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Determination of Δ resonance parameters from lattice QCD

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A method suitable for extracting resonance parameters of unstable baryons in lattice QCD is examined. The method is applied to the strong decay of the Δ to a pion-nucleon state, extracting the $\pi N\Delta$ coupling constant and Δ decay width.

INTRODUCTION

The investigation of resonances and hadronic decays using the underlying theory of the strong interactions, Quantum Chromodynamics (QCD), is of fundamental importance for nuclear and particle physics. Recent progress in the simulation of QCD on the lattice opens up the possibility of understanding from first principles the phenomenology of the hadronic spectra such as the Roper resonance, the pattern and nature of the other observed resonances, and identification of exotic states. Unlike the study of stable low lying hadrons where the lattice QCD formalism is well developed and most suitable, the investigation of resonances and decays is intrinsically more difficult and it is only very recently that numerical results mainly on the width of mesons have emerged. The basic difficulty lies in the fact that scattering states cannot be realized on a lattice with finite volume in Euclidean field theory. One approach to calculate the decay width of resonances is to make use of the dependence of the energy of interacting particles in the finite volume of the lattice. The energy shift in finite volume can be related to the scattering parameters and in particular to the decay width [1, 2]. This approach has been successfully applied mostly to calculate the width of the ρ-resonance and the scattering length of $\pi-\pi$ and other two-meson systems [3–5].

An alternative approach was proposed in Ref. [6]. This approach is based on the mixing of hadronic states on the lattice when their energies are close and the evaluation of the corresponding transition matrix element. This approach has been shown to work for the decay of the lightest vector meson on the lattice, $\rho\to\pi\,\pi$ [7] as well as for the B-meson [8]. In this letter we apply, for the first time, this method to hadronic decays in the baryon sector and show its applicability in the well-known case of the Δ decay to a pion-nucleon state.

Specifically, we choose the isospin channel $\Delta^{++}\to\pi^+\,p$. Thus we seek an evaluation of the Δ to πN transition amplitude $\langle\Delta,\,\vec{Q},\,\vec{q},\,t_f\,|\,\pi\,N,\,\vec{Q},\,\vec{q},\,t_i\rangle$ where t_f (t_i) is the final (initial) time.

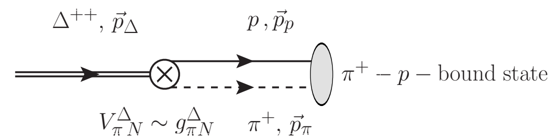

FIG. 1:

We consider a two-state transfer matrix T, which parameterizes the transition amplitude $x=\langle\Delta|\pi N\rangle$ of the form

$$\mathrm{T}\;=\mathrm{e}^{-a\bar{E}}\begin{pmatrix}\mathrm{e}^{-a\delta/2} & ax\\ ax & \mathrm{e}^{+a\delta/2}\end{pmatrix},$$

where $\bar{E}=(E_\Delta+E_{\pi N})/2$ and $\delta=E_{\pi N}-E_\Delta$. Restricting to the Δ and πN states, the matrix element can be written as

$$\begin{aligned}&\langle\Delta,\,t_f\,|\,\pi N,\,t_i\rangle=\langle\Delta\,|\,\mathrm{e}^{-H(t_f-t_i)}\,|\,\pi N\rangle=\langle\Delta\,|\,\mathrm{T}^{n_{fi}}\,|\,\pi N\rangle\\ =&\sum_{n=0}^{n_{fi}-1}\mathrm{e}^{-(\bar{E}-\delta/2)t_n}\,\langle\Delta\,|\,T\,|\,\pi N\rangle\,\mathrm{e}^{-(\bar{E}+\delta/2)(\Delta_{fi}-t_n-a)}\\ =&\;ax\,\frac{\sinh(\delta\,\Delta t_{fi}/2)}{\sinh(a\delta/2)}\,\mathrm{e}^{-\bar{E}\Delta t_{fi}}\end{aligned}$$

where a is the lattice spacing. We construct a ratio of 3-point to 2-point functions that for $t_f-t_i=\Delta t_{fi}\to\gg 1$ yields the Δ to πN overlap

$$R(\Delta t_{fi},\vec{Q},\vec{q})=\frac{C_\mu^{\Delta\to\pi N}(\Delta t_{fi},\,\vec{Q},\vec{q})}{\sqrt{C_\mu^\Delta(\Delta t_{fi},\,\vec{Q})\,C^{\pi N}(\Delta t_{fi},\,\vec{Q},\,\vec{q})}}\,,\quad(1)$$

where $C_\mu^\Delta(\Delta t_{fi},\,\vec{Q})$, $C^{\pi N}(\Delta t_{fi},\,\vec{Q},\vec{q})$ are the Δ and πN two-point correlation functions and $C_\mu^{\Delta\to\pi N}(\Delta t_{fi},\,\vec{Q},\vec{q})$ is the Δ to πN three-point function. For the Δ we use the standard interpolating field

$$J^\alpha_{\Delta\mu}(t,\vec{x})=\epsilon_{abc}\,u^{Ta}(t,\vec{x})\,C\gamma_\mu\,u^b(t,\vec{x})\,u^{\alpha c}(t,\vec{x})\qquad(2)$$

whereas we approximate the πN state with the product of the nucleon and pion interpolating fields given by

$$J^\alpha_{\pi N}(t,\vec{q},\vec{x}) = \sum_{\vec{y}} J_\pi(t,\vec{y}+\vec{x})\, J^\alpha_N(t,\vec{x})\, \mathrm{e}^{-i\vec{q}\vec{y}} \qquad (3)$$

where $J_\pi(t,\vec{y}) = \bar{d}(t,\vec{y})\,\gamma_5\, u(t,\vec{y})$ and $J^\alpha_N(t,\vec{x}) = \epsilon_{abc}\, u^{T\,a}(t,\vec{x})\, C\gamma_5\, d^b(t,\vec{x})\, u^{\alpha c}(t,\vec{x})$ are the standard pion and proton interpolating fields. Moreover, on the level of Wick contractions we keep only the dominant contribution, which corresponds to the product of the individually contracted $\pi-\pi-$current and $N-N-$ current. For large Euclidean time we expect the interpolating fields to dominantly generate the Δ and the $\pi - N$ state which have overlaps with the vacuum that cancel in the ratio of Eq. (1). The πN state is constructed to have a relative momentum of $\vec{q} \neq 0$. In this way the lattice ground state will have overlap with the two-particle state having orbital angular momentum $l = 1$. Together with the nucleon spin $s_N = 1/2$, we thus expect a dominant coupling $l + s_N \to J = 3/2 = J_\Delta$, which allows mixing with the Δ state.

LATTICE QCD CALCULATION

We perform a lattice calculation using a hybrid action of domain wall (DW) valence quarks and gauge configurations generated with two degenerate staggered up and down quarks and a strange staggered quark fixed to its physical value ($N_f = 2+1$) using the Asqtad improved action [9]. The light quark mass in the simulations corresponds to a lightest pion mass of $m_{PS} \approx 360\,\mathrm{MeV}$. The spatial length of the lattice is $L = 3.4\,\mathrm{fm}$ and the lattice spacing is $a \approx 0.124\,\mathrm{fm}$. The light quark mass in the hybrid theory is determined by matching the pion mass to that of the lightest pion generated by the Asqtad action as described in Ref. [10].

For kinematics, we choose $\vec{q} = \pm\, k\, \hat{e}_i$ with $k = 2\pi/L$ and $\hat{e}_i$ the unit vector in spatial direction $i = 1, 2, 3$ and we work in the rest frame of the Δ setting the total momentum $\vec{Q} = 0$. Each choice of a pair $\pm$, i defines a corresponding ratio $R_{\pm,i}$, that we label with these same indices. We evaluate $R_{\pm,i}$ by analyzing 210 gauge configurations with 4 randomly chosen source locations per configuration subject to the constraint of maximal distance $T/4$ between neighboring source time-slices. We optimize ground state dominance by using Gaussian smearing on the interpolating fields and by performing APE smearing on the gauge field configurations that enter the Gaussian smearing function.

In Table I we list the energies and the momenta corresponding to our kinematics. The energy of the pion-nucleon system labeled "$\pi + N$" is given by the sum of individual pion and nucleon energies. This corresponds to approximating the two-particle state as a product of one-particle states.

TABLE I: The energy and momentum of the states considered.

state	π	N	$\pi + N$	Δ
$\lvert\vec{q}\rvert$	$2\pi/L$	$2\pi/L$	$2\pi/L$	0
$aE(\vec{q})$	0.3170 (09)	0.7547 (72)	1.0717 (74)	0.965 (16)

In order to increase statistics we average over forward and backward propagating pion-nucleon and Δ as well as over all momentum directions. From the values of the mass and energy given in Table I we estimate the energy splitting between the Δ and pion-nucleon bound state to be,

$$a\delta \approx aE_{\pi+N} - aE_\Delta \;=\; 0.106\,(16)\,.$$

Note that the pion is so heavy that $\delta > 0$ and the Δ cannot decay since its mass is below that of the pion-nucleon bound state. However, for a Δ energy close to the energy of the pion-nucleon system one can evaluate the overlap x between the Δ and the pion-nucleon decay channel. Only in the limit $\Delta t_{fi} \to \infty$ and $a \to 0$ do we recover the Dirac-δ function of the continuum theory,

$$\delta_{\Delta t_{fi},a}(\delta) \;=\; a\,\frac{\sinh(\delta\,\Delta t_{fi}/2)}{\sinh(a\delta/2)} \xrightarrow[a\to 0]{\Delta t_{fi}\to\infty} 2\pi\,\delta(p^0_{\pi N} - p^0_\Delta)\,,$$

enforcing exact equality of the Δ and the pion-nucleon energies. Based on these observations we use two ansätze to fit the ratio R and extract the transition amplitude B,

$$f_1(t) \;=\; A + B\,a\,\frac{\sinh(\delta\,t/2)}{\sin(a\delta/2)}$$
$$f_2(t) \;=\; A + B\,t\,(+C\,t^3)\,.$$

The first version, f_1, corresponds to the expected functional dependence on the lattice given a non-zero splitting of the energy levels. The form f_2 is the linearized version of f_1 and represents the limit of f_1 for $\delta \to 0$, whereas by adding the term cubic in t we check for the significance of a potential curvature. Given the sizable energy gap we observe from the spectral analysis it will be interesting to check the impact of the splitting and thus the possible curvature on the fit.

As was done for the two-point functions, we improve the signal of the ratio by combining data from forward and backward propagation and average the ratio obtained for all six combinations ($\pm$, i). The resulting ratio is shown in Fig. [2]. In Fig. [2] we can indeed identify a region bounded by $t/a \sim 4$ and $t/a \sim 10$, where we find a dominating linear dependence on the source-sink time separation.

We list the results for the fit parameters for different choices of $f_{1/2}$ and fit intervals in Table II.

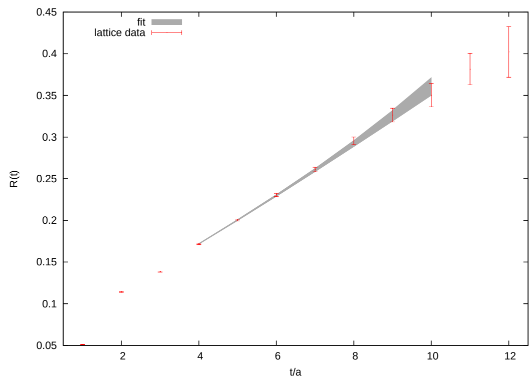

FIG. 2: Ratio for combined forward and backward propagation and averaged over six momentum directions. The shaded band shows the fit using f_1 in the interval $4 \leq t/a \leq 10$.

TABLE II: Parameters extracted using f_1, f_2 with no cubic term, and f_2 using different fits ranges $t_{\rm min}/a$ to $t_{\rm max}/a$. In the last column we give the χ^2 per degree of freedom (dof).

	$t_{\rm min}/a$	$t_{\rm max}/a$	$A \cdot 10^2$	$B \cdot 10^2$	$C \cdot 10^5/a\delta$	χ^2/dof
f_1	4	9	6.47 (49)	2.62 (15)	0.188 (68)	2.4/3
f_1	4	10	6.24 (47)	2.69 (14)	0.156 (79)	4.3/4
f_1	5	9	5.62 (103)	2.82 (26)	0.140 (104)	1.8/2
f_1	5	10	5.05 (84)	2.98 (21)	0.074 (122)	2.9/3
f_2	4	9	5.62 (25)	2.89 (06)		6.0/4
f_2	4	10	5.63 (25)	2.89 (06)		6.5/5
f_2	5	9	4.75 (51)	3.05 (10)		2.4/3
f_2	5	10	4.78 (52)	3.05 (11)		3.0/4
f_2	4	9	6.51 (53)	2.60 (16)	4.1 (22)	2.4/3
f_2	4	10	6.27 (52)	2.68 (16)	2.9 (21)	4.3/4
f_2	5	9	5.64 (128)	2.82 (33)	2.4 (32)	1.8/2
f_2	5	10	5.05 (117)	2.98 (30)	0.7 (28)	2.9/3

An important outcome is that the value for the slope B is stable when using different fits ranges and the two fitting ansätze. We also find a positive value for the energy splitting, as expected, although the statistical error is large not allowing a precise determination of the splitting δ from these fits. This is a consequence of the observation that already the linearized fit gives a satisfactory description of the data as indicated by the value of the χ^2/dof. Although allowing for deviations from the linear dependence tends to decrease the value extracted for the slope, it also leads to an increase of the statistical uncertainty. In fact, the difference between the values of the slopes determined using f_1 and f_2 agree within two standard deviations. The variation of the fit parameters with the lower boundary of the fit interval probes the sensitivity to the influence of excited states. As can be seen, the changes in the slope are within the statistical uncertainty and thus to the accuracy of our measurements we do not see excited state contamination.

EXTRACTION OF THE COUPLING

The value of the slope B is connected with the asymptotic behavior of the correlator at large Euclidean time. In this limit we expect the slope to be associated with the following expression:

$$B_i \;=\; \sum_{\sigma_3,\,\tau_3} \frac{\mathcal{M}(\vec{Q},\,\vec{q},\,\sigma_3,\,\tau_3)}{\sqrt{N_\Delta\,N_{\pi N}}}\, V\,\delta_{\vec{Q}\vec{Q}}\, \frac{\Xi_i(\sigma_3,\,\tau_3)}{\sqrt{\Sigma_i^\Delta\,\Sigma^N}}\,,$$

where Σ_i^Δ and Σ^N denote the spin sums arising from the corresponding 2-point functions of the Δ and the nucleon in the denominator. We include the index i denoting the component of the momentum vector $\vec{q}$, which is non-zero. We also have

$$\Xi_i(\sigma_3,\,\tau_3) \;=\; \Gamma^{(4)}_{\beta\alpha}\, u_\Delta^{i\,\alpha}(\vec{Q}=0,\,\sigma_3)\, u_N^\beta(\vec{q},\,\tau_3)\,.$$

We use the standard normalization for the fermionic and bosonic states given by

$$N_\Delta \;=\; V\,\frac{E_\Delta}{m_\Delta}$$

$$N_{\pi N} \;=\; N_\pi \times N_N = 2V\,E_\pi \times V\,\frac{E_N}{m_N}\,.$$

Note that in accordance with our approximation of the pion-nucleon bound state we normalize the latter as a product of single particle states. The factor $V\,\delta_{\vec{Q}\vec{Q}}$ with a lattice Kronecker-δ reflects the conservation of spatial momentum for our lattice kinematics, where we match exactly $\vec{p}_\Delta = \vec{p}_{\pi N}$. Finally, $\mathcal{M}$ is the transition matrix element, which to leading order we connect using effective field theory [11] to the coupling constant by the relation

$$\mathcal{M}(\vec{Q},\,\vec{q},\,\sigma_3,\,\tau_3) \;=\; \frac{g_{\pi N}^\Delta}{2m_N}\,\bar{u}_\Delta^{\mu\,\alpha}(\vec{Q},\sigma_3)\,q_\mu\,u_N^\alpha(\vec{Q}+\vec{q},\tau_3)\,,$$

where we consider the isospin channel $I_3 = +3/2$. With our specific choice $\vec{Q}=0$ and $\vec{q} \propto \hat{e}_i$ we find

$$B_i\,\frac{\sqrt{N_\Delta\,N_{\pi N}}}{V} \;=\; g_{\pi N}^\Delta\,\frac{q_i}{2m_N}\,\sqrt{\frac{1}{3}\,\frac{E_N+m_N}{m_N}}\,.$$

We solve this relation to extract the coupling $g_{\pi N}^\Delta$. Combining the results from all fits we find

$$g_{\pi N}^\Delta(\mathrm{lat}) \;=\; 27.0\,(6)\,(15)\,.$$

We estimate the systematic error from the variance of the results for $g_{\pi N}^\Delta$ from the individual fits.

DISCUSSION AND OUTLOOK

The method outlined here is based on the mixing of hadronic states on the lattice provided their energies are close. This enables us to compute the overlap of these states even if the particle is above the decay threshold. This overlap can then be related to the coupling constant by connecting to the effective field theory to leading order. Although we have only obtained results for one lattice spacing and volume, our previous studies of the properties of these hadrons have not shown large lattice artifacts and therefore we expect the same to hold true for this calculation [12, 13]. The pion mass dependence of the $\pi N\Delta$ form factor at non-zero momentum transfer was studied using DW fermions with smallest pion mass about 300 MeV and with the same hybrid ensemble as this work [12]. Within this range of pion mass no large variation was observed. However, one would need to perform the same calculation closer to the physical pion mass in order to access the pion mass dependence of the coupling. Nevertheless we can compare with other determinations bearing in mind that our value holds for a pion mass of about 360 MeV.

First, let us look at the result for the width in leading order continuum effective field theory (cf. [11]),

$$\Gamma \;=\; \frac{g_{\pi N\Delta}^2}{48\pi}\,\frac{1}{m_N^2}\,\frac{E_N+m_N}{E_N+E_\pi}\,q^3\,.$$

Together with the PDG value for the width $\Gamma = 118\,(3)\,\mathrm{MeV}$ this leads to a coupling

$$g_{\pi N}^{\Delta}(\text{lo eft}) \;=\; 29.4\,(4)\,.$$

Secondly, in Ref. [14] an experimental value was derived based on a model-independent K-matrix analysis, which reads

$$g_{\pi N}^{\Delta}(\text{exp}) \;=\; 28.6\,(3)\,.$$

We find that our result is compatible with both these values, which is remarkable given that this is a determination at higher than physical pion mass.

In order to calculate the width we will consider the continuum expressions. We note here that the width cannot be calculated by simply using lattice results in e.g. the leading order effective field theory formula since the lattice setup differs from the physical decay process. The problem is rooted in the non-conservation of energy in the lattice setup. Note for instance, that for an experimental decay, the relative momentum in the pion-nucleon system is fixed at $k_{\text{exp}} \approx 227\,\mathrm{MeV}$, while in our lattice setup with $a^{-1} \approx 1.6\,\mathrm{GeV}$ one unit of momentum corresponds to a much larger value $k_{\text{lat}} \approx 360\,\mathrm{MeV}$. Assuming a flat scaling of the dimensionless $g_{\pi N}^{\Delta}(\text{lat})$ given in Eq. (4), neglecting volume dependence and using the continuum relation we obtain an estimate of the decay width of the Δ:

$$\Gamma_\Delta \;=\; 99\,(12)\,\mathrm{MeV}\,.$$

An alternative method to evaluate the width is based on the Lüscher approach that measures the energies as a function of the lattice spatial length L. Although this method has been applied successfully to calculate the decay width of mesons [15] application to baryon decays is still limited by the accuracy attainable in baryon systems. The generalization of Lüscher's formulation in the case of the Δ has been given in Ref. [16] and some preliminary results have been obtained [15]. A matrix Hamiltonian method applicable to $\Delta \to \pi N$ has also been developed [17]. The method presented here is thus a valuable alternative that yields a reliable result for the coupling constant, which can then be related to the width using continuum relations. This first application to the Δ width has demonstrated the applicability of the method. Future work will aim at quantifying systematic uncertainties by performing analyses for different lattice spacings, spatial volumes and pion masses as well as studying the impact of a pion-nucleon state beyond the non-interacting case.

The method presented here can immediately be applied to a number of other hadronic decays, such as $\Sigma^{\frac{3}{2}+} \to \Lambda\pi$ or $\Sigma^{\frac{1}{2}-} \to \bar{K}N$ as well as for the corresponding charmed baryons. A calculation of the widths of these baryons is feasible using this method and work in this direction is underway. Moreover, given the statistical accuracy achieved for the coupling compared to the systematic error, even with the moderate ensemble size used for this work, tackling excited baryonic state decays may become feasible within the same approach.

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