Primordial tensor power spectrum in holonomy corrected Ω loop quantum cosmology

Linda Linsefors, Thomas Cailleteau, Aurelien Barrau, and Julien Grain

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Primordial tensor power spectrum in holonomy corrected $\Omega-$LQC

Linda Linsefors*
Laboratoire de Physique Subatomique et de Cosmologie, UJF, INPG, CNRS, IN2P3
53, avenue des Martyrs, 38026 Grenoble cedex, France

Thomas Cailleteau†
Institute for Gravitation & the Cosmos, Penn State
University Park, PA 16802, USA

Aurelien Barrau‡
Laboratoire de Physique Subatomique et de Cosmologie, UJF, INPG, CNRS, IN2P3
53,avenue des Martyrs, 38026 Grenoble cedex, France

Julien Grain§
Université Paris-Sud 11, Institut d'Astrophysique Spatiale, UMR8617, Orsay, France, F-91405 and
CNRS, Orsay, France, F-91405

The holonomy correction is one of the main terms arising when implementing loop quantum gravity ideas at an effective level in cosmology. The recent construction of an anomaly free algebra has shown that the formalism used, up to now, to derive the primordial spectrum of fluctuations was not correct. This article aims at computing the tensor spectrum in a fully consistent way within this deformed and closed algebra.

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I. INTRODUCTION

Nonperturbatively quantizing General Relativity (GR) in a background-invariant way is obviously an outstanding open problem of theoretical physics. Loop Quantum Gravity (LQG) is a promising framework to perform this program (see [1] for introductory reviews). Although this is still to be demonstrated, there are evidences that different approaches, based either on quantizations (covariant or canonical) of GR, or on a formal quantization of geometry lead to the same LQG theory. Experimental tests are, however, still missing. Trying to find possible observational signatures is a key challenge and cosmological footprints are known for being one of the only possible paths toward a real experimental test of LQG. It is very hard to make clear predictions in Loop Quantum Cosmology (LQC) using the full "mother" LQG theory. General introductions to LQC can be found in [2]. This study focuses on an effective treatment taking into account recent results on the correct algebra of constraints. We first review the theoretical framework. The spectrum is then derived. Some conclusions and consequences are finally underlined.

*Electronic address: linsefors@lpsc.in2p3.fr
†Electronic address: thomas@gravity.psu.edu
‡Electronic address: Aurelien.Barrau@cern.ch
§Electronic address: julien.grain@ias.u-psud.fr

II. THEORETICAL FRAMEWORK

One of the fundamental quantum corrections expected from the Hamiltonian of LQG arises from the fact that loop quantization is based on holonomies, *i.e.* exponentials of the connection, rather than direct connection components. Based on a canonical approach, the theory uses Ashtekar variables, namely SU(2) valued connections and conjugate densitized triads. The quantization is obtained through holonomies of the connections and fluxes of the densitized triads. This is the key ingredient of the effective approach. The cosmological equations are modified so as to account for the loop basis of the theory.

The main consequence of the holonomy correction on the cosmological background is to induce a bounce. The evolution is not singular anymore and the Big Bang is replaced by a Big Bounce. The next step consists in studying the propagation of perturbations within this modified background. In cosmology, perturbations are of three different types : scalar, vector and tensor. We focus here on the tensor modes that are directly gauge-invariant. Quite a lot of works have already been devoted to tensor modes in this framework [3]. Beyond, the phenomenology of LQG is now a well established field (see [4] for a review). Unfortunately, a recent study [5] has shown that the previously derived spectra are most probably incorrect.

The key issue relies in the closure of the algebra of constraints. Due to general covariance, the canonical Hamiltonian is a combination of constraints $\mathcal{C}_I$. Consistency requires that the constraints are preserved under the evolution they generate. This is ensured in the classical theory by the closure of the Poisson algebra of constraints

$$\{\mathcal{C}_I, \mathcal{C}_J\} = f^K{}_{IJ}(A^j_b, E^a_i)\mathcal{C}_K, \tag{1}$$

where $\mathcal{C}_I$, $I = 1, 2, 3$, are the Gauss, diffeomorphism and Hamiltonian constraints and $f^K{}_{IJ}(A^j_b, E^a_i)$ are structure functions which, in general, depend on the phase space (Ashtekar) variables (A^j_b, E^a_i). They form a first class set. Otherwise stated, the gauge transformations and evolution generated by the constraints define vector fields which are tangent to the sub-manifold defined by the vanishing of constraints.

In LQC, quantum corrections are introduced as effective modifications of the Hamiltonian constraint. This generates anomalies: the modified constraints $\mathcal{C}^Q_I$ do not form a closed algebra anymore:

$$\{\mathcal{C}^Q_I, \mathcal{C}^Q_J\} = f^K{}_{IJ}(A^j_b, E^a_i)\,\mathcal{C}^Q_K + \mathcal{A}_{IJ}. \tag{2}$$

The anomalous terms $\mathcal{A}_{IJ}$ are removed by carefully adjusting the form of the quantum correction to the Hamiltonian constraint through the addition of suitable "counterterms" that vanish in the classical limit. This has been done in [5], following the approach of [6].

In the classical case, the Poisson brackets between the constraints read as:

$$\{D_{(m+g)}[N^a_1], D_{(m+g)}[N^a_2]\} = 0\,, \tag{3}$$

$$\{H_{(m+g)}[N], D_{(m+g)}[N^a]\} = -H_{(m+g)}[\delta N^a\partial_a\delta N]\,, \tag{4}$$

$$\{H_{(m+g)}[N_1], H_{(m+g)}[N_2]\} = D_{(m+g)}\Big[\tfrac{\bar{N}}{\bar{p}}\partial^a(\delta N_2 - \delta N_1)\Big], \tag{5}$$

where $(m+g)$ stands for gravity and matter. The quantum corrections are included at the effective level by replacing, as usual, in the Hamiltonian constraint

$$\bar{k} \to \frac{\sin(\bar{\mu}\gamma\bar{k})}{\bar{\mu}\gamma}. \tag{6}$$

The important result of [5] is that the quantum-corrected algebra is described by a single modification:

$$\{H_{(m+g)}[N_1], H_{(m+g)}[N_2]\} = \mathbf{\Omega}\; D_{(m+g)}\left[\frac{\bar{N}}{\bar{p}}\partial^a(\delta N_2 - \delta N_1)\right] \tag{7}$$

where

$$\mathbf{\Omega} = \cos(2\bar{\mu}\gamma\bar{k}) = 1 - 2\frac{\rho}{\rho_c}\ . \tag{8}$$

The $\mathbf{\Omega}$ factor encodes the quantum correction, $\bar{k}$ being the homogeneous Ashtekar connection and $\bar{\mu}$ being proportional to the ratio between the Planck length and the scale factor. The Mukhanov-Sasaki [7] equation of motion for gauge-invariant perturbations of scalar and tensor types $v_{\mathrm{S(T)}}$ can be explicitly derived. In conformal time, the propagation of tensor modes is given by

$$v''_{\mathrm{T}} - \mathbf{\Omega}\,\nabla^2 v_{\mathrm{T}} - \frac{z''_{\mathrm{T}}}{z_{\mathrm{T}}}v_{\mathrm{T}} = 0\ ;\ z_{\mathrm{T}} = \frac{a}{\sqrt{\mathbf{\Omega}}}, \tag{9}$$

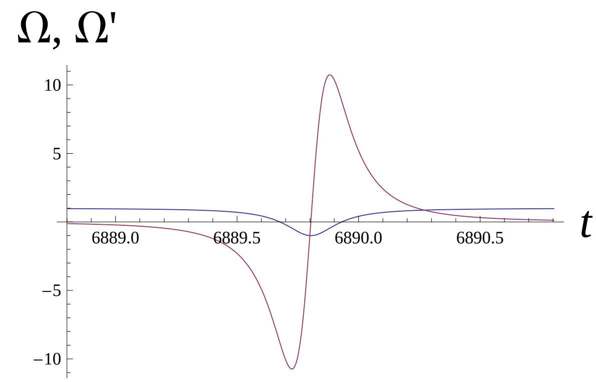

FIG. 1: Evolution of $\mathbf{\Omega}$ and its derivative with respect to conformal time. The density where $\mathbf{\Omega}$ vanishes is half the critical density whereas $\mathbf{\Omega}'$ vanishes at the bounce.

where *prime* means differentiation with respect to conformal time. This leads to the following equation of motion for tensor perturbations, defined via $v_{\mathrm{T}} = z_{\mathrm{T}} \times h^i_a$:

$$h^i_a{}'' + h^i_a{}'\left(2\mathscr{H} - \frac{\mathbf{\Omega}'}{\mathbf{\Omega}}\right) - \mathbf{\Omega}\,\nabla^2 h^i_a = 0. \tag{10}$$

where $\mathscr{H} := a'/a$ is the conformal Hubble parameter.

III. POWER SPECTRUM

This equation being known, it is possible to investigate the associated primordial power spectrum. This is the fundamental ingredient for phenomenology. The background dynamics is not modified by the $\mathbf{\Omega}$ term. However, the perturbations will of course undergo a different evolution.

We use the Fourier transformed version of Eq. (10):

$$h'' + \left(2\mathscr{H} - \frac{\mathbf{\Omega}'}{\mathbf{\Omega}}\right)h' + \mathbf{\Omega}k^2 h = 0, \tag{11}$$

where the indices have been skipped for simplicity. The behavior of $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ is displayed in Fig. 1. One can immediately see that $\mathbf{\Omega}$ vanishes for $\rho = \rho_c/2$, where

$$\rho_{\mathrm{c}} = \frac{\sqrt{3}}{32\pi^2\gamma^3}m^4_{\mathrm{Pl}} \simeq 0.41 m^4_{\mathrm{Pl}}. \tag{12}$$

In addition, $\mathbf{\Omega}$ becomes negative-valued, leading to an effective change of signature of the metric (Euclidean phase) around the bounce. The interested reader will find a technical discussion in [8] and some qualitative speculations in [9]. Intuitively, this signature change can be straightforwardly interpreted as a change of sign of the Poisson bracket between Hamiltonian constraints. Equation (11) is apparently ill-defined as $\mathbf{\Omega}'/\mathbf{\Omega} \to \infty$ at $\eta = \eta^{(-)}$ and $\eta = \eta^{(+)}$, the values of conformal time when $\rho = \rho_c/2$ before and after the bounce, respectively.

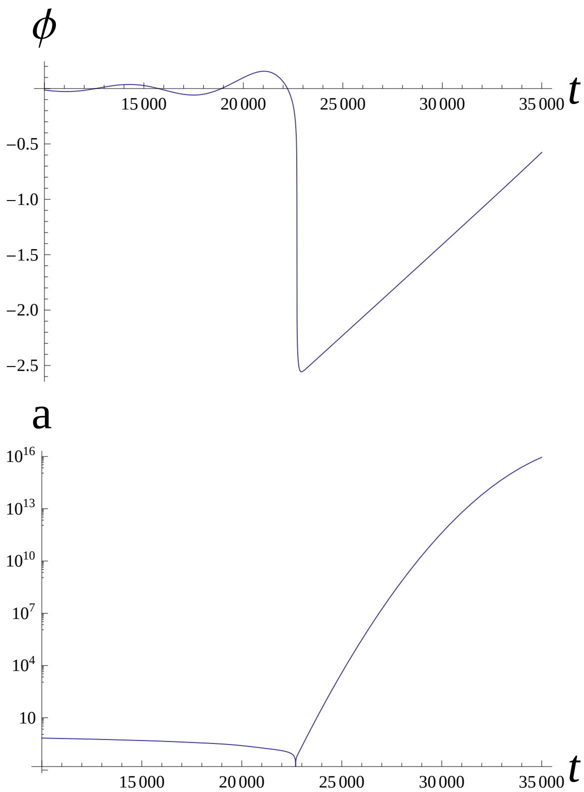

FIG. 2: Evolution of the scalar field (upper panel) and the scale factor (lower panel) as a function of cosmic time (the bounce corresponds to $t = 22693$). The parameters are $m = 10^{-3}M_{\rm Pl}$ and $x_B = -1.5 \times 10^{-3}$.

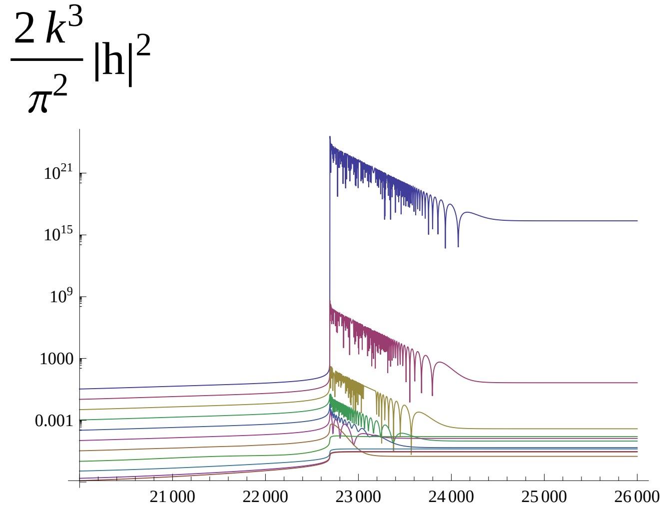

FIG. 3: Mode amplitudes as a function of time, corresponding (from top to bottom at $t = 20000$) to $k = 10^2, 10^{1.5}, 10^1, 10^{0.5}, 1, 10^{-0.5}, 10^{-1}, 10^{-1.5}, 10^{-2}, 10^{-2.5}, 10^{-3}$. The parameters are $m = 10^{-3}M_{\rm Pl}$ and $x_B = -1.5 \times 10^{-3}$. It should be noticed that the initial conditions fore each mode are specified long before the time interval of this plot.

However, regular solutions do exist by rewriting Eq. (11) as:

$$h' = \mathbf{\Omega} g \ ; \ g' = -2\mathscr{H} g - k^2 h, \tag{13}$$

which is regular.

The same set of equations in cosmic time are:

$$\dot{h} = \frac{\mathbf{\Omega}}{a} g \ ; \ \dot{g} = -2Hg - \frac{k^2}{a} h, \tag{14}$$

where *dot* means differentiation with respect to cosmic time and H is the usual Hubble parameter. The dynamics can also be recast in a single second order equation:

$$g'' + 2\mathscr{H} g' + (2\mathscr{H}' + \mathbf{\Omega} k^2) g = 0. \tag{15}$$

Whatever the chosen form, either (13), (14) or (15), the evolution can be computed numerically. Of course, the propagation of modes has to be coupled with the background evolution which is drastically modified by the holonomy corrections that are at the origin of the bounce. The cosmological background evolution is basically driven by a single scalar massive matter field of mass m. We define

$$x := \frac{m\phi}{\sqrt{2\rho_c}} \text{ and } y := \frac{\dot{\phi}}{\sqrt{2\rho_c}}, \tag{16}$$

which respectively represent the density of potential and kinetic energy normalized so that $x_B^2 + y_B^2 = 1$ at the bounce. The free parameters of the study are therefore m, x_B (the value of x at the bouce) and the relative sign of x_B and y_B. Interestingly, if the initial conditions for the background are specified at any time, long enough before the bounce, the probability of $|x_B|$ is strongly peeked around a given value of order m (in Planck units), with $\text{sign}(x_B) = \text{sign}(y_B)$ (the detailed probability distribution for x_B will be studied somewhere else [10]). For numerical reasons it is better to specify computational initial conditions for the background before the bounce rather than at the bounce. Because of the peaked probability, the resulting x_B is always close to the same value.

It is also necessary to assign a numerical value to the scale factor, a at some point. This choice has of course no physical consequences but has to be taken into account for the interpretation of the meaning of the wave vectors k, since they are expressed in the coordinate space and not in the physical space. The explicit choice made was $a = 1$ at the bounce, which is numerically easier than the usual normalization at the nowadays value.

In Fig. 2, the evolution of the scalar field and scale factor are shown for some typical parameters. As expected, the oscillations of the scalar field are amplified before the bounce, because the negative Hubble parameter acts as an anti-friction term. Then, just after the bounce, the Hubble parameters becomes positive and large, acting as a huge friction and therefore leading to slow roll inflation.

The amplitudes of some Fourier modes of h are plotted

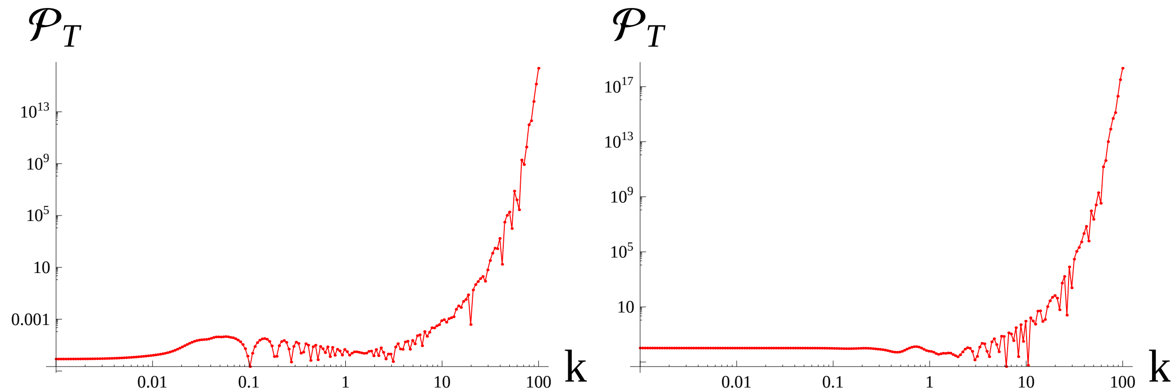

FIG. 4: Power spectrum of tensor perturbations after inflation for $m = 10^{-3}M_{\rm Pl}$ and (left panel), and for $m = 10^{-1}M_{\rm Pl}$ (right panel).

in Fig. 3. They are obtained by chosing the Minkowski vacuum as the initial state, since $z''/z \to 0$ in the remote past.

Before the bounce, for $k^2 \gg z''/z$, $|h|^2 = 1/(2ka^2)$. When $z''/z \approx k^2$ or $z''/z > k^2$, $|h|^2$ grows quicker. Since the amplitudes of smaller k start growing quicker before the amplitudes of larger k, this adds up to a collecting effect that brings all modes up to a certain $k \approx \max_{t<t_B}(\sqrt{z''/z})$ up to the same amplitude. After the bounce, the amplitudes oscillate until $k^2 \gg z''/z$ when we get $v \propto a$ (as can bee seen from Eq. (9)) and therefore $h =$ constant.

Finally, the power spectra for different cases are presented in Fig. 4. The main features are the following:

- a flat (scale invariant) infrared limit,
- an oscillating intermediary part,
- an exponential behavior in the ultraviolet limit (starting around $k = 2$ independantly of m).

This obviously exhibits important deviations, with respect both to the standard GR case and with respect to previous LQC computations without the $\mathbf{\Omega}$ term. Although surprising at first sight, the exponential divergence in the UV limit might not be catastrophic as physics at very small scale is anyway not described by the primordial power spectrum.

Furthermore, this ultraviolet behavior can be checked analytically. In the large k limit of Eq. (9), the WKB conditions are satisfied in the euclidean phase around the bounce. More precisely, those WKB conditions are met for $\eta \in [\eta^{(-)} + \epsilon_k^{(-)}, \eta^{(+)} - \epsilon_k^{(+)}]$ with $\epsilon_k^{(\pm)} \sim (k^2 \left|\Omega'(\eta = \eta^{(\pm)}\right|)^{-1/3}$. The Mukhanov-Sasaki function can be approximated by

$$v_{\rm T} = v_+ e^{ik\int\sqrt{\mathbf{\Omega}}d\eta} + v_- e^{-ik\int\sqrt{\mathbf{\Omega}}d\eta}. \qquad (17)$$

As $\mathbf{\Omega}$ is negative-valued during the euclidean phase, the tensor mode is dominated by its exponentially-growing solution

$$h \propto \exp\left(k\int_{\eta^{(-)}+\epsilon_k^{(-)}}^{\eta^{(+)}-\epsilon_k^{(+)}} \sqrt{|\mathbf{\Omega}|}d\eta\right). \qquad (18)$$

This can also be seen in Fig. 3 where the amplitude of large k modes grows rapidly in the vicinity of the bounce, where $\mathbf{\Omega} < 0$.

IV. DICUSSION

This study implements in a consistent way the modified algebra induced by holonomy corrections in the calculation of the primordial tensor power spectrum. Thanks to numerical calculations, it was possible to solve the equation of motion for gravitational waves. The resulting spectrum exhibits specific features. Of course, this raises important questions. First, the well known problem of trans-planckian modes in inflation (see, *e.g.*, [11]) should be treated with a specific care in LQG where the very meaning of a length smaller than the Planck length is dubious. If the number of e-folds of inflation is chosen (by appropriately setting a very small fraction of potential energy density at the bounce) to be just above the minimum required value, then modes relevant for the CMB are still sub-planckian and the approach makes sense anyway. In other cases, the effective theory might just breakdown. With the normalization chosen in this work the trans-planckian window corresponds to $k > 1$. Second, the propagation of modes through the euclidean phase is not straitghforward [8]. Strictly speaking, there is no "time" in that region and the concept of evolution is not well defined. In this work, we have deliberately chosen to withdraw the conceptual issues associated with the transition between hyperbolic and elliptic solutions and to focus on a well defined mathematical solution. An alternative approach, based on the BKL conjecture, will be studied later [12]. An analogous study should also be performed for scalar modes. The regularization trick used here, however, does not apply directly, and other methods have to be constructed. We stress out that the case of scalar modes with holonomy corrections has been studied in [13] and [14] but in different settings for the background ; for the study of [13] is restricted to super-inflation while the study of [14] considered a dust-like bouncing Universe. Finally, those results will have to be compared with forthcoming studies based on other very recent approaches to LQC [15].

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