposed into its irreducible parts

$$\nabla_{\alpha}u_{\beta} = D_{\alpha}u_{\beta} - u_{\alpha}\dot{u}_{\beta} = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + w_{\alpha\beta} - u_{\alpha}\dot{u}_{\beta}$$

with

$$\sigma_{\alpha\beta} = \sigma_{(\alpha\beta)}; \quad \sigma_{\alpha}^{\alpha} = 0; \quad \sigma^2 = \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}; \quad \sigma_{\alpha\beta}u^{\beta} = 0; \quad \omega_{\alpha\beta} = \omega_{[\alpha\beta]}; \quad \omega_{\alpha\beta}u^{\beta} = 0, \quad (61)$$

where the curly (resp. squared) brackets denote symmetrization (resp. anti-symmetrization) of a tensor and

$$\begin{aligned} \theta &= \nabla_{\alpha}u^{\alpha} \\ \dot{u}_{\alpha} &= \nabla_{\beta}u_{\alpha}u^{\beta} \\ \sigma_{\alpha\beta} &= \nabla_{(\beta}u_{\alpha)} - \frac{1}{3}\theta h_{\alpha\beta} + \dot{u}_{(\alpha}u_{\beta)} \\ \omega_{\alpha\beta} &= \nabla_{[\beta}u_{\alpha]} + \dot{u}_{[\alpha}u_{\beta]}. \end{aligned} \quad (62)$$

The tensor $\omega_{\alpha\beta}$ is called the vorticity, $\sigma_{\alpha\beta}$ the shear and the expansion tensor is defined as

$$\theta_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}. \quad (63)$$

In the following, we shall also use the *Hubble function* defined by

$$H = \frac{1}{3}\theta. \quad (64)$$

To deduce the propagation equations for the gauge-invariant perturbation variables in Section 4 it is useful to recall the following relations between commutators of spatial and time derivatives acting on scalars [40]

$$D_{\alpha}\dot{f} = \frac{1}{3}\theta D_{\alpha}f + (\sigma_{\alpha}^{\beta} + \omega_{\alpha}^{\beta})D_{\beta}f + h_{\alpha}^{\mu}(D_{\mu}\dot{f}) - \dot{f}\dot{u}_{\alpha} \quad (65)$$

$$D_{[\alpha}D_{\beta]}f = -\omega_{\alpha\beta}\dot{f}. \quad (66)$$

The more general decomposition of the energy-momentum tensor field with respect to \mathbf{u} , is given by

$$T_{\alpha\beta} = \rho u_{\alpha}u_{\beta} + p h_{\alpha\beta} + 2u_{(\alpha}q_{\beta)} + \pi_{\alpha\beta}, \quad (67)$$

where q_{α} is the energy-transfer function and $\pi_{\alpha\beta}$ is the anisotropic stress with $u^{\alpha}q_{\alpha} = 0$, $u^{\alpha}\pi_{\alpha\beta} = 0$, $\pi_{\alpha\beta} = \pi_{(\alpha\beta)}$, $\pi^{\alpha}_{\alpha} = 0$. In the multicomponent case, we assume that the total matter energy-momentum tensor is the sum of the individual energy-momentum tensors for the components plus an interaction term Π between these components:

$$T_{\alpha\beta} := \sum_{A=1}^N T_{\alpha\beta}^A + g_{\alpha\beta}\Pi. \quad (68)$$

Moreover, given the preferred future directed time-like vector field \mathbf{u} , the EFEs are expressed through the Ricci identities applied to \mathbf{u} and the Bianchi identities in terms of the kinematic quantities, see e.g. [24].

3.1 Scalar fields

It was shown by Madsen [51] that, if we require ϕ_A to be locally constant on a spacelike hypersurface, $D_A^{\mu}\phi_A = 0$ and $\nabla^{\mu}\phi_A \neq 0$, such that $\nabla^{\mu}\phi_A$ defines uniquely a time-like vector field orthogonal to the surfaces $\phi_A = \text{const}$ with

$$(\nabla_{\lambda}\phi_A)(\nabla^{\lambda}\phi_A) < 0. \quad (69)$$

From the local decomposition of the covariant derivative ($\nabla^{\mu}\phi_A = -u_A^{\mu}\psi_A$), we find that

$$u_A^{\mu} := -\frac{1}{\psi_A}(\nabla^{\mu}\phi_A) \quad (70)$$

is a unitary time-like vector field, with ψ_A the momentum-density defined by

$$\psi_A := \dot{\phi}_A = u_A^{\lambda}\nabla_{\lambda}\phi_A. \quad (71)$$

Due to the uniqueness of u_A^{μ} , we can use the 1 + 3 covariant decomposition, in this case, taking the local projector on the spacelike hypersurfaces of constant ϕ_A , in the form

$$h_{\alpha\beta}^A \equiv g_{\alpha\beta} + \frac{1}{\psi_A^2}(\nabla_{\alpha}\phi_A)(\nabla_{\beta}\phi_A). \quad (72)$$

Then, the energy-momentum tensor of each scalar field has the perfect fluid form

$$T_{\mu\nu}^A = \left[\frac{1}{2}\psi_A^2 + \mathcal{V}(\phi_A) \right] u_\mu^A u_\nu^A + \left[\frac{1}{2}\psi_A^2 - \mathcal{V}(\phi_A) \right] h_{\mu\nu}^A \quad (73)$$

with the identifications

$$\begin{aligned} \rho_A &= \frac{1}{2}\psi_A^2 + \mathcal{V}_A \\ p_A &= \frac{1}{2}\psi_A^2 - \mathcal{V}_A \end{aligned} \quad (74)$$

and, from the total energy-momentum tensor (68), we have $\Pi = -\mathcal{W}$. Finally, decomposing each \mathbf{u}_A into components orthogonal and parallel to \mathbf{u} , it follows that [64]

$$u_A^\mu = \Gamma_A (u^\mu + v_A^\mu) \quad , \quad \Gamma_A = \frac{1}{\sqrt{1 - v_A^2}} \quad , \quad u^\mu v_\mu = 0 . \quad (75)$$

3.2 Characterization of FLRW models

The particular case of FLRW models is characterized by

$$\dot{u}_\mu = 0, \quad \sigma_{\mu\nu} = 0, \quad \omega_{\mu\nu} = 0, \quad (76)$$

and by the fact that spatial gradients of scalars are zero, in particular

$$D_\mu \phi = 0 \quad , \quad D_\mu \psi = 0 \quad , \quad D_\mu H = 0. \quad (77)$$

Furthermore, the symmetry of the spacetime forces the energy-momentum tensor (67) to have the algebraic form of a perfect fluid with

$$q_\alpha = 0, \quad \pi_{\alpha\beta} = 0. \quad (78)$$

4 Evolution of inhomogeneity variables

In this section, we shall construct a system of differential equations governing the dynamical behaviour of linear scalar perturbations of FLRW models with N interacting scalar fields which generalises the system derived in [50]. We shall take the perturbative covariant and gauge-invariant approach [39, 40, 41] which is constructed through the 1+3 covariant formalism, see e.g. [65] and references therein.

4.1 Covariant and gauge-invariant variables

We will use the following definitions for the covariant and gauge-invariant variables [50]

$$\Delta_\alpha := a(t) \frac{D_\alpha \psi}{\psi}, \quad \Delta_\alpha^A := a(t) \frac{D_\alpha \psi_A}{\psi_A}, \quad \mathcal{Z}_\alpha := a(t) D_\alpha \theta, \quad v_\alpha^A = -\frac{D_\alpha \phi_A}{\psi_A} \quad (79)$$

which represent, respectively, the total and each scalar field *comoving fractional momentum-density spatial gradients*, the *comoving spatial gradient of the expansion*, and the *velocity perturbations*. From (71), we can define the *effective scalar field momentum-density*

$$\psi^2 = \sum_{A=1}^N \psi_A^2 \quad (80)$$

which leads to the following relations between the perturbations variables

$$\Delta_\alpha = \frac{1}{\psi^2} \sum_{A=1}^N \psi_A^2 \Delta_\alpha^A. \quad (81)$$

The variables (79)-(81) contain information about three types of inhomogeneities and, similarly to the standard non-local decomposition, we follow [40] defining a local decomposition for comoving vector gradients as

$$a(t) D_\alpha X_\beta := X_{\alpha\beta} = \frac{1}{3} X h_{\alpha\beta} + \Sigma_{\alpha\beta} + \Omega_{\alpha\beta} , \quad (82)$$

where $X = a(t)D^\alpha X_\alpha$, $\Sigma_{\alpha\beta} = a(t)D_{(\alpha}X_{\beta)}$ and $\Omega_{\alpha\beta} = a(t)D_{[\alpha}X_{\beta]}$. In this way, local scalar variables can be obtained by taking the divergence of quantities (79) as

$$\Delta := a(t)D^\alpha \Delta_\alpha, \quad \mathcal{Z} := a(t)D^\alpha \mathcal{Z}_\alpha, \quad v_A := a(t)D^\alpha v_\alpha^A. \quad (83)$$

Also, we shall refer to the cosmological model with a self-interacting scalar field of potential $\mathcal{V}(\phi)$ as *close to a FLRW-nonlinear scalar field universe* in some open set if, for some suitably small constants $\varepsilon_1 \ll 1$, $\varepsilon_2 \ll 1$ and $\varepsilon_3 \ll 1$, the following inequalities hold

$$\frac{|D_\alpha \psi|}{H\psi} < \varepsilon_1, \quad \frac{|D_\alpha \phi|}{H^2 \psi} < \varepsilon_2, \quad \frac{|D_\alpha H|}{H^2} < \varepsilon_3, \quad (84)$$

where $|D_\alpha \psi| = (D_\alpha \psi D^\alpha \psi)^{\frac{1}{2}}$. We note that the constants ε_1 , ε_2 and ε_3 are taken to be different since the perturbation variables don't have the same dimensions.

4.2 Linearised equations

Let \mathbf{u} be a time-like future-directed vector-field associated with the 4-velocity field of the total matter and \mathbf{u}_A the orthogonal vectors to the surfaces $\phi_A = \text{const.}$, which are tilted with respect to \mathbf{u} by a small angle so that in (75)

$$\Gamma_A \approx 1.$$

Then, to first order, the relation between each \mathbf{u}_A and \mathbf{u} in the local rest frame defined by the latter vector field is given by [47]

$$\mathbf{u}_A \approx \mathbf{u} + \mathbf{v}_A. \quad (85)$$

By a small angle it is meant that \mathbf{u}_A is time-like, which validates the space-like vector field \mathbf{v}_A as being a small deviation from the background solution. Thus, in a FLRW background, $\mathbf{v}_A = 0$, so that \mathbf{v}_A will be a gauge-invariant perturbation variable. The total energy-momentum tensor for N minimally coupled scalar-fields, in this frame, is given by [47, 48]

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p h_{\alpha\beta} + 2u_{(\alpha} q_{\beta)} + \pi_{\alpha\beta} - g_{\alpha\beta} \mathcal{W}, \quad (86)$$

where $\mathcal{W} = -\Pi$, u_α denotes the components of \mathbf{u} and, to first order,

$$\rho = \sum_{A=1}^N \left[\frac{1}{2} \psi_A^2 + \mathcal{V}_A \right], \quad p = \sum_{A=1}^N \left[\frac{1}{2} \psi_A^2 - \mathcal{V}_A \right], \quad q_\alpha = \sum_{A=1}^N \psi_A^2 v_\alpha^A, \quad \pi_{\alpha\beta} = 0. \quad (87)$$

Also, to first order, we have that

$$\begin{aligned} \dot{q}^{<\mu>} &= \sum_{A=1}^N \left[2 \frac{\dot{\psi}_A}{\psi_A} v_A^\mu + \dot{v}_A^{<\mu>} \right] \psi_A^2 \\ &= -6H \sum_{A=1}^N \psi_A^2 v_A^\mu - 2 \sum_{A=1}^N \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right] v_A^\mu + \sum_{A=1}^N \psi_A^2 \dot{v}_A^{<\mu>} \end{aligned} \quad (88)$$

and

$$\nabla^\mu \mathcal{W} = \sum_{A=1}^N \frac{\partial \mathcal{W}}{\partial \phi_A} [-u^\mu \psi_A + D_\mu \phi_A] = - \sum_{A=1}^N \psi_A \frac{\partial \mathcal{W}}{\partial \phi_A} [u^\mu + v_A^\mu]. \quad (89)$$

Then, the exact linearised evolution equations around a FLRW-scalar field model in the frame defined by \mathbf{u} , are given by a wave equation in the 1 + 3 covariant form for the effective momentum-density

$$\dot{\psi} = -3H\psi - \frac{1}{\psi} \sum_{A=1}^N \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right] - \frac{1}{\psi} \sum_{A=1}^N \psi_A^2 v_A \quad , \quad \psi \neq 0. \quad (90)$$

The momentum conservation equation is

$$\psi^2 \dot{u}^\mu = -\psi (D^\mu \psi) - \sum_{A=1}^N \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right] v_A^\mu - 4H \sum_{A=1}^N \psi_A^2 v_A^\mu - \dot{q}^{<\mu>}$$

which, after multiplying by the scale factor $a(t)$ and using (88), simplifies to

$$a(t)\dot{u}^\mu = -\Delta^\mu + \frac{a(t)}{\psi^2} \sum_{A=1}^N \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A} \right] v_A^\mu + 2H \frac{a(t)}{\psi^2} \sum_{A=1}^N \psi_A^2 v_A^\mu - \frac{a(t)}{\psi^2} \sum_{A=1}^N \psi_A^2 \dot{v}_A^{<\mu>}. \quad (91)$$

In turn, the linearised Raychaudhuri equation for scalar fields is (see also e.g. [50])

$$3\dot{H} = -3H^2 - \psi^2 + \sum_{A=1}^N \mathcal{V}_A + \mathcal{W} + D_\sigma \dot{u}^\sigma. \quad (92)$$

The N linearised wave equations in the 1+3 covariant form, for each scalar field, are

$$\dot{\psi}_A = -3H\psi_A - \frac{\partial\mathcal{V}_A}{\partial\phi_A} - \frac{\partial\mathcal{W}}{\partial\phi_A} - \psi_A v_A \quad , \quad \psi_A \neq 0 \quad , \quad A = 1, \dots, N \quad (93)$$

and the first order equation associated with the momentum conservation equation, for each scalar field, is

$$a(t)\dot{u}_\alpha + \Delta_\alpha^A + \left\{ 2\frac{\dot{\psi}_A}{\psi_A} + 4H + \frac{1}{\psi_A} \left[\frac{\partial\mathcal{W}}{\partial\phi_A} + \frac{d\mathcal{V}_A}{d\phi_A} \right] \right\} a(t)v_\alpha^A + a(t)\dot{v}_{<\alpha>}^A = 0. \quad (94)$$

4.2.1 Evolution equation for Δ_μ

To obtain the evolution equation for Δ_μ , we take spatial gradients of equation (90) and keep the first order terms to get

$$\begin{aligned} a(t)\frac{D_\mu\dot{\psi}}{\psi} = & -\mathcal{Z}_\mu - \left[3H - \frac{1}{\psi^2} \sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A} \right) \right] \Delta_\mu - \frac{1}{\psi^2} \sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A} \right) \Delta_\mu^A \\ & - \frac{a(t)}{\psi^2} D_\mu \sum_{A=1}^N \psi_A^2 v_A + \frac{a(t)}{\psi^2} \sum_{A=1}^N \psi_A^2 \frac{d^2\mathcal{V}_A}{d\phi_A^2} v_\mu^A + \frac{a(t)}{\psi^2} \sum_{A,B=1}^N \psi_A \psi_B \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} v_\mu^B. \end{aligned} \quad (95)$$

Then, using the linearised relation (65) for the effective momentum-density variable ψ , we find

$$D_\mu\dot{\psi} = HD_\mu\psi + h_\mu^\nu(D_\nu\dot{\psi}) - \dot{\psi}\dot{u}_\mu$$

together with the relation

$$a(t)\frac{h_\mu^\nu(D_\nu\dot{\psi})}{\psi} = \dot{\Delta}_{<\mu>} - \left[H - \frac{\dot{\psi}}{\psi} \right] \Delta_\mu$$

giving

$$a(t)\frac{D_\mu\dot{\psi}}{\psi} = \dot{\Delta}_{<\mu>} + \frac{\dot{\psi}}{\psi}\Delta_\mu - a(t)\frac{\dot{\psi}}{\psi}\dot{u}_\mu,$$

which, after inserting into (95), finally gives

$$\begin{aligned} \dot{\Delta}_{<\mu>} = & -\mathcal{Z}_\mu - \left[\frac{\dot{\psi}}{\psi} + 3H - \frac{1}{\psi^2} \sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A} \right) \right] \Delta_\mu - \frac{1}{\psi^2} \sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A} \right) \Delta_\mu^A \\ & - \frac{a(t)}{\psi^2} D_\mu \sum_{A=1}^N \psi_A^2 v_A + \frac{a(t)}{\psi^2} \sum_{A=1}^N \psi_A^2 \frac{d^2\mathcal{V}_A}{d\phi_A^2} v_\mu^A + \frac{a(t)}{\psi^2} \sum_{A,B=1}^N \psi_A \psi_B \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} v_\mu^B + a(t)\frac{\dot{\psi}}{\psi}\dot{u}_\mu. \end{aligned} \quad (96)$$

4.2.2 Evolution equation for \mathcal{Z}_μ

The evolution equation for the perturbation variable \mathcal{Z}_μ is found by taking spatial gradients of the linearized Raychaudhuri equation (92) which, after multiplication by the scale factor $a(t)$, reads

$$3a(t)D_\mu\dot{H} = -2H\mathcal{Z}_\mu - 2\psi^2\Delta_\mu - a(t)\sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A} \right) v_\mu^A + a(t)D_\mu D_\sigma \dot{u}^\sigma. \quad (97)$$

Now, using (65) we get

$$D_\mu \dot{H} = HD_\mu H + h_\mu^\nu (D_\nu H) - \dot{H} \dot{u}_\mu ,$$

which together with

$$3a(t)h_\mu^\nu (D_\nu H) = \dot{Z}_{<\mu>} - HZ_\mu ,$$

gives

$$3a(t)D_\mu \dot{H} = \dot{Z}_{<\mu>} - 3\dot{H}a(t)\dot{u}_\mu .$$

Finally, inserting the last equation into (97) gives

$$\dot{Z}_{<\mu>} = -2HZ_\mu - 2\psi^2\Delta_\mu - a(t) \sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A} \right) v_\mu^A + a(t)D_\mu D_\sigma \dot{u}^\sigma + 3\dot{H}a(t)\dot{u}_\mu . \quad (98)$$

4.2.3 Evolution equation for Δ_μ^A and v_μ^A

The evolution equation for each variable Δ_μ^A is obtained by taking spatial gradients of equation (93) and keeping first order terms as

$$a(t) \frac{D_\mu \dot{\psi}_A}{\psi_A} = -3H\Delta_\mu^A - Z_\mu - a(t)D_\mu v_A + a(t) \frac{d^2\mathcal{V}_A}{d\phi_A^2} v_\mu^A + a(t) \sum_{B=1}^N \frac{\psi_B}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} v_\mu^B$$

and

$$\begin{aligned} \dot{\Delta}_{<\mu>}^A = & - \left[\frac{\dot{\psi}_A}{\psi_A} + 3H \right] \Delta_\mu^A - Z_\mu - a(t)D_\mu D_\sigma v_A^\sigma + a(t) \frac{d^2\mathcal{V}_A}{d\phi_A^2} v_\mu^A \\ & + a(t) \sum_{B=1}^N \frac{\psi_B}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} v_\mu^B + a(t) \frac{\dot{\psi}_A}{\psi_A} \dot{u}_\mu . \end{aligned} \quad (99)$$

To get the evolution equation for the velocity perturbation variables, we can use the relation (65) for each scalar field ϕ_A and get

$$a(t)\dot{v}_{<\mu>}^A = -a(t) \left[H + \frac{\dot{\psi}_A}{\psi_A} \right] v_\mu^A - \Delta_\mu^A - a(t)\dot{u}_\mu \quad , \quad \psi_A \neq 0. \quad (100)$$

This equation is identical to the momentum conservation equation for the A th component (94), after using the background nonlinear wave equation in the 1 + 3 form.

4.3 Equations in the Energy frame

In order to close the system of evolution and constraint equations, we need to fix the frame for which we are constructing perturbation variables. Furthermore, this choice of frame must ensure that the perturbation variables are gauge invariant. A suitable choice is the energy frame defined through

$$q^\mu = 0 \quad , \quad \dot{q}^{<\mu>} = 0. \quad (101)$$

Thus, if we choose $\mathbf{u} = \mathbf{u}_E$ to be the energy frame, then

$$\sum_{A=1}^N \psi_A^2 v_A^\mu = 0$$

and

$$\sum_{A=1}^N \psi_A^2 \dot{v}_A^{<\mu>} = 2 \sum_{A=1}^N \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A} \right] v_A^\mu .$$

In this frame, (91) reads

$$a(t)\dot{u}^\mu = -\Delta^\mu - \frac{a(t)}{\psi^2} \sum_{A=1}^N \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial\mathcal{W}}{\partial\phi_A} \right] v_A^\mu \quad (102)$$

and the first order equation for the divergence of the acceleration is

$$a^2 D_\sigma \dot{u}^\sigma = -\Delta - \frac{a(t)}{\psi^2} \sum_{A=1}^N \psi_A \left[\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right] v_A. \quad (103)$$

By using (102) and (103) into (96), (98), (99) and (100), and after multiplying the resulting equation by the scale factor and taking the spatial divergence of these equations, we finally obtain:

Proposition 4.1. *The evolution of first order scalar perturbations on FLRW-scalar fields background with arbitrary smooth potentials in the energy frame, is given by the following system of equations for the variables Δ , \mathcal{Z} , Δ_A and v_A :*

$$\begin{aligned} \dot{\Delta} = & -\mathcal{Z} + \left[3H + \frac{3}{\psi^2} \sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right) \right] \Delta - \frac{1}{\psi^2} \sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right) \Delta_A \\ & + a(t) \left[3H + \frac{1}{\psi^2} \sum_{C=1}^N \psi_C \left(\frac{d\mathcal{V}_C}{d\phi_C} + \frac{\partial \mathcal{W}}{\partial \phi_C} \right) \right] \frac{1}{\psi^2} \sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right) v_A \\ & + \frac{a(t)}{\psi^2} \sum_{A=1}^N \psi_A^2 \frac{d^2 \mathcal{V}_A}{d\phi_A^2} v_A + \frac{a(t)}{\psi^2} \sum_{A,B=1}^N \psi_A \psi_B \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} v_B \end{aligned} \quad (104)$$

$$\dot{\mathcal{Z}} = -2H\mathcal{Z} - \left[3\dot{H} + 2\psi^2 + D^2 \right] \Delta - \left[\psi^2 + 3\dot{H} + D^2 \right] \frac{a(t)}{\psi^2} \sum_{A=1}^N \psi_A \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right) v_A \quad (105)$$

$$\begin{aligned} \dot{\Delta}_A = & -\mathcal{Z} + \left[3H + \frac{1}{\psi_A} \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right) \right] \Delta - \left[\frac{1}{\psi_A} \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right) \right] \Delta_A \\ & - a(t) D^2 v_A + a(t) \frac{d^2 \mathcal{V}_A}{d\phi_A^2} v_A + a(t) \sum_{B=1}^N \frac{\psi_B}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} v_B \\ & + \left[3H + \frac{1}{\psi_A} \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right) \right] \frac{a(t)}{\psi^2} \sum_{B=1}^N \psi_B \left[\frac{d\mathcal{V}_B}{d\phi_B} + \frac{\partial \mathcal{W}}{\partial \phi_B} \right] v_B \end{aligned} \quad (106)$$

$$a(t) \dot{v}_A = \Delta - \Delta_A + a(t) \left[2H + \frac{1}{\psi_A} \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right) \right] v_A + \frac{a(t)}{\psi^2} \sum_{B=1}^N \psi_B \left[\frac{d\mathcal{V}_B}{d\phi_B} + \frac{\partial \mathcal{W}}{\partial \phi_B} \right] v_B, \quad (107)$$

together with the background equations (6)-(8).

4.4 Equations in Relative variables

In order to simplify the notation and the calculations in the multiple scalar field case, let

$$\alpha_A := \frac{1}{\psi_A} \left(\frac{d\mathcal{V}_A}{d\phi_A} + \frac{\partial \mathcal{W}}{\partial \phi_A} \right), \quad \beta_A^2 = \frac{\psi_A^2}{\psi^2} \quad (108)$$

so that

$$\dot{\alpha}_A = (3H + \alpha_A) \alpha_A + \frac{d^2 \mathcal{V}_A}{d\phi_A^2} + \frac{\partial^2 \mathcal{W}}{\partial \phi_A^2} + \sum_{B \neq A}^N \frac{\psi_B}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} \quad (109)$$

$$(\dot{\beta}_A^2) = -2 \sum_{B \neq A}^N (\alpha_A - \alpha_B) \beta_A^2 \beta_B^2 \quad (110)$$

and, from the fact that β_A verifies $\sum_{A=1}^N \beta_A^2 = 1$, we get

$$\sum_{A=1}^N (\dot{\beta}_A^2) = 0. \quad (111)$$

Defining *relative perturbation variables* as

$$\Delta_{[AB]} = \Delta_A - \Delta_B \quad , \quad v_{[AB]} = v_A - v_B, \quad (112)$$

then (81) reads

$$\Delta_A = \Delta + \sum_{B \neq A}^N \beta_B^2 \Delta_{[AB]}. \quad (113)$$

For the relative velocity perturbation variables, using the energy-frame condition (101), it follows that

$$v_A = \sum_{B \neq A}^N \beta_B^2 v_{[AB]}, \quad (114)$$

and, using the above variables, we obtain from equations (104) and (105),

$$\dot{\Delta} = A(t)\Delta - \mathcal{Z} - \sum_{A=1}^N \sum_{B>A}^N B_{AB}(t)\Delta_{[AB]} + a(t) \sum_{A=1}^N \sum_{B>A}^N C_{AB}(t)v_{[AB]} \quad (115)$$

$$\dot{\mathcal{Z}} = -2H\mathcal{Z} - \left[3\dot{H} + 2\psi^2 + D^2\right] \Delta \quad (116)$$

$$- a(t) \sum_{A=1}^N \sum_{B>A}^N (\alpha_A - \alpha_B) \beta_A^2 \beta_B^2 \left[\psi^2 + 3\dot{H} + D^2\right] v_{[AB]}. \quad (117)$$

To get the evolution equation for $\Delta_{[AB]}$ we take the difference $\dot{\Delta}_{[AB]} = \dot{\Delta}_A - \dot{\Delta}_B$, which upon using (113) and the following relation

$$\begin{aligned} (\alpha_A - \alpha_B)\Delta - \alpha_A\Delta_A + \alpha_B\Delta_B &= -(\alpha_A\beta_B^2 + \alpha_B\beta_A^2)\Delta_{[AB]} \\ &\quad - \alpha_A \sum_{C \neq A,B}^N \beta_C^2 \Delta_{[AC]} - \alpha_B \sum_{C \neq A,B}^N \beta_C^2 \Delta_{[BC]} \end{aligned}$$

gives

$$\begin{aligned} \dot{\Delta}_{[AB]} &= -D_{AB}(t)\Delta_{[AB]} + \sum_{C \neq A,B}^N (\alpha_A\beta_C^2\Delta_{[AC]} - \alpha_B\beta_C^2\Delta_{[BC]}) \\ &\quad + a(t) \sum_{C \neq A,B}^N [F_{AC}(t)v_{[AC]} - F_{BC}(t)v_{[BC]}] \\ &\quad + a(t) \sum_{C \neq A,B}^N \sum_{D \neq A,B,C}^N \left(\frac{\psi_C}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} - \frac{\psi_C}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) \beta_D^2 v_{[CD]} \end{aligned} \quad (118)$$

as well as

$$a\dot{v}_{[AB]} = -\Delta_{[AB]} + a(t)(2H + D_{AB})v_{[AB]} + a(t) \sum_{C \neq A,B}^N (\alpha_A\beta_C^2 v_{[AC]} - \alpha_B\beta_C^2 v_{[BC]}), \quad (119)$$

where

$$\begin{aligned}
A(t) &= 3H + 2(\alpha_A\beta_A^2 + \alpha_B\beta_B^2) + 2\sum_{C \neq A,B}^N \alpha_C\beta_C^2 \\
B_{AB}(t) &= (\alpha_A - \alpha_B)\beta_A^2\beta_B^2 \\
C_{AB}(t) &= (3H + (\alpha_A\beta_A^2 + \alpha_B\beta_B^2))B_{AB} \\
&\quad + \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2} \right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} \right) - \left(\frac{\psi_A}{\psi_B} - \frac{\psi_B}{\psi_A} \right) \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} \right] \beta_A^2\beta_B^2 \\
&\quad + \sum_{C \neq A,B}^N \left[\alpha_C\beta_C^2 B_{AB} + \left(\frac{\psi_C}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} - \frac{\psi_C}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C} \right) \right] \\
D_{AB}(t) &= \alpha_A\beta_B^2 + \alpha_B\beta_A^2 \\
E_{AB}(t) &= \left(\frac{d^2\mathcal{V}}{d\phi_A^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\psi_A}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B} \right) \beta_B^2 + \left(\frac{d^2\mathcal{V}}{d\phi_B^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} - \frac{\psi_B}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} \right) \beta_A^2 \\
&\quad + (\alpha_A - \alpha_B)B_{AB} \\
F_{AB}(t) &= \alpha_A(\alpha_A + \alpha_B)\beta_A^2\beta_B^2.
\end{aligned} \tag{120}$$

We note that the system (118)-(119) is closely related to the system obtained by Bruni et al. in [50, 48].

Now, as usual, we can decouple the evolution equation for \mathcal{Z} by differentiating (115) with respect to time t and using (116), to obtain

$$\begin{aligned}
\ddot{\Delta} &= (A - 2H)\dot{\Delta} + \left(\dot{A} + 2HA + 3\dot{H} + 2\psi^2 + D^2 \right) \Delta \\
&\quad - \sum_{A=1}^N \sum_{B>A}^N \left[\dot{B}_{AB} + (2H - D_{AB})B_{AB} + C_{AB} \right] \Delta_{[AB]} \\
&\quad - \sum_{A=1}^N \sum_{B>A}^N \sum_{C \neq A,B}^N B_{AB} (\alpha_A\beta_C^2 \Delta_{[AC]} - \alpha_B\beta_C^2 \Delta_{[BC]}) \\
&\quad + a(t) \sum_{A=1}^N \sum_{B>A}^N \left[\dot{C}_{AB} + (5H + D_{AB})C_{AB} + B_{AB} (3\dot{H} + \psi^2 - E_{AB} + 2D^2) \right] v_{[AB]} \\
&\quad - a(t) \sum_{A=1}^N \sum_{B>A}^N \sum_{C \neq A,B}^N \left[(B_{AB}F_{AC} - C_{AB}\alpha_A\beta_C^2) v_{[AC]} + (B_{AB}F_{BC} + C_{AB}\alpha_B\beta_C^2) v_{[BC]} \right] \\
&\quad - a(t) \sum_{A=1}^N \sum_{B>A}^N \sum_{C \neq A,B}^N \sum_{D \neq A,B,C}^N B_{AB} \left(\frac{\psi_C}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} - \frac{\psi_C}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B} \right) \beta_D^2 v_{[CD]}.
\end{aligned} \tag{121}$$

where the evolution equations for A, B_{AB} and C_{AB} are given in section 6.1 of the appendix. Due to its generality, the system composed by equations (121) and (169)-(171) is quite long. We shall see ahead a simplified version of this system in the case of two scalar fields.

4.5 Decomposition into scalar harmonics and particular solutions

A common procedure to analyse the PDE system of equations derived above is to transform it into a system of ODEs by doing a harmonic decomposition. This is done by expanding the first order gauge invariant scalars Δ in terms of scalar harmonics¹ $Q_{(n)}$ as [66],

$$\Delta = \sum_n \Delta_{(n)} Q_{(n)}, \tag{122}$$

which are comoving eigenfunctions of the operator

$$D^2 Q_{(n)} = -\frac{n^2}{a^2} Q_{(n)}, \quad \text{with } \dot{Q}_{(n)} = 0. \tag{123}$$

¹The wave number n shouldn't be confused with the exponent n in the scalar field potentials.

This is a useful procedure that has been followed many times in the literature, such as in models of structure formation in the universe [67, 41] and in the study of perturbations of spherically symmetric models, see e.g. [68] and references therein.

Using the time variable τ defined in (11) and the above harmonic decomposition, the system of equations (118), (119) and (121) reads

$$\begin{aligned}
\Delta'' &= \left[q + 2 + 2 \left(\frac{\alpha_A}{H} \beta_A^2 + \frac{\alpha_B}{H} \beta_B^2 \right) + 2 \sum_{C \neq A, B}^N \beta_C^2 \frac{\alpha_C}{H} \right] \Delta' + \left[\frac{2HA + 3HH' + 2\psi^2 + HA'}{H^2} - \frac{n^2}{a^2 H^2} \right] \Delta \\
&- \sum_{A=1}^N \sum_{B>A}^N \left[\frac{HB'_{AB} + (2H - D_{AB}) B_{AB} + C_{AB}}{H^2} \right] \Delta_{[AB]} \\
&+ \sum_{A=1}^N \sum_{B>A}^N \sum_{C \neq A, B}^N \frac{B_{AB}}{H} \beta_C^2 \left(\frac{\alpha_A}{H} \Delta_{[AC]} - \frac{\alpha_B}{H} \Delta_{[BC]} \right) \\
&+ a(t) \sum_{A=1}^N \sum_{B>A}^N \left[\frac{HC'_{AB} + (5H + D_{AB}) C_{AB}}{H^2} + B_{AB} \left(\frac{3HH' + \psi^2 - E_{AB}}{H^2} - 2 \frac{q^2}{a^2 H^2} \right) \right] v_{[AB]} \\
&- a(t) \sum_{A=1}^N \sum_{B>A}^N \sum_{C \neq A, B}^N \left[(B_{AB} F_{AC} - C_{AB} \alpha_A \beta_C^2) v_{[AC]} + (B_{AB} F_{BC} + C_{AB} \alpha_B \beta_C^2) v_{[BC]} \right] \\
&- a(t) \sum_{A=1}^N \sum_{B>A}^N \sum_{C \neq A, B}^N \sum_{D \neq A, B, C}^N B_{AB} \left(\frac{\psi_C}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} - \frac{\psi_C}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) \beta_D^2 v_{[CD]} \\
\Delta'_{[AB]} &= - \frac{D_{AB}}{H} \Delta_{[AB]} + \left(\frac{E_{AB}}{H^2} + \frac{q^2}{a^2 H^2} \right) v_{[AB]} + \sum_{C \neq A, B}^N \beta_C^2 \left(\frac{\alpha_A}{H} \Delta_{[AC]} - \frac{\alpha_B}{H} \Delta_{[BC]} \right) \\
&+ a(t) H \sum_{C \neq A, B}^N \left[\frac{F_{AC}}{H^2} v_{[AC]} - \frac{F_{BC}}{H^2} v_{[BC]} \right] \\
&+ a(t) H \sum_{C \neq A, B}^N \left[\frac{1}{H^2} \sum_{D \neq A, B, C}^N \left(\frac{\psi_C}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} - \frac{\psi_C}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) \beta_D^2 v_{[CD]} \right] \\
aH v'_{[AB]} &= - \Delta_{[AB]} + a(2H + D_{AB}) v_{[AB]} + a(t) H \sum_{C \neq A, B}^N \left(\frac{\alpha_A}{H} \beta_C^2 v_{[AC]} - \frac{\alpha_B}{H} \beta_C^2 v_{[BC]} \right),
\end{aligned} \tag{124}$$

where q is the wave number² associated to the velocity scalar perturbations. In the case where the only matter present is a single nonlinear scalar field ϕ , then the relative density perturbations are identically zero $\Delta_{[AB]} = 0$ and the velocity perturbations vanish since $D^2\phi = 0$, by construction. In this case, the system (124) reduces to the linear second order ODE

$$\begin{aligned}
\Delta''_{(n)} &- \left[1 - \frac{\dot{H}}{H^2} + \frac{2}{H\psi} \frac{d\mathcal{V}}{d\phi} \right] \Delta'_{(n)} \\
&- \left[6 + 6 \frac{\dot{H}}{H^2} + 2 \frac{\psi^2}{H^2} + \frac{2}{H\psi} \frac{d\mathcal{V}}{d\phi} \left(5 + \frac{1}{H\psi} \frac{d\mathcal{V}}{d\phi} \right) + \frac{2}{H^2} \frac{d^2\mathcal{V}}{d\phi^2} - \frac{n^2}{a^2 H^2} \right] \Delta_{(n)} = 0,
\end{aligned} \tag{125}$$

which depends on the background quantities H , ψ and on the first and second derivatives of the potential \mathcal{V} with respect to ϕ which, in turn, depends on time.

In the case of a perfect fluid with a linear equation of state $p = (1 - \gamma)\rho$, the analogous differential equation can be found in Chapter 14 of [24], as equation (14.32). The case of dust in a flat background can be solved explicitly, with the particularity that the solution is independent of n , while for general γ , the solution can be written in terms of Bessel functions (see Goode [69] for the corresponding equations using Bardeen variables [43]).

It is also possible to get explicit solutions in the so-called *long-wavelength limit*, which amounts to consider solutions with wavelengths larger than the Hubble distance, or equivalently

$$\frac{n}{aH} \ll 1. \tag{126}$$

²The wave number q shouldn't be confused with the deceleration parameter q .

For example, for the exact massless scalar field background solution, the general solution to the perturbation equations (125) can be written in terms of special functions, depending on the wave number n , while in the long-wavelength limit the solution is

$$\Delta(\tau) = \frac{C_1}{4} e^{4\tau} + C_2 \quad (127)$$

or, in terms of cosmic time t ,

$$\Delta = C_2 + \frac{3}{4} C_1 t^{\frac{4}{3}}. \quad (128)$$

Even when it is possible to solve explicitly the background equations for a given potential, the perturbed system is, in general, impossible to solve. In [70], Zimdahl used the slow-roll approximation in order to simplify the coefficients and studied the behaviour of density inhomogeneities during slow-roll inflation. In what follows, we shall apply a dynamical systems' approach to perform a qualitative analysis of the evolution of density inhomogeneities.

5 Dynamical systems' approach to density inhomogeneities

We shall now use the system of equations derived in last section and employ a dynamical systems' approach. Our approach follows the methods of Woszczyna [55, 56, 57, 58] to study the evolution of inhomogeneities, which was also used to study stability problems in a universe with dust and radiation [59], magnetized cosmologies [60] and locally rotational symmetric Bianchi I models [61]. Following these works, we introduce the dimensionless variables

$$\mathcal{U}_{(n)} := \frac{\Delta'_{(n)}}{\Delta_{(n)}} \quad , \quad \mathcal{X}_{[AB]} := \frac{\Delta_{[AB]}}{\Delta} \quad , \quad \mathcal{Y}_{[AB]} := \frac{aHv_{[AB]}}{\Delta} \quad , \quad (129)$$

and we arrive at the following result:

Proposition 5.1. *The evolution for the first order scalar perturbations on a FLRW-scalar fields background, with arbitrary smooth potentials, is given by the following system of differential equations for the*

state vector $((\Psi_A, \Phi_A, \Theta_A), \mathcal{U}, \mathcal{X}_{[AB]}, \mathcal{Y}_{[AB]})$:

$$\begin{aligned}
\mathcal{U}' &= -\mathcal{U}^2 - \xi\mathcal{U} - \zeta + \sum_{A=1}^N \sum_{B>A}^N \gamma_{AB} \mathcal{X}_{[AB]} + \sum_{A=1}^N \sum_{B>A}^N \eta_{AB} \mathcal{Y}_{[AB]} \\
&+ \sum_{A=1}^N \sum_{B>A}^N \sum_{C \neq A, B}^N \frac{\Psi_C^2}{\Psi^2} \frac{B_{AB}}{H} \left(\frac{\alpha_A}{H} \mathcal{X}_{[AC]} - \frac{\alpha_B}{H} \mathcal{X}_{[BC]} \right) \\
&- \sum_{A=1}^N \sum_{B>A}^N \sum_{C \neq A, B}^N \left(\frac{(B_{AB} F_{AC} - C_{AB} \alpha_A \beta_C^2)}{H^3} \mathcal{Y}_{[AC]} + \frac{(B_{AB} F_{BC} - C_{AB} \alpha_B \beta_C^2)}{H^3} \mathcal{Y}_{[BC]} \right) \\
&- \sum_{A=1}^N \sum_{B>A}^N \sum_{C \neq A, B}^N \sum_{D \neq A, B, C}^N \frac{B_{AB}}{H^3} \left(\frac{\Psi_C}{\Psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} - \frac{\Psi_C}{\Psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) \beta_D^2 \mathcal{Y}_{[CD]} \\
\mathcal{X}'_{[AB]} &= -\mathcal{U} \mathcal{X}_{[AB]} + \varsigma_{AB} \mathcal{X}_{[AB]} + \varpi_{AB} \mathcal{Y}_{[AB]} + \sum_{C \neq A, B} \beta_C^2 \left(\frac{\alpha_A}{H} \mathcal{X}_{[AC]} - \frac{\alpha_B}{H} \mathcal{X}_{[BC]} \right) \\
&+ \sum_{C \neq A, B}^N \left[\frac{F_{AC}}{H^2} \mathcal{Y}_{[AC]} - \frac{F_{BC}}{H^2} \mathcal{Y}_{[BC]} + \frac{1}{H^2} \sum_{D \neq A, B, C}^N \left(\frac{\Psi_C}{\Psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} - \frac{\Psi_C}{\Psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) \beta_D^2 \mathcal{Y}_{[CD]} \right] \quad (130) \\
\mathcal{Y}'_{[AB]} &= -\mathcal{U} \mathcal{Y}_{[AB]} + \iota_{AB} \mathcal{Y}_{[AB]} - \mathcal{X}_{[AB]} \\
&+ \sum_{C \neq A, B}^N \left(\frac{\alpha_A}{H} \beta_C^2 \mathcal{Y}_{[AC]} - \frac{\alpha_B}{H} \beta_C^2 \mathcal{Y}_{[BC]} \right) \\
\Psi'_A &= (q-2)\Psi_A - n\sqrt{6} \left[\Phi_A^{2n-1} \frac{\partial \Phi_A}{\partial \phi_A} + \Theta_A^{2n-1} \frac{\partial \Theta}{\partial \phi_A} \right] \\
\Phi'_A &= \frac{1}{n}(q+1)\Phi_A + \sqrt{6} \frac{\partial \Phi_A}{\partial \phi_A} \Psi_A \\
\Theta' &= \frac{1}{n}(q+1)\Theta + \sqrt{6} \sum_{A=1}^N \frac{\partial \Theta}{\partial \phi_A} \Psi_A
\end{aligned}$$

subject to the background constraint equation

$$\sum_{A=1}^N \Psi_A^2 + \sum_{A=1}^N \Phi_A^{2n} + \Theta^{2n} = 1 - K, \quad (131)$$

where

$$q = 2 \sum_{A=1}^N \Psi_A^2 - \sum_{A=1}^N \Phi_A^{2n} - \Theta^{2n}$$

and the coefficients $\xi, \zeta, \gamma_{AB}, \eta_{AB}, \varsigma, \varpi$ and ι_{AB} are given in appendix.

We note that, in terms of the quantities (10), we have

$$\begin{aligned}
\frac{d\mathcal{V}_A}{d\phi_A} &= 6nH^2 \Phi_A^{2n-1} \frac{\partial \Phi_A}{\partial \phi_A}, \quad \frac{d^2 \mathcal{V}_A}{d\phi_A^2} = 6nH^2 \left[(2n-1) \Phi_A^{2(n-1)} \left(\frac{\partial \Phi_A}{\partial \phi_A} \right)^2 + \Phi_A^{2n-1} \frac{\partial^2 \Phi_A}{\partial \phi_A^2} \right] \\
\alpha_A &= \frac{\sqrt{6}nH}{\Psi_A} \left[\Phi_A^{2n-1} \frac{\partial \Phi_A}{\partial \phi_A} + \Theta^{2n-1} \frac{\partial \Theta}{\partial \phi_A} \right], \quad \beta_A^2 = \frac{\Psi_A^2}{\Psi^2} \quad (132)
\end{aligned}$$

where

$$\Psi^2 = \sum_{A=1}^N \Psi_A^2$$

and

$$\frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} = 6nH^2 \left[(2n-1) \Theta^{2(n-1)} \left(\frac{\partial \Theta}{\partial \phi_B} \right) \left(\frac{\partial \Theta}{\partial \phi_A} \right) + \Theta^{2n-1} \frac{\partial^2 \Theta}{\partial \phi_B \partial \phi_A} \right]$$

$$\frac{d^3\mathcal{V}_A}{d\phi_A^3} = 6nH^2(2n-1) \left[3\Phi^{2(n-1)} \frac{\partial\Phi_A}{\partial\phi_A} \frac{\partial^2\Phi_A}{\partial\phi_A^2} + 2(n-1)\Phi_A^{2(n-\frac{3}{2})} \left(\frac{\partial\Phi_A}{\partial\phi_A} \right)^3 \right] + 6nH^2\Phi^{2n-1} \frac{\partial^3\Phi_A}{\partial\phi_A^3}$$

which may be substituted in the above equations.

As explained by Dunsby in Chapter 14 of [24], the variable $\mathcal{U}_{(n)}$ should be viewed as $\tan(\theta_{(n)})$, where $0 \leq \theta_{(n)} < 2\pi$ is the polar angle in the plane $(\Delta_{(n)}, \Delta'_{(n)})$. For scalar fields in a flat background, we consider the variables subset defined by

$$\sum_{A=1}^N \Psi_A^2 + \sum_{A=1}^N \Phi_A^2 = 1, \quad -\infty < \mathcal{U}_n < +\infty, \quad -\infty < \mathcal{X}_{[AB]} < +\infty, \quad -\infty < \mathcal{Y}_{[AB]} < +\infty \quad (133)$$

and regard the state space as $\mathbb{S}^N \times \mathbb{S}^1 \times \mathbb{R}^{N(N-1)}$. As we shall see ahead, in the case of assisted power-law inflation, the fixed points are restricted to the compact subset $\mathbb{S}^N \times \mathbb{S}^1$. When only one scalar field is present, we can use either the variable Ψ or Φ since they are related through the flat Friedman constraint (16), and the state space is regarded as the *cylinder* $[0, 1] \times \mathbb{S}^1$.

The use of the variable $\mathcal{U}_{(n)}$ makes the analysis of the system's stability quite transparent: If an orbit is asymptotic to an equilibrium point, the perturbation approaches a stationary state either: decaying to zero if $\mathcal{U} < 0$, growing if $\mathcal{U} > 0$ or having a constant value $\mathcal{U} = 0$. If the orbit is asymptotic to a periodic orbit in the cylinder, the perturbation propagates as waves (see pgs. 296-297 of Chapter 14 in [24]).

We shall now investigate, separately, the cases of one and two scalar fields.

5.1 Single scalar field

In the case of a single scalar field, we obtain from Proposition 5.1 the following result:

The evolution of the phase of first order scalar perturbations of FLRW-nonlinear scalar field models, is given by the following system of differential equations for the state vector $((\Psi, \Phi), \mathcal{U}_{(n)})$:

$$\begin{aligned} \mathcal{U}'_{(n)} &= -\mathcal{U}_{(n)}^2 - \xi(\Psi, \Phi)\mathcal{U}_{(n)} - \zeta(\Psi, \Phi) \\ \Psi' &= 2\Psi^3 - (2 + \Phi^{2n})\Psi - n\sqrt{6}\Phi^{2n-1} \frac{\partial\Phi}{\partial\phi} \\ \Phi' &= -\frac{\Phi^{2n+1}}{n} + \frac{(1 + 2\Psi^2)}{n}\Phi + \sqrt{6}\Psi \frac{\partial\Phi}{\partial\phi} \end{aligned} \quad (134)$$

subject to the background constraint equation

$$\Psi^2 + \Phi^{2n} + K = 1 \quad (135)$$

and with the coefficients given by

$$\begin{aligned} \xi(\Phi, \Psi) &= - \left[2 + 2\Psi^2 - \Phi^{2n} + \frac{12n}{\sqrt{6}} \frac{\Phi^{2n-1}}{\Psi} \frac{\partial\Phi}{\partial\phi} \right] \\ \zeta(\Phi, \Psi) &= -6\Phi^{2n} - \frac{6n}{\sqrt{6}} \frac{\Phi^{2n-1}}{\Psi} \frac{\partial\Phi}{\partial\phi} \left(10 + \frac{12n}{\sqrt{6}} \frac{\Phi^{2n-1}}{\Psi} \frac{\partial\Phi}{\partial\phi} \right) \\ &\quad - 12n \left((2n-1)\Phi^{2n-2} \left(\frac{\partial\Phi}{\partial\phi} \right)^2 + \Phi^{2n-1} \frac{\partial^2\Phi}{\partial\phi^2} \right) + \frac{n^2}{a^2 H^2}. \end{aligned} \quad (136)$$

Equations (134)-(136) generalise and correct Eqs. (4) of [71, 72]. In fact, a term was missing in the coefficients $\xi(\Phi, \Psi)$ and $\zeta(\Phi, \Psi)$ of the latter equations which, as shall see ahead, affects the results quantitatively, but not qualitatively.

As in the qualitative analysis of the background spacetime in Section 2, we are interested in the dynamics in the invariant set of $K = 0$ models, corresponding to the background of spatially flat hypersurfaces. In particular, since the background evolution equations forms an autonomous subsystem, the background fixed points, given by Lemma 1, are also fixed points of (134) and the following result holds:

Lemma 2. For $K = 0$, the fixed points of system (134) are given by the conditions:

$$\begin{aligned} \mathcal{P} : \quad \frac{\partial\Phi}{\partial\phi} &= -\frac{\sqrt{6}}{2n}\Phi\Psi \\ \mathcal{U}_{(n)}^{\pm}(\mathcal{P}) &= \frac{1}{2} \left(-\xi(\mathcal{P}) \pm \sqrt{\xi^2(\mathcal{P}) - 4\zeta(\mathcal{P})} \right) \end{aligned} \quad (137)$$

subject to $\Psi^2 + \Phi^{2n} = 1$, where

$$\begin{aligned} \xi(\mathcal{P}) &= -[4 - 9\Phi_{\mathcal{P}}^{2n}] \\ \zeta(\mathcal{P}) &= 18 \left(\frac{1}{n} - \frac{2}{3} \right) \Phi_{\mathcal{P}}^{2n} - 18 \left(\frac{1}{n} - 1 \right) \Phi_{\mathcal{P}}^{4n} - 12n\Phi_{\mathcal{P}}^{2n-1} \left(\frac{\partial^2\Phi}{\partial\phi^2} \right)_{\mathcal{P}} + \frac{n^2}{H^2 a^2}. \end{aligned} \quad (138)$$

From the above considerations, and from the fact that in a flat background we can reduce the linearised matrix at \mathcal{P} given by (18), it also follows that the eigenvalues of the linearized system around the fixed points are the ones given by (19) together with

$$\omega_{\mathcal{U}^{\pm}(\mathcal{P})} = -2\mathcal{U}^{\pm}(\mathcal{P}) - \xi(\mathcal{P}) = \mp \sqrt{\xi^2(\mathcal{P}) - 4\zeta(\mathcal{P})}. \quad (139)$$

Thus, from (137), the fixed points exist if

$$\xi^2(\mathcal{P}) - 4\zeta(\mathcal{P}) \geq 0 \quad (140)$$

and reduce to a single point when the equality is verified. In that case, the eigenvalues coincide and, from (139), are identically zero, resulting in a saddle point. If this is not the case, and if \mathcal{P} is an attractor point of the background dynamical system, then it follows that $\mathcal{U}^+(\mathcal{P})$ is the late time attractor of (134) having the following properties:

$$\begin{aligned} \mathcal{U}_{(n)}^+(\mathcal{P}) &< 0 \quad \text{if} \quad \zeta(\mathcal{P}) > 0 \quad \text{and} \quad \xi(\mathcal{P}) > 0, \\ \mathcal{U}_{(n)}^+(\mathcal{P}) &= 0 \quad \text{if} \quad \zeta(\mathcal{P}) = 0, \\ \mathcal{U}_{(n)}^+(\mathcal{P}) &> 0 \quad \text{if} \quad \xi(\mathcal{P}) < 0. \end{aligned} \quad (141)$$

5.1.1 Exponential potential: Power-law inflation

We have seen in Section 2.1 that, for an exponential potential, the background subsystem has two fixed points, \mathcal{P}_0 and \mathcal{P}_1 given by (24) and (25), respectively. Then, for the perturbed system (134) it follows from Lemma 2 that there are four fixed points

$$\left(\mathcal{P}_0, \mathcal{U}_{(n)}^{\pm}(\mathcal{P}_0) \right), \quad \left(\mathcal{P}_1, \mathcal{U}_{(n)}^{\pm}(\mathcal{P}_1) \right) \quad (142)$$

where, in this case, the coefficients (138) are

$$\xi(\mathcal{P}_0) = -4 \quad \text{and} \quad \xi(\mathcal{P}_1) = 5 - \frac{3}{2}\lambda^2 \quad (143)$$

$$\zeta(\mathcal{P}_0) = \frac{n^2}{a^2 H^2} \quad \text{and} \quad \zeta(\mathcal{P}_1) = 6 - 4\lambda^2 + \frac{\lambda^4}{2} + \frac{n^2}{a^2 H^2}. \quad (144)$$

The fixed point solutions are real if (140) is satisfied which, in turn, implies that the wave number satisfies

$$\mathcal{P}_0 : \quad n^2 \leq n_{crit}^2 = \frac{a^2 H^2}{4}, \quad (145)$$

$$\mathcal{P}_1 : \quad n^2 \leq n_{crit}^2 = \frac{a^2 H^2}{4} \left(1 + \lambda^2 + \frac{\lambda^4}{4} \right). \quad (146)$$

When $n = n_{crit}$, the points $\mathcal{U}_{(n)}^{\pm}$ merge into a single saddle point. For $n > n_{crit}$, the fixed points cease to exist, the orbit is periodic and the perturbations behave as waves. We also saw that \mathcal{P}_1 was the late time

attractor corresponding to the flat power-law solution of the background dynamical system (12). Thus, the attractor point of the dynamical system (134) is $\mathcal{U}_n^+(\mathcal{P}_1)$ and we obtain, from (141), that there exists

$$n_{(-)}^2 = a^2 H^2 \left(-6 + 4\lambda^2 - \frac{\lambda^4}{2} \right)$$

such that, if $n_{(-)}^2(\lambda) < n^2 < n_{crit}^2(\lambda)$, then $\mathcal{U}_{(n)}^+(\mathcal{P}_1) < 0$ and the perturbations decay. When $n^2 = n_{(-)}^2(\lambda)$, then $\zeta(\mathcal{P}_1) = 0$ and the perturbations tend to a constant. Therefore, when the slope parameter satisfies $0 < \lambda < \sqrt{2}$, the density perturbation modes decay for all wavelengths in the range for which there exist fixed points, i.e. for $0 \leq n^2 < n_{crit}^2(\lambda)$. In particular, for $\lambda = \sqrt{2}$ the modes decay, except in the long wavelength limit, for which the perturbations tend to constant since $n_{(-)}^2(\sqrt{2}) = 0$ (see Figs. 5, 6, 7, 8 and 9). Moreover, if $\sqrt{2} < \lambda < \sqrt{10/3}$, there are perturbation modes which can either decay, tend to

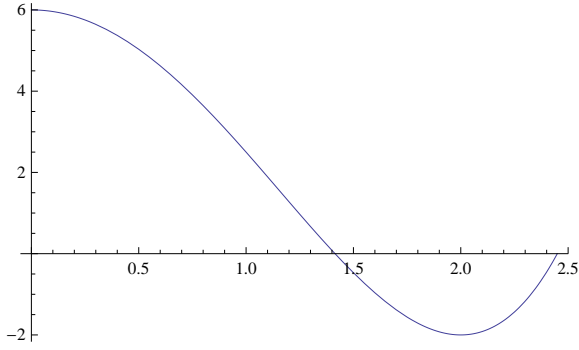


Figure 5: Plot of $\zeta(\lambda)$ using the long wavelength limit. The positive region gives the values of λ for which the perturbations decay, for an exponential potential. The zeros are at $\lambda = \sqrt{2}$ and $\lambda = \sqrt{6}$. Notice that if $n^2 > 0$ the graph is shifted upwards along the ζ axis and, consequently, the interval for which ζ is positive gets bigger.

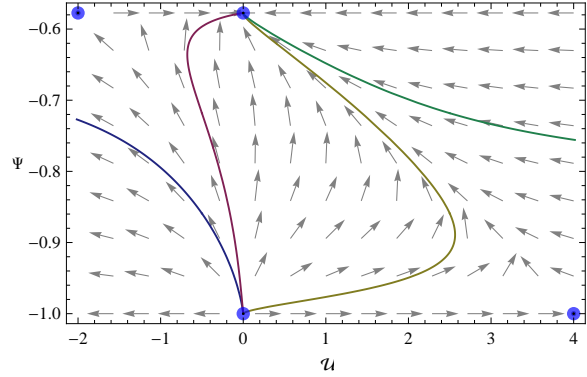


Figure 6: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{2}$ in the long wavelength limit. The figure shows the equilibrium points, one of which is the future attractor $(\mathcal{P}, \mathcal{U}_{(n)}^+)$ having $\mathcal{U}_{(n)} = 0$.

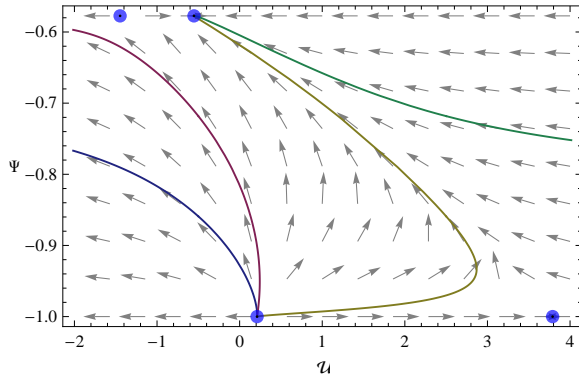


Figure 7: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{2}$ for $\frac{n^2}{a^2 H^2} = 0.8 > \frac{n_{(-)}^2(\sqrt{2})}{a^2 H^2} = 0$. The figure shows the equilibrium points, one of which is the future attractor $(\mathcal{P}, \mathcal{U}_{(n)}^+)$ having $\mathcal{U}_{(n)} < 0$.

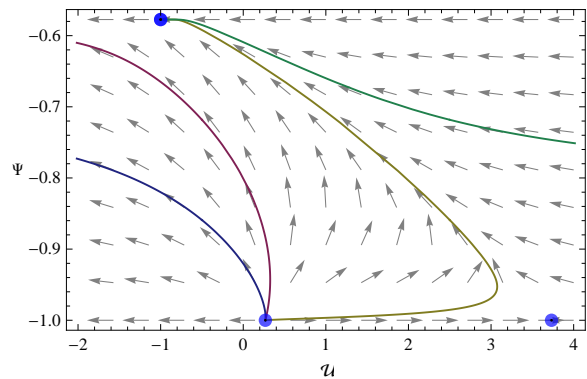


Figure 8: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{2}$ and $n^2 = n_{crit}^2(\sqrt{2}) = 1$. The figure shows the saddle point in the region $\mathcal{U}_{(n)} < 0$.

constant or grow, depending on the value of n^2 (see Figs. 10, 11). For $\lambda = \sqrt{10/3}$, all perturbations grow, while n reaches its critical value. In this case, the saddle point is at $\mathcal{U}_{n_{crit}} = 0$ (see Figs. 12, 13,

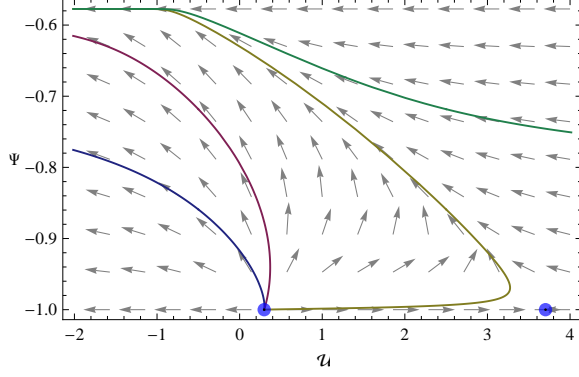


Figure 9: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{2}$ and $n^2 > n_{crit}^2(\sqrt{2}) = 1$. For this value of the slope parameter the orbits are periodic.

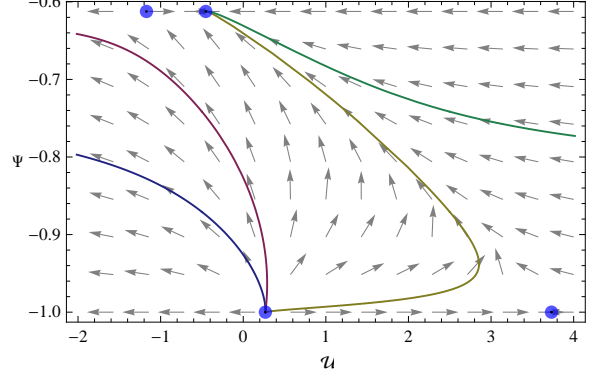


Figure 10: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = 1.5$ and $\frac{n^2}{a^2 H^2} = 1$. The figure shows the equilibrium points, one of which is the future attractor $(\mathcal{P}, \mathcal{U}_{(n)}^+)$ having $\mathcal{U}_{(n)} < 0$.

14). Finally, for $\sqrt{\frac{10}{3}} \leq \lambda < \sqrt{6}$, all modes grow for all wavelengths in the range for which there exist fixed points.

Notice that, by taking the long wavelength limit $\mathcal{U}^-(\mathcal{P}_0) = 0$ (see Figs 6 and 11), which corresponds to the constant mode of (128), while $\mathcal{U}^+(\mathcal{P}_0) > 0$ corresponds to the growing mode. Our results are summarised in Table 1 which corrects and generalises Table 1 of [72].

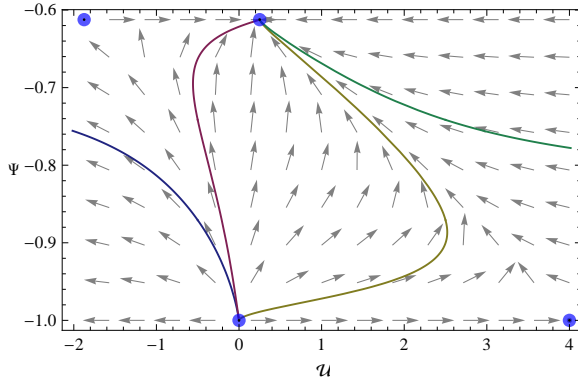


Figure 11: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = 1.5$ taking the long wavelength limit.

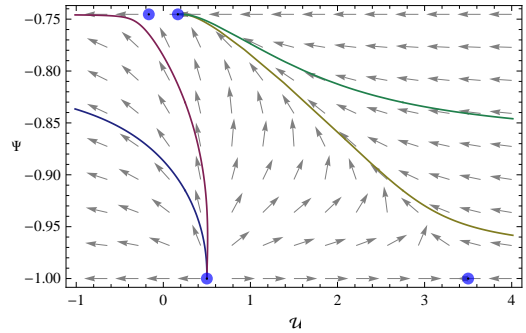


Figure 12: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{\frac{10}{3}}$ and $\frac{n^2}{a^2 H^2} = 1.75$.

5.1.2 Polynomial potentials: Chaotic inflation

(i) Quadratic potential

We have seen, in Section 2.2, that for a quadratic potential the method of [35] gives two fixed points \mathcal{P}_0 and \mathcal{P}_1 given by (45) and (46), respectively. Then, for the perturbed system (134), it follows from Lemma 2 that there are four fixed points

$$\left(\mathcal{P}_0, \mathcal{U}_{(n)}^\pm(\mathcal{P}_0)\right), \quad \left(\mathcal{P}_1, \mathcal{U}_{(n)}^\pm(\mathcal{P}_1)\right) \quad (147)$$

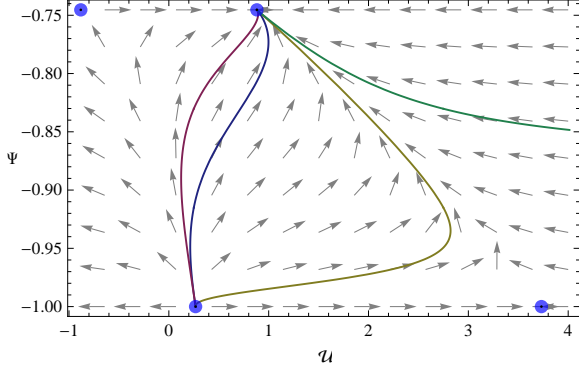


Figure 13: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{\frac{10}{3}}$ and $\frac{n^2}{a^2 H^2} = 1$. The figure shows the equilibrium points, one of which is the future attractor $(\mathcal{P}, \mathcal{U}_{(n)}^+)$ having $\mathcal{U}_{(n)} > 0$.

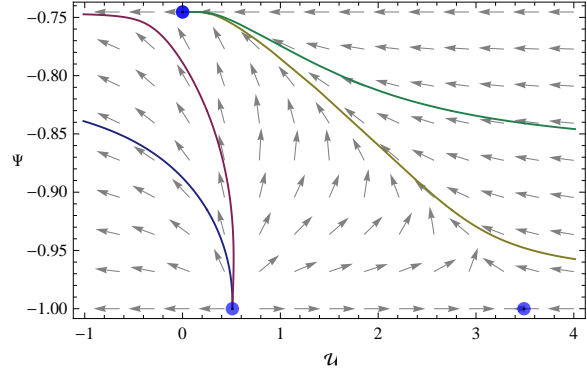


Figure 14: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an exponential potential with $\lambda = \sqrt{\frac{10}{3}}$ and $\frac{n^2}{a^2 H^2} = \frac{16}{9}$. The figure shows the equilibrium points. The saddle point has $\mathcal{U}_{(n)} = 0$.

where the coefficients (138) are

$$\begin{aligned}\xi(\mathcal{P}_{0,1}) &= \frac{1}{2} \mp \frac{9}{2} \sqrt{1 - \frac{4}{9} \mathcal{M}^2} \\ \zeta(\mathcal{P}_{0,1}) &= 3 \mp 3 \sqrt{1 - \frac{4}{9} \mathcal{M}^2 + \frac{n^2}{H^2 a^2}}.\end{aligned}\quad (148)$$

We also saw that \mathcal{P}_1 is the late time attractor of the background subsystem. Therefore, the late time attractor of the perturbed dynamical system (130) is $(\mathcal{P}_1, \mathcal{U}_{(n)}^+(\mathcal{P}_1))$. From (141) and (148), we easily see that this fixed point always lies in the region $\mathcal{U}_{(n)} < 0$ of the phase-space, and it only exists if (140) is satisfied (see Fig. 15), i.e. for values of the wave number satisfying

$$n^2 \leq a^2 H^2 \left(\frac{17}{8} - \frac{15}{8} \sqrt{1 - \frac{4}{9} \mathcal{M}^2} - \frac{9}{4} \mathcal{M}^2 \right), \quad (149)$$

which, in turn, implies

$$\mathcal{M} \leq \frac{2}{9} \sqrt{16 - 5\sqrt{7}}. \quad (150)$$

Thus, when $\mathcal{M} = \frac{2}{9} \sqrt{16 - 5\sqrt{7}}$, the fixed point only exists in the long wavelength limit and it is a saddle point (see Fig. 16), while for $\mathcal{M} > \frac{2}{9} \sqrt{16 - 5\sqrt{7}}$, the orbits are periodic. See Figs. 17, 18, 19 and 20 for the case $\mathcal{M} = 10^{-1} < \frac{2}{9} \sqrt{16 - 5\sqrt{7}}$ and Table 1 for a summary of results.

(ii) Quartic potential

We have seen, in Section 2.2, that for a quartic potential the method of [35], used in [37], showed the existence of two fixed points \mathcal{P}_0 and \mathcal{P}_1 given by (50). Then, for the perturbed system (134) it follows, from (137), that there are four fixed points

$$\left(\mathcal{P}_0, \mathcal{U}_{(n)}^\pm(\mathcal{P}_0) \right), \quad \left(\mathcal{P}_1, \mathcal{U}_{(n)}^\pm(\mathcal{P}_1) \right) \quad (151)$$

with the coefficients (138) given by

$$\begin{aligned}\xi(\mathcal{P}_0) &= 2 \left(1 - 3 \cos \left(\frac{2}{3} \chi \right) \right) \quad \text{and} \quad \xi(\mathcal{P}_1) = 2 \left(1 - 3 \cos \left(\frac{2}{3} (\chi - \pi) \right) \right) \\ \zeta(\mathcal{P}_0) &= \xi(\mathcal{P}_0) + 4 \cos^2 \left(\frac{2}{3} \chi \right) + \frac{n^2}{H^2 a^2} \quad \text{and} \quad \zeta(\mathcal{P}_1) = \xi(\mathcal{P}_1) + 4 \cos^2 \left(\frac{2}{3} (\chi - \pi) \right) + \frac{n^2}{H^2 a^2}.\end{aligned}\quad (152)$$

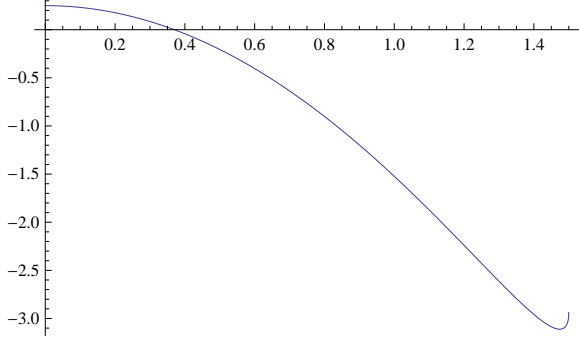


Figure 15: The non-negative region shows the set of values of \mathcal{M} for which there are fixed points.

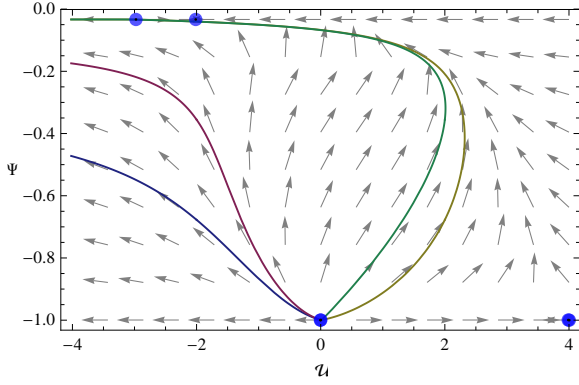


Figure 17: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quadratic potential with $\mathcal{M} = 10^{-1}$ in the long wavelength limit. The figure shows the attractor point $\mathcal{U}^+(\mathcal{P}_1)$ in the region $\mathcal{U}_{(n)} < 0$. See also Fig. 18.

From (140), the fixed points exist if

$$n^2 \leq n_{crit}^2 = a^2 H^2 \left(5 \cos^2 \left(\frac{2}{3} (\chi - \pi) \right) - 1 \right), \quad (153)$$

i.e. for χ satisfying (see also Fig. 23)

$$0 < \chi \leq \pi - \frac{3}{2} \arccos \left(-\frac{1}{\sqrt{5}} \right) \quad \text{or} \quad \pi - \frac{3}{2} \arccos \left(\frac{1}{\sqrt{5}} \right) \leq \chi \leq \frac{\pi}{2}. \quad (154)$$

Furthermore, we find that $\zeta(\mathcal{P}_1) > 0$ for all values of $0 < \chi < \pi/2$ and $\zeta(\mathcal{P}_1) = 0$ for $\chi = \pi/2$, see Fig. 22. We also find that (see also Fig. 21)

$$\begin{aligned} \text{For } 0 < \chi < \pi - \frac{3}{2} \arccos \left(\frac{1}{\sqrt{3}} \right) \quad \text{then} \quad \xi(\mathcal{P}_1) > 0 \\ \text{For } \chi = \pi - \frac{3}{2} \arccos \left(\frac{1}{\sqrt{3}} \right) \quad \text{then} \quad \xi(\mathcal{P}_1) = 0 \\ \text{For } \pi - \frac{3}{2} \arccos \left(\frac{1}{\sqrt{3}} \right) < \chi \leq \frac{\pi}{2} \quad \text{then} \quad \xi(\mathcal{P}_1) < 0. \end{aligned} \quad (155)$$

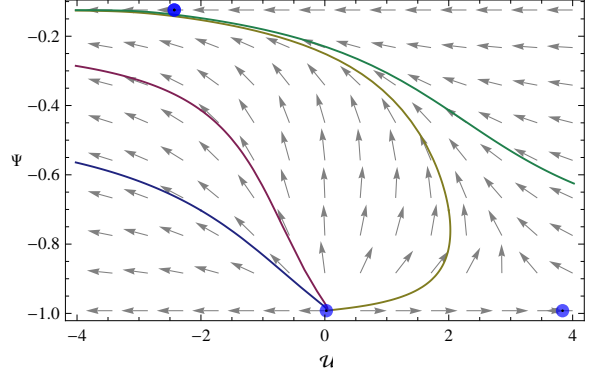


Figure 16: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quadratic potential with $\mathcal{M} = \frac{2}{9} \sqrt{16 - 5\sqrt{7}}$ in the long wavelength limit. The figure shows the saddle point in the region $\mathcal{U}_{(n)} < 0$.

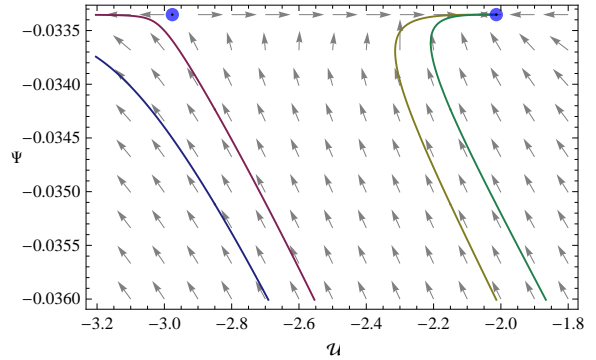


Figure 18: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quadratic potential with $\mathcal{M} = 10^{-1}$ in the long wavelength limit. The figure shows in more detail the attractor point in the region $\mathcal{U}_{(n)} < 0$ of Fig. 17.

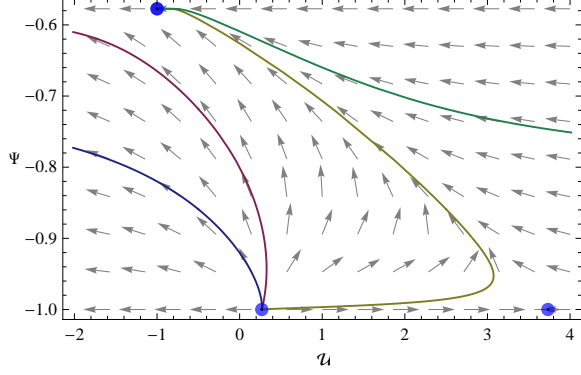


Figure 19: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quadratic potential with $\mathcal{M} = 10^{-1}$ for $n^2 = n_{crit}^2$. The figure shows the saddle point, in the region $\mathcal{U}_{(n)} < 0$.

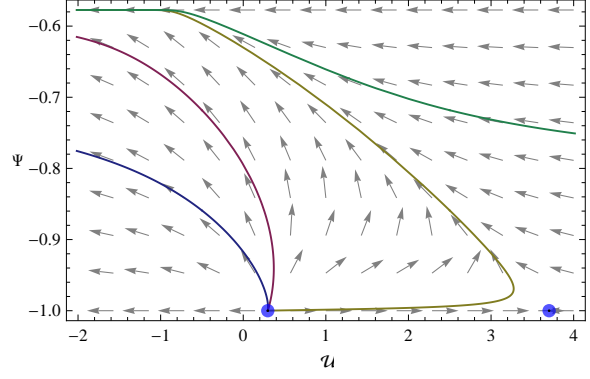


Figure 20: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quadratic potential with $\mathcal{M} = 10^{-1}$ and $n^2 > n_{crit}^2$. The figure shows the existence of periodic orbits, and the perturbations behave as waves.

Therefore, there are perturbation modes which decay for $0 < \chi < \pi - \frac{3}{2} \arccos\left(-\frac{1}{\sqrt{5}}\right)$, see Figs. 24, 25 and 26, and which grow for $\pi - \frac{3}{2} \arccos\left(\frac{1}{\sqrt{5}}\right) < \chi < \pi/2$, see Figs. 27, 28, 29 and 30. From (52), the solution is inflationary for $\chi < \pi/4$ and the instabilities (in the sense of growing modes) occur when the solution is non-inflationary while, when inflation occurs, there are only decaying modes present.

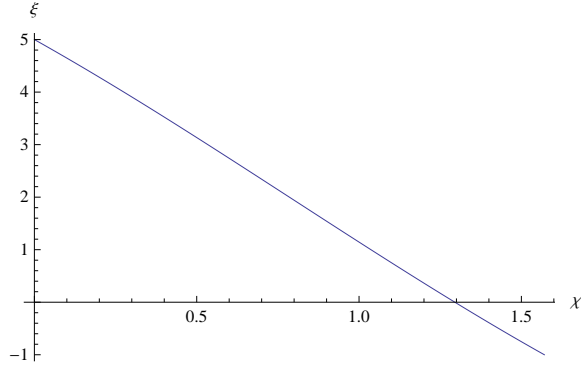


Figure 21: Plot of $\xi(\mathcal{P}_1)$ in the interval $0 \leq \chi \leq \frac{\pi}{2}$. The figure shows a zero at $\chi = \pi - \frac{3}{2} \arccos\left(\frac{1}{\sqrt{3}}\right)$. The positive region shows the admitted values for decay as long as there exists fixed points.

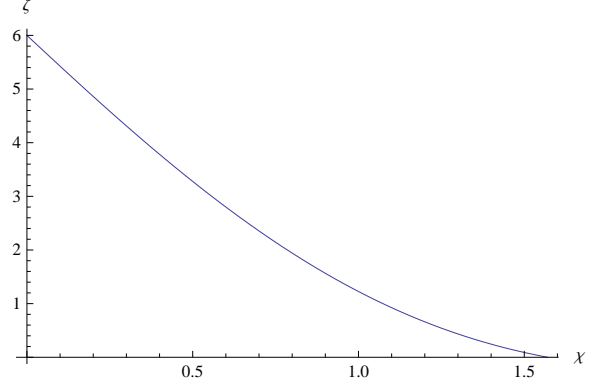


Figure 22: Plot of $\zeta(\mathcal{P}_1)$ in the interval $0 \leq \chi \leq \frac{\pi}{2}$ taking the long-wavelength limit. The figure shows that ζ is a positive monotone function of χ achieving the value zero at $\chi = \frac{\pi}{2}$.

(iii) New inflation

In the linearly perturbed case, we find that the fixed points of the system (130) are

$$\left(\mathcal{P}, \mathcal{U}_{(n)}^{\pm}(\mathcal{P})\right), \quad (156)$$

where $\mathcal{U}_{(n)}^{\pm}(\mathcal{P})$ are given by Eqs. (137) with

$$\begin{aligned} \xi(\mathcal{P}) &= -[4 - 9\Phi_{\mathcal{P}}^4] \\ \zeta(\mathcal{P}) &= -3\Phi_{\mathcal{P}}^4(1 - 3\Phi_{\mathcal{P}}^4) + \mathcal{M}^4 \mathcal{N}^2 + \frac{n^2}{H^2 a^2} \end{aligned}$$

and are investigated numerically, see Figs. 31 and 32. See Table 1 for a summary of the results of Section 5.1.

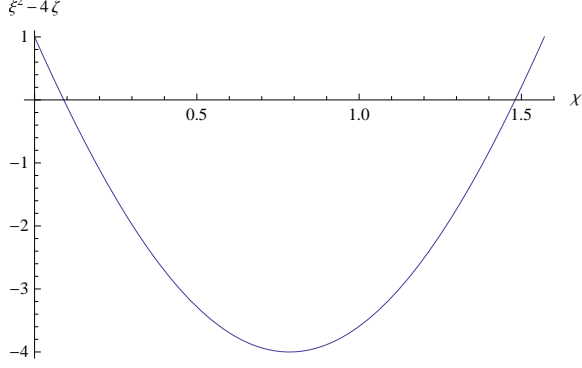


Figure 23: Plot of $\xi^2(\mathcal{P}_1) - 4\zeta(\mathcal{P}_1)$ in the interval $0 \leq \chi \leq \frac{\pi}{2}$. The figure shows two zeros at $\chi = \pi - \frac{3}{2} \arccos\left(\pm \frac{1}{\sqrt{5}}\right)$. The nonnegative region shows the admitted values of χ for which there exists fixed points.

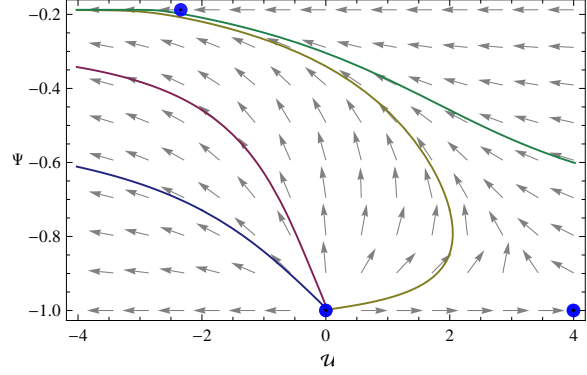


Figure 24: Phase portrait for $\chi = \pi - \frac{3}{2} \arccos\left(-\frac{1}{\sqrt{5}}\right)$ taking the long wavelength limit. The figure shows the saddle point in the region $\mathcal{U} < 0$.

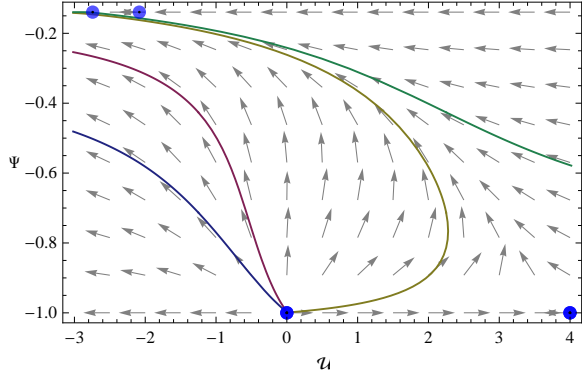


Figure 25: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quartic potential with $\chi = 20^{-1}$ taking the long wavelength limit. The figure shows the late time attractor in the region $\mathcal{U} < 0$.

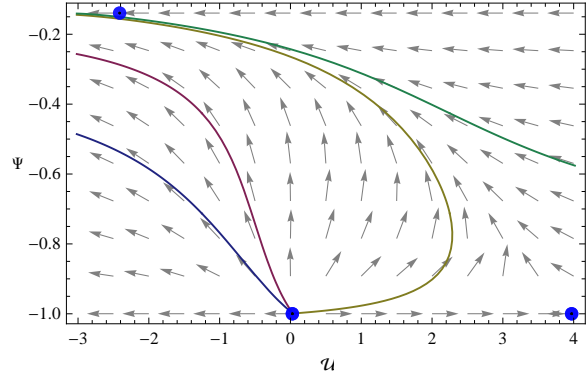


Figure 26: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quartic potential with $\chi = 20^{-1}$ for $n^2 = n_{crit}^2(20^{-1})$. The figure shows the saddle point in the region $\mathcal{U} < 0$.

5.2 Two scalar fields

In this section, we consider the simplified case where two scalar fields ϕ_1 and ϕ_2 do not interact with each other, i.e. with $\mathcal{W}(\phi_1, \phi_2) = 0$. In this case, we find from Proposition 5.1 the following result:

Corollary 1. *The evolution for the phase of first order scalar perturbations on a FLRW background, with two nonlinear smooth scalar fields, is given by the following system of differential equations for the state*

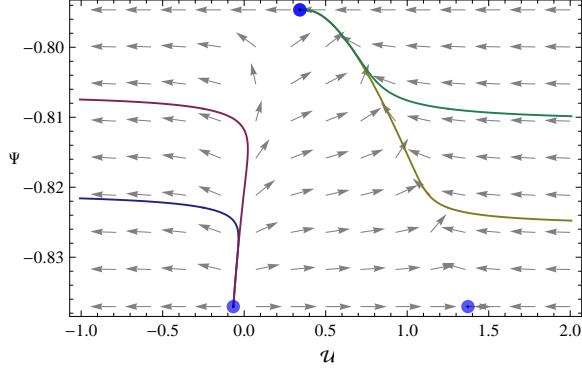


Figure 27: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quartic potential with $\chi = \pi - \frac{3}{2} \arccos\left(\frac{1}{\sqrt{5}}\right)$ taking the long wavelength limit. The figure shows the saddle point in the region $\mathcal{U} > 0$.

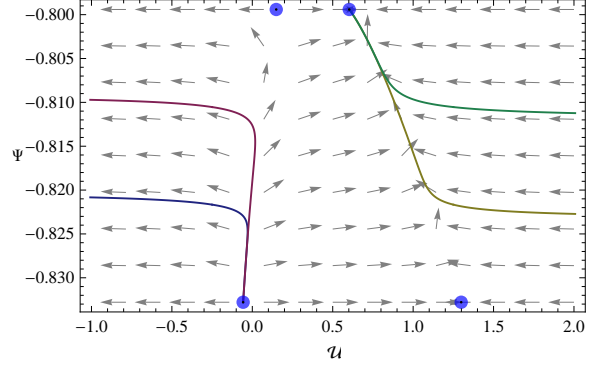


Figure 28: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for an quartic potential with $\chi = 3/2$ taking the long wavelength limit. The figure shows the late time attractor in the region $\mathcal{U} > 0$.

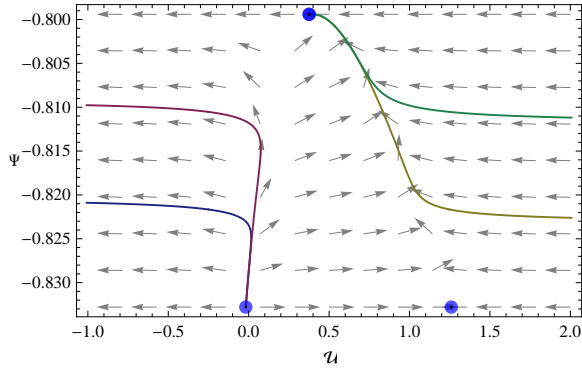


Figure 29: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quartic potential with $\chi = 3/2$ for $n^2 = n_{crit}^2(3/2)$. The figure shows the saddle point in the region $\mathcal{U} > 0$.

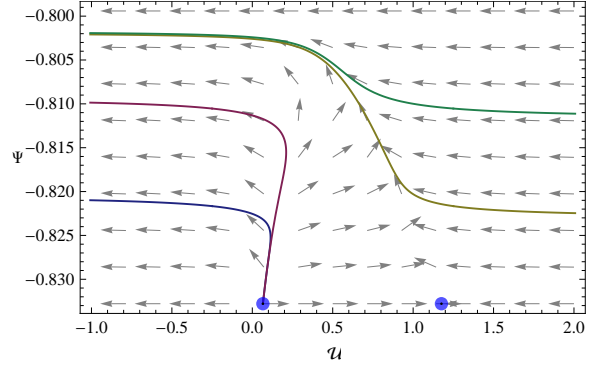


Figure 30: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for a quartic potential with $\chi = 3/2$, for $n^2 > n_{crit}^2(3/2)$. The figure shows periodic orbits and the perturbations behave as waves.

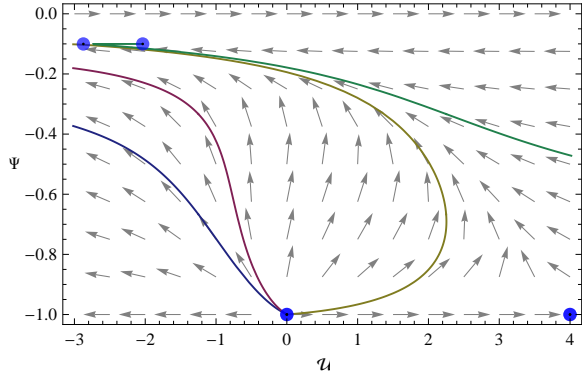


Figure 31: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for new inflation with $\mathcal{M} = 0.3$ and $\mathcal{N} = 0.1$ in the long wavelength limit.

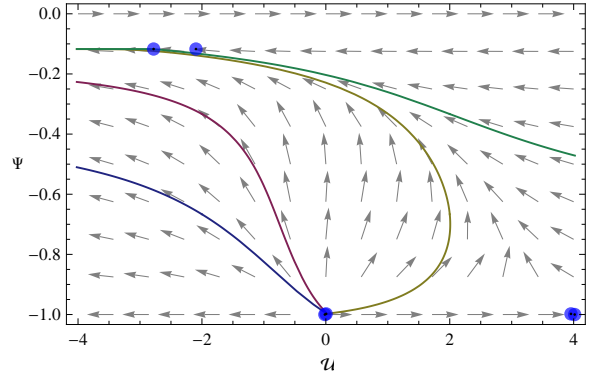


Figure 32: Density perturbations described by orbits in the phase plane $(\Psi, \mathcal{U}_{(n)})$ for new inflation with $\mathcal{M} = 0.3$ and $\mathcal{N} = 2$ in the long wavelength limit.

vector $((\Psi_1, \Psi_2, \Phi_1, \Phi_1), \mathcal{U}, \mathcal{X}_{[12]}, \mathcal{Y}_{[12]})$:

$$\begin{aligned} \mathcal{U}' &= -\mathcal{U}^2 - \xi\mathcal{U} - \zeta + \gamma_{12}\mathcal{X}_{[12]} + \eta_{12}\mathcal{Y}_{[12]} \\ \mathcal{X}'_{[12]} &= (-\mathcal{U} + \varsigma_{12})\mathcal{X}_{[12]} + \varpi_{12}\mathcal{Y}_{[12]} \\ \mathcal{Y}'_{[12]} &= (-\mathcal{U} + \iota_{12})\mathcal{Y}_{[12]} - \mathcal{X}_{[12]} \end{aligned}$$

$\mathcal{V}(\phi)$	Fut. Att.: $(\mathcal{P}_1, \mathcal{U}_n^+(\mathcal{P}_1))$	Parameter Space	Phys. Mean.	Inf. Sol.
$\Lambda e^{\lambda\phi}$	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}_n^+ < 0$	$0 < \lambda < \sqrt{2}, 0 \leq n^2 < n_{crit}^2(\lambda)$	decays	$0 < \lambda < \sqrt{2}$
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}_n^+ < 0$	$\lambda = \sqrt{2}, 0 < n^2 < n_{crit}^2(\lambda)$	decays	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}_n^+ = 0$	$\lambda = \sqrt{2}, n^2 = 0$	tend to a const.	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}_n^+ < 0$	$\sqrt{2} < \lambda < \sqrt{\frac{10}{3}}, 0 < n_-^2(\lambda) < n^2 < n_{crit}^2(\lambda)$	decays	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}_n^+ = 0$	$\sqrt{2} < \lambda < \sqrt{\frac{10}{3}}, n^2 = n_-^2(\lambda)$	tend to a const.	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}_n^+ > 0$	$\sqrt{2} < \lambda < \sqrt{\frac{10}{3}}, 0 \leq n^2 < n_-^2(\lambda)$	grows	
	Pt $\Psi = -\frac{\lambda}{\sqrt{6}}, \mathcal{U}_n^+ > 0$	$\frac{10}{3} \leq \lambda < \sqrt{6}, 0 \leq n^2 < n_{crit}^2(\lambda)$	grows	
	Per. orb. $\Psi = -\frac{\lambda}{\sqrt{6}}$	for all $\lambda, n^2 > n_{crit}^2(\lambda)$	Pert. is a wave	
$\frac{m^2}{2}\phi^2$	Point $\Psi = \Psi_{\mathcal{P}_1}, \mathcal{U}_n^+ < 0$	$0 < \mathcal{M} < \frac{2}{9}\sqrt{16 - 5\sqrt{7}}, n^2 < n_{crit}^2$	Pert. decays	$0 < \mathcal{M} < \sqrt{2}$
	Periodic orbit $\Psi = \Psi_{\mathcal{P}_1}$	$\mathcal{M} > \frac{2}{9}\sqrt{16 - 5\sqrt{7}}$	Pert. is a wave	
$\frac{\lambda^4}{4}\phi^4$	Point $\Psi = \Psi_{\mathcal{P}_1}, \mathcal{U}_n^+ < 0$	See (155)	Pert. decays	$0 < \mathcal{M} < 6^{\frac{1}{4}}$
	Periodic orbit $\Psi = \Psi_{\mathcal{P}_1}$		Pert. is a wave	
	Point $\Psi = \Psi_{\mathcal{P}_1}, \mathcal{U}_n^+ > 0$		Pert. grows	

Table 1: Summary of the results of Section 5.1 about the behaviour of density perturbations in flat FLRW scalar field backgrounds with exponential, quadratic and quartic potentials.

subject to the background constraint equation

$$\Psi_1^2 + \Psi_2^2 + \Phi_1^{2n} + \Phi_2^{2n} = 1 - K, \quad (158)$$

where

$$q = 2(\Psi_1^2 + \Psi_2^2) - (\Phi_1^{2n} + \Phi_2^{2n}) \quad (159)$$

and the coefficients are given in Section 6.2 of the appendix.

We also prove from (157) and Lemma 1:

Lemma 3. For $K = 0$, the fixed points of system (130) are given by Lemma 1 together with:

$$-U^2 - \xi(\mathcal{P})\mathcal{U} - \zeta(\mathcal{P}) = 0 \quad \text{and} \quad \gamma_{12}(\mathcal{P})\mathcal{X}_{[12]} + \eta_{12}(\mathcal{P})\mathcal{Y}_{[12]} = 0$$

or

$$-U^2 - (\varsigma_{12}(\mathcal{P}) + \iota_{12}(\mathcal{P}))\mathcal{U} + (\varsigma_{12}(\mathcal{P})\iota_{12}(\mathcal{P}) + \varpi_{12}(\mathcal{P})) = 0 \quad \text{and} \quad \gamma_{12}(\mathcal{P})\mathcal{X}_{[12]} + \eta_{12}(\mathcal{P})\mathcal{Y}_{[12]} \neq 0,$$

where the coefficients $\xi, \zeta, \gamma_{12}, \eta_{12}, \varsigma_{12}, \iota_{12}$ and ϖ_{12} at \mathcal{P} are given in appendix.

In particular, if

$$\gamma_{12}(\mathcal{P}) = \eta_{12}(\mathcal{P}) = 0$$

then

$$\mathcal{Y}_{[12]}(\mathcal{P}) = \mathcal{X}_{[12]}(\mathcal{P}) = 0 \quad \text{and} \quad \mathcal{U}_n^\pm(\mathcal{P}) = \frac{1}{2} \left(-\xi(\mathcal{P}) \pm \sqrt{\xi(\mathcal{P})^2 - 4\zeta(\mathcal{P})} \right). \quad (160)$$

and the linearised matrix of the system at the fixed points has eigenvalues given by the background eigenvalues together with

$$-2\mathcal{U}^\pm(\mathcal{P}) - \xi(\mathcal{P}) = \mp \sqrt{\xi^2(\mathcal{P}) - 4\zeta(\mathcal{P})} \quad (161)$$

and

$$\frac{1}{2} \left[-2\mathcal{U}^\pm + \iota_{12} + \varsigma_{12} \pm \sqrt{(2\mathcal{U}^\pm - \iota_{12} - \varsigma_{12})^2 - 4((\mathcal{U}^\pm)^2 - (\iota_{12} + \varsigma_{12})\mathcal{U}^\pm + \varpi_{12} + \iota_{12}\varsigma_{12})} \right]. \quad (162)$$

In the next section, we shall apply Lemma 3 to the case of two scalar fields with independent exponential potentials.

5.2.1 Assisted power-law inflation

In this case, we recall from Section 2.12 that the fixed points are characterised by

$$\mathcal{P} : \quad \frac{\partial^2 \Phi_A}{\partial \phi_A^2} = -\frac{\sqrt{6}}{4} \lambda_A \frac{\Phi_A}{\Psi_A} \Psi^2 \quad , \quad \frac{\partial^3 \Phi_A}{\partial \phi_A^3} = -\frac{\sqrt{6}}{8} \lambda_A^2 \frac{\Phi_A}{\Psi_A} \Psi^2 \quad (163)$$

so that, using equations (27)-(30), the coefficients of the system (157) become

$$\begin{aligned} \xi(\mathcal{P}_0) &= -4 \quad , \quad \xi(\mathcal{P}_{1,2}) = 5 - \frac{3}{2} \lambda_{1,2}^2 \quad , \quad \xi(\mathcal{P}_3) = 5 - \frac{3}{2} \lambda^2 \\ \zeta(\mathcal{P}_0) &= \frac{n^2}{a^2 H^2} \quad , \quad \zeta(\mathcal{P}_{1,2}) = 6 - 4\lambda_{1,2}^2 + \frac{\lambda_{1,2}^4}{2} + \frac{n^2}{a^2 H^2} \quad , \quad \zeta(\mathcal{P}_3) = 6 - 4\lambda^2 + \frac{\lambda^4}{2} + \frac{n^2}{a^2 H^2} \\ \gamma_{12}(\mathcal{P}_0) &= 0 \quad , \quad \gamma_{12}(\mathcal{P}_{1,2}) = 0 \quad , \quad \gamma_{12}(\mathcal{P}_3) = 0 \\ \eta_{12}(\mathcal{P}_0) &= 0 \quad , \quad \eta_{12}(\mathcal{P}_{1,2}) = 0 \quad , \quad \eta_{12}(\mathcal{P}_3) = 0 \\ \varsigma_{12}(\mathcal{P}_0) &= 0 \quad , \quad \varsigma_{12}(\mathcal{P}_{1,2}) = 0 \quad , \quad \varsigma_{12}(\mathcal{P}_3) = \frac{6 - \lambda^2}{2} \\ \iota_{12}(\mathcal{P}_0) &= 0 \quad , \quad \iota_{12}(\mathcal{P}_{1,2}) = \frac{6 - \lambda_{1,2}^2}{2} \quad , \quad \iota_{12}(\mathcal{P}_3) = 0 \\ \varpi_{12}(\mathcal{P}_0) &= \frac{q^2}{a^2 H^2} \quad , \quad \varpi_{12}(\mathcal{P}_{1,2}) = \frac{q^2}{a^2 H^2} \quad , \quad \varpi_{12}(\mathcal{P}_3) = \frac{\lambda^2}{2} (6 - \lambda^2) + \frac{q^2}{a^2 H^2} \end{aligned}$$

where q is the wave number associated with the velocity scalar perturbations. Then, from equations (160)-(162), the fixed points of the system reduce to

$$((\Psi_A, \Phi_A), \mathcal{U}, \mathcal{X}_{[12]}, \mathcal{Y}_{[12]}) = (\mathcal{P}_i, \mathcal{U}_n^\pm(\mathcal{P}_i), 0, 0) \quad , \quad i = 0, 1, 2, 3, \quad (164)$$

with eigenvalues given by the background solution (31)-(33) together with

$$\omega^\pm(\mathcal{P}_0) = \mp 2 \sqrt{4 - \frac{n^2}{a^2 H^2}} \quad , \quad -2 \mp \sqrt{4 - \frac{n^2}{a^2 H^2}} \pm \sqrt{-\frac{q^2}{a^2 H^2}} \quad , \quad (165)$$

$$\omega^\pm(\mathcal{P}_{1,2}) = \mp 2 \sqrt{\frac{1}{4} \left(1 + \lambda_{1,2}^2 + \frac{\lambda_{1,2}^4}{4} \right) - \frac{n^2}{a^2 H^2}} \quad , \quad (166)$$

$$(4 - \lambda_{1,2}^2) \mp \sqrt{\frac{1}{4} \left(1 + \lambda_{1,2}^2 + \frac{\lambda_{1,2}^4}{4} \right) - \frac{n^2}{a^2 H^2}} \pm \sqrt{\frac{1}{4} \left(\frac{6 - \lambda_{1,2}^2}{2} \right)^2 - \frac{q^2}{a^2 H^2}} \quad ,$$

$$\omega^\pm(\mathcal{P}_3) = \mp 2 \sqrt{\frac{1}{4} \left(1 + \lambda^2 + \frac{\lambda^4}{4} \right) - \frac{n^2}{a^2 H^2}} \quad , \quad (167)$$

$$(4 - \lambda^2) \mp \sqrt{\frac{1}{4} \left(1 + \lambda^2 + \frac{\lambda^4}{4} \right) - \frac{n^2}{a^2 H^2}} \pm \sqrt{\frac{1}{4} \left(\frac{6 - \lambda^2}{2} \right)^2 - \frac{\lambda^2}{2} (6 - \lambda^2) - \frac{q^2}{a^2 H^2}} \quad .$$

Thus, the behaviour of \mathcal{U}^+ , being the future attractor, is similar to the case of a single scalar field solution with more restrictions due to the new eigenvalue (162). In particular, the parameter space constraints of Table 1 also apply to this case for λ defined by $1/\lambda^2 = 1/\lambda_1^2 + 1/\lambda_2^2$. However, there are further constraints to the parameters due to the velocity perturbations (which were not present in the single scalar field case) that can be inferred, in each case, from the sign of the second eigenvalue of $\omega^+(\mathcal{P}_3)$. For instance, the fixed points for the perturbations of the massless scalar field solution only exist if $q/(aH) \ll 1$.

The results of this section appear to be easily extendible to the case of an arbitrary number of scalar fields with this type of potentials.

Acknowledgments

We thank Claes Uggla for very helpful comments and Ureña-López for pointing out useful references. We acknowledge support from CMAT, Univ. Minho (through the FEDER Funds COMPETE and FCT Project Est-C/MAT/UI0013/2011), as well as from FCT projects PTDC/MAT/108921/2008 and CERN/FP/123609/2011.

6 Appendix

In this appendix, we present the explicit expressions for the derivatives of the coefficients in equations (121) and for the coefficients of (130).

6.1 Evolution equations for A, B_{AB} and C_{AB}

From

$$\begin{aligned}
\dot{\alpha}_A \beta_A^2 + \dot{\alpha}_B \beta_B^2 + \alpha_A (\dot{\beta}_A^2) + \alpha_B (\dot{\beta}_B^2) = \\
= 3H (\alpha_A \beta_A^2 + \alpha_B \beta_B^2) + (\alpha_A^2 \beta_A^2 + \alpha_B^2 \beta_B^2) - 2(\alpha_A - \alpha_B) B_{AB} \\
+ \left(\frac{d^2 \mathcal{V}}{d\phi_A^2} + \frac{\partial^2 \mathcal{W}}{\partial \phi_A^2} + \frac{\psi_B}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} \right) \beta_A^2 + \left(\frac{d^2 \mathcal{V}}{d\phi_B^2} + \frac{\partial^2 \mathcal{W}}{\partial \phi_B^2} + \frac{\psi_A}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_A \partial \phi_B} \right) \beta_B^2 \\
+ \sum_{C \neq A, B}^N \psi_C \left[\left(\frac{\beta_A^2}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} + \frac{\beta_B^2}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) - 2(\alpha_A B_{AC} + \alpha_B B_{BC}) \right]
\end{aligned} \tag{168}$$

and equations (109)-(110) we get

$$\begin{aligned}
\dot{A} = 3\dot{H} + 6H (\alpha_A \beta_A^2 + \alpha_B \beta_B^2) + 2(\alpha_A^2 \beta_A^2 + \alpha_B^2 \beta_B^2) - 4(\alpha_A - \alpha_B) B_{AB} \\
+ 2 \left(\frac{d^2 \mathcal{V}}{d\phi_A^2} + \frac{\partial^2 \mathcal{W}}{\partial \phi_A^2} + \frac{\psi_B}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} \right) \beta_A^2 + 2 \left(\frac{d^2 \mathcal{V}}{d\phi_B^2} + \frac{\partial^2 \mathcal{W}}{\partial \phi_B^2} + \frac{\psi_A}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_A \partial \phi_B} \right) \beta_B^2 \\
+ 2 \sum_{C \neq A, B}^N [-2(\alpha_C - \alpha_A) B_{CA} - 2(\alpha_C - \alpha_B) B_{CB}] \\
+ 2 \sum_{C \neq A, B}^N \left[\frac{\psi_A}{\psi_C} \beta_C^2 \frac{\partial^2 \mathcal{W}}{\partial \phi_A \partial \phi_C} + \frac{\psi_B}{\psi_C} \beta_C^2 \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_C} + \psi_C \left(\frac{\beta_A^2}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} + \frac{\beta_B^2}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) \right] \\
+ 2 \sum_{C \neq A, B}^N \sum_{D \neq A, B, C}^N \left[(3H + \alpha_C) \alpha_C + \frac{d^2 \mathcal{V}}{d\phi_C^2} + \frac{\partial^2 \mathcal{W}}{\partial \phi_C^2} + \frac{\psi_D}{\psi_C} \frac{\partial^2 \mathcal{W}}{\partial \phi_D \partial \phi_C} - 2\alpha_C (\alpha_C - \alpha_D) \beta_D^2 \right] \beta_C^2
\end{aligned} \tag{169}$$

and

$$\begin{aligned}
\dot{B}_{AB} = [3H + (\alpha_A + \alpha_B) + 2(\alpha_A - \alpha_B) (\beta_A^2 - \beta_B^2)] B_{AB} \\
+ \left[\left(\frac{d^2 \mathcal{V}}{d\phi_A^2} - \frac{d^2 \mathcal{V}}{d\phi_B^2} \right) + \left(\frac{\partial^2 \mathcal{W}}{\partial \phi_A^2} - \frac{\partial^2 \mathcal{W}}{\partial \phi_B^2} \right) + \left(\frac{\psi_B}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} - \frac{\psi_A}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_A \partial \phi_B} \right) \right] \beta_A^2 \beta_B^2 \\
+ \sum_{C \neq A, B}^N \left[-2B_{AB} (\alpha_A + \alpha_B - 2\alpha_C) \beta_C^2 + \left(\frac{\psi_C}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} - \frac{\psi_C}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) \beta_A^2 \beta_B^2 \right],
\end{aligned} \tag{170}$$

$$\begin{aligned}
\dot{C}_{AB} = & \left[3\dot{H} + 3H(\alpha_A\beta_A^2 + \alpha_B\beta_B^2) + (\alpha_A^2\beta_A^2 + \alpha_B^2\beta_B^2) - 2(\alpha_A - \alpha_B)B_{AB} \right] B_{AB} \\
& + \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_A^2} + \frac{\psi_B}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} \right) \beta_A^2 + \left(\frac{d^2\mathcal{V}}{d\phi_B^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} + \frac{\psi_A}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B} \right) \beta_B^2 \right] B_{AB} \\
& + \left(\frac{\psi_A}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} - \frac{\psi_B}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B} \right) B_{AB} + (3H + (\alpha_A\beta_A^2 + \alpha_B\beta_B^2)) \dot{B}_{AB} \\
& + 2 \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2} \right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} \right) - \left(\frac{\psi_A}{\psi_B} - \frac{\psi_B}{\psi_A} \right) \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B} \right] (\beta_A^2 - \beta_B^2) B_{AB} \\
& + \left[\left(\frac{d^3\mathcal{V}}{d\phi_A^3} + \frac{\partial^3\mathcal{W}}{\partial\phi_A^3} - \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_B^2\partial\phi_A} - \frac{\psi_A}{\psi_B} \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_A} \right) \psi_A \right] \beta_A^2\beta_B^2 \\
& - \left[\left(\frac{d^3\mathcal{V}}{d\phi_B^3} + \frac{\partial^3\mathcal{W}}{\partial\phi_B^3} - \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_A^2\partial\phi_B} - \frac{\psi_B}{\psi_A} \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_B} \right) \psi_B \right] \beta_A^2\beta_B^2 \\
& + 2 \sum_{C \neq A, B}^N \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2} \right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} \right) \right] (B_{CA}\beta_B^2 + B_{CB}\beta_A^2) \\
& - 2 \sum_{C \neq A, B}^N \left[\left(\frac{\psi_A}{\psi_B} - \frac{\psi_B}{\psi_A} \right) \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} \right] (B_{CA}\beta_B^2 + B_{CB}\beta_A^2) \\
& + \sum_{C \neq A, B}^N \left[(3H + \alpha_C)\alpha_C + \frac{d^2\mathcal{V}}{d\phi_C^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_C^2} + \frac{\psi_A}{\psi_C} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} + \frac{\psi_B}{\psi_C} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C} \right] \beta_A^2\beta_B^2 \\
& + \sum_{C \neq A, B}^N \left[-2(\alpha_A B_{AC} + \alpha_B B_{BC}) B_{AB} - 2\alpha_C B_{AB} (B_{CA} + B_{CB}) + \alpha_C \beta_C^2 \dot{B}_{AB} \right] \\
& + \sum_{C \neq A, B}^N \left[\psi_C \left(\frac{\beta_A^2}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} + \frac{\beta_B^2}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B} \right) B_{AB} \right] \\
& + \sum_{C \neq A, B}^N \left[(\alpha_A - \alpha_C) \frac{\psi_C}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} - (\alpha_B - \alpha_C) \frac{\psi_C}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C} \right] \\
& + \sum_{C \neq A, B}^N \psi_C \left[\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B^2} - \left(\frac{\psi_A}{\psi_B} \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_C} - \frac{\psi_B}{\psi_A} \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_C} \right) \right] \beta_A^2\beta_B^2 \\
& + \sum_{C \neq A, B}^N \left[\frac{\psi_C}{\psi_A} \left(\psi_A \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A^2} + \psi_B \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_C} + \psi_C \frac{\partial^3\mathcal{W}}{\partial\phi_C^2\partial\phi_A} \right) \right] \\
& - \sum_{C \neq A, B}^N \left[\frac{\psi_C}{\psi_B} \left(\psi_A \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_C} + \psi_B \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B^2} + \psi_C \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B\partial\phi_C} \right) \right] \\
& + \sum_{C \neq A, B}^N \sum_{D \neq A, B, C}^N \left[-2\alpha_C B_{AB} B_{CD} + B_{AB} \beta_C^2 \frac{\psi_D}{\psi_C} \frac{\partial^2\mathcal{W}}{\partial\phi_D\partial\phi_C} \right] \\
& + \sum_{C \neq A, B}^N \sum_{D \neq A, B, C}^N \left[\psi_D \left(\frac{\psi_C}{\psi_A} \frac{\partial^3\mathcal{W}}{\partial\phi_D\partial\phi_A\partial\phi_C} - \frac{\psi_C}{\psi_B} \frac{\partial^3\mathcal{W}}{\partial\phi_D\partial\phi_B\partial\phi_C} \right) \right].
\end{aligned} \tag{171}$$

6.2 Coefficients $\xi, \zeta, \gamma_{AB}, \eta_{AB}, \varsigma_{AB}, \varpi_{AB}$ and ι_{AB}

The coefficients $\xi, \zeta, \gamma_{AB}, \eta_{AB}, \varsigma_{AB}, \varpi_{AB}$ and ι_{AB} are given by

$$\xi = - \left[q + 2 + 2 \left(\frac{\alpha_A}{H} \beta_A^2 + \frac{\alpha_B}{H} \beta_B^2 \right) + 2 \sum_{C \neq A, B}^N \beta_C^2 \frac{\alpha_C}{H} \right] \tag{172}$$

$$\begin{aligned}
\zeta &= - \left[\frac{2HA + 3\dot{H} + 2\psi^2 + \dot{A}}{H^2} - \frac{n^2}{a^2 H^2} \right] \\
&= - \left[6 \left(1 + \frac{\dot{H}}{H^2} \right) + 2 \frac{\psi^2}{H^2} + 2 \left[\frac{\alpha_A}{H} \beta_A^2 \left(5 + \frac{\alpha_A}{H} \right) + \frac{\alpha_B}{H} \beta_B^2 \left(5 + \frac{\alpha_B}{H} \right) \right] - 4 \frac{(\alpha_A - \alpha_B)}{H} B_{AB} \right] \\
&\quad - \frac{2}{H^2} \left[\left(\frac{d^2 \mathcal{V}}{d\phi_A^2} + \frac{\partial^2 \mathcal{W}}{\partial \phi_A^2} + \frac{\psi_B}{\psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} \right) \beta_A^2 + \left(\frac{d^2 \mathcal{V}}{d\phi_B^2} + \frac{\partial^2 \mathcal{W}}{\partial \phi_B^2} + \frac{\psi_A}{\psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_A \partial \phi_B} \right) \beta_B^2 \right] \\
&\quad - 2 \sum_{C \neq A, B}^N \left[-2 \left(\frac{\alpha_C}{H} - \frac{\alpha_A}{H} \right) \frac{B_{CA}}{H} - 2 \left(\frac{\alpha_C}{H} - \frac{\alpha_B}{H} \right) \frac{B_{CB}}{H} \right] \tag{173} \\
&\quad - \frac{2}{H^2} \sum_{C \neq A, B}^N \left[\frac{\Psi_A}{\Psi_C} \beta_C^2 \frac{\partial^2 \mathcal{W}}{\partial \phi_A \partial \phi_C} + \frac{\Psi_B}{\Psi_C} \beta_C^2 \frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_C} + \Psi_C \left(\frac{\beta_A^2}{\Psi_A} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} + \frac{\beta_B^2}{\Psi_B} \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) \right] \\
&\quad - 2 \sum_{C \neq A, B}^N \sum_{D \neq A, B, C}^N \left[\left(3 + \frac{\alpha_C}{H} \right) \frac{\alpha_C}{H} + \frac{1}{H^2} \frac{d^2 \mathcal{V}}{d\phi_C^2} + \frac{1}{H^2} \frac{\partial^2 \mathcal{W}}{\partial \phi_C^2} + \frac{1}{H^2} \frac{\Psi_D}{\Psi_C} \frac{\partial^2 \mathcal{W}}{\partial \phi_D \partial \phi_C} \right] \beta_C^2
\end{aligned}$$

$$\begin{aligned}
\gamma_{AB} &= - \left[\frac{\dot{B}_{AB} + (2H - D_{AB}) B_{AB} + C_{AB}}{H^2} \right] \\
&= - \left[8 + \frac{(\alpha_A + \alpha_B)}{H} + 3 \frac{(\alpha_A - \alpha_B)}{H} (\beta_A^2 - \beta_B^2) \right] \frac{B_{AB}}{H} \\
&\quad - \frac{1}{H^2} \left[2 \left(\frac{d^2 \mathcal{V}}{d\phi_A^2} - \frac{d^2 \mathcal{V}}{d\phi_B^2} \right) + 2 \left(\frac{\partial^2 \mathcal{W}}{\partial \phi_A^2} - \frac{\partial^2 \mathcal{W}}{\partial \phi_B^2} \right) + \left(\frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_A} + \frac{\partial^2 \mathcal{W}}{\partial \phi_A \partial \phi_B} \right) \left(\frac{\psi_B}{\psi_A} - \frac{\psi_A}{\psi_B} \right) \right] \beta_A^2 \beta_B^2 \\
&\quad + \frac{B_{AB}}{H} \sum_{C \neq A, B}^N \left[\left(2 \frac{\alpha_A}{H} + 2 \frac{\alpha_B}{H} - 3 \frac{\alpha_C}{H} \right) \beta_C^2 \right] \\
&\quad - \frac{1}{H^2} \sum_{C \neq A, B}^N \left[\frac{\psi_C}{\psi_A} \left(\frac{\partial^2 \mathcal{W}}{\partial \phi_A \partial \phi_C} + \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_A} \right) - \frac{\psi_C}{\psi_B} \left(\frac{\partial^2 \mathcal{W}}{\partial \phi_B \partial \phi_C} + \frac{\partial^2 \mathcal{W}}{\partial \phi_C \partial \phi_B} \right) \right]
\end{aligned}$$

$$\begin{aligned}
\eta_{AB} &= \left[\frac{\dot{C}_{AB} + (5H + D_{AB})C_{AB}}{H^3} + \frac{B_{AB}}{H} \left(\frac{3\dot{H} + \psi^2 - E_{AB}}{H^2} - 2\frac{q^2}{a^2H^2} \right) \right] \\
&= \left\{ 6\frac{\dot{H}}{H^2} + 24 + \frac{\psi^2}{H^2} + \frac{(\alpha_A^2\beta_A^2 + \alpha_B^2\beta_B^2)}{H^2} + 3\frac{D_{AB}}{H} - \frac{E_{AB}}{H^2} - \frac{q^2}{a^2H^2} \right\} \frac{B_{AB}}{H} \\
&\quad + \left\{ 2\frac{(\alpha_A - \alpha_B)}{H} \left[3 + \left(\frac{\alpha_A}{H}\beta_A^2 + \frac{\alpha_B}{H}\beta_B^2 \right) \right] (\beta_A^2 - \beta_B^2) - \frac{B_{AB}}{H} \right\} \frac{B_{AB}}{H} \\
&\quad + \left\{ \frac{(\alpha_A + \alpha_B)}{H} \left[3 + \left(\frac{\alpha_A}{H}\beta_A^2 + \frac{\alpha_B}{H}\beta_B^2 \right) \right] + \left[11 + \frac{D_{AB}}{H} \right] \left(\frac{\alpha_A}{H}\beta_A^2 + \frac{\alpha_B}{H}\beta_B^2 \right) \right\} \frac{B_{AB}}{H} \\
&\quad + \frac{1}{H^2} \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2} \right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} \right) - \left(\frac{\Psi_A}{\Psi_B} - \frac{\Psi_B}{\Psi_A} \right) \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B} \right] \left[\left(8 + \left(\frac{\alpha_A}{H}\beta_A^2 + \frac{\alpha_B}{H}\beta_B^2 \right) \frac{D_{AB}}{H} \right) \beta_A^2\beta_B^2 \right] \\
&\quad + \frac{1}{H^2} \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2} \right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} \right) - \left(\frac{\Psi_A}{\Psi_B} - \frac{\Psi_B}{\Psi_A} \right) \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B} \right] \left[2(\beta_A^2 - \beta_B^2) \frac{B_{AB}}{H} \right] \\
&\quad + \frac{1}{H^2} \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_A^2} + \frac{\Psi_B}{\Psi_A} \frac{\partial^2\mathcal{W}_{AB}}{\partial\phi_A\partial\phi_B} \right) \beta_A^2 + \left(\frac{d^2\mathcal{V}}{d\phi_B^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} + \frac{\Psi_A}{\Psi_B} \frac{\partial^2\mathcal{W}_{AB}}{\partial\phi_A\partial\phi_B} \right) \beta_B^2 + \left(\frac{\Psi_A}{\Psi_B} - \frac{\Psi_B}{\Psi_A} \right) \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B} \right] \frac{B_{AB}}{H} \\
&\quad + \frac{1}{H^3} \left[\left(\frac{d^3\mathcal{V}}{d\phi_A^3} + \frac{\partial^3\mathcal{W}}{\partial\phi_A^3} - \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_B^2\partial\phi_A} - \frac{\psi_A}{\psi_B} \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_A} \right) \psi_A \right] \beta_A^2\beta_B^2 \\
&\quad - \frac{1}{H^3} \left[\left(\frac{d^3\mathcal{V}}{d\phi_B^3} + \frac{\partial^3\mathcal{W}}{\partial\phi_B^3} - \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_A^2\partial\phi_B} - \frac{\psi_B}{\psi_A} \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_B} \right) \psi_B \right] \beta_A^2\beta_B^2 \\
&\quad + \left(\frac{5H + D_{AB}}{H} \right) \sum_{C \neq A, B}^N \left[\alpha_C \beta_C^2 B_{AB} + \left(\frac{\psi_C}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} - \frac{\psi_C}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C} \right) \right] \\
&\quad + \left(\frac{3H + \alpha_A\beta_A^2 + \alpha_B\beta_B^2}{H} \right) \sum_{C \neq A, B}^N \left[-2B_{AB}(\alpha_A + \alpha_B - 2\alpha_C)\beta_C^2 + \left(\frac{\psi_C}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} - \frac{\psi_C}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B} \right) \beta_A^2\beta_B^2 \right] \\
&\quad + 2 \sum_{C \neq A, B}^N \left[\left(\frac{d^2\mathcal{V}}{d\phi_A^2} - \frac{d^2\mathcal{V}}{d\phi_B^2} \right) + \left(\frac{\partial^2\mathcal{W}}{\partial\phi_A^2} - \frac{\partial^2\mathcal{W}}{\partial\phi_B^2} \right) - \left(\frac{\psi_A}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_A} - \frac{\psi_B}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_B} \right) \right] (B_{CA}\beta_B^2 + B_{CB}\beta_A^2) \\
&\quad + \sum_{C \neq A, B}^N \left[(3H + \alpha_C)\alpha_C + \frac{d^2\mathcal{V}}{d\phi_C^2} + \frac{\partial^2\mathcal{W}}{\partial\phi_C^2} + \frac{\psi_A}{\psi_C} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} + \frac{\psi_B}{\psi_C} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C} \right] \beta_A^2\beta_B^2 \\
&\quad + \sum_{C \neq A, B}^N \left[-2(\alpha_A B_{AC} + \alpha_B B_{BC}) B_{AB} - 2\alpha_C B_{AB} (B_{CA} + B_{CB}) + \alpha_C \beta_C^2 \dot{B}_{AB} \right] \\
&\quad + \sum_{C \neq A, B}^N \left[\psi_C \left(\frac{\beta_A^2}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_A} + \frac{\beta_B^2}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_C\partial\phi_B} \right) B_{AB} + (\alpha_A - \alpha_C) \frac{\psi_C}{\psi_A} \frac{\partial^2\mathcal{W}}{\partial\phi_A\partial\phi_C} - (\alpha_B - \alpha_C) \frac{\psi_C}{\psi_B} \frac{\partial^2\mathcal{W}}{\partial\phi_B\partial\phi_C} \right] \\
&\quad + \sum_{C \neq A, B}^N \psi_C \left[\frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A^2} - \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B^2} - \left(\frac{\psi_A}{\psi_B} \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_C} - \frac{\psi_B}{\psi_A} \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_C} \right) \right] \beta_A^2\beta_B^2 \\
&\quad + \sum_{C \neq A, B}^N \left[\frac{\psi_C}{\psi_A} \left(\psi_A \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A^2} + \psi_B \frac{\partial^3\mathcal{W}}{\partial\phi_B\partial\phi_A\partial\phi_C} + \psi_C \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_A\partial\phi_C} \right) \right] \\
&\quad - \sum_{C \neq A, B}^N \left[\frac{\psi_C}{\psi_B} \left(\psi_A \frac{\partial^3\mathcal{W}}{\partial\phi_A\partial\phi_B\partial\phi_C} + \psi_B \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B^2} + \psi_C \frac{\partial^3\mathcal{W}}{\partial\phi_C\partial\phi_B\partial\phi_C} \right) \right] \\
&\quad + \sum_{C \neq A, B}^N \sum_{D \neq A, B, C}^N \left[-2\alpha_C B_{AB} B_{CD} + B_{AB} \beta_C^2 \frac{\psi_D}{\psi_C} \frac{\partial^2\mathcal{W}}{\partial\phi_D\partial\phi_C} + \psi_D \left(\frac{\psi_C}{\psi_A} \frac{\partial^3\mathcal{W}}{\partial\phi_D\partial\phi_A\partial\phi_C} - \frac{\psi_C}{\psi_B} \frac{\partial^3\mathcal{W}}{\partial\phi_D\partial\phi_B\partial\phi_C} \right) \right].
\end{aligned}$$

$$s_{AB} = -\frac{D_{AB}}{H} = -\frac{1}{H} (\alpha_A\beta_B^2 + \alpha_B\beta_A^2)$$

$$\varpi_{AB} = \frac{E_{AB}}{H^2} + \frac{q^2}{a^2H^2}$$

$$\iota_{AB} = \left[1 - (1 + q) + 2 + \frac{D_{AB}}{H} \right]$$

In the case of two scalar fields which do not interact with each other we have $\mathcal{W}(\phi_1, \phi_2) = 0$ and, using expansion normalised variables (10), the above coefficient read

$$\xi = - \left[2 + 2 (\Psi_1^2 + \Psi_2^2) - (\Phi_1^{2n} + \Phi_2^{2n}) + 2\sqrt{6}n \left[\frac{\Phi_1^{2n-1}}{\Psi_1} \left(\frac{\Psi_1^2}{\Psi^2} \frac{\partial \Phi_1}{\partial \phi_1} \right) + \frac{\Phi_2^{2n-1}}{\Psi_2} \left(\frac{\Psi_2^2}{\Psi^2} \frac{\partial \Phi_2}{\partial \phi_2} \right) \right] \right] \quad (174)$$

$$\begin{aligned} \zeta = & - \left[6\Phi^{2n} + 2\sqrt{6}n \left[\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} \frac{\Psi_1^2}{\Psi^2} \left(5 + \sqrt{6}n \frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} \right) + \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \frac{\Psi_2^2}{\Psi^2} \left(5 + \sqrt{6}n \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right) \right] \right] \\ & + 24n^2 \left[\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right]^2 \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & - 12n(2n-1) \left[\Phi_1^{2(n-1)} \left(\frac{\partial \Phi_1}{\partial \phi_1} \right)^2 \frac{\Psi_1^2}{\Psi^2} + \Phi_2^{2(n-1)} \left(\frac{\partial \Phi_2}{\partial \phi_2} \right)^2 \frac{\Psi_2^2}{\Psi^2} \right] \\ & - 12n \left[\Phi_1^{2n-1} \frac{\partial^2 \Phi_1}{\partial \phi_1^2} \frac{\Psi_1^2}{\Psi^2} + \Phi_2^{2n-1} \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \frac{\Psi_2^2}{\Psi^2} \right] + \frac{n^2}{a^2 H^2} \end{aligned} \quad (175)$$

$$\begin{aligned} \gamma_{12} = & - 8\sqrt{6}n \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right) \frac{\Psi_1^2 \Psi_2^2}{\Psi^2 \Psi^2} \\ & - 18n^2 \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right)^2 \frac{(\Psi_1^2 - \Psi_2^2)}{\Psi^2} \frac{\Psi_1^2 \Psi_2^2}{\Psi^2 \Psi^2} \\ & - 6n^2 \left[\left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} \right)^2 - \left(\frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right)^2 \right] \frac{\Psi_1^2 \Psi_2^2}{\Psi^2 \Psi^2} \\ & - 12n(2n-1) \left(\Phi_1^{2(n-1)} \left(\frac{\partial \Phi_1}{\partial \phi_1} \right)^2 - \Phi_2^{2(n-1)} \left(\frac{\partial \Phi_2}{\partial \phi_2} \right)^2 \right) \frac{\Psi_1^2 \Psi_2^2}{\Psi^2 \Psi^2} \\ & - 12n \left(\Phi_1^{2n-1} \frac{\partial^2 \Phi_1}{\partial \phi_1^2} - \Phi_2^{2n-1} \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \right) \frac{\Psi_1^2 \Psi_2^2}{\Psi^2 \Psi^2} \end{aligned}$$

$$\begin{aligned} \eta_{12} = & \left[18\sqrt{6}n - 6\sqrt{6}n (\Psi^2 - \Phi^{2n}) - 2\sqrt{6}n \frac{a^2}{a^2 H^2} \right] \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right) \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & + 48n^2 \left[\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\Psi_1^2}{\Psi^2} \frac{\partial \Phi_1}{\partial \phi_1} + \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\Psi_2^2}{\Psi^2} \frac{\partial \Phi_2}{\partial \phi_2} \right] \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right) \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & + 18\sqrt{6}n^3 \left[\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\Psi_1^2}{\Psi^2} \frac{\partial \Phi_1}{\partial \phi_1} + \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\Psi_2^2}{\Psi^2} \frac{\partial \Phi_2}{\partial \phi_2} \right]^2 \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right) \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & + 6\sqrt{6}n^3 \left[\left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} \right)^2 \frac{\Psi_1^2}{\Psi^2} + \left(\frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right)^2 \frac{\Psi_2^2}{\Psi^2} \right] \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right) \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & + 36n^2 \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right)^2 \frac{\Psi_1^2 - \Psi_2^2}{\Psi^2} \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & + 24n^2 \left[\left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} \right)^2 - \left(\frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right)^2 \right] \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & - 18\sqrt{6}n^3 \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right)^3 \frac{\Psi_1^4 \Psi_2^4}{\Psi^8} \\ & + 6n(2n-1) \left[8 + \sqrt{6}n \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} + \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right) \right] \left(\Phi_1^{2(n-1)} \left(\frac{\partial \Phi_1}{\partial \phi_1} \right)^2 - \Phi_2^{2(n-1)} \left(\frac{\partial \Phi_2}{\partial \phi_2} \right)^2 \right) \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & + 6n \left[8 + \sqrt{6}n \left(\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} + \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right) \right] \left(\Phi_1^{2n-1} \frac{\partial^2 \Phi_1}{\partial \phi_1^2} - \Phi_2^{2n-1} \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \right) \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & + 18\sqrt{6}n(2n-1) \left[\Phi_1^{2(n-1)} \Psi_1 \frac{\partial \Phi_1}{\partial \phi_1} \frac{\partial^2 \Phi_1}{\partial \phi_1^2} - \Phi_2^{2(n-1)} \Psi_2 \frac{\partial \Phi_2}{\partial \phi_2} \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \right] \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & + 12\sqrt{6}n(2n-1)(n-1) \left[\Phi_1^{2n-3} \Psi_1 \left(\frac{\partial \Phi_1}{\partial \phi_1} \right)^3 - \Phi_2^{2n-3} \Psi_2 \left(\frac{\partial \Phi_2}{\partial \phi_2} \right)^3 \right] \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\ & + 6\sqrt{6}n \left[\Phi_1^{2n-1} \Psi_1 \frac{\partial^3 \Phi_1}{\partial \phi_1^3} - \Phi_2^{2n-1} \Psi_2 \frac{\partial^3 \Phi_2}{\partial \phi_2^3} \right] \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \end{aligned}$$

$$\begin{aligned}
\varsigma_{12} &= -\sqrt{6}n \left[\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} \frac{\Psi_2^2}{\Psi^2} + \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \frac{\Psi_1^2}{\Psi^2} \right] \\
\varpi_{12} &= 6n^2 \left[\frac{\Phi_1^{2n-1}}{\Psi_1} \frac{\partial \Phi_1}{\partial \phi_1} - \frac{\Phi_2^{2n-1}}{\Psi_2} \frac{\partial \Phi_2}{\partial \phi_2} \right]^2 \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\
&\quad + 6n \left[(2n-1) \Phi_1^{2(n-1)} \left(\frac{\partial \Phi_1}{\partial \phi_1} \right)^2 + \Phi_1^{2n-1} \frac{\partial^2 \Phi_1}{\partial \phi_1^2} \right] \frac{\Psi_2^2}{\Psi^2} \\
&\quad + 6n \left[(2n-1) \Phi_2^{2(n-1)} \left(\frac{\partial \Phi_2}{\partial \phi_2} \right)^2 + \Phi_2^{2n-1} \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \right] \frac{\Psi_1^2}{\Psi^2} + \frac{q^2}{a^2 H^2} \\
\iota_{12} &= 2 - 2(\Psi_1^2 + \Psi_2^2) + (\Phi_1^{2n} + \Phi_2^{2n}) - \varsigma_{12}.
\end{aligned}$$

At \mathcal{P} , it follows from Lemma 1 that

$$\begin{aligned}
\xi(\mathcal{P}) &= -[4 - 9\Phi_1^{2n}(\mathcal{P}) - 9\Phi_2^{2n}(\mathcal{P})] \\
\zeta(\mathcal{P}) &= \left[24 - 9 \left(4 - \frac{2}{n} \right) \Psi^2 \right] (\Phi_1^{2n} + \Phi_2^{2n}) - 18 \left[\frac{\Phi_1^{4n}}{\Psi_1^2} + \frac{\Phi_2^{4n}}{\Psi_2^2} \right] \Psi^2 \\
&\quad + 36 \left[\Phi_1^{2n} \frac{\Psi_2}{\Psi_1} - \Phi_2^{2n} \frac{\Psi_1}{\Psi_2} \right]^2 - 12n \left[\Phi_1^{2n-1} \frac{\Psi_1^2}{\Psi^2} \frac{\partial^2 \Phi_1}{\partial \phi_1^2} + \Phi_2^{2n-1} \frac{\Psi_2^2}{\Psi^2} \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \right] + \frac{n^2}{a^2 H^2} \\
\gamma_{12}(\mathcal{P}) &= 24 \left[\Phi_1^{2n} \frac{\Psi_2^2}{\Psi_1^2 + \Psi_2^2} - \Phi_2^{2n} \frac{\Psi_1^2}{\Psi_1^2 + \Psi_2^2} \right] - 9 \left[\Phi_1^{4n} \frac{\Psi_2^2}{\Psi_1^2} - \Phi_2^{4n} \frac{\Psi_1^2}{\Psi_2^2} \right] \\
&\quad - 24 \left[\Phi_1^{2n} \frac{\Psi_2}{\Psi_1} - \Phi_2^{2n} \frac{\Psi_1}{\Psi_2} \right]^2 \frac{\Psi_1^2 - \Psi_2^2}{\Psi^2} - 9 \left(4 - \frac{2}{n} \right) [\Phi_1^{2n} \Psi_2^2 - \Phi_2^{2n} \Psi_1^2] \\
&\quad - 12n \left[\Phi_1^{2n-1} \frac{\partial^2 \Phi_1}{\partial \phi_1^2} - \Phi_2^{2n-1} \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \right] \frac{\Psi_1^2 \Psi_2^2}{(\Psi_1^2 + \Psi_2^2)^2} \\
\eta_{12}(\mathcal{P}) &= - \left[36\Psi^2 + 81\Phi^{4n} + 24 \left(\frac{\Phi_1^{4n}}{\Psi_1^2} + \frac{\Phi_2^{4n}}{\Psi_2^2} \right) \Psi^2 - 6 \frac{q^2}{a^2 H^2} \right] \left[\Phi_1^{2n} \frac{\Psi_2^2}{\Psi^2} - \Phi_2^{2n} \frac{\Psi_1^2}{\Psi^2} \right] \\
&\quad + 54 \left[\Phi_1^{2n} \frac{\Psi_2}{\Psi_1} - \Phi_2^{2n} \frac{\Psi_1}{\Psi_2} \right]^2 \frac{\Psi_1^2 - \Psi_2^2}{\Psi^2} + 36 \left[\Phi_1^{4n} \frac{\Psi_2^2}{\Psi_1^2} - \Phi_2^{4n} \frac{\Psi_1^2}{\Psi_2^2} \right] \\
&\quad + 81 \left[\Phi_1^{2n} \frac{\Psi_2}{\Psi_1} - \Phi_2^{2n} \frac{\Psi_1}{\Psi_2} \right]^3 \frac{\Psi_1 \Psi_2}{\Psi^4} \\
&\quad + 9 \left(2 - \frac{1}{n} \right) \left[8 - 3 \left(\frac{\Phi_1^{2n}}{\Psi_1^2} + \frac{\Phi_2^{2n}}{\Psi_2^2} \right) \Psi^2 \right] (\Phi_1^{2n} \Psi_2^2 - \Phi_2^{2n} \Psi_1^2) \\
&\quad - \left(\frac{\sqrt{6}}{2n} \right)^3 12\sqrt{6}n(2n-1)(n-1) [\Phi_1^{2n} \Psi_2^2 - \Phi_2^{2n} \Psi_1^2] \Psi^2 \\
&\quad + 6n \left[8 - 3 \left(\frac{\Phi_1^{2n}}{\Psi_1^2} + \frac{\Phi_2^{2n}}{\Psi_2^2} \right) \Psi^2 \right] \left[\Phi_1^{2n-1} \frac{\partial^2 \Phi_1}{\partial \phi_1^2} - \Phi_2^{2n-1} \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \right] \frac{\Psi_1^2 \Psi_2^2}{(\Psi_1^2 + \Psi_2^2)^2} \\
&\quad - 18 \left(\frac{\sqrt{6}}{2n} \right) \sqrt{6}n(2n-1) \left[\Phi_1^{2n-1} \frac{\partial^2 \Phi_1}{\partial \phi_1^2} - \Phi_2^{2n-1} \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \right] \frac{\Psi_1^2 \Psi_2^2}{\Psi^4} \\
&\quad + 6\sqrt{6}n \left[\Phi_1^{2n-1} \Psi_1 \frac{\partial^3 \Phi_1}{\partial \phi_1^3} - \Phi_2^{2n-1} \Psi_2 \frac{\partial^3 \Phi_2}{\partial \phi_2^3} \right] \frac{\Psi_1^2 \Psi_2^2}{\Psi^4}
\end{aligned}$$

$$\begin{aligned}
\varsigma_{12}(\mathcal{P}) &= \frac{3}{\Psi_1^2 \Psi_2^2} [\Phi_1^{2n} \Psi_2^4 + \Phi_2^{2n} \Psi_1^4] \\
\varpi_{12}(\mathcal{P}) &= 9 \frac{(\Phi_1^{2n} \Psi_2^2 - \Phi_2^{2n} \Psi_1^2)^2}{\Psi_1^2 \Psi_2^2} + \frac{9}{n} (2n-1) \Psi^2 \left(\Phi_1^{2n} \left(\frac{\Psi_2}{\Psi_1} \right)^2 + \Phi_2^{2n} \left(\frac{\Psi_1}{\Psi_2} \right)^2 \right) \\
&\quad + \frac{6n}{\Psi^2} \left(\Phi_1^{2n-1} \Psi_2^2 \frac{\partial^2 \Phi_1}{\partial \phi_1^2} + \Phi_2^{2n-1} \Psi_1^2 \frac{\partial^2 \Phi_2}{\partial \phi_2^2} \right) + \frac{q^2}{a^2 H^2} \\
\iota_{12}(\mathcal{P}) &= 3 (\Phi_1^2 + \Phi_2^2) - \varsigma_{12}(\mathcal{P})
\end{aligned}$$

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