Holographic mutual information at finite temperature

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Holographic Mutual Information at Finite Temperature

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Using the Ryu-Takayanagi conjectured formula for entanglement entropy in the context of gauge-gravity duality, we investigate properties of mutual information between two disjoint rectangular sub-systems in finite temperature relativistic conformal field theories in d-spacetime dimensions and non-relativistic scale-invariant theories in some generic examples. In all these cases mutual information undergoes a transition beyond which it is identically zero. We study this transition in details and find universal qualitative features for the above class of theories which has holographic dual descriptions. We also obtain analytical results for mutual information in specific regime of the parameter space. This demonstrates that mutual information contains the quantum entanglement part of the entanglement entropy, which is otherwise dominated by the thermal entropy at large temperatures.

1. Introduction and a summary Entanglement entropy measures the quantum entanglement between two sub-systems of a given system. In a QFT, entanglement entropy of a region A contains short-distance divergence which scales like the area[1, 2]. Holography[3, 4] has emerged to be a powerful technique to analyze strongly coupled large N gauge theories. Within this context, entanglement entropy can be computed using the Ryu-Takayanagi (RT) conjectured formula proposed in [5]: entanglement entropy of a region A is given by the area of a minimal area surface, denoted by γ_A, whose boundary coincides with the boundary of the region A: $\partial\gamma_A = \partial A$. As described in [6, 7], the RT formula has passed several non-trivial checks.

Due to its short distance divergence structure, entanglement entropy is a scheme-dependent quantity. This issue can be avoided by introducing a new concept named mutual information: $I(A,B) = S_A + S_B - S_{A\cup B}$, where S_Y denotes the entanglement entropy of the region Y with surroundings. Mutual information has certain advantages over entanglement entropy. It is (i) finite, (ii) positive semi-definite, (iii) measures the total correlations between the two sub-systems A and B and (iv) it is proportional to the entanglement entropy when $B \equiv A^c$, where A^c denotes the complement of A, such that $S_{A\cup A^c} = 0$. As showed in [8], given an operator $\mathcal{O}_A$ in the region A and $\mathcal{O}_B$ in the region B, mutual information sets an upper bound on the connected correlator

$$I(A,B) \geq \frac{\left(\langle \mathcal{O}_A \mathcal{O}_B\rangle - \langle \mathcal{O}_A\rangle\langle \mathcal{O}_B\rangle\right)^2}{2||\mathcal{O}_A||^2||\mathcal{O}_B||^2} \tag{1}$$

and thus encodes all possible correlations, both classical and quantum, between the two sub-systems. Note that, in view of eqn (1) mutual information obeys a fundamental bound whereas entanglement entropy may not. Furthermore, it was also proved in [8] that mutual information follows an area law even at finite temperature.

Let us also remark that it is possible to consider similar physics in strongly coupled *non-local* field theories,

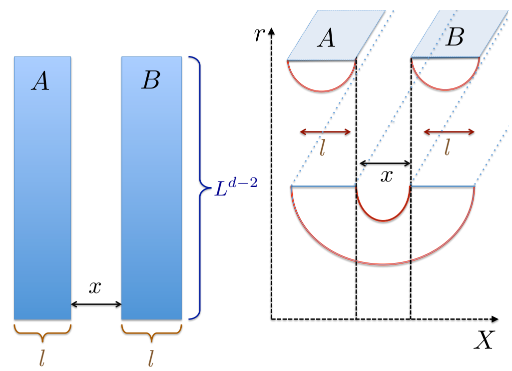

FIG. 1. The two disjoint sub-systems A and B, each of length l along X-direction and separated by a distance x. The schematic diagram on the right shows the possible candidates for minimal area surfaces which is relevant for computing $S_{A\cup B}$. The choice on top gives $S_{A\cup B} = S_A + S_B = 2S(l)$; and the choice at the bottom gives $S_{A\cup B} = S(2l+x) + S(x)$.

e.g. a non-commutative gauge theory. In a particular non-commutative gauge theory that has a gravity dual, we find[9] that mutual information is still a well-behaved quantity but the entanglement entropy is not.

Let us imagine two disjoint sub-systems A and B, each of "rectangular" shape with one dimension of length l and the other as L^{d-2}, are separated by a distance x along one of the spatial directions of a given CFT. This is schematically shown in fig. 1. One can use the RT formula to compute the entanglement entropy and hence the mutual information between A and B. Depending on the ratio x/l, the computation of $S_{A\cup B}$ is determined by different minimal area surfaces as shown in fig. 1.

This results in an intriguing first order phase transition for the resulting mutual information for such systems[7]:

$$\begin{aligned} I(A,B) \neq& 0\ , \quad x/l \leq a_d,\\ =& 0\ , \quad x/l > a_d. \end{aligned} \tag{2}$$

Here a_d is a number that depends on the dimension of the CFT. This transition implies that for $x/l > a_d$, the two sub-systems A and B completely disentangle[8]. Similar phenomenon persists at finite temperature. In the limit $l \to \infty$, the disentangling transition takes place as a function of temperature

$$\begin{aligned} I(A,B) &\neq 0\ , \quad xT \leq b_d, \\ &= 0 \quad xT > b_d, \end{aligned} \tag{3}$$

where b_d is a constant and T denotes the backgrounds temperature. Our goal here is to study this disentangling transition for a class of strongly coupled conformal (or scale-invariant) large N gauge theories in a general dimension using holography. We will use the analytical techniques developed in [10] and also use numerical methods to explore the disentangling transition in the (x/l) vs (Tx) plane.

For relativistic CFTs, the area law for mutual information at finite temperature along with dimensional analysis suggests that

$$I(A,B) = \left(\frac{L}{l}\right)^{d-2} F\left(x/l, xT\right), \tag{4}$$

where $F(x/l, xT)$ is some function that depends on the CFT. At vanishing temperature, we recover the well-known form[11]. The two regimes where we are able to obtain analytical results are $lT \ll 1$, $xT \ll 1$ and $lT \gg 1$, $xT \ll 1$ respectively. When both $lT \ll 1$, $xT \ll 1$, we can make a formal expansion of form

$$F\left(x/l, xT\right) = \sum_i (xT)^i g_i(x/l). \tag{5}$$

In the limit $lT \gg 1$, $xT \ll 1$, we can make the following expansion

$$F\left(x/l, xT\right) = (lT)^{d-2} \sum_\alpha (xT)^\alpha \tilde{g}_\alpha(lT), \tag{6}$$

where $g_i(x/l)$ and $\tilde{g}_\alpha(lT)$ are hitherto undetermined functions that depend on the underlying theory. We will find that generally $i \geq 0$, but α can range over positive and negative numbers. Finally, when both $lT \gg 1$ and $xT \gg 1$, $I(A,B) = 0$. Note that, here we are excluding the possibility of any logarithmic term. In general, such logarithmic contributions can arise; see *e.g.* the example of $(1+1)$-dim CFT and the special case of hyperscaling-violating background in later sections. Also note that, for non-relativistic scale-invariant theories, we have to replace with $T \to T^{1/z}$, where z denotes the dynamical exponent of the theory.

the two regimes with $lT \ll 1$ and $lT \gg 1$ contain distinct physics, and it is particularly interesting to consider the case $lT \gg 1$. Here we get[10]

$$S_A = S_B = S_{\rm div} + S_{\rm thermal} + S_{\rm ent} + S_{\rm corr}, \tag{7}$$

where $S_{\rm div}$ denotes the divergent piece, $S_{\rm thermal}$ denotes the purely thermal entropy that goes as the volume, $S_{\rm ent}$ denotes the next leading order contribution that also follows an area law and finally $S_{\rm corr}$ denotes corrections suppressed by exponentials of (lT). In this limit, mutual information behaves in the following manner:

$$I(A,B)|_{x\to 0} = I_{\rm div} + S_{\rm ent} + I_{\rm corr}, \tag{8}$$

where $I_{\rm div}$ is the divergent piece that emerges in the limit $x \to 0$ and $I_{\rm div} = S_{\rm div}$ similar to what is observed in [11] and $I_{\rm corr}$ are correction terms in powers of (xT) and e^{-lT}. From (7) and (8), we see that apart from the diverging piece as $x \to 0$, mutual information does coincide with the thermal-part-subtracted entanglement entropy at the leading order. Thus, it truly measures quantum entanglement by discarding the volume-worth thermal contribution in the entanglement entropy.

There are perhaps a couple of non-trivialities associated with this observation: First, note that *a priori* there is no reason for the sub-leading terms of entanglement entropy to follow an area law in the large temperature regime. Second, there is a precise match between the numerical factors as well. For a more detailed discussion of the same physics, we will refer the reader to [12].

2. Mutual information in relativistic CFTs Let us begin by considering a class of large N gauge theories in d-dimensions whose dual is given by an asymptotically AdS$_{d+1}$-background. The generic bulk spacetime is given by the AdS-Schwarzschild metric of the form

$$ds^2 = \frac{r^2}{R^2}\left(-f dt^2 + d\vec{x}^2\right) + \frac{R^2 dr^2}{r^2 f}, f = 1 - \frac{r_H^d}{r^d}, \tag{9}$$

where r_H is the location of the black hole horizon, R is the AdS radius, $\vec{x}$ is a $(d-1)$-dimensional vector and the boundary of the spacetime is located at $r \to \infty$. The temperature of the black hole is given by: $T = \frac{r_H d}{4\pi R^2}$. We will set $R = 1$.

To obtain mutual information for an arrangement schematically shown in fig. 1, we specify the strip by

$$X \equiv x^1 \in \left[-\frac{l}{2}, \frac{l}{2}\right], x^i \in \left[-\frac{L}{2}, \frac{L}{2}\right], \tag{10}$$

with $L \to \infty$. Extremal surface is translationally invariant along $x^i, i = 2, ..., d-1$ and the profile of the surface in the bulk is given by $X(r)$. Area of this surface is given by

$$A = L^{d-2} \int dr r^{d-2} \sqrt{r^2 X'^2 + \frac{1}{r^2\left(1 - \frac{r_H^d}{r^d}\right)}}. \tag{11}$$

This action leads to the equation of motion

$$\frac{dX}{dr} = \pm \frac{r_c^{d-1}}{r^{d+1}\sqrt{\left(1 - \frac{r_c^{2d-2}}{r^{2d-2}}\right)\left(1 - \frac{r_H^d}{r^d}\right)}}, \tag{12}$$

where, r_c is an integral of motion and $r = r_c$ represents the point of closest approach of the extremal surface. Such surfaces have two branches, joined smoothly at $(r = r_c, X = 0)$ and r_c can be determined using the boundary conditions: $X(\infty) = \pm\frac{l}{2}$, which leads to

$$\frac{l}{2} = \int_{r_c}^{\infty} \frac{r_c^{d-1} dr}{r^{d+1}\sqrt{\left(1 - \frac{r_c^{2d-2}}{r^{2d-2}}\right)}} \left(1 - \frac{r_H^d}{r^d}\right)^{-1/2} . \quad (13)$$

Special case: $d = 2$

In this case, it is possible to evaluate the integrals in (13) and (11) in closed forms. This eventually leads to the following expression for mutual information:

$$I(A, B) = \frac{c}{3} \log\left[\frac{(\sinh(\pi l T))^2}{\sinh(\pi x T)\sinh(\pi(2l + x)T)}\right], (14)$$

with $c = 3/(2G_N^{(2+1)})$. In the low temperature limit, when $lT \ll 1$ and $xT \ll 1$, we get

$$I(A, B) = \frac{c}{3}\left[\log\left(\frac{l^2}{x(2l + x)}\right) - \frac{1}{3}\pi^2 T^2 (l + x)^2 + \ldots\right], \quad (15)$$

where the first term in the square bracket is just the zero temperature mutual information. In view of (4), we observe that there is no linear term in T. We also observe that finite temperature reduces mutual information and therefore promotes disentangling between the two subsystems.

On the other hand, in the intermediate temperature regime, where $lT \gg 1$ and $xT \ll 1$, we get

$$I(A, B) = \frac{-c}{3}\left[\log\left(\frac{2\pi x T}{\tanh(\pi l T)}\right) + (\pi x T) + \ldots\right], (16)$$

$$S_A = S_{\text{div}} + \frac{c}{3}\log(\sinh(\pi l T)) + \ldots . \quad (17)$$

In the limit $x \to 0$, defining $\epsilon = x/2$ we get that the large temperature expansion of mutual information given in (16) coincides exactly with the leading order large temperature expansion of the entanglement entropy given in (17). The mismatch is suppressed in exponentials of (lT). This is an example of what we discussed in equations (7) and (8).

We have pictorially shown a "phase diagram" in fig. 2 corresponding to either $I(A, B) \neq 0$ or the $I(A, B) = 0$ phase. The blue-shaded region represents the regime of parameters where there is non-vanishing correlation between the two sub-systems. From this phase diagram it is evident that increasing temperature does indeed disentangle the two sub-systems and entanglement reduces monotonically for increasing temperature. Similar physics was studied earlier in [13].

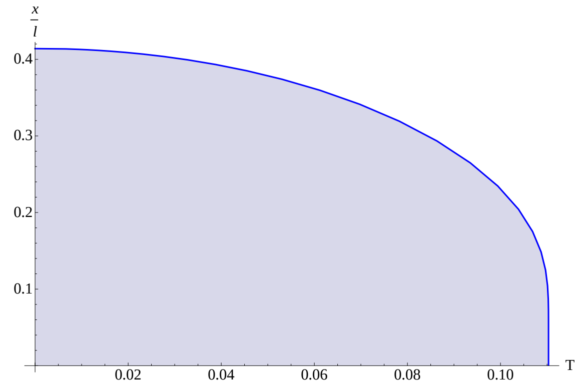

FIG. 2. 2-dimensional parameter space for the (1+1)-dimensional boundary theory. The mutual informational is non-zero only in the shaded region. A similar qualitative result holds for all other cases discussed later.

General case: $d > 2$

We now move on to discussing the general case of $d > 2$. In this case, it is not possible to evaluate the integrals in (11) and (12) in closed forms. We will use the approximation scheme outlined in [10]. Now we will define: $c = \frac{R^{d-1}}{4G_N^{(d+1)}}$, where $G_N^{(d+1)}$ is the Newton's constant in $(d+1)$-dimensional bulk theory.

In the limit $T \ll \frac{1}{l}, \frac{1}{x}$, when the mutual information is non-zero, it is given by,

$$I(A, B) = I(A, B)|_{T=0} - 2c\,\mathcal{S}_0\mathcal{S}_1 \left(\frac{4\pi}{d}\right)^d L^{d-2}\, T^d\, (l - x)^2 . \quad (18)$$

Here $\mathcal{S}_0$ and $\mathcal{S}_1$ are negative real numbers that depends on the dimensions[12]. In this case, the finite temperature correction obeys an area law as generally proved in [8].

In the limit $\frac{1}{l} \ll T \ll \frac{1}{x}$, when the mutual information is non-zero, it is given by,

$$I(A, B) = c\, L^{d-2} T^{d-2}\left[-\mathcal{S}_0 \frac{1}{(xT)^{d-2}} + \left(\frac{4\pi}{d}\right)^{d-2} \mathcal{S}_{\text{high}} + \mathcal{O}(xT) + \ldots\right]. \quad (19)$$

$\mathcal{S}_{\text{high}}$ is a numerical constant that depends on the dimensions[12]. Mutual information indeed captures the area-worth entanglement by subtracting off the thermal contribution. It is an example of the generic observation mentioned in (7) and (8). Finally for $xT \gg 1$, the two sub-systems are completely disentangled and mutual information vanishes. The corresponding "phase diagram" looks qualitatively similar to fig. 2.

3. Other backgrounds We will now consider generic examples of scale-invariant (but not conformal) theories, which are known to have gravity dual descriptions. Such field theories with gravity duals, assuming they exist, are non-relativistic. Examples include the so called Lifshitz geometry introduced in [14]; and more recently the background with hyperscaling violation in [15]. Note that, in both [14] and [15] the approach is *phenomenological* or the so called *bottom-up*.

Lifshitz background

In this case, the background metric is invariant under the following scale transformation: $t \to \lambda^z t, \{x, r\} \to \lambda\{x, r\}$; where λ is a real number and r is the radial coordinate, in which the boundary is located at $r \to 0$. An analytic finite temperature Lifshitz background in (3+1)-dimensions is obtained in [16], and is given by

$$ds^2 = R^2\left(-\frac{f dt^2}{r^{2z}} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{fr^2}\right), f = 1 - \frac{r^2}{r_H^2}, \quad (20)$$

for $z = 2$. The temperature in the dual field theory is given by the Hawking temperature of the black hole: $T = \frac{R}{2\pi r_H^2}$.

At zero temperature, the mutual information is given by

$$I(A, B) = -c\,\mathcal{L}_0 L\left[\frac{2}{l} - \frac{1}{x} - \frac{1}{(2l + x)}\right], c = \frac{R^2}{4G_N^{(4)}}, \quad (21)$$

for $x/l \leq 0.618$. Here $\mathcal{L}_0$ is a positive real number. In the low temperature range: $\sqrt{T/R} \ll \frac{1}{l}, \frac{1}{x}$, we get

$$I(A, B) = I(A, B)|_{T=0} - 2c\,\mathcal{L}_0\mathcal{L}_1\frac{L\,T}{R}x + \ldots, \quad (22)$$

where $\mathcal{L}_1$ is a positive real number. In this case, in addition to the familiar area law, we do observe a linear correction in temperature as a small temperature is introduced. In the limit $\frac{1}{l} \ll \sqrt{T/R} \ll \frac{1}{x}$, we get:

$$\begin{aligned} I(A, B) &= c\,L\sqrt{\frac{T}{R}}\left[\mathcal{L}_0\sqrt{\frac{R}{T}}\frac{1}{x} - \mathcal{L}_{\rm high}\right], \sqrt{\frac{T}{R}}x \leq 0.261, \\ &= 0\ , \quad \sqrt{\frac{T}{R}}x > 0.261. \end{aligned} \quad (23)$$

Here $\mathcal{L}_{\rm high} = 2.671$ is a numerical constant. We find that in this regime mutual information coincides with the thermal-part-subtracted entanglement entropy. Finally for $\sqrt{T/R} \gg \frac{1}{x}$, $I(A, B) = 0$ identically. The corresponding 2-dimensional "phase diagram" takes a similar form as observed in fig. 2.

Hyperscaling-violating background

In this case, the metric is covariant under the Lifshitz-type scale transformation and the metric[15] has the following property: $ds^2 \to \lambda^{2\theta/(d-1)} ds^2$, where θ is known as the hyperscaling violation exponent. In the presence of a black hole, the metric takes the following form[17]

$$\begin{aligned} ds^2 &= r^{2\theta/(d-1)}\left(-\frac{f dt^2}{r^{2z}} + \frac{dr^2}{r^2 f} + \frac{d\vec{x}^2}{r^2}\right), \\ f &= 1 - \left(\frac{r}{r_H}\right)^\gamma, \end{aligned} \quad (24)$$

where γ is a real-valued constant which we will keep unspecified for now, r_H is the location of the horizon and d is the spacetime dimension of the boundary dual theory. We have also set the curvature of the space $R = 1$. The boundary is located at $r \to 0$. The backgrounds temperature is given by: $T = \frac{\gamma}{4\pi r_H^z}$. We will consider the case when $d - \theta - 2 \geq 0$, which typically exhibits an area law for entanglement entropy with the exception of logarithmic violation for $\theta = d - 2$. We will present the results in the intermediate temperature regime only.

General case: $\theta \neq d - 2$

In the regime $xT^{1/z} \ll 1, lT^{1/z} \gg 1$, when the mutual information is non-zero, it is given by

$$I(A, B) = -c\,L^{d-2}\left[\frac{\mathcal{C}(\theta, d)}{x^{d-\theta-2}} - h_3 T^{\frac{d-\theta-2}{z}} + \ldots\right], \quad (25)$$

where $\mathcal{C}$ and h_3 are numerical constants. We note that mutual information coincides with the thermal-part-subtracted entanglement entropy at large temperature. The large temperature behaviour is similar to the ones observed before.

Special case: $\theta = d - 2$

Let us now consider the special case of $\theta = d-2$, where logarithmic violation of the area law shows up. In the regime $xT^{1/z} \ll 1, lT^{1/z} \gg 1$, the mutual information for the $d-$dimensional boundary theory (for $d - \theta - 2 = 0$), is given by

$$I(A, B) = cL^{d-2}\left[2\ln\left(\frac{1}{xT^{1/z}}\right) + k_2 + \ldots\right], \quad (26)$$

where k_1, k_2 and k_3 are three real numbers. In this case, irrespective of the value of γ, mutual information does indeed capture the thermal-part-subtracted entanglement entropy. Finally, this also yields a similar "phase transition".

4. Conclusions and Outlook In this article, we have explored the disentangling transition between two sub-systems by studying mutual information in the context of holography. We have considered a class of large N relativistic gauge theories as well as generic examples of non-relativistic scale-invariant theories. We have found an universal qualitative behaviour in the corresponding "phase diagram" in (x/l) vs (xT)-plane. We have also observed that at large temperature, when the mutual information is non-zero, it coincides with the thermal-part-subtracted entanglement entropy for all the cases considered above.

Let us note that the sharp transition of mutual information is a consequence of large N limit. In this limit, the inequality in (1) is trivially satisfied since the right hand side is always $1/N$-suppressed[7]. At finite N, however, mutual information should not vanish identically. Hence, the $1/N$-corrections to the RT formula perhaps do not contain a simple geometric interpretation as an area functional in the bulk geometry.

In quantum many-body systems, mutual information is emerging as an useful order parameter for certain phase transitions, such as the ones described in [18]. Within the context of AdS/CFT correspondence or the gauge-gravity duality, it will be interesting to explore what role mutual information plays in similar contexts.

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