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Gravitational collapse in Hořava-Lifshitz theory

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We study gravitational collapse of a spherical fluid in nonrelativistic general covariant theory of the Hořava-Lifshitz gravity with the projectability condition and an arbitrary coupling constant λ , where $|\lambda - 1|$ characterizes the deviation of the theory from general relativity in the infrared limit. The junction conditions across the surface of a collapsing star are derived under the (minimal) assumption that the junctions be mathematically meaningful in terms of distribution theory. When the collapsing star is made of a homogeneous and isotropic perfect fluid, and the external region is described by a stationary spacetime, the problem reduces to the matching of six independent conditions. If the perfect fluid is pressureless (a dust fluid), it is found that the matching is also possible. In particular, in the case $\lambda = 1$, the external spacetime is described by the Schwarzschild (anti-) de Sitter solution written in Painlevé-Gullstrand coordinates. In the case $\lambda \neq 1$, the external spacetime is static but not asymptotically flat. Our treatment can be easily generalized to other versions of Hořava-Lifshitz gravity or, more generally, to any theory of higher-order derivative gravity.

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I. INTRODUCTION

The study of gravitational collapse provides useful insights into the final fate of a massive star [1]. Within the framework of general relativity, the dynamical collapse of a homogeneous spherical dust cloud under its own gravity was first considered by Datt [2] and Oppenheimer and Snyder [3]. It was shown that it always leads to the formation of singularities. However, in a theory of quantum gravity, it is expected that the formation of singularities in a gravitational collapse is prevented by short-distance quantum effects.

In this paper, we study this phenomenon (classically) in the context of the Hořava theory of gravity [4]. Since Hořava's theory is motivated by the Lifshitz theory in solid state physics [5], it is often referred to as Hořava-Lifshitz (HL) theory. One of the essential ingredients of the theory is the inclusion of higher-dimensional spatial derivative operators which dominate in the ultraviolet, making the theory power-counting renormalizable. The exclusion of higher-dimensional time derivative operators, on the other hand, guarantees that the theory is unitary (the problem of non-unitarity has plagued the quantization of gravity for a long time [6]). However, this asymmetrical treatment of the space and time variables inevitably leads to the breaking of Lorentz symmetry. Although such a breaking is much less restricted by experiments in the gravitational sector than it is in the matter sector [7, 8], the question of how to prevent the

propagation of the Lorentz violations into the Standard Model of particle physics remains challenging [9].

The breaking of Lorentz symmetry in the ultraviolet manifests itself in strongly anisotropic scalings of space and time,

$$\mathbf{x} \rightarrow \ell \mathbf{x}, \quad t \rightarrow \ell^z t. \quad (1.1)$$

In (3 + 1)-dimensional spacetimes, HL theory is power-counting renormalizable provided that $z \geq 3$ [4, 10]. In this paper, we will assume that $z = 3$. At low energies, the theory is expected to flow to $z = 1$. In this limit the Lorentz invariance is “accidentally restored.”

The anisotropy between time and space mentioned above is conveniently expressed in terms of the Arnowitt-Deser-Misner (ADM) decomposition [11], N , N^i , g_{ij} , ($i, j = 1, 2, 3$), which are, respectively, the lapse function, shift vector, and the three-dimensional metric defined on the leaves of constant time. The requirement that the foliation defined by these leaves be preserved by any gauge symmetry implies that the theory is covariant only under the action of the group $\text{Diff}(M, \mathcal{F})$ of foliation-preserving diffeomorphisms,

$$\delta t = -f(t), \quad \delta x^i = -\zeta^i(t, \mathbf{x}). \quad (1.2)$$

As a consequence, an additional degree of freedom appears in the gravitational sector – the spin-0 graviton. In order to be consistent with observations, this degree of freedom needs to decouple in the infrared (IR). Whether this decoupling takes place or not is still an open question [12]. Let us point out that the spin-0 mode is unstable in the Minkowski background in the original incarnation of HL theory [4]. If the projectability condition

$$N = N(t) \quad (1.3)$$

remains imposed, this instability persists in the generalization of HL theory in which additional higher-order

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operators are included [13, 14] (although in this case the de Sitter spacetime is stable [15]). Another potential complication of HL theory is that the theory becomes strongly coupled when energy is very low [16]. However, as long as the theory is consistent with observations when the nonlinear effects are taken into account, this is not necessarily a problem (at least not classically). A careful analysis shows that the theory is consistent with observations in the vacuum spherically symmetry static case [17] and in the cosmological setting [18–20].

One way to overcome the above problems is to introduce an extra local $U(1)$ symmetry, so that the total symmetry of the theory is enlarged to [21]

$$U(1) \times \text{Diff}(M, \mathcal{F}). \quad (1.4)$$

This is achieved by introducing a gauge field A and a Newtonian prepotential φ . One consequence of the $U(1)$ symmetry is that the spin-0 gravitons are eliminated [21, 22]. As a result, all problems related to them, such as instability, strong coupling, and different propagation speeds in the gravitational sector, are resolved. The $U(1)$ symmetry was initially introduced in the case of $\lambda = 1$, but the formalism was soon extended to the case of any λ [23–25]. In the presence of a $U(1)$ symmetry, the consistency of HL theory with solar system tests and cosmology was systematically studied in [26–28]. In particular, it was shown in [29] that in order for the theory to be consistent with solar system tests, the gauge field A and the Newtonian prepotential φ must be part of the metric in the IR limit (this ensures that the line element ds^2 is a scalar not only under $\text{Diff}(M, \mathcal{F})$ but also under the local $U(1)$ symmetry).

Another possibility is to give up the projectability condition (1.3). This opens up for new operators to be included in the action, in particular, operators involving $a_i \equiv N_{,i}/N$ [16]. In this way, all the problems mentioned above can be avoided by properly choosing the coupling constants. However, since this leads to a theory with more than 70 independent coupling constants [30], it makes the theory’s predictive power questionable, although only five coupling constants are relevant in the infrared.

A non-trivial generalization of the enlarged symmetry (1.4) to the nonprojectable case $N = N(t, x)$ was recently presented in [31, 32]. It was shown that, as in general relativity, the only degree of freedom of the model in the gravitational sector is the spin-2 massless graviton. Moreover, thanks to the elimination of the spin-0 gravitons, the physically viable range for the coupling constants is considerably enlarged, in comparison with the healthy extension [16], where the extra $U(1)$ symmetry is absent. Furthermore, the number of independent coupling constants is dramatically reduced from more than 70 to 15. The consistency of the model with cosmology was recently established in [32–34]. In the case with spherical symmetry, the model was shown to be consistent with solar system tests [35]. In contrast to the projectable case, the consistency can be achieved without

taking the gauge field A and Newtonian prepotential φ to be part of the metric. Finally, the duality between this version of HL theory and a non-relativistic quantum field theory was analyzed in [37], and its embedding in string theory were constructed in [38] (For other examples, see for example, [39]).

In this paper, we study gravitational collapse of a spherical star with a finite radius in the HL theory with the projectability condition, an arbitrary coupling constant λ , and the extra $U(1)$ symmetry [21–24]. In general relativity, there are two common approaches for such studies. One approach relies on Israel’s junction conditions [40], which are essentially obtained by using the Gauss and Codazzi equations. An advantage of this method is that it can be applied to the case where the coordinate systems inside and outside a collapsing body are different ¹. The other approach is originally due to Taub [43] and relies on distribution theory. In this approach, although the coordinate systems inside and outside the collapsing stars are taken to be the same, the null-hypersurface case can be easily included. Taub’s approach was widely used to study colliding gravitational waves and other related issues in general relativity [44].

In this paper, we follow Taub’s approach, as it turns out to be more convenient when dealing with higher-order derivatives. Moreover, in contrast to the case of general relativity, the foliation structure of the HL theory implies that the coordinate systems inside and outside of the collapsing star are unique. Thus, also from a technical point of view, Taub’s method seems a natural choice for the study of a collapsing star with a finite radius in the HL theory.

The paper is organized as follows: In Sec. II, we give a brief introduction to the HL theory with the projectability condition, an arbitrary coupling constant λ , and an extra $U(1)$ symmetry. In Sec. III, we write down the field equations relevant for a spherical spacetime filled with a fluid. In Sec. IV, we generalize these equations to include the case where an infinitesimal thin matter shell appears on the surface of a collapsing star, and give explicitly all the necessary junction conditions. This generalization is carried out under the only assumption that the junctions should be mathematically meaningful in terms of generalized functions; therefore, in this sense the generalization is the most general. In Sec. V, we apply the junction conditions to the case where the collapsing star is made of a homogeneous and isotropic perfect fluid, while the external region is described by a stationary spacetime. When the perfect fluid is pressureless (a dust fluid), we find that matching is possible for any choice of λ , but with different external spacetimes. In particular, when $\lambda = 1$, the external spacetime is described by the Schwarzschild

¹ Although Israel’s method was initially developed only for non-null hypersurfaces, it was later generalized to the null hypersurface case [41]. For a recent review of this method, we refer to [42] and references therein.

(anti-) de Sitter solution written in Painlevé-Gullstrand coordinates [45]. In Sec. VI, we present our main results and conclusions. Two appendices are also included. In Appendix A, some relevant functions are given for the spherical case considered here, while in Appendix B, proof of Eqs.(4.11)-(4.12) is provided.

We would like to emphasize that our approach can be easily generalized to other versions of HL gravity or, more generally, to any model of a higher-order derivative gravity theory.

II. GENERAL COVARIANT HL THEORY

In this section, we give a brief introduction to HL theory with the projectability condition (1.3), an arbitrary coupling constant λ and the enlarged symmetry (1.4). For details, we refer readers to [24]. The fundamental variables are $(N, N^i, g_{ij}, A, \varphi)$, which transform as

$$\begin{aligned}\delta N &= \zeta^k \nabla_k N + \dot{N} f + N \dot{f}, \\ \delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik} \dot{\zeta}^k + \dot{N}_i f + N_i \dot{f}, \\ \delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + f \dot{g}_{ij}, \\ \delta A &= \zeta^i \partial_i A + \dot{A} f + f \dot{A}, \\ \delta \varphi &= f \dot{\varphi} + \zeta^i \partial_i \varphi,\end{aligned}\quad (2.1)$$

under $\text{Diff}(M, \mathcal{F})$, and as

$$\begin{aligned}\delta_\alpha A &= \dot{\alpha} - N^i \nabla_i \alpha, \quad \delta_\alpha \varphi = -\alpha, \\ \delta_\alpha N_i &= N \nabla_i \alpha, \quad \delta_\alpha g_{ij} = 0 = \delta_\alpha N,\end{aligned}\quad (2.2)$$

under the local $U(1)$ symmetry, where α is the generator of the $U(1)$ symmetry. The total action is given by

$$S = \zeta^2 \int dt d^3x N \sqrt{g} \left(\mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_\varphi + \mathcal{L}_A + \mathcal{L}_\lambda + \zeta^{-2} \mathcal{L}_M \right), \quad (2.3)$$

where $g = \det g_{ij}$, and

$$\begin{aligned}\mathcal{L}_K &= K_{ij} K^{ij} - \lambda K^2, \\ \mathcal{L}_\varphi &= \varphi \mathcal{G}^{ij} \left(2K_{ij} + \nabla_i \nabla_j \varphi \right), \\ \mathcal{L}_A &= \frac{A}{N} \left(2\Lambda_g - R \right), \\ \mathcal{L}_\lambda &= (1 - \lambda) \left[(\nabla^2 \varphi)^2 + 2K \nabla^2 \varphi \right].\end{aligned}\quad (2.4)$$

Here the coupling constant Λ_g , which acts like a three-dimensional cosmological constant, has the dimension of $(\text{length})^{-2}$. The Ricci and Riemann terms all refer to the three-metric g_{ij} . K_{ij} is the extrinsic curvature, and \mathcal{G}_{ij} is the 3-dimensional ‘‘generalized’’ Einstein tensor defined by

$$\begin{aligned}K_{ij} &= \frac{1}{2N} \left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i \right), \\ \mathcal{G}_{ij} &= R_{ij} - \frac{1}{2} g_{ij} R + \Lambda_g g_{ij}.\end{aligned}\quad (2.5)$$

\mathcal{L}_M is the matter Lagrangian density and \mathcal{L}_V denotes the potential part of the action given by

$$\begin{aligned}\mathcal{L}_V &= \zeta^2 g_0 + g_1 R + \frac{1}{\zeta^2} \left(g_2 R^2 + g_3 R_{ij} R^{ij} \right) \\ &+ \frac{1}{\zeta^4} \left(g_4 R^3 + g_5 R R_{ij} R^{ij} + g_6 R_j^i R_k^j R_i^k \right) \\ &+ \frac{1}{\zeta^4} \left[g_7 R \nabla^2 R + g_8 (\nabla_i R_{jk}) (\nabla^i R^{jk}) \right],\end{aligned}\quad (2.6)$$

which preserves the parity, where the coupling constants g_s ($s = 0, 1, 2, \dots, 8$) are all dimensionless. The relativistic limit in the IR requires that

$$g_1 = -1, \quad \zeta^2 = \frac{1}{16\pi G}, \quad (2.7)$$

where G denotes the Newtonian constant.

Variation of the total action (2.3) with respect to the lapse function $N(t)$ yields the Hamiltonian constraint

$$\begin{aligned}\int d^3x \sqrt{g} \left[\mathcal{L}_K + \mathcal{L}_V - \varphi \mathcal{G}^{ij} \nabla_i \nabla_j \varphi - (1 - \lambda) (\nabla^2 \varphi)^2 \right] \\ = 8\pi G \int d^3x \sqrt{g} J^t,\end{aligned}\quad (2.8)$$

where

$$J^t = 2 \frac{\delta(N\mathcal{L}_M)}{\delta N}.\quad (2.9)$$

Variation of the action with respect to the shift N^i yields the super-momentum constraint

$$\nabla^j \left[\pi_{ij} - \varphi \mathcal{G}_{ij} - (1 - \lambda) g_{ij} \nabla^2 \varphi \right] = 8\pi G J_i, \quad (2.10)$$

where the super-momentum π^{ij} and matter current J^i are defined as

$$\pi^{ij} \equiv -K^{ij} + \lambda K g^{ij}, \quad J^i \equiv -N \frac{\delta \mathcal{L}_M}{\delta N_i}.\quad (2.11)$$

Similarly, variations of the action with respect to φ and A yield, respectively,

$$\begin{aligned}\mathcal{G}^{ij} \left(K_{ij} + \nabla_i \nabla_j \varphi \right) + (1 - \lambda) \nabla^2 \left(K + \nabla^2 \varphi \right) \\ = 8\pi G J_\varphi,\end{aligned}\quad (2.12)$$

$$R - 2\Lambda_g = 8\pi G J_A, \quad (2.13)$$

where

$$J_\varphi \equiv -\frac{\delta \mathcal{L}_M}{\delta \varphi}, \quad J_A \equiv 2 \frac{\delta(N\mathcal{L}_M)}{\delta A}.\quad (2.14)$$

On the other hand, variation with respect to g_{ij} leads to the dynamical equations

$$\frac{1}{N\sqrt{g}} \left\{ \sqrt{g} \left[\pi^{ij} - \varphi \mathcal{G}^{ij} - (1 - \lambda) g^{ij} \nabla^2 \varphi \right] \right\}_{,t}$$

$$\begin{aligned}
&= -2(K^2)^{ij} + 2\lambda K K^{ij} \\
&\quad + \frac{1}{N} \nabla_k \left[N^k \pi^{ij} - 2\pi^{k(i} N^{j)} \right] \\
&\quad - 2(1-\lambda) \left[(K + \nabla^2 \varphi) \nabla^i \nabla^j \varphi + K^{ij} \nabla^2 \varphi \right] \\
&\quad + (1-\lambda) \left[2\nabla^{(i} F_\varphi^{j)} - g^{ij} \nabla_k F_\varphi^k \right] \\
&\quad + \frac{1}{2} (\mathcal{L}_K + \mathcal{L}_\varphi + \mathcal{L}_A + \mathcal{L}_\lambda) g^{ij} \\
&\quad + F^{ij} + F_\varphi^{ij} + F_A^{ij} + 8\pi G \tau^{ij}, \tag{2.15}
\end{aligned}$$

where $(K^2)^{ij} \equiv K^{il} K_l^j$, $f_{(ij)} \equiv (f_{ij} + f_{ji})/2$, and

$$\begin{aligned}
F_A^{ij} &= \frac{1}{N} \left[A R^{ij} - (\nabla^i \nabla^j - g^{ij} \nabla^2) A \right], \\
F_\varphi^{ij} &= \sum_{n=1}^3 F_{(\varphi, n)}^{ij}, \\
F^{ij} &\equiv \frac{1}{\sqrt{g}} \frac{\delta(-\sqrt{g} \mathcal{L}_V)}{\delta g_{ij}} = \sum_{s=0}^8 g_s \zeta^{n_s} (F_s)^{ij}, \tag{2.16}
\end{aligned}$$

with $n_s = (2, 0, -2, -2, -4, -4, -4, -4, -4)$. The 3-tensors $(F_s)_{ij}$ and $F_{(\varphi, n)}^{ij}$ are given by Eqs.(2.21)-(2.23) in [22], which, for the sake of the readers' convenience, are reproduced in Eqs.(A.1) and (A.2) of this paper. The stress 3-tensor τ^{ij} is defined as

$$\tau^{ij} = \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g} \mathcal{L}_M)}{\delta g_{ij}}. \tag{2.17}$$

The matter quantities $(J^t, J^i, J_\varphi, J_A, \tau^{ij})$ satisfy the conservation laws

$$\begin{aligned}
\int d^3x \sqrt{g} \left[\dot{g}_{kl} \tau^{kl} - \frac{1}{\sqrt{g}} (\sqrt{g} J^t)_{,t} + \frac{2N_k}{N\sqrt{g}} (\sqrt{g} J^k)_{,t} \right. \\
\left. - 2\dot{\varphi} J_\varphi - \frac{A}{N\sqrt{g}} (\sqrt{g} J_A)_{,t} \right] = 0, \tag{2.18}
\end{aligned}$$

$$\begin{aligned}
\nabla^k \tau_{ik} - \frac{1}{N\sqrt{g}} (\sqrt{g} J_i)_{,t} - \frac{J^k}{N} (\nabla_k N_i - \nabla_i N_k) \\
- \frac{N_i}{N} \nabla_k J^k + J_\varphi \nabla_i \varphi - \frac{J_A}{2N} \nabla_i A = 0. \tag{2.19}
\end{aligned}$$

In general relativity, the four-dimensional energy-momentum tensor is defined as

$$T^{\mu\nu} = \frac{1}{\sqrt{-g^{(4)}}} \frac{\delta(\sqrt{-g^{(4)}} \mathcal{L}_M)}{\delta g_{\mu\nu}^{(4)}}, \tag{2.20}$$

where $\mu, \nu = 0, 1, 2, 3$, and

$$g_{00}^{(4)} = -N^2 + N^i N_i, \quad g_{0i}^{(4)} = N_i, \quad g_{ij}^{(4)} = g_{ij}. \tag{2.21}$$

Introducing the normal vector n_μ to the hypersurface $t = \text{constant}$ by

$$n_\mu = N \delta_\mu^t, \quad n^\mu = \frac{1}{N} (-1, N^i), \tag{2.22}$$

one can decompose $T_{\mu\nu}$ as follows [46]:

$$\begin{aligned}
\rho_H &\equiv T_{\mu\nu} n^\mu n^\nu, \quad s_i \equiv -T_{\mu\nu} h_i^{(4)\mu} n^\nu, \\
s_{ij} &\equiv T_{\mu\nu} h_i^{(4)\mu} h_j^{(4)\nu}, \tag{2.23}
\end{aligned}$$

where $h_{\mu\nu}^{(4)}$ is the projection operator defined by $h_{\mu\nu}^{(4)} \equiv g_{\mu\nu}^{(4)} + n_\mu n_\nu$. In the relativistic limit, one may make the following identification:

$$(J^t, J_i, \tau_{ij}) = (-2\rho_H, -s_i, s_{ij}). \tag{2.24}$$

III. SPHERICAL SPACETIMES FILLED WITH A FLUID

Spherically symmetric static spacetimes in the framework of the HL theory with U(1) symmetry with or without the projectability condition are studied systematically in [26, 27, 29, 35, 36, 47, 48]. In particular, the ADM variables for spherically symmetric spacetimes with the projectability condition take the forms

$$\begin{aligned}
N &= 1, \quad N^i = \delta_r^i e^{\mu(r,t) - \nu(r,t)}, \\
g_{ij} dx^i dx^j &= e^{2\nu(r,t)} dr^2 + r^2 d\Omega^2, \tag{3.1}
\end{aligned}$$

in the spherical coordinates $x^i = (r, \theta, \phi)$, where $d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2$. The diagonal case $N^i = 0$ corresponds to $\mu(t, r) = -\infty$. On the other hand, using the U(1) gauge freedom (2.2), without loss of generality, we set

$$\varphi = 0, \tag{3.2}$$

which uniquely fixes the gauge. Then, we find that

$$\begin{aligned}
\mathcal{L}_\varphi &= 0 = \mathcal{L}_\lambda, \quad F_\varphi^{ij} = 0, \\
K_{ij} &= e^{\mu+\nu} \left((\mu' - \nu e^{-\mu+\nu}) \delta_i^r \delta_j^r + r e^{-2\nu} \Omega_{ij} \right), \\
R_{ij} &= \frac{2\nu'}{r} \delta_i^r \delta_j^r + e^{-2\nu} \left[r\nu' - (1 - e^{2\nu}) \right] \Omega_{ij}, \\
\mathcal{L}_K &= (1-\lambda) \left[\dot{\nu}^2 - 2\dot{\nu} \mu' e^{\mu-\nu} + \left(\mu'^2 + \frac{2}{r^2} \right) e^{2(\mu-\nu)} \right] \\
&\quad + \lambda \left[\frac{4}{r} \dot{\nu} e^{\mu-\nu} - \frac{2}{r^2} e^{2(\mu-\nu)} (2r\mu' + 1) \right] \\
\mathcal{L}_A &= \frac{2A}{r^2} \left[e^{-2\nu} (1 - 2r\nu') + \Lambda_g r^2 - 1 \right], \\
\mathcal{L}_V &= \sum_{s=0}^3 \mathcal{L}_V^{(s)}, \tag{3.3}
\end{aligned}$$

where a prime denotes the partial derivative with respect to r , $\Omega_{ij} \equiv \delta_i^\theta \delta_j^\theta + \sin^2 \theta \delta_i^\phi \delta_j^\phi$, and $\mathcal{L}_V^{(s)}$'s are given by Eq.(A1) in [49]. The Hamiltonian constraint (2.8) reads

$$\int (\mathcal{L}_K + \mathcal{L}_V - 8\pi G J^t) e^\nu r^2 dr = 0, \tag{3.4}$$

while the momentum constraint (2.10) yields

$$\begin{aligned} (1-\lambda) & \left\{ e^{\mu-\nu} [r^2(\mu'' + \mu'^2 - \mu'\nu') + 2(\mu'r - 1)] \right. \\ & \left. - \dot{\nu}' r^2 \right\} + 2r(\lambda\nu' e^{\mu-\nu} - \dot{\nu}) \\ & = -8\pi G r^2 e^{-\mu+\nu} v, \end{aligned} \quad (3.5)$$

where

$$J^i \equiv e^{-(\mu+\nu)}(v, 0, 0).$$

It can also be shown that Eqs.(2.12) and (2.13) now read

$$\begin{aligned} & \left[e^{2\nu} (\Lambda_g r^2 - 1) + 1 \right] \left(e^{\mu+\nu} \mu' - e^{2\nu} \dot{\nu} \right) \\ & - 2 \left(\nu' - \Lambda_g r e^{2\nu} \right) e^{\mu+\nu} \\ & + (1-\lambda) \left\{ e^{2\nu} (-r^2 \dot{\nu}'' + r^2 \dot{\nu}' \nu' - 2r \dot{\nu}') \right. \\ & \left. + e^{\mu+\nu} [r^2(\mu''' + 3\mu'\mu'' - \mu'\nu'' - 3\mu''\nu' + \mu'^3 - 3\nu'\mu'^2 \right. \\ & \left. + 2\mu'\nu'^2) + 2r(2\mu'' - \nu'' + 2(\nu' - \mu')^2) + 2\nu'] \right\} \\ & = 8\pi G r^2 e^{4\nu} J_\varphi, \end{aligned} \quad (3.6)$$

$$2r\nu' - \left[e^{2\nu} (\Lambda_g r^2 - 1) + 1 \right] = 4\pi G r^2 e^{2\nu} J_A. \quad (3.7)$$

The dynamical equations (2.15), on the other hand, yield

$$\begin{aligned} & (1-\lambda)r \left[e^{\nu+\mu} (\dot{\mu}\mu' + \dot{\mu}' - \dot{\nu}') - e^{2\nu} (\ddot{\nu} + \frac{1}{2}\dot{\nu}^2) \right. \\ & \left. + e^{2\mu} (\mu'' + \frac{1}{2}\mu'^2 - \mu'\nu') \right] \\ & + \left[2(\mu' + \lambda\nu') + (4\lambda - 3)\frac{1}{r} \right] e^{2\mu} - 2e^{\nu+\mu} (\lambda\dot{\mu} + \dot{\nu}) \\ & + \frac{1}{2} r e^{2\nu} \mathcal{L}_A = -r (F_{rr} + F_{rr}^A + 8\pi G e^{2\nu} p_r), \quad (3.8) \\ & \left[\lambda r (\mu'' - \mu'\nu') + (2\lambda - 1)(2\mu' - \nu') \right. \\ & \left. + \frac{1}{2}(3\lambda + 1)r\mu'^2 \right] e^{2\mu} + \frac{1}{2} r e^{2\nu} \mathcal{L}_A \\ & + \left(\lambda\ddot{\nu} + \frac{1}{2}(\lambda + 1)\dot{\nu}^2 \right) r e^{2\nu} \\ & - \left[(2\lambda - 1)\dot{\mu} + r\mu'(\dot{\nu} + \lambda\dot{\mu}) + \lambda r(\dot{\nu}' + \dot{\mu}') \right] e^{\nu+\mu} \\ & = -\frac{e^{2\nu}}{r} (F_{\theta\theta} + F_{\theta\theta}^A + 8\pi G r^2 p_\theta), \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} \tau_{ij} & = e^{2\nu} p_r \delta_i^r \delta_j^r + r^2 p_\theta \Omega_{ij}, \\ F_{ij}^A & = \frac{2}{r} (A' + A\nu') \delta_i^r \delta_j^r + e^{-2\nu} \left[r^2 (A'' - \nu' A') \right. \\ & \left. + r(A' + A\nu') - A(1 - e^{2\nu}) \right] \Omega_{ij}, \end{aligned} \quad (3.10)$$

and F_{ij} is given by Eq.(A4) in Appendix A. We define a fluid with $p_r = p_\theta$ as a perfect fluid, which in general allows energy flow along a radial direction, i.e., v does not not necessarily vanish [50].

The energy conservation law (2.18) now reads

$$\int dr e^\nu r^2 \left[\dot{\rho}_H + (\rho_H + 4p_r) \dot{\nu} \right. \\ \left. + 4(\dot{\nu} - \nu\dot{\mu}) - 2(\dot{J}_A + \dot{\nu} J_A) \right] = 0, \quad (3.11)$$

while the momentum conservation (2.19) yields

$$\begin{aligned} \nu\mu' - (\nu' - p_r') - \frac{2}{r}(\nu - p_r + p_\theta) - \frac{1}{2} J_A A' \\ - e^{\nu-\mu} \left[\dot{\nu} + \nu(2\dot{\nu} - \dot{\mu}) \right] = 0. \end{aligned} \quad (3.12)$$

To relate the quantities J^t , J^i and τ_{ij} to the ones often used in general relativity, in addition to the normal vector n_μ defined in Eq.(2.22), we also introduce the spacelike unit vectors χ_μ , θ_μ and ϕ_μ by

$$\begin{aligned} n_\mu & = \delta_\mu^t, \quad n^\mu = -\delta_t^\mu + e^{\mu-\nu} \delta_r^\mu, \\ \chi^\mu & = e^{-\nu} \delta_r^\mu, \quad \chi_\mu = e^\mu \delta_\mu^t + e^\nu \delta_\mu^r, \\ \theta_\mu & = r \delta_\mu^\theta, \quad \phi_\mu = r \sin \theta \delta_\mu^\phi. \end{aligned} \quad (3.13)$$

In terms of these four unit vectors, the energy-momentum tensor for an anisotropic fluid can be written as

$$\begin{aligned} T_{\mu\nu} & = \rho_H n_\mu n_\nu + q(n_\mu \chi_\nu + n_\nu \chi_\mu) \\ & \quad + p_r \chi_\mu \chi_\nu + p_\theta (\theta_\mu \theta_\nu + \phi_\mu \phi_\nu), \end{aligned} \quad (3.14)$$

where ρ_H , q , p_r and p_θ denote, respectively, the energy density, heat flow along radial direction, radial, and tangential pressures, as measured by the observer with the four-velocity n_μ . This decomposition is consistent with the quantities J^t and J^i defined by

$$\rho_H = -\frac{1}{2} J^t, \quad v = e^\mu q. \quad (3.15)$$

It should be noted that the definitions of the energy density ρ_H , the radial pressure p_r and the heat flow q are different from the ones defined in a comoving frame in general relativity. We refer readers to Appendix B of [49] for details.

IV. JUNCTION CONDITION ACROSS THE SURFACE OF A COLLAPSING SPHERE

The surface Σ of a spherically symmetric collapsing star naturally divides the spacetime M into two regions, the internal and the external regions, denoted by M^- and M^+ respectively, as shown schematically in Fig. 1. The surface $\Sigma = \partial M^- = -\partial M^+$ is described by

$$\Phi(t, r) = 0, \quad (4.1)$$

where $\Phi(t, r) \equiv r - \mathcal{R}(t)$. The spherical symmetry implies that the ADM variables on M take the form (3.1).

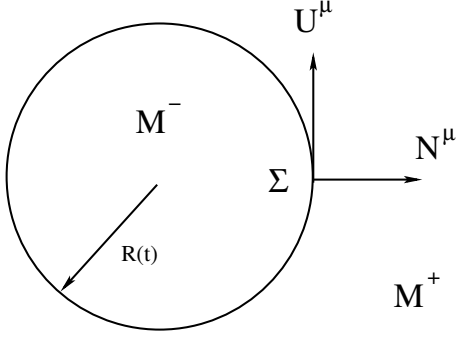


FIG. 1: The spacetime is divided into two regions, the internal M^- and external M^+ , where $M^- = \{x^\mu : r < \mathcal{R}(t)\}$, and $M^+ = \{x^\mu : r > \mathcal{R}(t)\}$. The surface $r = \mathcal{R}(t)$ is denoted by Σ .

A. Preliminaries

We assume that the normal vector $\nabla\Phi$ to the hypersurface Σ with components

$$\begin{aligned}\Phi_{,\lambda} &= \delta_\lambda^r - \dot{\mathcal{R}}\delta_\lambda^t, \\ \Phi^{,\lambda} &= e^{-2\nu}(1 - e^{2\mu} - \dot{\mathcal{R}}e^{\mu+\nu})\delta_r^\lambda + (e^{\mu-\nu} + \dot{\mathcal{R}})\delta_t^\lambda,\end{aligned}\quad (4.2)$$

is everywhere spacelike, i.e.

$$\Phi^{,\lambda}\Phi_{,\lambda} = e^{-2\nu}[1 - (e^\mu + e^\nu\dot{\mathcal{R}})^2] > 0. \quad (4.3)$$

This is the case if $\dot{\mathcal{R}}$ is small enough. We may then define the vector field $N = \nabla\Phi/\|\nabla\Phi\|_g$ in a neighborhood of Σ .² N has length one, i.e. $N_\lambda N^\lambda = 1$, and the restriction of N to Σ is the outward pointing unit normal vector field on Σ .

Let $H(\Phi)$ denote the Heaviside function defined by

$$H(\Phi) = \begin{cases} 1, & \Phi > 0, \\ \frac{1}{2}, & \Phi = 0, \\ 0, & \Phi < 0, \end{cases} \quad (4.4)$$

and let $\delta(\Phi)$ denote the delta distribution with support on Σ . By definition, $\delta(\Phi)$ acts on a smooth test function $\varphi \in C^\infty(M)$ of compact support by

$$(\delta(\Phi), \varphi) = \int_\Sigma \varphi d\Sigma, \quad (4.5)$$

where $d\Sigma = \iota_N \text{Vol}_g$ is the volume three-form induced by g on Σ and ι_N denotes interior multiplication by N . The derivatives $\delta^{(n)}(\Phi)$, $n \geq 1$, of $\delta(\Phi)$ are defined in a standard way and the following relations are valid [51]:

$$\begin{aligned}\frac{\partial H(\Phi)}{\partial x^\lambda} &= \frac{\partial\Phi}{\partial x^\lambda} \delta(\Phi), \\ \frac{\partial}{\partial x^\lambda} \delta^{(n)}(\Phi) &= \frac{\partial\Phi}{\partial x^\lambda} \delta^{(n+1)}(\Phi), \quad n = 0, 1, 2, \dots, \\ \Phi \delta^{(n)}(\Phi) &= -n \delta^{(n-1)}(\Phi), \quad n = 1, 2, \dots\end{aligned} \quad (4.6)$$

² It must not be confused with the lapse function, as in the present case it is set to one, as one can see from Eq.(3.1).

If f is a function defined in a neighborhood of Σ , we define the distribution $f\delta^{(n)}(\Phi)$ by letting it act on a test function φ by

$$(f\delta^{(n)}(\Phi), \varphi) = (\delta^{(n)}(\Phi), f\varphi). \quad (4.7)$$

The product $f\delta(\Phi)$ is well defined whenever f is C^0 and it depends only on the restriction $f|_\Sigma$ of f to Σ . More generally, the product $f\delta^{(n)}(\Phi)$ is well defined provided that f is C^n and it depends only on the values of f and its partial derivatives of order $\leq n$ evaluated on Σ .

Let F be a distribution on M of the form

$$F = F^+ H(\Phi) + F^- [1 - H(\Phi)] + \sum_{k=0}^n F^{Im(k)} \delta^{(k)}(\Phi), \quad (4.8)$$

where the F_n 's are functions defined in a neighborhood of Σ while F^+ and F^- are sufficiently smooth functions defined on M^+ and M^- respectively. We define the function F^D on M by

$$F^D = F^+ H(\Phi) + F^- [1 - H(\Phi)], \quad (4.9)$$

and we define the jump $[F]^-$ of F across Σ by

$$[F]^- (x) = F^+(x) - F^-(x), \quad x \in \Sigma. \quad (4.10)$$

We will also need the fact that the equation $F = 0$ is equivalent to the equations

$$F^\pm(x) = 0, \quad x \in M^\pm, \quad (4.11)$$

and

$$\sum_{k=0}^j (-1)^k \frac{(n-k)! j!}{(j-k)!} \frac{\partial^{j-k}}{\partial \Phi^{j-k}} F^{Im(n-k)} \Big|_\Sigma = 0, \quad 0 \leq j \leq n, \quad (4.12)$$

where $\frac{\partial}{\partial \Phi}$ acts on a function f by

$$\frac{\partial f}{\partial \Phi} = \frac{1}{\|\nabla\Phi\|_g} df \cdot N, \quad (4.13)$$

and, more generally, for any $j \geq 1$,

$$\frac{\partial^j f}{\partial \Phi^j} = \left(\frac{1}{\|\nabla\Phi\|_g} \iota_N d \right)^j f. \quad (4.14)$$

A proof of this fact is given in Appendix B.

For $n = 3$, the conditions in (4.12) are

$$\begin{aligned}F^{Im(3)}|_\Sigma &= 0, \\ \left(3 \frac{\partial F^{Im(3)}}{\partial \Phi} - F^{Im(2)} \right) \Big|_\Sigma &= 0, \\ \left(3 \frac{\partial^2 F^{Im(3)}}{\partial \Phi^2} - 2 \frac{\partial F^{Im(2)}}{\partial \Phi} + F^{Im(1)} \right) \Big|_\Sigma &= 0, \\ \left(\frac{\partial^3 F^{Im(3)}}{\partial \Phi^3} - \frac{\partial^2 F^{Im(2)}}{\partial \Phi^2} + \frac{\partial F^{Im(1)}}{\partial \Phi} - F^{Im(0)} \right) \Big|_\Sigma &= 0.\end{aligned} \quad (4.15)$$

B. Distributional metric functions

The field equations (2.8) - (2.15) involve second-order derivatives of the metric coefficients with respect to t and sixth-order derivatives with respect to x^i . Thus, one might require that the metric coefficients be C^1 with respect to t and C^5 with respect to x^i , where C^n indicates that the first n derivatives exist and are continuous across the hypersurface $\Phi = 0$. However, this assumption eliminates the important case of an infinitely thin shell of matter supported on Σ . Therefore, we will instead make weaker assumptions, so that a thin shell located on the hypersurface $\Phi = 0$ is in general allowed, and consider the case without a thin shell only as a particular case of our general treatment to be provided below. In fact, we shall impose the minimal requirement that *the corresponding problem be mathematically meaningful in terms of distribution theory*. Then, in review of Eqs.(3.4)-(3.12), we find that the cases $\lambda = 1$ and $\lambda \neq 1$ have different dependencies on the derivatives of μ . In particular, the term $\mu'\mu''$ appears when $\lambda \neq 1$. Thus, in the following we consider the two cases separately.

1. $\lambda = 1$

In this case, we assume that: (a) μ and ν are C^5 in each of the regions M^+ and M^- up to the boundary Σ ; (b) μ is C^0 across Σ ; (c) ν is C^0 with respect to t and C^2 with respect to r across Σ .

The above regularity assumptions ensure that the mathematically ill-defined products $\delta(\Phi)^2$ and $\delta(\Phi)H(\Phi)$ do not appear in the field equations. Indeed, the terms in the field equations (3.4) - (3.12) that could lead to products of this type are

$$\mu'^2, \quad \dot{\mu}\mu', \quad \dot{\nu}^2, \quad \nu'^2, \quad \nu''\nu'''. \quad (4.16)$$

Our assumptions imply that these terms may contain $H(\Phi)^2$ but not $\delta(\Phi)^2$ or $\delta(\Phi)H(\Phi)$.

In order to compute the derivatives of μ and ν , we note that

$$\begin{aligned} \mu &= \mu^D = \mu^+ H(\Phi) + \mu^- [1 - H(\Phi)], \\ \nu &= \nu^D = \nu^+ H(\Phi) + \nu^- [1 - H(\Phi)], \end{aligned} \quad (4.17)$$

where the functions μ^+ and ν^+ are C^5 on M^+ , while the functions μ^- and ν^- are C^5 on M^- . Let V_Σ denote an open neighborhood of Σ . Let $\tilde{\mu}^+$ and $\tilde{\nu}^+$ denote C^5 -extensions of μ^+ and ν^+ to $M^+ \cup V_\Sigma$. Let $\tilde{\mu}^-$ and $\tilde{\nu}^-$ denote C^5 -extensions of μ^- and ν^- to $M^- \cup V_\Sigma$. Then the functions

$$\hat{\mu} \equiv \tilde{\mu}^+ - \tilde{\mu}^-, \quad \hat{\nu} \equiv \tilde{\nu}^+ - \tilde{\nu}^-, \quad (4.18)$$

are defined on V_Σ and the following relations are valid on Σ whenever $\alpha + \beta \leq 5$:

$$\hat{\mu} = [\mu]^- , \quad \frac{\partial^{\alpha+\beta}}{\partial t^\alpha \partial r^\beta} \hat{\mu} = \left[\frac{\partial^{\alpha+\beta}}{\partial t^\alpha \partial r^\beta} \mu \right]^-, \quad (4.19)$$

$$\hat{\nu} = [\nu]^- , \quad \frac{\partial^{\alpha+\beta}}{\partial t^\alpha \partial r^\beta} \hat{\nu} = \left[\frac{\partial^{\alpha+\beta}}{\partial t^\alpha \partial r^\beta} \nu \right]^-. \quad (4.20)$$

Since μ is C^0 across Σ , we find

$$\begin{aligned} \mu_{,t} &= (\mu_{,t})^D, \\ \mu_{,r} &= (\mu_{,r})^D, \\ \mu_{,tr} &= (\mu_{,tr})^D + \hat{\mu}_{,t} \delta(\Phi), \\ \mu_{,rt} &= (\mu_{,rt})^D - \hat{\mathcal{R}} \hat{\mu}_{,r} \delta(\Phi), \\ \mu_{,rr} &= (\mu_{,rr})^D + \hat{\mu}_{,r} \delta(\Phi), \\ \mu_{,rrr} &= (\mu_{,rrr})^D + 2\hat{\mu}_{,rr} \delta(\Phi) + \hat{\mu}_{,r} \delta'(\Phi). \end{aligned} \quad (4.21)$$

Since μ is C^0 across Σ , the derivatives of μ^+ and μ^- in any direction tangential to Σ must coincide when evaluated on Σ . In particular, since the vector U defined by

$$U^\lambda \equiv \delta_t^\lambda + \hat{\mathcal{R}} \delta_r^\lambda, \quad (4.22)$$

is tangential to Σ (i.e. $U^\lambda N_\lambda = 0$), we obtain

$$U^\lambda [\mu_{,\lambda}]^- = [\mu_{,t}]^- + \hat{\mathcal{R}} [\mu_{,r}]^- = 0,$$

that is,

$$\hat{\mu}_{,t} = -\hat{\mathcal{R}} \hat{\mu}_{,r}, \quad (4.23)$$

after Eq.(4.19) is taken into account. Then, from Eq.(4.21) one finds $\mu_{,tr} = \mu_{,rt}$, as it is expected.

Similarly, since ν is C^0 across Σ , we also have

$$0 = U^\lambda [\nu_{,\lambda}]^- = [\nu_{,t}]^- + \hat{\mathcal{R}} [\nu_{,r}]^-. \quad (4.24)$$

But $[\nu_{,r}]^- = 0$, because ν is assumed to be C^2 with respect to r . Thus $[\nu_{,t}]^- = 0$. Therefore, ν is in fact C^1 across Σ . The same argument applied to $\nu_{,t}$ and $\nu_{,r}$ now implies that ν is in fact C^2 across Σ . We find

$$\begin{aligned} \nu_{,t} &= (\nu_{,t})^D, \\ \nu_{,r} &= (\nu_{,r})^D, \\ \nu_{,rr} &= (\nu_{,rr})^D, \\ \nu^{(3)} &= (\nu^{(3)})^D, \\ \nu^{(4)} &= (\nu^{(4)})^D + \hat{\nu}^{(3)} \delta(\Phi), \\ \nu^{(5)} &= (\nu^{(5)})^D + 2\hat{\nu}^{(4)} \delta(\Phi) + \hat{\nu}^{(3)} \delta'(\Phi), \end{aligned} \quad (4.25)$$

where $\nu^{(n)} \equiv \partial^n \nu / \partial r^n$. We emphasize that the expressions on the right-hand sides of (4.21) and (4.25) are independent of the extensions used to define $\hat{\mu}$ and $\hat{\nu}$ in (4.18), because the values of $\hat{\mu}$, $\hat{\nu}$, and their partial derivatives of order ≤ 5 are uniquely prescribed on Σ in view of (4.20).

We will find the junction conditions across Σ by substituting the expressions (4.21) and (4.25) for the derivatives of μ and ν into the field equations (3.4) - (3.12).

Suppose that the energy density $\rho_H = -2J^t$ has the form

$$\rho_H = (\rho_H)^D + \sum_{n=0}^{\infty} \rho_H^{Im(n)} \delta^{(n)}(\Phi), \quad (4.26)$$

Variation w.r.t.	Name of equation	General version	Spherically symmetric version	Associated junction condition
lapse $N(t)$	Hamiltonian constraint	(2.8)	(3.4)	(4.27)
shift N^i	Momentum constraint	(2.10)	(3.5)	(4.28)
φ	-	(2.12)	(3.6)	(4.28)
gauge field A	-	(2.13)	(3.7)	(4.28)
metric g_{ij}	Dynamical equations	(2.15)	(3.8) and (3.9)	(4.29) and (4.30)
-	Energy conservation law	(2.18)	(3.11)	(4.31)
-	Momentum conservation law	(2.19)	(3.12)	(4.32)

TABLE I: A list of all field equations for $\lambda = 1$.

where it is understood that only finitely many of the $\rho_H^{Im(n)}$'s are nonzero. Since, by (3.3),

$$\mathcal{L}_K = (\mathcal{L}_K)^D, \quad \mathcal{L}_V = (\mathcal{L}_V)^D,$$

the Hamiltonian constraint (3.4) reads

$$\begin{aligned} & \int_{r < \mathcal{R}(t)} (\mathcal{L}_K^- + \mathcal{L}_V^- + 4\pi G \rho_H^-) e^\nu r^2 dr \\ & + \int_{r > \mathcal{R}(t)} (\mathcal{L}_K^+ + \mathcal{L}_V^+ + 4\pi G \rho_H^+) e^\nu r^2 dr \quad (4.27) \\ & + 4\pi G \sum_{n=0}^{\infty} (-1)^n \frac{\partial^n}{\partial r^n} \Big|_{r=\mathcal{R}(t)} (\rho_H^{Im(n)} e^\nu r^2) = 0. \end{aligned}$$

The left-hand sides of Eqs.(3.5), (3.6) and (3.7) have no supports on the hypersurface $r = \mathcal{R}(t)$. Thus, these equations remain unchanged in the regions M^+ and M^- , while on the hypersurface Σ they yield

$$v = v^D, \quad J_\varphi = (J_\varphi)^D, \quad J_A = (J_A)^D. \quad (4.28)$$

In fact, in order to avoid that the ill-defined product $H(\Phi)\delta(\Phi)$ arises from the term $J_A A'$ in (3.12), we will assume that J_A is C^0 .

The gauge field A has dimension $[A] = 4$, so the action cannot contain terms like A^n with $n \geq 2$, that is, it must be linear in A . We therefore assume that A has the form

$$A(t, r) = A^D + \sum_{n=0}^{\infty} A^{Im(n)} \delta^{(n)}(\Phi).$$

It follows that

$$\begin{aligned} A_{,r} &= (A_{,r})^D + [\hat{A} + A_{,r}^{Im(0)}] \delta(\Phi) \\ &+ \sum_{n=1}^{\infty} [A_{,r}^{Im(n)} + A^{Im(n-1)}] \delta^{(n)}(\Phi), \\ A_{,rr} &= (A_{,rr})^D + [2\hat{A}_{,r} + A_{,rr}^{Im(0)}] \delta(\Phi) \\ &+ [\hat{A} + 2A_{,r}^{Im(0)} + A_{,rr}^{Im(1)}] \delta'(\Phi) \\ &+ \sum_{n=2}^{\infty} [A_{,rr}^{Im(n)} + 2A_{,r}^{Im(n-1)} + A^{Im(n-2)}] \delta^{(n)}(\Phi). \end{aligned}$$

Thus,

$$\begin{aligned} F_{rr}^A &= \frac{2}{r} \left\{ (A_{,r})^D + \nu_{,r} A^D \right. \\ &+ [\hat{A} + A_{,r}^{Im(0)} + \nu_{,r} A^{Im(0)}] \delta(\Phi) \\ &+ \sum_{n=1}^{\infty} [(A_{,r}^{Im(n)} + A^{Im(n-1)}) \delta^{(n)}(\Phi) \\ &\left. + \nu_{,r} A^{Im(n)} \delta^{(n)}(\Phi) \right\}, \end{aligned}$$

$$F_{\theta\theta}^A = (F_{\theta\theta}^A)^D + \sum_{n=0}^{\infty} F_{\theta\theta}^{A,Im(n)} \delta^{(n)}(\Phi),$$

where

$$\begin{aligned} (F_{\theta\theta}^A)^D &= e^{-2\nu} [r^2 (A_{,rr})^D - \nu_{,r} r^2 (A_{,r})^D + r (A_{,r})^D \\ &+ r \nu_{,r} A^D - (1 - e^{2\nu}) A^D], \\ F_{\theta\theta}^{A,Im(0)} &= e^{-2\nu} [r^2 (2\hat{A}_{,r} + A_{,rr}^{Im(0)}) \\ &- r^2 \nu_{,r} (\hat{A} + A_{,r}^{Im(0)}) \\ &+ r (\hat{A} + A_{,r}^{Im(0)}) + r \nu_{,r} A^{Im(0)} \\ &- (1 - e^{2\nu}) A^{Im(0)}], \\ F_{\theta\theta}^{A,Im(1)} &= e^{-2\nu} [r^2 (\hat{A} + 2A_{,r}^{Im(0)} + A_{,rr}^{Im(1)}) \\ &- r^2 \nu_{,r} (A^{Im(0)} + A_{,r}^{Im(1)}) \\ &+ r (A^{Im(0)} + A_{,r}^{Im(1)}) \\ &+ r \nu_{,r} A^{Im(1)} - (1 - e^{2\nu}) A^{Im(1)}], \\ F_{\theta\theta}^{A,Im(n)} &= e^{-2\nu} [r^2 (A^{Im(n-2)} + 2A_{,r}^{Im(n-1)} \\ &+ A_{,rr}^{Im(n)}) - r^2 \nu_{,r} (A^{Im(n-1)} + A_{,r}^{Im(n)}) \\ &+ r (A^{Im(n-1)} + A_{,r}^{Im(n)}) \\ &+ r \nu_{,r} A^{Im(n)} - (1 - e^{2\nu}) A^{Im(n)}], \quad n \geq 2. \end{aligned}$$

From Eq.(A.4) we find that the functions $\{F_n\}_{n=1}^6$ contain no delta functions whereas

$$\begin{aligned} (F_7)_{rr} &= (F_7)_{rr}^D - \frac{16e^{-4\nu}}{r^2} \dot{\nu}^{(3)} \delta(\Phi), \\ (F_8)_{rr} &= (F_8)_{rr}^D - \frac{6e^{-4\nu}}{r^2} \dot{\nu}^{(3)} \delta(\Phi), \\ (F_7)_{\theta\theta} &= (F_7)_{\theta\theta}^D - 8re^{-6\nu} [(2\dot{\nu}^{(4)} - 16\nu_{,r} \dot{\nu}^{(3)}) \delta(\Phi) \end{aligned}$$

$$\begin{aligned} & +\hat{\nu}^{(3)}\delta'(\Phi)], \\ (F_8)_{\theta\theta} &= (F_8)_{\theta\theta}^D - 3re^{-6\nu}[(2\hat{\nu}^{(4)} - 16\nu_{,r}\hat{\nu}^{(3)})\delta(\Phi) \\ & +\hat{\nu}^{(3)}\delta'(\Phi)]. \end{aligned}$$

Thus, (2.16) gives

$$\begin{aligned} F_{rr} &= (F_{rr})^D - (16g_7 + 6g_8)\frac{e^{-4\nu}}{r^2\zeta^4}\hat{\nu}^{(3)}\delta(\Phi), \\ F_{\theta\theta} &= (F_{\theta\theta})^D - (8g_7 + 3g_8)\frac{re^{-6\nu}}{\zeta^4} \\ & \times [(2\hat{\nu}^{(4)} - 16\nu_{,r}\hat{\nu}^{(3)})\delta(\Phi) + \hat{\nu}^{(3)}\delta'(\Phi)]. \end{aligned}$$

Writing p_r in the form

$$p_r(t, r) = p_r^D + \sum_{n=0}^{\infty} p_r^{Im(n)}\delta^{(n)}(\Phi),$$

we find that Eq.(3.8) remains unchanged in the regions M^+ and M^- , while on the hypersurface Σ it yields

$$\begin{aligned} & \sum_{n=0}^{\infty} \left\{ \frac{e^{2\nu}}{r} [e^{-2\nu}(1 - 2r\nu') + \Lambda_g r^2 - 1] A^{Im(n)} \right. \\ & \left. + r \left[F_{rr}^{Im(n)} + F_{rr}^{AIm(n)} + 8\pi G e^{2\nu} p_r^{Im(n)} \right] \right\} \\ & \times \delta^{(n)}(\Phi) = 0. \end{aligned} \quad (4.29)$$

Using (4.12), Eq.(4.29) can be rewritten as a hierarchy of scalar equations on Σ .

Similarly, Eq.(3.9) remains unchanged in the regions M^+ and M^- , while on the hypersurface Σ it yields

$$\begin{aligned} & r(\hat{\mu}_{,r}e^{2\mu} + \hat{\mathcal{R}}\hat{\mu}_{,r}e^{\nu+\mu})\delta(\Phi) \\ & + \sum_{n=0}^{\infty} \left\{ \frac{e^{2\nu}}{r} [e^{-2\nu}(1 - 2r\nu') + \Lambda_g r^2 - 1] A^{Im(n)} \right. \\ & \left. + \frac{e^{2\nu}}{r} \left[F_{\theta\theta}^{Im(n)} + F_{\theta\theta}^{AIm(n)} + 8\pi G r^2 p_{\theta}^{Im(n)} \right] \right\} \delta^{(n)}(\Phi) = 0. \end{aligned} \quad (4.30)$$

Note that

$$\begin{aligned} \rho_{H,t} &= (\rho_{H,t})^D + [\rho_{H,t}^{Im(0)} - \hat{\mathcal{R}}\hat{\rho}_H]\delta(\Phi) \\ & + \sum_{n=1}^{\infty} (\rho_{H,t}^{Im(n)} - \hat{\mathcal{R}}\rho_H^{Im(n-1)})\delta^{(n)}(\Phi), \end{aligned}$$

and, by (4.28),

$$v_{,t} = (v_{,t})^D - \hat{\mathcal{R}}\hat{v}\delta(\Phi).$$

Thus, in view of (4.28), the energy conservation law (3.11) takes the form

$$\begin{aligned} & \int dr e^{\nu} r^2 \left\{ (\rho_{H,t})^D + [\rho_{H,t}^{Im(0)} - \hat{\mathcal{R}}\hat{\rho}_H]\delta(\Phi) \right. \\ & \left. + \sum_{n=1}^{\infty} (\rho_{H,t}^{Im(n)} - \hat{\mathcal{R}}\rho_H^{Im(n-1)})\delta^{(n)}(\Phi) \right\} \end{aligned}$$

$$\begin{aligned} & + \left((\rho_H)^D + \sum_{n=0}^{\infty} \rho_H^{Im(n)}\delta^{(n)}(\Phi) \right. \\ & \left. + 4(p_r)^D + 4 \sum_{n=0}^{\infty} p_r^{Im(n)}\delta^{(n)}(\Phi) \right) \nu_{,t} \\ & + 4((v_{,t})^D - \hat{\mathcal{R}}\hat{v}\delta(\Phi) - v^D \mu_{,t}) \\ & - 2((J_{A,t})^D + \nu_{,t}(J_A)^D) \Big\} = 0, \end{aligned}$$

that is,

$$\begin{aligned} & \left(\int_{r < \mathcal{R}(t)} + \int_{r > \mathcal{R}(t)} \right) e^{\mu} r^2 (\rho_{H,t} + \nu_{,t}(\rho_H + 4p_r) \\ & + 4v_{,t} - 4v\mu_{,t} - 2(J_{A,t} + \nu_{,t}J_A)) dr + \left[e^{\nu} r^2 (\rho_{H,t}^{Im(0)} \right. \\ & - \hat{\mathcal{R}}\hat{\rho}_H + \nu_{,t}(\rho_H^{Im(0)} + 4p_r^{Im(0)}) - 4\hat{\mathcal{R}}\hat{v}) \Big] \Big|_{r=\mathcal{R}(t)} \\ & + \sum_{n=1}^{\infty} (-1)^n \frac{\partial^n}{\partial r^n} \Big|_{r=\mathcal{R}(t)} \left[e^{\nu} r^2 (\rho_{H,t}^{Im(n)} - \hat{\mathcal{R}}\rho_H^{Im(n-1)} \right. \\ & \left. + \nu_{,t}(\rho_H^{Im(n)} + 4p_r^{Im(n)})) \right] = 0. \end{aligned} \quad (4.31)$$

The momentum conservation law (3.12) remains unchanged in M^+ and M^- while on the hypersurface Σ it yields

$$\begin{aligned} & -\hat{v}\delta(\Phi) + \hat{p}_r\delta(\Phi) \\ & + \sum_{n=0}^{\infty} [(p_r^{Im(n)})_{,r}\delta^{(n)}(\Phi) + p_r^{Im(n)}\delta^{(n+1)}(\Phi)] \\ & + \frac{2}{r} \sum_{n=0}^{\infty} (p_r^{Im(n)} - p_{\theta}^{Im(n)})\delta^{(n)}(\Phi) \\ & - \frac{1}{2} J_A \left[(\hat{A} + A_{,r}^{Im(0)})\delta(\Phi) \right. \\ & \left. + \sum_{n=1}^{\infty} (A_{,r}^{Im(n)} + A^{Im(n-1)})\delta^{(n)}(\Phi) \right] \\ & + e^{\nu-\mu} \hat{\mathcal{R}}\hat{v}\delta(\Phi) = 0, \end{aligned} \quad (4.32)$$

where we have used that

$$p'_r = \hat{p}_r\delta(\Phi) + \sum_{n=0}^{\infty} [(p_r^{Im(n)})_{,r}\delta^{(n)}(\Phi) + p_r^{Im(n)}\delta^{(n+1)}(\Phi)].$$

This completes the general description of the junction conditions for the case $\lambda = 1$, which are summarized in Table 1.

2. $\lambda \neq 1$

In this case, the nonlinear terms

$$\mu'^2, \quad \hat{\mu}\mu', \quad \mu'\mu'', \quad \hat{\nu}^2, \quad \nu'^2, \quad \nu''\nu''', \quad (4.33)$$

appear in the field equations (3.4) - (3.12). Thus, to ensure these field equations are well-defined, we assume

that: (a) μ and ν are C^5 in each of the regions M^+ and M^- up to the boundary Σ ; (b) μ is C^0 with respect to t and C^1 with respect to r across Σ ; (c) ν is C^0 with respect to t and C^2 with respect to r across Σ .

The same argument as above shows that ν is C^2 and that μ is C^1 across Σ . Equations (4.21) and (4.25) for the derivatives of μ and ν are still valid, but since μ now is C^1 , we have $\hat{\mu}_{,t} = \hat{\mu}_{,r} = 0$. It follows that all the junction conditions (4.27) - (4.32) remain unchanged, except that the presence of the term μ''' in (3.6) implies that the expression for J_φ now may include a delta function:

$$J_\varphi = (J_\varphi)^D + (1 - \lambda) \frac{e^{\mu - 3\nu}}{4\pi G} \hat{\mu}_{,rr} \delta(\Phi). \quad (4.34)$$

In what follows, we will consider some specific models of gravitational collapse for which the spacetime inside the collapsing sphere is described by the Friedman-Lemaitre-Robertson-Walker (FLRW) universe.

V. GRAVITATIONAL COLLAPSE OF HOMOGENEOUS AND ISOTROPIC PERFECT FLUID

In this section, we consider the gravitational collapse of a spherical cloud consisting of a homogeneous and isotropic perfect fluid³, described by the FLRW universe,

$$ds^2 = -d\bar{t}^2 + a^2(\bar{t}) \left(\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d^2\Omega \right),$$

where $k = 0, \pm 1$. Letting $r = a(\bar{t})\bar{r}$, $t = \bar{t}$, the corresponding ADM variables take the form (3.1) with $N^- = 1$, and

$$\begin{aligned} \nu^-(t, r) &= -\frac{1}{2} \ln \left(1 - k \frac{r^2}{a^2(t)} \right), \\ \mu^-(t, r) &= \ln \left(\frac{-\dot{a}(t)r}{\sqrt{a^2(t) - kr^2}} \right), \end{aligned} \quad (5.1)$$

where $\dot{a} \leq 0$ for a collapsing cloud. For a perfect fluid, we assume that

$$p_\theta^- = p_r^- = p^-(t), \quad v = 0. \quad (5.2)$$

We anticipate that the junction condition for ν requires $k = 0$. Then, we find that

$$\nu^-(t, r) = 0, \quad \mu^-(t, r) = \ln(-rH), \quad (k = 0), \quad (5.3)$$

where $H \equiv \dot{a}(t)/a(t)$, and that

$$\begin{aligned} \mathcal{L}_K^- &= 3(1 - 3\lambda)H^2, \quad \mathcal{L}_V^- = 2\Lambda, \\ \mathcal{L}_\varphi^- &= \mathcal{L}_\lambda^- = 0, \quad \mathcal{L}_A^- = 2\Lambda_g A^-. \end{aligned} \quad (5.4)$$

It is easy to verify that the momentum constraint (3.5) is satisfied, whereas the equations (3.6) and (3.7) obtained by variation with respect to φ and A respectively, reduce to

$$\begin{aligned} 3\Lambda_g H + 8\pi G J_\varphi^- &= 0, \\ 4\pi G J_A^- + \Lambda_g &= 0. \end{aligned} \quad (5.5)$$

Since $\nu^- = 0$, we have $F_{ij}^- = -\Lambda g_{ij}^-$, and the first dynamical equation (3.8) reduces to the condition

$$\begin{aligned} \frac{4}{r} a^2 A_{,r}^- + 2a^2 \Lambda_g A^- + 2(3\lambda - 1)a\dot{a} + (3\lambda - 1)\dot{a}^2 \\ + 2a^2(8\pi G p^- - \Lambda) = 0. \end{aligned}$$

If this condition is satisfied the second dynamical equation (3.8) also holds provided that $A_{,r}^- - rA_{,rr}^- = 0$. On the other hand, the momentum conservation law (3.12) reduces to $J_A^- A_{,r}^- = 0$. We conclude that the general solution when $k = 0$ is given by

$$J_\varphi^- = -\frac{3\Lambda_g H}{8\pi G}, \quad J_A^- = -\frac{\Lambda_g}{4\pi G}, \quad (5.6)$$

with $A^- = A^-(t)$ being given by

$$\Lambda_g A^- + (3\lambda - 1) \left(\frac{\ddot{a}}{a} + \frac{H^2}{2} \right) - \Lambda = -8\pi G p^-. \quad (5.7)$$

In the rest of this section, we consider only the case where $\Lambda_g = 0$. Then, Eq.(5.6) yields

$$J_A^- = J_\varphi^- = 0, \quad (5.8)$$

for which Eq.(5.7) shows that now $A^-(t)$ is an arbitrary function of t , and $a(t)$ is given by

$$(3\lambda - 1) \left(\frac{\ddot{a}}{a} + \frac{H^2}{2} \right) - \Lambda = -8\pi G p^-. \quad (5.9)$$

It is interesting to note that, since the Hamiltonian constraint is global, there is no analog of the Friedman equation in the current situation. This is in contrast to the case of HL cosmology [24], where a Friedman-like equation still exists, because of the homogeneity and isotropy of the whole universe.⁴ Although there is no analog of the Birkhoff theorem in HL theory, so that the spacetime outside the collapsing cloud can be either static or dynamical, we assume in this paper that the exterior solution is a static spherically symmetric vacuum spacetime. We also assume that the value of Λ_g is the same in the exterior and interior regions, i.e.

$$\Lambda_g^+ = \Lambda_g^- = 0. \quad (5.10)$$

³ Gravitational collapse of a homogeneous and isotropic dust fluid filled in the whole space-time was considered in [52], using a method proposed in [53].

⁴ Considering that homogeneity and isotropy are good approximations for our observational universe, this global Hamiltonian constraint allows dust-like fluid to exist; this was first realized in [54] where it was considered as a candidate of dark matter.

It is convenient to consider the cases $\lambda = 1$ and $\lambda \neq 1$ separately.

A. Gravitational Collapse with $\lambda = 1$

We first consider the case of $\lambda = 1$. In this case, the static spherically symmetric exterior vacuum solution has the form [26]

$$\begin{aligned} \mu^+ &= \mu^+(r) = \frac{1}{2} \ln \left(\frac{2m^+}{r} + \frac{1}{3} \Lambda r^2 - 2A^+(r) \right. \\ &\quad \left. + \frac{2}{r} \int_{r_0}^r A^+(r') dr' \right), \\ \nu^+ &= 0, \end{aligned} \quad (5.11)$$

for which we find that

$$\begin{aligned} \mathcal{L}_K^+ &= \frac{4}{r} A_{,r}^+ - 2\Lambda, \quad \mathcal{L}_V^+ = 2\Lambda, \quad \mathcal{L}_A^+ = 0, \\ v^+ &= J_A^+ = J_\varphi^+ = \rho_H^+ = 0, \end{aligned} \quad (5.12)$$

where m^+ , r_0 are constants and $A^+ = A^+(r)$ is a function of r only, yet to be determined.

As mentioned previously, the condition that ν be continuous across Σ implies that $k = 0$. We let the interior solution be of the form (5.3), and assume that the thin shell of matter separating the interior and exterior solutions is such that

$$\begin{aligned} p &= p_r^-, \quad v = 0, \quad J_\varphi = J_\varphi^{Im(0)} \delta(\Phi), \\ p_\theta &= p_\theta^- + p_\theta^{Im(0)} \delta(\Phi), \quad \rho_H = \rho_H^- + \rho_H^{Im(0)} \delta(\Phi), \\ A &= A^D + A^{Im(0)} \delta(\Phi), \quad J_A = 0, \end{aligned} \quad (5.13)$$

where $\rho_H^\pm = J_\varphi^\pm = p_\theta^\pm = p_r^\pm = 0$.

Proposition V.1 *For the spacetime defined by (5.3), (5.9), (5.11), the six junction conditions (4.27)-(4.32) reduce to the following six conditions:*

$$\begin{aligned} &(-6H^2 + 2\Lambda + 4\pi G \rho_H^-(t)) \frac{\mathcal{R}(t)^3}{3} \\ &+ 4 \int_{\mathcal{R}(t)}^\infty A_{,r}^+ r dr + 4\pi G \rho_H^{Im(0)} r^2 \Big|_{r=\mathcal{R}(t)} = 0, \end{aligned} \quad (5.14)$$

$$J_\varphi^{Im(0)} = 0, \quad (5.15)$$

$$A(t, r) \text{ is continuous across } \Sigma, \quad (5.16)$$

$$A_{,t}^- = \mathcal{R} \left(\frac{\Lambda}{2} - H^2 \right) (\dot{\mathcal{R}} - H\mathcal{R}) - 8\pi G p_\theta^{Im(0)} H\mathcal{R}, \quad (5.17)$$

$$\begin{aligned} \rho_H^{Im(0)}(t, \mathcal{R}(t)) &= e^{-\int_0^t \frac{2\dot{\mathcal{R}}(\tau)}{\mathcal{R}(\tau)} d\tau} \left[\rho_H^{Im(0)}(0, \mathcal{R}(0)) \right. \\ &+ \int_0^t e^{\int_0^s \frac{2\dot{\mathcal{R}}(\tau)}{\mathcal{R}(\tau)} d\tau} \left(\frac{1}{4} H(s) \mathcal{R}(s)^2 \rho_{H,t}^-(s) \right. \\ &\quad \left. \left. - \dot{\mathcal{R}}(s) \rho_H^-(s) \right) ds \right], \end{aligned} \quad (5.18)$$

$$rp + 2p_\theta^{Im(0)} = 0 \text{ on } \Sigma. \quad (5.19)$$

Moreover, the condition that μ be continuous across Σ implies that

$$A_{,t}^- = \frac{\Lambda - 3H^2}{2} \mathcal{R} \dot{\mathcal{R}} - HH_{,t} \mathcal{R}^2. \quad (5.20)$$

Proof. For the spacetime defined by (5.12)-(5.13), condition (4.27) reduces to

$$\begin{aligned} &(-6H(t)^2 + 2\Lambda + 4\pi G \rho_H^-(t)) \int_0^{\mathcal{R}(t)} r^2 dr \\ &+ 4 \int_{\mathcal{R}(t)}^\infty A_{,r}^+ r dr + 4\pi G \rho_H^{Im(0)} r^2 \Big|_{r=\mathcal{R}(t)} = 0, \end{aligned}$$

which yields (5.14). Moreover, condition (4.28) reduces immediately to (5.15).

Conditions (4.29) and (4.30) reduce to

$$F_{rr}^{AI(0)} \delta(\Phi) + F_{rr}^{AI(1)} \delta'(\Phi) = 0, \quad (5.21)$$

and

$$\begin{aligned} &r(\hat{\mu}_{,r} e^{2\mu} + \dot{\mathcal{R}} \hat{\mu}_{,r} e^\mu) \delta(\Phi) \\ &+ \frac{1}{r} F_{\theta\theta}^{AI(0)} \delta(\Phi) + \frac{1}{r} F_{\theta\theta}^{AI(1)} \delta'(\Phi) \\ &+ \frac{1}{r} F_{\theta\theta}^{AI(2)} \delta''(\Phi) + 8\pi G r p_\theta^{Im(0)} \delta(\Phi) = 0, \end{aligned} \quad (5.22)$$

respectively, where we have used that

$$F_{rr} = (F_{rr})^D, \quad F_{\theta\theta} = (F_{\theta\theta})^D. \quad (5.23)$$

Now

$$\begin{aligned} F_{rr}^A &= \frac{2}{r} \left\{ (A_{,r})^D + [\hat{A} + A_{,r}^{Im(0)}] \delta(\Phi) \right. \\ &\quad \left. + A^{Im(0)} \delta'(\Phi) \right\}, \end{aligned} \quad (5.24)$$

$$F_{\theta\theta}^A = (F_{\theta\theta}^A)^D + \sum_{n=0}^2 F_{\theta\theta}^{A,Im(n)} \delta^{(n)}(\Phi), \quad (5.25)$$

where

$$\begin{aligned} (F_{\theta\theta}^A)^D &= r^2 (A_{,rr})^D + r (A_{,r})^D, \\ F_{\theta\theta}^{A,Im(0)} &= r^2 (2\hat{A}_{,r} + A_{,rr}^{Im(0)}) + r (\hat{A} + A_{,r}^{Im(0)}), \\ F_{\theta\theta}^{A,Im(1)} &= r^2 (\hat{A} + 2A_{,r}^{Im(0)}) + r A^{Im(0)}, \\ F_{\theta\theta}^{A,Im(2)} &= r^2 A^{Im(0)}. \end{aligned}$$

Thus, equation (5.21) can be written as

$$[\hat{A} + A_{,r}^{Im(0)}] \delta(\Phi) + A^{Im(0)} \delta'(\Phi) = 0. \quad (5.26)$$

Thus, by (4.15), $A^{Im(0)}|_\Sigma = 0$. Hence, $A^{Im(0)} \delta(\Phi) = 0$ which gives

$$0 = (A^{Im(0)} \delta(\Phi))_{,r} = A_{,r}^{Im(0)} \delta(\Phi) + A^{Im(0)} \delta'(\Phi).$$

Equation (5.26) then gives $\hat{A}|_{\Sigma} = 0$ so that in fact A is continuous across Σ , which proves (5.16). Equation (5.22) can now be written as

$$\begin{aligned} & \left[\hat{\mu}_{,r}(e^{2\mu} + \dot{\mathcal{R}}e^{\mu}) + 2\hat{A}_{,r} + 8\pi G p_{\theta}^{Im(0)} \right] \delta(\Phi) \\ & + \hat{A}\delta'(\Phi) = 0. \end{aligned} \quad (5.27)$$

In view of (4.15) this yields

$$\hat{\mu}_{,r}(e^{2\mu} + \dot{\mathcal{R}}e^{\mu}) + 2\hat{A}_{,r} + 8\pi G p_{\theta}^{Im(0)} = \frac{\partial \hat{A}}{\partial \Phi} \text{ on } \Sigma. \quad (5.28)$$

Now observe that if a function $f(t, r)$ is C^0 across Σ , then

$$\frac{\partial \hat{f}}{\partial \Phi} = \frac{\partial \hat{f}}{\partial r} \text{ on } \Sigma. \quad (5.29)$$

Indeed, the continuity of f implies that the derivative of \hat{f} in any direction tangential to Σ must vanish when evaluated on Σ ; thus $\hat{f}_{,t} + \dot{\mathcal{R}}\hat{f}_{,r} = 0$ on Σ . A computation using (4.2), (4.3), and (4.13) now gives (5.29).

On the other hand, since

$$\mu_{,r}^+ = \frac{1}{2}(\Lambda r - 2A_{,r}^+)e^{-2\mu^+}, \quad \mu_{,r}^- = \frac{1}{r},$$

we find

$$\hat{\mu}_{,r} = \frac{1}{2}(\Lambda r - 2A_{,r}^+)e^{-2\mu^+} - \frac{1}{r}. \quad (5.30)$$

Inserting the equations (5.29) and (5.30) into (5.28), we find

$$\begin{aligned} & \left(\frac{1}{2}(\Lambda r - 2A_{,r}^+)e^{-2\mu} - \frac{1}{r} \right) (e^{2\mu} + \dot{\mathcal{R}}e^{\mu}) + \hat{A}_{,r} \\ & + 8\pi G p_{\theta}^{Im(0)} = 0 \text{ on } \Sigma. \end{aligned}$$

Since $\hat{A}_{,r} = A_{,r}^+ = A_{,t}^- \dot{\mathcal{R}}^{-1}$, simplification yields (5.17).

Condition (4.31) reduces to

$$\begin{aligned} & \int_0^{\mathcal{R}(t)} e^{\mu} r^2 \rho_{H,t}^- dr + r^2 \left[\rho_{H,t}^{Im(0)} - \dot{\mathcal{R}}\hat{\rho}_H \right] \Big|_{r=\mathcal{R}(t)} \\ & + \frac{\partial}{\partial r} \Big|_{r=\mathcal{R}(t)} \left[r^2 \dot{\mathcal{R}} \rho_H^{Im(0)} \right] = 0. \end{aligned} \quad (5.31)$$

That is,

$$\begin{aligned} & - \frac{\dot{a}(t)\rho_{H,t}^-(t)}{a(t)} \int_0^{\mathcal{R}(t)} r^3 dr \\ & + \mathcal{R}(t)^2 \left[\rho_{H,t}^{Im(0)}(t, \mathcal{R}(t)) + \dot{\mathcal{R}}(t)\rho_H^-(t) \right] \\ & + 2\mathcal{R}(t)\dot{\mathcal{R}}(t)\rho_H^{Im(0)}(t, \mathcal{R}(t)) \\ & + \mathcal{R}(t)^2 \dot{\mathcal{R}}(t)\rho_{H,r}^{Im(0)}(t, \mathcal{R}(t)) = 0. \end{aligned} \quad (5.32)$$

Consequently,

$$\begin{aligned} & - \frac{H(t)\rho_{H,t}^-(t)\mathcal{R}(t)^2}{4} + \frac{d}{dt} \left[\rho_H^{Im(0)}(t, \mathcal{R}(t)) \right] \\ & + 2 \frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)} \rho_H^{Im(0)}(t, \mathcal{R}(t)) + \dot{\mathcal{R}}(t)\rho_H^-(t) = 0. \end{aligned} \quad (5.33)$$

Solving this differential equation for $\rho_H^{Im(0)}$, we find (5.18).

Condition (4.32) reduces to

$$\left(p + \frac{2}{r} p_{\theta}^{Im(0)} \right) \delta(\Phi) = 0.$$

This yields (5.19).

Finally, the condition that μ be continuous across Σ can be written as

$$\begin{aligned} & \frac{2m^+}{\mathcal{R}(t)} + \frac{1}{3}\Lambda\mathcal{R}(t)^2 - 2A^+(\mathcal{R}(t)) \\ & + \frac{2}{\mathcal{R}(t)} \int_{r_0}^{\mathcal{R}(t)} A^+(r') dr' = H^2 \mathcal{R}^2. \end{aligned} \quad (5.34)$$

Since A is continuous across Σ , we have $A^+(\mathcal{R}(t)) = A^-(t)$. Hence, multiplying (5.34) by \mathcal{R} and then differentiating with respect to t , we find

$$\Lambda\mathcal{R}^2\dot{\mathcal{R}} - 2A_{,t}^-\mathcal{R} = 2HH_{,t}\mathcal{R}^3 + 3H^2\mathcal{R}^2\dot{\mathcal{R}}.$$

Solving this equation for $A_{,t}^-$, we find (5.20). \square

The conditions (5.17) and (5.20) imply that

$$\begin{aligned} & \left(\frac{\Lambda}{2} - H^2 \right) (\dot{\mathcal{R}} - H\mathcal{R}) - 8\pi G p_{\theta}^{Im(0)} H \\ & = \frac{\Lambda - 3H^2}{2} \dot{\mathcal{R}} - HH_{,t}\mathcal{R}, \end{aligned}$$

i.e.

$$H\dot{\mathcal{R}} + (2H^2 + 2H_{,t} - \Lambda)\mathcal{R} - 16\pi G p_{\theta}^{Im(0)} = 0.$$

Solving this equation for $\mathcal{R}(t)$ we find the following equation which expresses $\mathcal{R}(t)$ in terms of $H(t)$ and the pressure $p_{\theta}^{Im(0)}$ on the shell:

$$\begin{aligned} \mathcal{R}(t) = & e^{-\int_0^t I(s) ds} \left\{ \mathcal{R}(0) \right. \\ & \left. + 16\pi G \int_0^t e^{\int_0^s I(\tau) d\tau} \frac{p_{\theta}^{Im(0)}(s, \mathcal{R}(s))}{H(s)} ds \right\}, \end{aligned} \quad (5.35)$$

where $I(t)$ is defined by

$$I = 2H + \frac{2H_{,t}}{H} - \frac{\Lambda}{H}. \quad (5.36)$$

B. Dust Collapse with $\lambda = 1$

Suppose now that the perfect fluid in the interior region consists of dust, i.e.

$$p_r^- = p_{\theta}^- = 0. \quad (5.37)$$

Then, the condition (5.19) implies that

$$p_{\theta}^{Im(0)} = 0. \quad (5.38)$$

Solving equation (5.9) for $a(t)$ we find

$$a(t) = \begin{cases} a_0 \cosh^{\frac{2}{3}}\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right), & \Lambda \neq 0, \\ a_0(t_0-t)^{2/3}, & \Lambda = 0, \end{cases} \quad (5.39)$$

where a_0 and t_0 are constants. In the following, let us consider the cases $\Lambda \neq 0$ and $\Lambda = 0$, separately.

1. $\Lambda > 0$

In this case, substituting the expression for $a(t)$ into (5.36) we obtain

$$I(t) = \sqrt{\frac{\Lambda}{3}} \tanh\left(\frac{\sqrt{3\Lambda}}{2}(t_0-t)\right),$$

and then (5.35) yields

$$\mathcal{R}(t) = \mathcal{R}_0 \cosh^{\frac{2}{3}}\left(\frac{\sqrt{3\Lambda}}{2}(t_0-t)\right), \quad (5.40)$$

where \mathcal{R}_0 is a constant. Condition (5.20) now implies that $A_{,t}^- = 0$, i.e. $A^-(t) = A_0$ for some constant A_0 . Then, by (5.16), $A^+(\mathcal{R}(t)) = A_0$. That is, $A^+(r) = A_0$ for all r such that $r = \mathcal{R}(t)$ for some t . Hence, the form of (5.40) implies that $A^+ = A_0$ for all (t, r) in the exterior region. This gives

$$A(t, r) = A_0. \quad (5.41)$$

Condition (5.14) now implies

$$\begin{aligned} \rho_H^{Im(0)}(t, \mathcal{R}(t)) &= -\frac{(-6H^2 + 2\Lambda + 4\pi G \rho_H^-(t))\mathcal{R}(t)}{12\pi G} \\ &= -\mathcal{R}_0 \frac{\Lambda + \pi G[1 + \cosh(\frac{\sqrt{3\Lambda}}{2}(t_0-t))]\rho_H^-(t)}{6\pi G \cosh^{\frac{4}{3}}(\frac{\sqrt{3\Lambda}}{2}(t_0-t))}. \end{aligned} \quad (5.42)$$

Substituting this into condition (5.18), or its equivalent form (5.33), we infer that $\rho_H^-(t)$ satisfies:

$$\begin{aligned} &-\frac{\mathcal{R}_0}{12} \cosh^{\frac{1}{3}}\left(\frac{\sqrt{3\Lambda}}{2}(t_0-t)\right) \left\{ 4 \cosh^{\frac{1}{3}}\left(\frac{\sqrt{3\Lambda}}{2}(t_0-t)\right) \right. \\ &\left. - \mathcal{R}_0 \sqrt{3\Lambda} \sinh\left(\frac{\sqrt{3\Lambda}}{2}(t_0-t)\right) \right\} \rho_{H,t}^-(t) = 0, \end{aligned}$$

i.e.

$$\rho_H^-(t) = \rho_H^{(0)}, \quad (5.43)$$

where $\rho_H^{(0)}$ is a constant. All the conditions of Proposition V.1 are now satisfied. It only remains to consider the condition that μ be continuous across Σ . This condition reduces to

$$\begin{aligned} 0 &= \frac{2m^+}{\mathcal{R}} + \frac{1}{3}\Lambda\mathcal{R}^2 - 2A_0 + \frac{2}{\mathcal{R}}A_0(\mathcal{R}-r_0) - H^2\mathcal{R}^2 \\ &= \frac{6m^+ - 6A_0r_0 + \mathcal{R}_0^3\Lambda}{3\mathcal{R}_0 \cosh^{2/3}(\frac{\sqrt{3\Lambda}}{2}(t_0-t))}. \end{aligned}$$

That is, the parameter r_0 is fixed by

$$r_0 = \frac{6m^+ + \mathcal{R}_0^3\Lambda}{6A_0}. \quad (5.44)$$

This implies that

$$\mu^+ = \frac{1}{2} \ln\left(\frac{\Lambda r^2}{3} - \frac{\Lambda \mathcal{R}_0^3}{3r}\right). \quad (5.45)$$

Since all the field equations and junction conditions are now satisfied we have proved the following result.

Proposition V.2 *Hořava-Lifshitz gravity admits the following explicit solution when $\lambda = 1$ and $\Lambda > 0$:*

$$\begin{aligned} \mu^+ &= \frac{1}{2} \ln\left(\frac{\Lambda r^2}{3} - \frac{\mathcal{R}_0^3\Lambda}{3r}\right), \quad \mu^- = \ln(-H(t)r), \\ \nu &= 0, \quad H(t) = -\sqrt{\frac{\Lambda}{3}} \tanh\left(\frac{\sqrt{3\Lambda}}{2}(t_0-t)\right), \\ \mathcal{R}(t) &= \mathcal{R}_0 \cosh^{\frac{2}{3}}\left(\frac{\sqrt{3\Lambda}}{2}(t_0-t)\right), \\ p_r = p_\theta &= 0, \quad \rho_H^-(t) = \rho_H^{(0)}, \quad A(t, r) = A_0, \\ \rho_H^{Im(0)} &\text{ is given by (5.42),} \end{aligned} \quad (5.46)$$

where t_0 , \mathcal{R}_0 , A_0 , and $\rho_H^{(0)}$ are constants.

For $t < t_0$ the dust cloud is contracting. As $t \nearrow t_0$, the radius of the dust sphere approaches its minimal value of $\mathcal{R} = \mathcal{R}_0$ at $t = t_0$, and the function e^{μ^+} approaches zero:

$$\lim_{t \nearrow t_0} \mathcal{R}(t) = \mathcal{R}_0, \quad \lim_{t \nearrow t_0} e^{\mu^+(t)} = 0,$$

as shown schematically in Fig. 2. After the star collapses to this point, it is not clear how spacetime evolves, because μ_+ becomes unbounded as one can see from Eq.(5.46), for which the extrinsic scalar K^+ ,

$$K^+(r) = e^{\mu^+(r)} \left(\mu_{,r}^+(r) + \frac{2}{r} \right), \quad (5.47)$$

also becomes unbounded, which indicates the existence of a scalar singularity at this point [55]. However, such a singularity is weak. In particular, the corresponding four-dimensional Ricci scalar remains finite, ${}^{(4)}R = 4\Lambda$. Thus, it is not clear whether the spacetime across this point is extendable or not.

In addition, Eq.(5.42) shows that $\rho_H^{Im(0)}$ and ρ_H^- cannot both be positive. To understand this, letting $M = -\Lambda\mathcal{R}_0^3/6$ we can write μ^+ in the form,

$$\mu^+ = \frac{1}{2} \ln\left(\frac{2M}{r} + \frac{\Lambda r^2}{3}\right). \quad (5.48)$$

However, this is nothing but the Schwarzschild-de Sitter solution with mass M and a cosmological constant Λ , where M is negative.

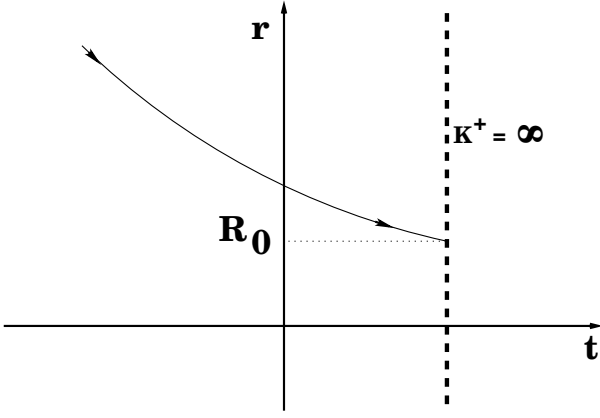


FIG. 2: The evolution of the surface of the collapsing star for $\lambda = 1$ and $\Lambda > 0$, given by Eq.(5.46). At the moment $t = t_0$, the star collapses to its minimal radius $\mathcal{R}(t_0) = \mathcal{R}_0$, at which the extrinsic curvature K^+ becomes unbounded, while the four-dimensional Ricci scalar remains finite.

2. $\Lambda < 0$

In this case, substituting the expression for $a(t)$ into (5.36) we obtain

$$I(t) = \sqrt{\frac{|\Lambda|}{3}} \tan\left(\frac{\sqrt{3|\Lambda|}}{2}(t - t_0)\right),$$

and then (5.35) yields

$$\mathcal{R}(t) = \mathcal{R}_0 \cos^{\frac{2}{3}}\left(\frac{\sqrt{3|\Lambda|}}{2}(t - t_0)\right), \quad (5.49)$$

where \mathcal{R}_0 is another constant. Condition (5.20) now implies that $A_{,t}^- = 0$, i.e. $A^-(t) = A_0$ for some constant A_0 . Then, by (5.16), $A^+(\mathcal{R}(t)) = A_0$. That is, $A^+(r) = A_0$ for all r such that $r = \mathcal{R}(t)$ for some t . We will assume that $A^+ = A_0$ for all (t, r) in the exterior region, i.e. $A(t, r) = A_0$. Condition (5.14) now implies

$$\begin{aligned} \rho_H^{Im(0)}(t, \mathcal{R}(t)) &= -\frac{(-6H^2 + 2\Lambda + 4\pi G\rho_H^-(t))\mathcal{R}(t)}{12\pi G} \\ &= \mathcal{R}_0 \frac{|\Lambda| - \pi G[1 + \cos(\frac{\sqrt{3|\Lambda|}}{2}(t - t_0))]\rho_H^-(t)}{6\pi G \cos^{\frac{4}{3}}(\frac{\sqrt{3|\Lambda|}}{2}(t - t_0))}. \end{aligned} \quad (5.50)$$

Substituting this into condition (5.18), or its equivalent form (5.33), we infer that $\rho_H^-(t)$ satisfies

$$\rho_H^-(t) = \rho_H^{(0)}, \quad (5.51)$$

where $\rho_H^{(0)}$ is a constant. All the conditions of Proposition V.1 are now satisfied, while the condition that μ be continuous across Σ reduces to

$$\begin{aligned} 0 &= \frac{2m^+}{\mathcal{R}} + \frac{1}{3}\Lambda\mathcal{R}^2 - 2A_0 + \frac{2}{\mathcal{R}}A_0(\mathcal{R} - r_0) - H^2\mathcal{R}^2 \\ &= \frac{6m^+ - 6A_0r_0 + \mathcal{R}_0^3\Lambda}{3\mathcal{R}_0 \cos^{2/3}(\frac{\sqrt{3|\Lambda|}}{2}(t - t_0))}. \end{aligned}$$

Thus, the parameter r_0 is fixed by

$$r_0 = \frac{6m^+ + \mathcal{R}_0^3\Lambda}{6A_0}. \quad (5.52)$$

This implies that

$$\mu^+ = \frac{1}{2} \ln\left(\frac{2M}{r} - \frac{|\Lambda|}{3}r^2\right), \quad (5.53)$$

where $M \equiv |\Lambda|\mathcal{R}_0^3/6$. Clearly, this corresponds to the Schwarzschild-anti-de Sitter solution. For μ^+ to be real, we must assume that $r \leq \mathcal{R}_0$. Similar to the last case, the extrinsic curvature K^+ at $r = \mathcal{R}_0$ becomes unbounded, while the four-dimensional Ricci scalar ${}^{(4)}R$ remains constant. Thus, in this case it is also not clear whether or not the spacetime is extendable cross $r = \mathcal{R}_0$.

In any case, all the field equations and junction conditions are now satisfied for $r \leq \mathcal{R}_0$, and we have proved the following result.

Proposition V.3 *Hořava-Lifshitz gravity admits the following explicit solution when $\lambda = 1$ and $\Lambda < 0$:*

$$\begin{aligned} \mu^+ &= \frac{1}{2} \ln\left(\frac{|\Lambda|}{3r}(\mathcal{R}_0^3 - r^3)\right), \quad \mu^- = \ln(-H(t)r), \\ \nu &= 0, \quad H(t) = -\sqrt{\frac{|\Lambda|}{3}} \tan\left(\frac{\sqrt{3|\Lambda|}}{2}(t - t_0)\right), \\ \mathcal{R}(t) &= \mathcal{R}_0 \cos^{\frac{2}{3}}\left(\frac{\sqrt{3|\Lambda|}}{2}(t - t_0)\right), \\ p_r &= p_\theta = 0, \quad \rho_H^-(t) = \rho_H^{(0)}, \quad A(t, r) = A_0, \\ \rho_H^{Im(0)} &\text{ is given by (5.50),} \end{aligned} \quad (5.54)$$

where t_0 , \mathcal{R}_0 , A_0 , and $\rho_H^{(0)}$ are constants.

The evolution of the surface of the collapsing star is illustrated in Fig. 3. The collapse starts at an initial time $t_i \leq t_0$, and at time $t = t_s$, the star collapses to a central singularity at which we have $\mathcal{R}(t_s) = 0$, where $t_s \equiv t_0 + \pi/\sqrt{3|\Lambda|}$. Equation (5.50) shows that now both $\rho_H^{Im(0)}$ and ρ_H^- can be positive, provided that $|\Lambda| > 2\pi G\rho_H^{(0)}$.

3. $\Lambda = 0$

In this case, substituting the expression (5.39) for $a(t)$ into (5.36) we obtain

$$I(t) = \frac{2}{3(t_0 - t)},$$

and then (5.35) yields

$$\mathcal{R}(t) = \mathcal{R}_0(t_0 - t)^{\frac{2}{3}}, \quad (5.55)$$

where \mathcal{R}_0 is a constant. Condition (5.20) now implies that $A_{,t}^- = 0$, i.e. $A^-(t) = A_0$ for some constant A_0 .

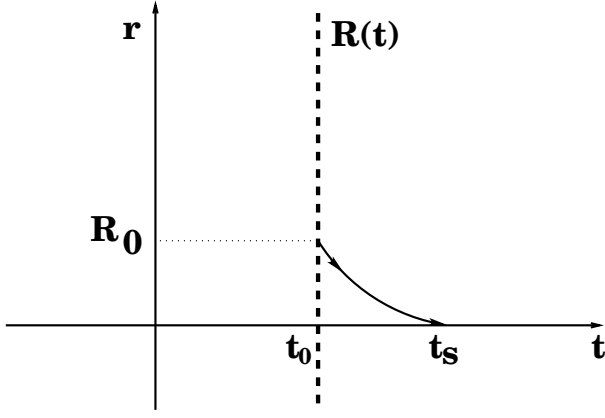


FIG. 3: The evolution of the surface of the collapsing star for $\lambda = 1$ and $\Lambda < 0$, given by Eq.(5.54). The star starts to collapse at a time $t = t_i \geq t_0$. At the later time $t = t_s$, at which $\mathcal{R}(t_s) = 0$, the star collapses and a central singularity is formed.

Then, by (5.16), $A^+(\mathcal{R}(t)) = A_0$. That is, $A^+(r) = A_0$ for all r such that $r = \mathcal{R}(t)$ for some t . Hence (5.55) implies that $A^+ = A_0$ for all (t, r) in the exterior region. Thus, in the present case we also have $A(t, r) = A_0$. Condition (5.14) now implies

$$\begin{aligned} \rho_H^{Im(0)}(t, \mathcal{R}(t)) &= -\frac{(-6H^2 + 4\pi G\rho_H^-(t))\mathcal{R}(t)}{12\pi G} \\ &= \mathcal{R}_0 \frac{2 - 3G\pi(t_0 - t)^2 \rho_H^-(t)}{9G\pi(t_0 - t)^{4/3}}. \end{aligned}$$

Substituting this into condition (5.18), or its equivalent form (5.33), we infer that

$$\rho_H^-(t) = \rho_H^{(0)}, \quad (5.56)$$

where $\rho_H^{(0)}$ is a constant. All the conditions of Proposition V.1 are now satisfied, and the condition that μ be continuous across Σ becomes

$$\begin{aligned} 0 &= \frac{2m^+}{\mathcal{R}} - 2A_0 + \frac{2}{\mathcal{R}}A_0(\mathcal{R} - r_0) - H^2\mathcal{R}^2 \\ &= 2\frac{9m^+ - 9A_0r_0 - 2\mathcal{R}_0^3}{9\mathcal{R}(t)}. \end{aligned}$$

Hence, the parameter r_0 is fixed to

$$r_0 = \frac{9m^+ - 2\mathcal{R}_0^3}{9A_0}, \quad (5.57)$$

which implies that

$$\mu^+ = \frac{1}{2} \ln\left(\frac{r_g}{r}\right), \quad \nu^+ = 0, \quad N^+ = 1, \quad (5.58)$$

where $r_g \equiv 4\mathcal{R}_0^3/9$. This is nothing but is the Schwarzschild solution written in the Painlevé-Gullstrand coordinates [45]. All the field equations and junction conditions are satisfied, so we have proved the following result.

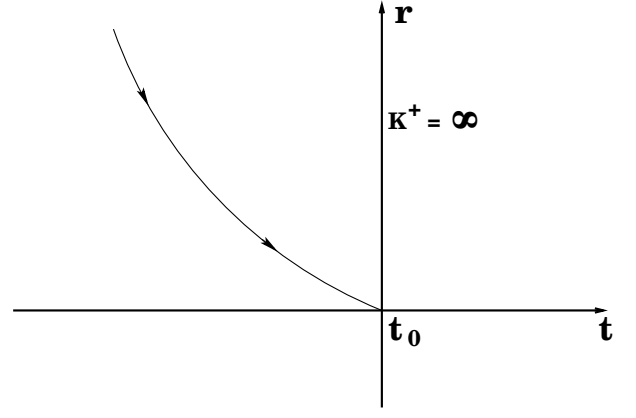


FIG. 4: The evolution of the surface of the collapsing star for $\lambda = 1$ and $\Lambda = 0$, given by Eq.(5.59). At the moment $t = t_i \leq t_0$, the star starts to collapse until the moment $t = t_0$, at which we have $\mathcal{R}(t_0) = 0$, whereby a central singularity is formed.

Proposition V.4 *Hořava-Lifshitz gravity admits the following explicit solution when $\lambda = 1$ and $\Lambda = 0$:*

$$\begin{aligned} \mu^+ &= \frac{1}{2} \ln\left(\frac{r_g}{r}\right), \quad \mu^- = \ln(-H(t)r), \\ \nu &= 0, \quad H(t) = -\frac{2}{3(t_0 - t)}, \\ \mathcal{R}(t) &= \mathcal{R}_0(t_0 - t)^{2/3}, \\ p_r = p_\theta &= 0, \quad \rho_H^-(t) = \rho_H^{(0)}, \quad A(t, r) = A_0, \\ \rho_H^{Im(0)} &= \mathcal{R}_0 \frac{2 - 3G\pi(t_0 - t)^2 \rho_H^{(0)}}{9G\pi(t_0 - t)^{4/3}}, \end{aligned} \quad (5.59)$$

where t_0, \mathcal{R}_0, A_0 , and $\rho_H^{(0)}$ are constants.

The evolution of the surface of the collapsing star is shown in Fig. 4. The star begins to collapse at the moment t_i with a radius $\mathcal{R}_i[\equiv \mathcal{R}(t_i)]$ until the moment $t = t_0$, at which we have $\mathcal{R}(t_0) = 0$ and a central singularity is formed. The spacetime outside of the star is given by the Schwarzschild solution. Thus, as in GR, the Schwarzschild spacetime can be formed by the collapse of a homogenous and isotropic dust perfect fluid [1]. We note that $\rho_H^{Im(0)} > 0$ for

$$t_0 - \sqrt{\frac{2}{3G\pi\rho_H^{(0)}}} < t < t_0.$$

C. Gravitational Collapse with $\lambda \neq 1$

We now consider the case of $\lambda \neq 1$. For an exterior static spherically symmetric vacuum spacetime with $\lambda \neq 1$ and $\Lambda_g = 0$, equation (3.7) implies that

$$\nu^+ = -\frac{1}{2} \ln\left(1 - \frac{2B}{r}\right), \quad (5.60)$$

where B is a constant. On the other hand, for the interior FLRW region, we have

$$\nu^- = -\frac{1}{2} \ln \left(1 - k \frac{r^2}{a^2(t)} \right). \quad (5.61)$$

Hence, the condition $\nu_{,t}^+ = \nu_{,t}^-$ on Σ implies that $0 = k\mathcal{R}(t)^2$. Consequently, in order for a solution with $\mathcal{R}(t) \neq 0$ to exist, we must have $k = 0$. The conditions that ν and $\nu_{,r}$ be continuous across Σ then reduce to $2B/\mathcal{R}(t) = 0$. Thus, in order for a nontrivial solution to exist we must have $k = B = 0$. Thus, we have

$$\nu^- = \nu^+ = 0, \quad \mu^- = \ln(-rH). \quad (5.62)$$

On the other hand, since $\lambda \neq 1$, the momentum constraint (3.5) yields

$$\mu^+(r) = \ln \left(C_1 r + \frac{C_2}{r^2} \right), \quad (5.63)$$

where C_1 and C_2 are constants. The field equations (3.6) - (3.9) are then satisfied provided that

$$A^+(r) = A_0^+ - \frac{3C_2^2}{8r^4} + \frac{3(1-3\lambda)C_1^2 + 2\Lambda}{8} r^2, \quad (5.64)$$

where A_0^+ is a constant. It is interesting to note that this class of solutions was first found in [29] in the IR limit. However, since the restriction of the spacetime to the leaves $t = \text{constant}$ is flat, we have $R_{ij} = 0$, and the higher-order derivative terms of R_{ij} vanish identically, so they are also solutions of the full theory. Moreover, since

$$\mu_{,r}^+ = \frac{C_1 r^3 - 2C_2}{C_1 r^4 + C_2 r}, \quad \mu_{,r}^- = \frac{1}{r},$$

we find

$$\hat{\mu}_{,r} = \frac{-3C_2}{C_1 r^4 + C_2 r}. \quad (5.65)$$

Thus, the requirement that μ is C^1 implies that $C_2 = 0$. The continuity of μ then requires that $H(t) = -C_1$ is a constant and so

$$a(t) = a_0 e^{-C_1 t}.$$

It follows that μ is smooth across Σ . Note also that the asymptotical-flatness condition requires $C_1 = 0$. However, in the following we leave the possibility of $C_1 \neq 0$ open.

We find that

$$\begin{aligned} \mathcal{L}_K^+ &= 3C_1^2(1-3\lambda), & \mathcal{L}_V^+ &= 2\Lambda, & \mathcal{L}_A^+ &= 0, \\ v^+ &= 0, & J_A^+ &= 0, & J_\varphi^+ &= 0, & \rho_H^+ &= 0. \end{aligned} \quad (5.66)$$

In order for the integral over the exterior region in the Hamiltonian constraint (4.27) to converge, we also need to assume that

$$3C_1^2(1-3\lambda) + 2\Lambda = 0. \quad (5.67)$$

Thus, $A^+(r) = A_0^+$ is a constant and equation (5.9) implies that $p^-(t) = 0$, that is, the perfect fluid in the interior region consists of dust.

Similar to the case with $\lambda = 1$, the interior solution is still of the form (5.8), i.e.

$$J_A^- = J_\varphi^- = 0, \quad A^- = A^-(t).$$

In view of (5.4), we have

$$\begin{aligned} \mathcal{L}_\varphi^- &= 0, & \mathcal{L}_\lambda^- &= 0, & \mathcal{L}_K^- &= 3(1-3\lambda)H^2, \\ \mathcal{L}_V^- &= 2\Lambda, & \mathcal{L}_A^- &= 0, & v^- &= 0. \end{aligned} \quad (5.68)$$

We assume that the thin shell of matter separating the interior and exterior solutions is such that

$$\begin{aligned} p_r &= 0, & v &= 0, & J_\varphi &= J_\varphi^{Im(0)} \delta(\Phi), \\ p_\theta &= p_\theta^{Im(0)} \delta(\Phi), & \rho_H &= \rho_H^- + \rho_H^{Im(0)} \delta(\Phi), \\ A &= A^D + A^{Im(0)} \delta(\Phi), & J_A &= 0, & \mathcal{L}_A &= \mathcal{L}_A^- \end{aligned} \quad (5.69)$$

with $\rho_H^- = \rho_H^-(t)$.

Proposition V.5 *For the spacetime defined by (5.62)-(5.69), the six junction conditions (4.27)-(4.32) reduce to the following six conditions:*

$$\rho_H^-(t) \frac{\mathcal{R}(t)}{3} + \rho_H^{Im(0)}(t, \mathcal{R}(t)) = 0, \quad (5.70)$$

$$J_\varphi^{Im(0)} = 0, \quad (5.71)$$

$$A(t, r) = A_0 \text{ is a constant}, \quad (5.72)$$

$$p_\theta^{Im(0)} = 0 \text{ on } \Sigma, \quad (5.73)$$

$$\begin{aligned} \frac{d}{dt} \left[\rho_H^{Im(0)}(t, \mathcal{R}(t)) \right] + 2 \frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)} \rho_H^{Im(0)}(t, \mathcal{R}(t)) \\ + \dot{\mathcal{R}}(t) \rho_H^-(t) + \frac{C_1 \rho_{H,t}^-(t) \mathcal{R}(t)^2}{4} = 0. \end{aligned} \quad (5.74)$$

Proof. For the spacetime defined by (5.62)-(5.69), condition (4.27) reduces to

$$\begin{aligned} (3(1-3\lambda)C_1^2 + 2\Lambda + 4\pi G \rho_H^-(t)) \int_0^{\mathcal{R}(t)} r^2 dr \\ + \int_{\mathcal{R}(t)}^\infty \left(3C_1^2(1-3\lambda) + 2\Lambda \right) r^2 dr \\ + 4\pi G \rho_H^{Im(0)} r^2 \Big|_{r=\mathcal{R}(t)} = 0, \end{aligned}$$

which, in view of (5.67), yields (5.70). Moreover, equation (4.34) reduces to (5.71).

The functions F and F^A are given by (5.23) - (5.25) also for $\lambda \neq 1$. Hence, condition (4.29) implies that A is continuous across Σ just like in the case of $\lambda = 1$. Since $A^+ = A_0^+$ is constant and $A^-(t)$ is independent of r , this gives (5.72). Condition (4.30) then reduces to

$$\begin{aligned} \frac{1}{r} F_{\theta\theta}^{AI m(0)} \delta(\Phi) + \frac{1}{r} F_{\theta\theta}^{AI m(1)} \delta'(\Phi) \\ + \frac{1}{r} F_{\theta\theta}^{AI m(2)} \delta''(\Phi) + 8\pi G r p_\theta^{Im(0)} \delta(\Phi) = 0. \end{aligned}$$

Since A is a constant, this yields (5.73).

Conditions (4.31) and (4.32) reduce to (5.74) and (5.73). \square

Conditions (5.70) and (5.74) imply that

$$\dot{\rho}_{\bar{H}}(t)\mathcal{R}(t)\left(\frac{C_1}{4}\mathcal{R}(t) - \frac{1}{3}\right) = 0.$$

Excluding the case of no collapse where $\mathcal{R}(t)$ is a constant, it follows that $\rho_{\bar{H}}$ must be a constant.

In summary, in the case $\lambda \neq 1$ a static spherical space-time can be produced by gravitational collapse of a homogeneous and isotropic dust fluid. However, the space-time outside of such a fluid is not asymptotically flat, as one can see from Eqs.(5.62) and (5.63) with $C_2 = 0$.

VI. CONCLUSIONS

In this paper, we have studied gravitational collapse of a spherical cloud of fluid with a finite radius in the framework of the nonrelativistic general covariant theory of the HL gravity with the projectability condition and an arbitrary coupling constant λ . Using distribution theory, we have developed the general junction conditions for such a collapsing spherical body, with the minimal requirement that *such junctions should be mathematically meaningful in the sense of generalized functions*. The general junction conditions have been summarized in Table I.

As one of the simplest applications, we have studied a collapsing star that is made of a homogeneous and isotropic perfect fluid, while the external region is described by a stationary spacetime. We have found that the problem reduces to the matching of six independent conditions that the Arnowitt-Deser-Misner variables (N, N^i, g_{ij}) and the gauge field A and Newtonian prepotential φ must satisfy.

For the case of a homogeneous and isotropic dust fluid (a perfect fluid with vanishing pressure), we have found explicitly the space-time outside of the collapsing sphere. In particular, in the case $\lambda = 1$, the external spacetimes are described by the Schwarzschild (anti-) de Sitter solutions, written in the Painlevé-Gullstrand coordinates [45]. It is remarkable that the collapse of a homogeneous and isotropic dust to a Schwarzschild black hole, studied by Oppenheimer and Snyder in general relativity more than 80 years ago [3], is a particular case. However, there are fundamental differences. First, in general relativity a thin shell does not necessarily appear on the surface of the collapsing sphere [3], while in the current case we have shown that such a thin shell must exist, as one can see from Propositions V.2 - 4 given in Section V. Second, in general relativity because of the local conservation of energy of the collapsing body, the energy density of the dust fluid is inversely proportional to the cube of the radius of the fluid, while in the current case it remains a constant, as now the conservation law becomes a global

one [cf. Eq.(2.8)], and the energy of the collapsing star is not necessarily conserved locally.

In the case $\lambda \neq 1$, the space-time outside of the homogeneous and isotropic dust fluid is described by Eqs.(5.62) and (5.63) with $C_2 = 0$. It is clear that such a space-time is not asymptotically flat. Therefore, in this case to obtain an asymptotically flat space-time outside of a collapsing dust fluid, it must not be homogeneous and/or isotropic.

From the above simple examples, one can already see the significant differences between the HL theory and general relativity in the strong gravitational field regime. Therefore, it is very interesting to study gravitational collapse of more general fluids, such as perfect fluids with different equations of state, anisotropic fluids with or without heat flows. Particular attentions should be paid on the roles that the equation of state and heat flows might play. It would be extremely interesting to study the implications to black hole physics, or more general to (observational) astrophysics and cosmology [1]. Since the general formulas have been already laid down in this paper, we expect that such studies can be carried easily.

As emphasized previously, our treatments for the junction conditions of a collapsing star presented in this paper can be easily generalized to other models of the Hořava-Lifshitz gravity, or more general to any model of high-order derivative gravity theories.

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Appendix A: Functions $(F_s)_{ij}$ and $F_{(\varphi,n)}^{ij}$

The geometric 3-tensors F^{ij} and $F_{(\varphi,n)}^{ij}$ defined in Eq.(2.16) are given by

$$\begin{aligned} (F_0)_{ij} &= -\frac{1}{2}g_{ij}, \\ (F_1)_{ij} &= -\frac{1}{2}g_{ij}R + R_{ij}, \\ (F_2)_{ij} &= -\frac{1}{2}g_{ij}R^2 + 2RR_{ij} - 2\nabla_{(i}\nabla_{j)}R \\ &\quad + 2g_{ij}\nabla^2 R, \\ (F_3)_{ij} &= -\frac{1}{2}g_{ij}R_{mn}R^{mn} + 2R_{ik}R_j^k - 2\nabla^k\nabla_{(i}R_{j)k} \\ &\quad + \nabla^2 R_{ij} + g_{ij}\nabla_m\nabla_n R^{mn}, \\ (F_4)_{ij} &= -\frac{1}{2}g_{ij}R^3 + 3R^2R_{ij} - 3\nabla_{(i}\nabla_{j)}R^2 \\ &\quad + 3g_{ij}\nabla^2 R^2, \\ (F_5)_{ij} &= -\frac{1}{2}g_{ij}RR^{mn}R_{mn} + R_{ij}R^{mn}R_{mn} \\ &\quad + 2RR_{ki}R_j^k - \nabla_{(i}\nabla_{j)}(R^{mn}R_{mn}) \end{aligned}$$

$$\begin{aligned}
& -2\nabla^n \nabla_{(i} R R_{j)n} + g_{ij} \nabla^2 (R^{mn} R_{mn}) \\
& + \nabla^2 (R R_{ij}) + g_{ij} \nabla_m \nabla_n (R R^{mn}), \\
(F_6)_{ij} &= -\frac{1}{2} g_{ij} R_n^m R_p^n R_m^p + 3R^{mn} R_{ni} R_{mj} \\
& + \frac{3}{2} \nabla^2 (R_{in} R_j^n) + \frac{3}{2} g_{ij} \nabla_k \nabla_l (R_n^k R^{ln}) \\
& - 3\nabla_k \nabla_{(i} (R_{j)n} R^{nk}), \\
(F_7)_{ij} &= \frac{1}{2} g_{ij} (\nabla R)^2 - (\nabla_i R) (\nabla_j R) + 2R_{ij} \nabla^2 R \\
& - 2\nabla_{(i} \nabla_{j)} \nabla^2 R + 2g_{ij} \nabla^4 R, \\
(F_8)_{ij} &= -\frac{1}{2} g_{ij} (\nabla_p R_{mn}) (\nabla^p R^{mn}) - \nabla^4 R_{ij} \\
& + (\nabla_i R_{mn}) (\nabla_j R^{mn}) + 2(\nabla_p R_{in}) (\nabla^p R_j^n) \\
& + 2\nabla^n \nabla_{(i} \nabla^2 R_{j)n} + 2\nabla_n (R_m^n \nabla_{(i} R_{j)}^m) \\
& - 2\nabla_n (R_{m(j} \nabla_{i)} R^{mn}) - 2\nabla_n (R_{m(i} \nabla^n R_{j)}^m) \\
& - g_{ij} \nabla^n \nabla^m \nabla^2 R_{mn}, \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
F_{(\varphi,1)}^{ij} &= \frac{1}{2} \varphi \left\{ (2K + \nabla^2 \varphi) R^{ij} - 2(2K_k^j + \nabla^j \nabla_k \varphi) R^{ik} \right. \\
& \quad - 2(2K_k^i + \nabla^i \nabla_k \varphi) R^{jk} \\
& \quad \left. - (2\Lambda_g - R) (2K^{ij} + \nabla^i \nabla^j \varphi) \right\}, \\
F_{(\varphi,2)}^{ij} &= \frac{1}{2} \nabla_k \left\{ \varphi \mathcal{G}^{ik} \left(\frac{2N^j}{N} + \nabla^j \varphi \right) \right. \\
& \quad \left. + \varphi \mathcal{G}^{jk} \left(\frac{2N^i}{N} + \nabla^i \varphi \right) - \varphi \mathcal{G}^{ij} \left(\frac{2N^k}{N} + \nabla^k \varphi \right) \right\}, \\
F_{(\varphi,3)}^{ij} &= \frac{1}{2} \left\{ 2\nabla_k \nabla_{(i} f_\varphi^{j)k} - \nabla^2 f_\varphi^{ij} - (\nabla_k \nabla_l f_\varphi^{kl}) g^{ij} \right\}, \tag{A.2}
\end{aligned}$$

where

$$f_\varphi^{ij} = \varphi \left\{ (2K^{ij} + \nabla^i \nabla^j \varphi) - \frac{1}{2} (2K + \nabla^2 \varphi) g^{ij} \right\}. \tag{A.3}$$

The F_{ij} for the spherical spacetime (3.1) are found to be

$$\begin{aligned}
(F_0)_{ij} &= -\frac{e^{2\nu}}{2} \delta_i^r \delta_j^r - \frac{r^2}{2} \Omega_{ij}, \\
(F_1)_{ij} &= \frac{1 - e^{2\nu}}{r^2} \delta_i^r \delta_j^r - e^{-2\nu} r \nu' \Omega_{ij}, \\
(F_2)_{ij} &= -\frac{2e^{-2\nu}}{r^4} \left[6e^{2\nu} + e^{4\nu} - 8r^2 \nu'' \right. \\
& \quad \left. + 12r^2 (\nu')^2 - 7 \right] \delta_i^r \delta_j^r \\
& \quad + \frac{2e^{-4\nu}}{r^2} \left[6e^{2\nu} + e^{4\nu} + 4\nu^{(3)} r^3 + 24r^3 (\nu')^3 \right. \\
& \quad \left. - 2r\nu' (-3e^{2\nu} + 14r^2 \nu'' + 7) - 7 \right] \Omega_{ij}, \\
(F_3)_{ij} &= -\frac{e^{-2\nu}}{r^4} \left[4e^{2\nu} + e^{4\nu} - 6r^2 \nu'' \right.
\end{aligned}$$

$$\begin{aligned}
& \quad \left. + 9r^2 (\nu')^2 - 5 \right] \delta_i^r \delta_j^r \\
& \quad + \frac{e^{-4\nu}}{r^2} \left[4e^{2\nu} + e^{4\nu} + 3\nu^{(3)} r^3 + 18r^3 (\nu')^3 \right. \\
& \quad \left. - r\nu' (-4e^{2\nu} + 21r^2 \nu'' + 10) - 5 \right] \Omega_{ij}, \\
(F_4)_{ij} &= -\frac{4e^{-4\nu}}{r^6} (e^{2\nu} + 2r\nu' - 1) \left[22e^{2\nu} + e^{4\nu} - 24r^2 \nu'' \right. \\
& \quad \left. + 40r^2 (\nu')^2 - 2(e^{2\nu} - 1) r\nu' - 23 \right] \delta_i^r \delta_j^r \\
& \quad + \frac{4e^{-6\nu}}{r^4} \left\{ 240r^4 (\nu')^4 + 4(18e^{2\nu} - 17) r^3 (\nu')^3 \right. \\
& \quad \left. - 12r^2 (\nu')^2 (-11e^{2\nu} + 22r^2 \nu'' + 15) \right. \\
& \quad \left. + 3r\nu' [-6e^{2\nu} + 7e^{4\nu} + 8\nu^{(3)} r^3 \right. \\
& \quad \quad \left. - 28(e^{2\nu} - 1) r^2 \nu'' - 1 \right] \\
& \quad + 2 \left[12r^4 (\nu'')^2 - 24(e^{2\nu} - 1) r^2 \nu'' \right. \\
& \quad \left. + (e^{2\nu} - 1) (22e^{2\nu} + e^{4\nu} + 6\nu^{(3)} r^3 - 23) \right] \Omega_{ij}, \\
(F_5)_{ij} &= -\frac{2e^{-4\nu}}{r^6} \left\{ 60r^3 (\nu')^3 \right. \\
& \quad \left. + (e^{2\nu} - 1) (16e^{2\nu} + e^{4\nu} - 14r^2 \nu'' - 17) \right. \\
& \quad \left. + (21e^{2\nu} - 17) r^2 (\nu')^2 \right. \\
& \quad \left. - 4r\nu' (-7e^{2\nu} + 9r^2 \nu'' + 7) \right\} \delta_i^r \delta_j^r \\
& \quad + \frac{2e^{-6\nu}}{r^4} \left\{ 18r^4 (\nu'')^2 + 180r^4 (\nu')^4 \right. \\
& \quad \left. + (e^{2\nu} - 1) (32e^{2\nu} + 2e^{4\nu} + 7\nu^{(3)} r^3 - 34) \right. \\
& \quad \left. + 21(2e^{2\nu} - 1) r^3 (\nu')^3 - 28(e^{2\nu} - 1) r^2 \nu'' \right. \\
& \quad \left. - r^2 (\nu')^2 (-77e^{2\nu} + 198r^2 \nu'' + 101) \right. \\
& \quad \left. + r\nu' \left[3(-8e^{2\nu} + 5e^{4\nu} + 6\nu^{(3)} r^3 + 3) \right. \right. \\
& \quad \quad \left. \left. - (49e^{2\nu} - 41) r^2 \nu'' \right] \right\} \Omega_{ij}, \\
(F_6)_{ij} &= \frac{e^{-4\nu}}{r^6} \left\{ -50r^3 (\nu')^3 \right. \\
& \quad \left. - (e^{2\nu} - 1) (13e^{2\nu} + e^{4\nu} - 6r^2 \nu'' - 14) \right. \\
& \quad \left. - 9e^{2\nu} r^2 (\nu')^2 + 6r\nu' (-2e^{2\nu} + 5r^2 \nu'' + 2) \right\} \delta_i^r \delta_j^r \\
& \quad + \frac{e^{-6\nu}}{r^4} \left\{ 15r^4 (\nu'')^2 + 150r^4 (\nu')^4 \right. \\
& \quad \left. + (e^{2\nu} - 1) (26e^{2\nu} + 2e^{4\nu} + 3\nu^{(3)} r^3 - 28) \right. \\
& \quad \left. + (18e^{2\nu} + 25) r^3 (\nu')^3 - 12(e^{2\nu} - 1) r^2 \nu'' \right. \\
& \quad \left. - 3r^2 (\nu')^2 (-11e^{2\nu} + 55r^2 \nu'' + 12) \right. \\
& \quad \left. + 3r\nu' \left[-12e^{2\nu} + 4e^{4\nu} + 5\nu^{(3)} r^3 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - (7e^{2\nu} - 1) r^2 \nu'' + 8 \Big] \Big\} \Omega_{ij}, \\
(F_7)_{ij} = & \frac{8e^{-4\nu}}{r^6} \left\{ 6e^{2\nu} + e^{4\nu} - 2\nu^{(4)} r^4 + 15r^4 (\nu'')^2 \right. \\
& + 40r^4 (\nu')^4 + 4r\nu' (2e^{2\nu} + 5\nu^{(3)} r^3 - 6) \\
& - 2r^2 (\nu')^2 (-3e^{2\nu} + 41r^2 \nu'' + 15) \\
& - 4 (e^{2\nu} - 3) r^2 \nu'' - 7 \Big\} \delta_i^r \delta_j^r \\
& - \frac{8e^{-6\nu}}{r^4} \left\{ 12e^{2\nu} + 2e^{4\nu} + \nu^{(5)} r^5 \right. \\
& + 120r^5 (\nu')^5 + 2e^{2\nu} \nu^{(3)} r^3 - 6\nu^{(3)} r^3 \\
& + r\nu' [24e^{2\nu} + e^{4\nu} - 16\nu^{(4)} r^4 + 127r^4 (\nu'')^2 \\
& - 2 (7e^{2\nu} - 33) r^2 \nu'' - 57] \\
& - r^2 \nu'' (8e^{2\nu} + 25\nu^{(3)} r^3 - 24) \\
& + r^2 (\nu')^2 (22e^{2\nu} + 101\nu^{(3)} r^3 - 102) \\
& \left. - 2r^3 (\nu')^3 (-6e^{2\nu} + 163r^2 \nu'' + 45) - 14 \right\} \Omega_{ij}, \\
(F_8)_{ij} = & \frac{e^{-4\nu}}{r^6} \left\{ 6 (2e^{2\nu} + e^{4\nu} - \nu^{(4)} r^4 - 3) \right. \\
& + 45r^4 (\nu'')^2 + 120r^4 (\nu')^4 + 10r^3 (\nu')^3 \\
& - 8 (e^{2\nu} - 4) r^2 \nu'' \\
& - r^2 (\nu')^2 (-12e^{2\nu} + 246r^2 \nu'' + 77) \\
& \left. + 2r\nu' (8e^{2\nu} + 30\nu^{(3)} r^3 - 3r^2 \nu'' - 32) \right\} \delta_i^r \delta_j^r \\
& + \frac{e^{-6\nu}}{r^4} \left\{ -24e^{2\nu} - 12e^{4\nu} - 3\nu^{(5)} r^5 \right. \\
& - 360r^5 (\nu')^5 - 3r^4 (\nu'')^2 - 30r^4 (\nu')^4 \\
& - 4e^{2\nu} \nu^{(3)} r^3 + 16\nu^{(3)} r^3 \\
& + r^2 \nu'' (16e^{2\nu} + 75\nu^{(3)} r^3 - 64) \\
& + 2r^3 (\nu')^3 (-12e^{2\nu} + 489r^2 \nu'' + 113) \\
& - r^2 (\nu')^2 (44e^{2\nu} + 303\nu^{(3)} r^3 - 33r^2 \nu'' - 269) \\
& - r\nu' [381r^4 (\nu'')^2 - 2 (14e^{2\nu} - 85) r^2 \nu'' \\
& + 6 (8e^{2\nu} + e^{4\nu} - 8\nu^{(4)} r^4 - 25) \\
& \left. + 3\nu^{(3)} r^3 \right\} + 36 \Big\} \Omega_{ij}. \tag{A.4}
\end{aligned}$$

where $\nu' = \partial\nu/\partial r$ and $\Omega_{ij} = \delta_i^\theta \delta_j^\theta + \sin^2 \theta \delta_i^\phi \delta_j^\phi$.

Appendix B: Proof of (4.11)-(4.12)

Let F be given by (4.8). We will show that the equation $F = 0$ is equivalent to the conditions (4.11) and

(4.12). It is clear that the equation $F = 0$ is equivalent to (4.11) together with the condition

$$\sum_{k=0}^n F^{Im(k)} \delta^{(k)}(\Phi) = 0. \tag{B.1}$$

It remains to show that (B.1) is equivalent to (4.12).

Suppose first that (B.1) holds. Then, multiplying (B.1) by Φ^{n-j} and using the recursion relation (4.6) repeatedly, we find

$$G\delta^{(j)}(\Phi) = 0, \quad 0 \leq j \leq n, \tag{B.2}$$

where the function G is defined in a neighborhood of Σ by

$$G(x) = \sum_{k=0}^n (-1)^k k! F^{Im(k)}(x) \Phi^{n-k}(x). \tag{B.3}$$

Equation (B.2) with $j = 0$ implies that the restriction of G to Σ vanishes, i.e. $G|_\Sigma = 0$. Equation (B.2) with $j = 1$ then gives

$$0 = G\delta'(\Phi) = \frac{G}{\Phi} \Phi\delta'(\Phi) = -\frac{G}{\Phi} \delta(\Phi) \quad \text{i.e.} \quad \frac{G}{\Phi} \Big|_\Sigma = 0.$$

In terms of local coordinates $\{u^j\}$ such that $u^1 = \Phi$ while the remaining coordinates $\{u^j\}_{j \geq 2}$ parametrize the level surfaces of Φ , we have

$$0 = \frac{G}{\Phi} \Big|_\Sigma = \frac{\partial G}{\partial \Phi} \Big|_{\Phi=0}.$$

Thus, G vanishes to the first order on Σ . Repeating the above procedure n times, we infer that G vanishes to the n th order on Σ :

$$\frac{G}{\Phi^j} \Big|_\Sigma = 0 \quad \text{i.e.} \quad \frac{\partial^j G}{\partial \Phi^j} \Big|_{\Phi=0} = 0, \quad 0 \leq j \leq n. \tag{B.4}$$

The partial derivatives denoted in the local coordinates (u^j) by $\frac{\partial^j}{\partial \Phi^j}$ can be expressed invariantly as in (4.14). Substituting the expression (B.3) for G into (B.4), we find

$$\begin{aligned}
0 &= \sum_{k=0}^n (-1)^k k! \sum_{r=0}^j \binom{j}{r} \frac{\partial^{j-r} F^{Im(k)}}{\partial \Phi^{j-r}} \frac{\partial^r \Phi^{n-k}}{\partial \Phi^r} \Big|_{\Phi=0} \\
&= \sum_{k=n-j}^n (-1)^k k! (n-k)! \binom{j}{n-k} \frac{\partial^{j-(n-k)} F^{Im(k)}}{\partial \Phi^{j-(n-k)}} \Big|_{\Phi=0}, \\
& \quad 0 \leq j \leq n.
\end{aligned}$$

Replacing k by $n-k$, we find (4.12).

Conversely, if (4.12) holds, then tracing the above steps backwards, we infer that (B.2), and hence also (B.1), holds.

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