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Detailing $\mathcal{N} = 1$ Seiberg's Duality through the Seiberg-Witten Solution of $\mathcal{N} = 2$

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Abstract

Starting from the Seiberg-Witten solution of $\mathcal{N} = 2$ SQCD with the U(N) gauge group and N_f quark flavors we construct the so-called μ-dual $\mathcal{N} = 1$ theory in the r vacua in the regime analogous to that existing to the left of the left edge of the Seiberg conformal window, where r is the number of condensed quarks. The strong-weak coupling duality is shown to exist in the so-called zero vacua which can be found at $r < N_f - N$. We show that the μ-dual theory matches the Seiberg dual in the zero vacua.

1 Introduction

Seiberg's duality in its original formulation [1, 2] relates $\mathcal{N} = 1$ supersymmetric QCD (SQCD) with the SU(N) gauge group and N_f quark flavors to a dual theory with the SU($\tilde{N}$) gauge group, the same number of dual quarks, plus a neutral meson field M. Here

$$\tilde{N} \equiv N_f - N \,. \tag{1.1}$$

These two theories forming the Seiberg pair are distinctly different in the ultraviolet (UV) domain, but describe exactly the same dynamics in the infrared (IR) domain. Later Seiberg's duality was generalized to other gauge groups and extended to other matter contents. Although Seiberg's duality was a conjecture it passed numerous tests both on the field and string theory sides, and is viewed as firmly established.

A breakthrough in understanding the strong coupling gauge dynamics was achieved with the Seiberg-Witten solution [3, 4] of $\mathcal{N} = 2$ SQCD. Combining the above two constructions together could shed light on the physical nature of Seiberg's dual quarks and provide us with an additional understanding of low-energy physics in $\mathcal{N} = 1$ SQCD, in particular, physics of confinement and screening in the regime where the dual theory is weakly coupled.

A crucial step in this direction was made in [5]. In this paper SU(N) $\mathcal{N} = 2$ theory deformed by the mass term $\mu \mathrm{Tr}\,\mathcal{A}^2$ for the adjoint matter was considered. At small μ this theory was described by the Seiberg-Witten solution [3, 4], while at large μ it obviously flows to $\mathcal{N} = 1$ SQCD. It was shown that the SU($\tilde{N}$) gauge group present at low energies at the root of a baryonic branch survives the large μ limit. This explains the emergence of the SU($\tilde{N}$) gauge group in the Seiberg's dual theory. The presence of a large number of distinct vacua in the IR, with different physical features, was not discussed in [5]. And understandably so, since the analysis of [5] was carried out with massless quarks in which case certain vacua coalesce, and Higgs branches develop from common roots.

Much later it was noted (in the framework of the U(N) gauge theories) that Seiberg's dual theory and the theory at the baryonic root are associated with different vacua [6]. To identify distinct vacua we introduced mass terms m_A, $A = 1, ..., N_f$ to the quark fields. It is known that the μ-deformed $\mathcal{N} = 2$ SQCD with generic quark masses has the so-called r vacua (they are isolated) in which r quark flavors condense,[1] $r \leq N$. The Seiberg $\mathcal{N} = 1$ duality was in fact formulated for monopole vacua with $r = 0$ in the limit $\mu \to \infty$. In the $r \neq 0$ vacua the condensates of r quark flavors are determined by the value of the effective parameters $\xi_A \sim \mu m_A$, hence, they are runaway vacua in the limit $\mu \to \infty$ corresponding to $\mathcal{N} = 1$. The root of the baryonic branch in the U(N) version of the theory corresponds to the $r = N$ isolated vacuum. In the limit $\mu \to \infty$ this vacuum becomes a runaway vacuum too.

The number of quark flavors N_f to be considered below is subject to the constraint

$$N + 1 < N_f < \frac{3}{2}\,N \,. \tag{1.2}$$

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[1] Note that $r = N$ is the maximum possible number of condensed quarks. The r counting is carried our at large m_A, see below.

This domain lies to the left of the left edge of the Seiberg conformal window. In this domain the original "electric" theory in the Seiberg pair is asymptotically free and strongly coupled in the IR, while its dual "magnetic" partner is infrared free and weakly coupled in the IR. This pattern will be preserved in our consideration.

In this paper we mostly consider r vacua with "small" r,

$$r < \frac{N_f}{2},\tag{1.3}$$

see [7, 14] and Sec. 6.1 for the discussion of $r > N_f/2$ vacua. Our strategy is as follows: we start from the original $U(N)$ theory in the $\mathcal{N} = 1$ large-μ limit, which is in fact the UV limit of the theory. Then we decrease μ approaching the $\mathcal{N} = 2$ limit. At this stage the mass parameters m_A are kept large. Then we use the Seiberg-Witten solution to analytically continue to the domain of small m_A. The theory obtained in this way still has $\mathcal{N} = 2$ supersymmetry. Then we increase μ to decouple the adjoint scalar superfield and return to $\mathcal{N} = 1$. In doing so we keep μ large but finite in order to keep track of all r vacua. In this limit we find an IR-free model, the dual partner to our original $\mathcal{N} = 1$ theory. At every stage of this road full theoretical control is maintained, including the IR domain. The Seiberg-Witten solution is combined with the powerful tools worked out by Cachazo, Seiberg, and Witten [9], and by Dijkgraaf and Vafa [10]. This allows us to identify, from the analysis of the dual partners, the relevant vacua and their dynamics. We are only interested in such dual partners that are at weak coupling in the IR, thus maintaining the same pattern as the one inherent to the Seiberg duality in the domain to the left from the conformal window. We will see that the original $U(N)$ theory can flow in the IR to the dual IR free theory, with the gauge group $U(\tilde{N})$ (i.e. exactly the same as in [1, 2]), possessing special r vacua, to be referred to as the *zero vacua*. We discuss the corresponding dynamics, as well as the nature of Seiberg's dual quarks.

To briefly explain the emergence and relevance of the zero vacua in the problem at hand we note that our starting point is $\mathcal{N} = 2$ SQCD with a small $\mu \mathrm{Tr}\, \mathcal{A}^2$ term. In r vacuum with $r < N_f/2$ at low energies, after developing condensates of r quarks, this theory reduces to the IR free $U(r) \times U(1)^{N-r}$ gauge theory with r light quarks and $(N - r - 1)$ Abelian monopoles.[2] Using the results of [3, 4, 9] one can detect a large number of various r vacua in the above low-energy theory. Among these vacua we identify a special set of the zero vacua, namely those, in which the gaugino condensate tends to zero in the small m_A limit. In all other r vacua (to be referred to as Λ vacua) it stays finite. In fact, the zero vacua exist only at

$$r < \tilde{N}.\tag{1.4}$$

The above theory can be "uplifted" (by increasing μ) to $\mathcal{N} = 1$. This uplift leads to the original $U(N)$ theory in UV (see Fig. 1). At the same time, at small m_A the uplift from the zero vacua leads us to an $\mathcal{N} = 1$ μ-dual theory weakly coupled in the IR and strongly coupled in the UV, with the enhanced $U(\tilde{N})$ gauge group and N_f flavors of quarks. The

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[2]The $U(r)$ gauge factor implies that all mass terms m_A are almost equal.

r quark flavors condensed in the vacuum trigger confinement of monopoles charged with respect to the Cartan generators of the SU(r) group. Thus the dual theory is in the mixed Coulomb/Higgs phase. The U($\tilde{N}$) gauge group of the μ-dual theory is the same as Seiberg's dual gauge group. We explicitly show that the μ-dual theory matches the generalized Seiberg dual in the zero vacua.[3] This match reveals the nature of Seiberg's dual quarks. They are just ordinary quarks of the original theory.
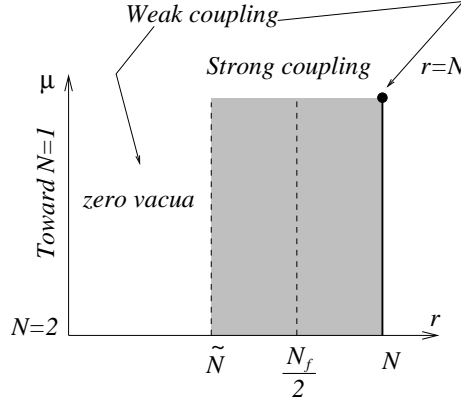

Figure 1: Uplifting $\mathcal{N} = 2$ theory to $\mathcal{N} = 1$. The zero vacua for which weakly coupled μ-dual theories exists can be found in the unshaded domain. The $r = N$ theory in the upper-right corner is exceptional. For $r = N$ weakly coupled dual theory exists, while all other theories in the shaded domain have strongly-coupled duals.

What happens to the Λ vacua, which exist both in the interval $N > r \geq \tilde{N}$ (populated exclusively by such vacua) and in the interval (1.4)? These vacua do not have IR weak coupling descriptions at large μ. Unfortunately, this was overlooked in [7, 8], where we claimed a discrepancy between the so-called r-dual theory and the generalized Seiberg dual at large μ.[4] Here we correct this claim. The only exception is the $r = N$ case, where a weakly coupled dual with the Seiberg U($N_f - N$) group does exist [14].

To conclude the introductory section we note that previously we discussed [15, 14] the $r > N_f/2$ vacua in the $\mathcal{N} = 2$ *limit* in some detail. In the $\mathcal{N} = 2$ limit (small μ) the strong-coupling domain of the original theory in the r vacua (with $r > N_f/2$) can be described in terms of a weakly coupled r-dual theory. The gauge group in this theory is U(ν)$\times$U(1)$^{N-\nu}$, $\nu = N_f - r$. Moreover, the r-dual theory has N_f flavors of quark-like dyons. Condensation of these dyons leads to the confinement of monopoles. Quarks and gauge bosons of the original theory are in the "instead-of-confinement" phase [7, 8, 15, 6]. However, as was already mentioned above, this weak-coupling r-dual description present at small μ becomes strongly coupled once we increase μ, see Fig. 1.

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[3]The generalization of the Seiberg duality for all r vacua in μ deformed U(N) SQCD (with finite μ) was worked out in [11], see also [12].

[4]A loophole was the assumption of weak coupling in the regime, which is a continuation of the Argyres-Douglas (AD) points [13] to large μ, while in fact the regime considered was at strong coupling.

The paper is organized as follows. In Sec. 2 we review μ-deformed $\mathcal{N} = 2$ supersymmetric QCD and its vacuum structure in the limit of small μ. In Sec. 3 we identify zero and Λ-vacua using Cachazo-Seiberg-Witten exact solution for chiral rings [9]. In Sec. 4 we describe μ-duality which relates the U(r)×U(1)$^{N-r}$ hybrid quark-monopole low energy theory present in zero vacua at small μ to μ-dual quark theory with U($\tilde{N}$) gauge group emerging at large μ. In Sec. 5 we discuss the generalization of Seiberg's duality to r vacua of the theory at large but finite μ and show the match of μ-dual theory with Seiberg's dual. Finally, in Sec. 6 we briefly describe r duality in $r > N_f/2$ vacua and summarize various phases of $\mathcal{N} = 1$ QCD present in the r vacua at strong coupling. In Sec. 8 we present our conclusions.

2 Preliminaries

2.1 $\mathcal{N} = 2$ SQCD (small μ)

Our basic "microscopic" (or UV) theory is described in detail in our previous publications (e.g. [17, 18] and review papers (e.g. [19]), where the reader can find all relevant notation. The gauge symmetry is U(N)=SU(N)×U(1), with the $\mu\mathrm{Tr}\,\mathcal{A}^2$ deformation term. We have N_f quark hypermultiplets generally speaking endowed with the mass terms m_A. The number of flavors is subject to the constraint (1.2) ensuring asymptotic freedom of the microscopic theory as well as IR freedom of the dual theory.

The superpotential of the undeformed $\mathcal{N} = 2$ theory has the form

$$\mathcal{W}_{\mathcal{N}=2} = \sqrt{2} \sum_{A=1}^{N_f} \left(\frac{1}{2} \tilde{q}_A \mathcal{A} q^A + \tilde{q}_A \mathcal{A}^a \, T^a q^A + m_A \, \tilde{q}_A q^A \right) , \qquad (2.1)$$

where $\mathcal{A}$ and $\mathcal{A}^a$ are chiral superfields, the $\mathcal{N} = 2$ superpartners of the U(1) and SU(N) gauge bosons. The deformation term

$$\mathcal{W}_{\mathrm{def}} = \mu \, \mathrm{Tr}\, \Phi^2, \qquad \Phi \equiv \frac{1}{2} \mathcal{A} + T^a \, \mathcal{A}^a \qquad (2.2)$$

does not break $\mathcal{N} = 2$ supersymmetry in the small-μ limit, see [20, 21, 17] (while at large μ this theory obviously flows to $\mathcal{N} = 1$). For small μ, i.e. $\mu \ll \Lambda_{\mathcal{N}=2}$, and if all quark masses are equal this term reduces to the Fayet-Iliopoulos F term which can be rotated [20, 21, 17, 18] into the D term [22].

2.2 Vacua

We define the r vacuum as a vacuum with r flavors of (s)quarks condensed It is assumed that the r counting is performed at large quark masses. As we will see in Sec. 3, effectively the value of r depends on the quark masses [8]. It is obvious that the maximal value of r is N. If $r = N$ the gauge group is fully Higgsed [19, 15].

For generic m_A the number of the isolated r vacua with $r < N$ is [11]

$$\mathcal{N}_{r<N} = \sum_{r=0}^{N-1} (N - r) \, C_{N_f}^r = \sum_{r=0}^{N-1} (N - r) \, \frac{N_f!}{r!(N_f - r)!} \,. \tag{2.3}$$

Consider a particular vacuum in which the first r quarks develop nonvanishing vacuum expectation values (VEVs). Quasiclassically, at large masses, the adjoint scalar VEVs are

$$\langle \Phi \rangle \approx -\frac{1}{\sqrt{2}} \, \mathrm{diag} \, [m_1, ..., m_r, 0, ..., 0] \,, \tag{2.4}$$

The last $(N-r)$ entries vanish at the classical level. In quantum theory these entries acquire values of the order of $\Lambda_{\mathcal{N}=2}$, generally speaking. In the classically unbroken $\mathrm{U}(N - r)$ pure gauge sector the gauge symmetry gets broken through the Seiberg–Witten mechanism [3]: first down to $\mathrm{U}(1)^{N-r}$ and then almost completely by condensation of $(N-r-1)$ monopoles. A single $\mathrm{U}(1)$ gauge factor survives, though, because monopoles are charged only with respect to the Cartan generators of the $\mathrm{SU}(N - r)$ group.

The presence of this unbroken $\mathrm{U}(1)$ factor in all $r < N$ vacua makes them different from the $r = N$ vacuum: in the latter there are no long-range forces.

In this paper we focus on the r vacua with $r < N_f/2$. Then the low-energy theory in the given r vacuum (following from the microscopic theory under consideration) has the

$$\mathrm{U}(r) \times \mathrm{U}(1)^{N-r} \,, \tag{2.5}$$

gauge group, assuming that the quark masses are almost equal. Moreover, N_f quarks are charged under the $\mathrm{U}(r)$ factor, while $(N-r-1)$ monopoles are charged under the $\mathrm{U}(1)$ factors. Note that the quarks and monopoles are charged with respect to orthogonal subgroups of $\mathrm{U}(N)$ and therefore are mutually local (i.e. can be described by a local Lagrangian). The low-energy theory is infrared-free and it is at weak coupling as long as VEVs of quarks and monopoles are small.

2.3 Large values of m_A

The quark VEVs in the large-mass limit can be read off from the superpotentials (2.1) and (2.2) using (2.4). They are given by

$$\langle q^{kA} \rangle = \langle \bar{\tilde{q}}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \sqrt{\xi_r} & 0 & \cdots & 0 \end{pmatrix},$$

$$k = 1, ..., r, \qquad A = 1, ..., N_f \,, \tag{2.6}$$

where the r parameters ξ are given quasiclassically by

$$\xi_P \approx 2 \, \mu m_P, \qquad P = 1, ..., r \,. \tag{2.7}$$

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These parameters can be made small in the large m_A limit if μ is sufficiently small.

In quantum theory all parameters ξ_P are determined by the roots of the Seiberg-Witten curve [18, 6, 7, 16] which in the case at hand takes the form [5]

$$y^2 = \prod_{P=1}^{N}(x - \phi_P)^2 - 4\left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{2N-N_f} \prod_{A=1}^{N_f}\left(x + \frac{m_A}{\sqrt{2}}\right). \tag{2.8}$$

Here ϕ_P are gauge invariant parameters on the Coulomb branch. Instead of (2.4) one can write

$$\Phi \approx \mathrm{diag}\left[\phi_1, ..., \phi_N\right], \tag{2.9}$$

where

$$\phi_P \approx -\frac{m_P}{\sqrt{2}}, \quad P = 1, ..., r; \qquad \phi_P \sim \Lambda_{\mathcal{N}=2}, \quad P = r+1, ..., N. \tag{2.10}$$

To identify the r vacuum in terms of the curve (2.8) it is necessary to find such values of ϕ_P which ensure the Seiberg-Witten curve to have $N-1$ double roots, while r parameters ϕ_P are *approximately* determined by the quark masses, see (2.10). Note that $(N-1)$ double roots are associated with r condensed quarks and $(N-r-1)$ condensed monopoles, altogether $N-1$ condensed states.

From this we deduce that the Seiberg–Witten curve factorizes [23],

$$y^2 = \prod_{P=1}^{r}(x - e_P)^2 \prod_{K=r+1}^{N-1}(x - e_K)^2 (x - e_N^+)(x - e_N^-). \tag{2.11}$$

The first r double roots are associated with the mass parameters in the large mass limit, $\sqrt{2}e_P \approx -m_P$, $P = 1, ..., r$. The subsequent $(N-r-1)$ double roots are associated with light monopoles are much smaller, and determined by $\Lambda_{\mathcal{N}=2}$. The last two roots are also much smaller. For the single-trace deformation superpotential (2.2) their sum vanishes [23],

$$e_N^+ + e_N^- = 0. \tag{2.12}$$

The root e_N^+ determines the value of the gaugino condensate [9],

$$e_N^2 = \frac{2S}{\mu}, \qquad S = \frac{1}{32\pi^2}\langle \mathrm{Tr}\, W_\alpha W^\alpha \rangle, \tag{2.13}$$

where the superfield W_α includes the gauge field strength tensor.

In terms of roots of the Seiberg-Witten curve the quark VEVs are given by the formula [7, 16]

$$\xi_P = -2\sqrt{2}\,\mu\,\sqrt{(e_P - e_N^+)(e_P - e_N^-)} \tag{2.14}$$

for $P = 1, ..., r$. At small ξ_P this theory is at weak coupling (IR free below $\Lambda_{\mathcal{N}=2}$) and supports non-Abelian magnetic strings [24, 25, 17, 26]. At $\mu \ll \Lambda_{\mathcal{N}=2}$ these strings are BPS-saturated and their tensions are determined by the ξ parameters, namely [19, 18]

$$T_P = 2\pi|\xi_P|. \tag{2.15}$$

Magnetic strings formed as a consequence of the quark condensation implement confinement of monopoles. The monopoles of the $SU(r)$ sector manifest themselves as two-string junctions [17, 26, 27].

Recently we demonstrated [16] that the monopole VEVs in either the monopole ($r = 0$) or the hybrid r vacua are determined by the same formula (2.14) with the substitutions of the quark double roots by the monopole double roots, so that the subscript P in (2.14) can run over the monopole double roots too,

$$\langle M_{P(P+1)} \rangle = \langle \tilde{\bar{M}}_{P(P+1)} \rangle = \sqrt{\frac{\xi_P}{2}}, \tag{2.16}$$

where ξ_P are determined by Eq. (2.14) and $P = (r+1), ..., (N-1)$. Here $M_{PP'}$ denotes the monopole with the charge given by the root $\alpha_{PP'} = w_P - w_{P'}$ of the $SU(N)$ algebra with weights w_P ($P < P'$).

Equation (2.14) is thus very general and determines VEVs of condensed states independently of their nature [16]. The monopole VEVs determine the tensions of the Abelian electric strings,

$$T_P = 2\pi |\xi_P|, \qquad P = (r+1), ..., (N-1). \tag{2.17}$$

In much in the same way as the magnetic non-Abelian strings in the r vacua, the electric strings are BPS-saturated to the leading order in μ [20, 21]. The electric strings confine quarks, while the magnetic strings confine monopoles.

2.4   Small quark mass limit

Now we turn to the opposite limit of small m_A which will be relevant to our discussion below. As we reduce the quark masses quantum numbers of the light states change due to monodromies [3, 4, 28]. In particular, quarks pick up root-like color-magnetic charges, in addition to their weight-like color-electric charges. If $r < N_f/2$ there is no crossover, the low-energy theory essentially remains the same as at large m_A, namely, infrared-free $U(r) \times U(1)^{N-r}$ gauge theory with N_f quarks (or, more exactly, what becomes of quarks) and $(N - r - 1)$ singlet monopoles [29]. It is at weak coupling provided the ξ_P parameters are small.

The quarks from the $U(r)$ sector and the monopoles form the orthogonal $U(1)^{N-r}$ sector still develop VEVs determined by Eq. (2.14). Physics of screening and confinement also remains intact at small m_A. Say, if a given monopole state (charged with respect to the $SU(r)$ Cartan generators) is confined through quark condensation at large m_A the the same applies to this state under the evolution into the domain of small m_A, although the quark color charges change [29]. If the quarks from the $U(r)$ sector are screened in the r vacuum at large m_A they (or what becomes of them) will still be screened in the same vacuum at small m_A. Monodromies just relabel the states, they do not change physics.

3 Λ vacua versus zero vacua

3.1 Consequences from the exact formulas

We will rely on exact results for the chiral condensates obtained by Cachazo, Seiberg and Witten [9] in μ-deformed $\mathcal{N} = 2$ QCD with the U(N) gauge group. In this section there is no need to assume μ small.

All chiral condensates are encoded in the following functions [9]:

$$
T(x) = \left\langle \mathrm{Tr} \, \frac{1}{x - \Phi} \right\rangle,
$$

$$
R(x) = \frac{1}{32\pi^2} \left\langle \mathrm{Tr} \, \frac{W_\alpha W^\alpha}{x - \Phi} \right\rangle,
$$

$$
M(x)_A^B = \left\langle \tilde{q}_A \, \frac{1}{x - \Phi} \, q^B \right\rangle. \tag{3.1}
$$

For the quadratic single-trace deformation (2.2) (the so-called "one-cut" model) the function $R(x)$ has the form

$$
R(x) = \frac{1}{2} \left(W'_{\mathrm{def}}(x) - \sqrt{W'_{\mathrm{def}}(x) + f(x)} \right) = \mu \left(x - \sqrt{x^2 - e_N^2} \right), \tag{3.2}
$$

where the unpaired root of the Seiberg–Witten curve $e_N = e_N^+$ (see (2.11)) is related to the gaugino condensate, see (2.13).

From the solution for the function $M_A^B(x)$ in [9] one can obtain the values of the quark VEVs in terms of the gaugino condensate S. In the r vacuum, when the function $M_A^B(x)$ has r poles on the first sheet,

$$
M_A = \frac{\mu}{2} \left(m_A + \sqrt{m_A^2 - \frac{4S}{\mu}} \right), \qquad A = 1, ..., r \,, ;
$$

$$
M_A = \frac{\mu}{2} \left(m_A - \sqrt{m_A^2 - \frac{4S}{\mu}} \right), \qquad A = (r+1), ..., N_f, \tag{3.3}
$$

where

$$
M_A^B = \left\langle \tilde{q}_A q^B \right\rangle, \tag{3.4}
$$

and we assume that the solution can be brought to the diagonal form

$$
M_A^B = \delta_A^B \, M_A. \tag{3.5}
$$

In the large quark mass limit, when $\frac{S}{\mu} \ll m_A$, we have r "large" values of M_A,

$$
M_A \approx \mu \, m_A \quad \text{for} \ A = 1, ..., r \,,
$$

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and $N_f - r$ "small" values. This pattern matches our definition of the r vacuum.

Note, that the quantum quark VEVs (3.4) in the microscopic U(N) theory are close to those obtained from the low-energy theory (see (2.6)) only in the limit of large m_A, although both are given by exact formulas. At $m_A \sim \Lambda_{\mathcal{N}=2}$ the difference is not small. In particular, the low-energy quark condensates (2.6) vanish at the Argyres-Douglas points [13] (where a double root e_P coincides with one of the unpaired roots $e_N^{\pm}$), see (2.14), while the values M_A remain finite. This was first noted in [30].

Now, to find the gaugino condensate S we use the glueball superpotential calculated in [9] from a matrix model [10]. For the quadratic deformation (2.2) it was studied in [31], see also [8]. Minimization of this superpotential gives the following equation for S:

$$S^N = \mu^N \Lambda_{\mathcal{N}=2}^{N-\tilde{N}} \left(\frac{m}{2} - \frac{1}{2}\sqrt{m^2 - \frac{4S}{\mu}} \right)^r \left(\frac{m}{2} + \frac{1}{2}\sqrt{m^2 - \frac{4S}{\mu}} \right)^{N_f - r}, \tag{3.6}$$

where for simplicity we assume quark mass equality. Using (3.3) we can rewrite the equation above as an equation for the quark condensate M_A [8],

$$\frac{1}{\mu} M_A = m - \frac{1}{\mu^{\frac{N}{N}} \Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{N}}} \frac{(\det M)^{\frac{1}{N}}}{M_A}, \tag{3.7}$$

where $\tilde{N}$ is defined in (1.1). Equation (3.7) obviously can be obtained from the following superpotential:

$$\mathcal{W}_{\mathrm{ADS}} = -\frac{1}{2\mu} \operatorname{Tr} M^2 + m_A \operatorname{Tr} M + (N - N_f) \frac{(\det M)^{\frac{1}{N_f - N}}}{\Lambda^{\frac{3N - N_f}{N_f - N}}}. \tag{3.8}$$

The first two terms in (3.8) can be obtained by integrating out the adjoint field $\mathcal{A}$ in the tree-level superpotential of the theory (2.1) and (2.2) in the large μ limit. The last term – obviously of the quantum nature – is nothing other than the continuation of the Afleck-Dine-Seiberg (ADS) superpotential [32] to $N_f > N$. This superpotential can be also derived from Seiberg's dual theory generalized to r vacua, see [8] and Sec. 5 below.

Thus, the Cachazo–Seiberg–Witten exact solution [9] produces the same equations for M's as the continuation of the ADS superpotential to $N_f > N$ in Eq. (3.8). The fact of coincidence was previously established in the SU(N) case in [33].

The superpotential (3.8) is exact and we can use it in any domain of the parameter space. In particular, for large masses ($m_A \gg \Lambda_{\mathcal{N}=2}$) the solution of Eq. (3.7) in the r vacuum is

$$M_A \approx \mu\, m, \qquad A = 1, ..., r\,;$$

$$M_A \approx \mu \Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{N-r}} m^{\frac{\tilde{N}-r}{N-r}} e^{\frac{2\pi k}{N-r} i}, \qquad A = (r+1), ..., N_f,$$

$$k = 1, ..., (N - r)\,. \tag{3.9}$$

As was anticipated, we have r large classical VEVs and $(N_f - r)$ small "quantum" VEVs.

The linear dependence of M on μ is exact and is fixed by the U(1) symmetries [30] after all condensates are expressed in terms of $\Lambda_{\mathcal{N}=2}$. The presence of $(N - r)$ distinct solutions ensures the total number of the $r < N$ vacua to coincide with (2.3) obtained at small μ.

3.2 Chiral condensates at small quark masses

Let us study the behavior of gaugino and quark condensates at small m_A. Most of the solutions of equations (3.6), (3.3) behave as $S \sim \mu \Lambda_{\mathcal{N}=2}^2$ and $M_A \sim \mu \Lambda_{\mathcal{N}=2}$. The r vacua with this behavior are referred to as the Λ vacua. However, there is a special set of vacua in which the gaugino and quark condensates tend to zero in the small quark mass limit. Namely, Eq. (3.7) has solutions [12, 8]

$$M_A \approx \mu\, m, \qquad A = 1, ..., p\,;$$

$$M_A \approx \mu\, \frac{m^{\frac{p-\tilde{N}}{p-N}}}{\Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{p-N}}}\, e^{\frac{2\pi k}{p-N}\, i}\,, \qquad A = (p+1), ..., N_f,$$

$$k = 1, ..., (p - N)\,, \tag{3.10}$$

where p is an integer. In other words, p eigenvalues of M are proportional to $\mu\, m$, while other eigenvalues are much smaller at $m_A \approx m \ll \Lambda_{\mathcal{N}=2}$. These solutions exist if $p > N$. We refer to the vacua with this behavior as the zero vacua.

At large m_A we start from an r vacuum, with r quarks (classically) condensed, hence $r \le N$. On the other hand, the integer p is defined as the number of "plus" signs in Eq. (3.3) for M_A, or the number of poles of $M_A^B(x)$ on the first sheet [9]. Then $(N_f - p)$ is the number of "minus" signs. In fact, p depends on the value of m_A. At large m_A we have

$$p(\infty) = r\,. \tag{3.11}$$

As we reduce m_A certain poles can and do pass through the cut from the first sheet to the second or vice versa [9]. When it happens $p(m_A)$ reduces by one unit or increases by one unit.

In Eq. (3.10) p is $p(m_A)$ in the small mass limit, i.e.

$$p \equiv p(0)\,. \tag{3.12}$$

Clearly, p can differ from r. The condition $r \le N$ applies only for $r = p(\infty)$ rather than for $p = p(0)$, instead $p > N$. In fact, $(p - r)$ is the net number of poles which pass through the cut from the second sheet to the first one as we reduce the quark masses from infinity to zero.

The relation between r and p was found in [8]. We look for a solution of (3.7) which has the pattern (3.9) at large m and (3.10) at small m. Translating this into the behavior of S given by Eq. (3.6) we arrive at [8]

$$p = N_f - r\,. \tag{3.13}$$

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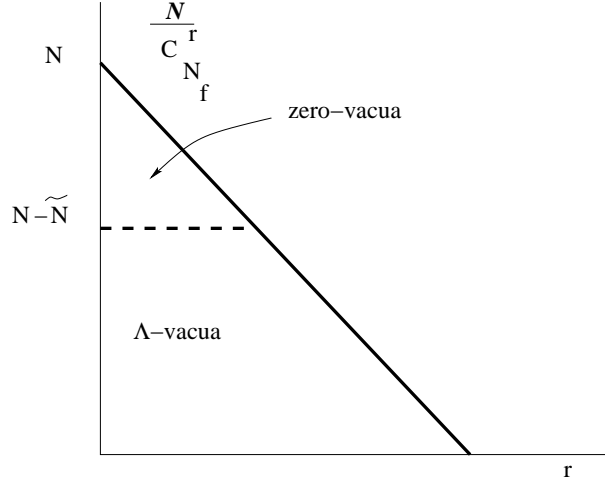
Figure 2: Multiplicity of zero-vacua and total multiplicity of $r < N$ vacua as a function of r.

Then constraint $p > N$ implies in turn that

$$r < \tilde{N} \,. \tag{3.14}$$

This is the domain of existence of the zero vacua.

Equation (3.10) for M_A in the zero vacua ensures the smallness of the gaugino condensate at small m_A,

$$S \approx \mu \, \frac{m^{\frac{N_f - 2r}{\tilde{N} - r}}}{\Lambda_{\mathcal{N}=2}^{\frac{N - \tilde{N}}{\tilde{N} - r}}} \, e^{\frac{2\pi k}{\tilde{N} - r} i} \,, \qquad k = 1, ..., (\tilde{N} - r) \,, \tag{3.15}$$

where we express p in terms of r using (3.13). The multiplicity of these solutions is $\tilde{N} - r$. In other words, for a given r the total number of the zero vacua is

$$\mathcal{N}_{0-\text{vac}} = \sum_{r=0}^{\tilde{N}-1} (\tilde{N} - r) \, C_{N_f}^r = \sum_{r=0}^{\tilde{N}-1} (\tilde{N} - r) \, \frac{N_f!}{r!(N_f - r)!} \,. \tag{3.16}$$

This number is smaller then the total number of the $r < N$ vacua (2.3). The multiplicity of the zero vacua as a function of r is depicted in Fig. 2.

We will show below that choosing any of the zero vacua we can pass from the weak coupling low-energy description of Sec. 2 at small μ (i.e. $\mathcal{N} = 2$) to a μ-dual $\mathcal{N} = 1$ theory which appears to be weakly coupled in the IR. At the same time, the zero vacua were shown [12, 8] to be precisely the vacua which are classically seen in the generalized Seiberg dual theory [1, 2]. Section 5 elucidates that these are two sides of the same coin.

4 Towards $\mathcal{N} = 1$ by increasing μ: μ Duality

4.1 Preliminaries

If the quark mass differences are small $(m_A - m_B) \ll m_{A,B} \sim m \ll \Lambda_{\mathcal{N}=2}$ then r parameters ϕ_P and the quark double roots e_P (in $r < N_f/2$ vacua) are exactly (rather than quasiclassically) determined by the quark masses [5, 15, 7],

$$\sqrt{2}\,\phi_P = -m_P, \qquad \sqrt{2}\,e_P = -m_P, \qquad P = 1, ..., r \tag{4.1}$$

(up to a small corrections of order of $(m_A - m_B)^2/\Lambda_{\mathcal{N}=2}$). The Seiberg-Witten curve factorizes as follows

$$y^2 = \left(x + \frac{m}{\sqrt{2}}\right)^{2r} \left\{ \prod_{P=r+1}^{N} (x - \phi_P)^2 - 4\left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{2N - N_f} \left(x + \frac{m}{\sqrt{2}}\right)^{N_f - 2r} \right\}. \tag{4.2}$$

This leads to the occurrence of the non-Abelian SU(r) gauge group in the low-energy theory in the limit of (almost) equal quark masses [5]. As a result, at small μ physics is described by weakly coupled IR free low-energy theory discussed in Sec. 2. It has the U$(r) \times$ U$(1)^{N-r}$ gauge group with r light quarks and $(N - r - 1)$ Abelian monopoles.

Our task is to increase μ and find a weakly coupled low-energy description of the theory at hand at large μ (i.e. $\mathcal{N} = 1$). However, this program runs onto an obstacle. At large μ the ξ parameters (2.14) generically become large forcing the infrared-free low-energy theory hit the strong coupling domain.

Previously we believed [7] that the problem could be overcome by approaching the Argyres-Douglas points [13] where r double roots come close to one of the unpaired roots $e_N^{\pm}$ and r parameters ξ remain small. It was overlooked, however, that the low-energy theory in this limit enters the AD strongly coupled regime, while our task was to find a weakly coupled dual.[5]

One exception where this problem does not appear is the $r = N$ vacuum. In the $r = N$ vacuum the gaugino condensate vanishes, and $\tilde{N} = N_f - N$ parameters ξ are determined by the quarks masses [6],

$$\xi_P = -2\sqrt{2}\,\mu m_P, \quad P = 1, ..., \tilde{N}.$$

This allows us to keep the ξ_P parameters small at large μ by making the quark masses sufficiently small, guaranteeing a weak coupling regime in the dual theory which in this case has the U$(\tilde{N})$ gauge group [6], in perfect agreement with Seiberg's duality.

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[5]In [7, 8] it was argued that the low-energy theory stays at weak coupling near the AD points at $\mu \not\to 0$ because the monopoles which become light are, in fact, confined and, therefore, do not contribute to the β function. The loophole in this argument is that at energies above the scale $\sqrt{\xi}$ the effect of confinement is negligible, and the light monopoles do cancel the logarithmic running of the coupling constant produced by the light quarks. We will discuss this issue in more detail elsewhere.

Now we want to demonstrate that the zero vacua provide us with additional exceptions. The gaugino condensate is very small in the limit of small masses, see (3.15). Therefore we do not need to approach the AD points to keep r parameters ξ small. What we need is to make the quark masses small as we increase μ.

4.2 The zero vacua from the Seiberg-Witten curve

We begin with identifying the zero vacua in terms of the Seiberg-Witten curve (4.2). The gaugino condensate is related to values of the unpaired roots of the curve (see (2.13)),

$$e_N^2 \approx 2\mu \frac{m^{\frac{N_f-2r}{N-r}}}{\Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{N-r}}} \, e^{\frac{2\pi k}{N-r} i}, \qquad k = 1, ..., (\tilde{N} - r), \tag{4.3}$$

in the small mass limit. All other roots of the curve (4.2) are doubled. In the zero vacua r parameters ϕ_P and the double roots e_P are given by the quark masses, $P = 1, ..., r$ (see (4.1)), while $(N - \tilde{N})$ parameters ϕ_P and the double roots e_P are of order of $\Lambda_{\mathcal{N}=2}$. The remaining $(\tilde{N} - r)$ parameters ϕ and $(\tilde{N} - r - 1)$ double roots are very small, of the order of $e_N^\pm$, see (4.3).

To find ϕ's and those double roots which are of the order of $\Lambda_{\mathcal{N}=2}$ (and for this purpose only) we consider $x \sim \Lambda_{\mathcal{N}=2}$ in (4.2) and neglect all parameters which are of the order of m or smaller (remember that $m \ll \Lambda_{\mathcal{N}=2}$). Then, Eq. (4.2) implies

$$y^2 = x^{2\tilde{N}} \left\{ \prod_{P=\tilde{N}+1}^{N} (x - \phi_P)^2 - 4 \left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}} \right)^{N-\tilde{N}} x^{N-\tilde{N}} \right\}. \tag{4.4}$$

We look for a solution with all $(N - \tilde{N})$ ϕ's being of the order of $\Lambda_{\mathcal{N}=2}$. Of course, there are solutions with smaller ϕ's given by the quark masses, but these solutions correspond to r'-vacua with larger r', i.e. $r' > \tilde{N}$.

The solution takes the form

$$\sqrt{2}\,\phi_P = -\Lambda_{\mathcal{N}=2}\, e^{\frac{2\pi i}{N-\tilde{N}}(P-\tilde{N}-1)}, \quad P = (\tilde{N}+1), ..., N, \quad \text{odd } (N - \tilde{N}), \tag{4.5}$$

and

$$\sqrt{2}\,\phi_P = -\Lambda_{\mathcal{N}=2}\, e^{\frac{2\pi i}{N-\tilde{N}}(P-\tilde{N}-\frac{1}{2})}, \quad P = (\tilde{N}+1), ..., N, \quad \text{even } (N - \tilde{N}). \tag{4.6}$$

The corresponding double roots are [6]

$$\sqrt{2}\,e_P = \Lambda_{\mathcal{N}=2}\, e^{\frac{2\pi i}{N-\tilde{N}}(P-\tilde{N})}, \qquad P = \tilde{N}, ..., (N-1). \tag{4.7}$$

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[6] Note a shift in the numbering of the "large" ϕ's and the double roots: the double root e_P corresponds to ϕ_{P+1}. This is because we use the notation $e_N^\pm$ for unpaired roots, which are small.

To find the remaining $(\tilde{N} - r)$ ϕ's and the roots that are much smaller than m we assume in (4.2) $x \ll m$. The Seiberg-Witten curve then takes the form

$$y^2 = \left(\frac{m}{\sqrt{2}}\right)^{2r} \left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{2(N-\tilde{N})} \left\{ \prod_{P=r+1}^{\tilde{N}} (x - \phi_P)^2 - 4\frac{\left(\frac{m}{\sqrt{2}}\right)^{N_f - 2r}}{\left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{N-\tilde{N}}} \right\}, \qquad (4.8)$$

where we use the fact that $(N - \tilde{N})$ ϕ's are given by (4.5) or (4.6). The curve in the curly brackets is the curve for pure Yang-Mills theory with the U$(\tilde{N} - r)$ gauge group. It has a very small scale Λ_0 defined as

$$\Lambda_0^{2(\tilde{N}-r)} = \frac{m^{N_f - 2r}}{\Lambda_{\mathcal{N}=2}^{N-\tilde{N}}}, \qquad \Lambda_0 \ll m. \qquad (4.9)$$

The relevant parameters ϕ as well as the roots in pure Yang-Mills theory were obtained in [34],

$$\phi_P = 2 \cos \frac{\pi(P - r - \frac{1}{2})}{\tilde{N} - r} \frac{\Lambda_0}{\sqrt{2}}, \qquad P = (r+1), ..., \tilde{N}, \qquad (4.10)$$

and

$$e_P = 2 \cos \frac{\pi(P - r)}{\tilde{N} - r} \frac{\Lambda_0}{\sqrt{2}}, \qquad P = (r+1), ..., (\tilde{N} - 1). \qquad (4.11)$$

The unpaired roots are

$$e_N^{\pm} = \pm 2 \frac{\Lambda_0}{\sqrt{2}}. \qquad (4.12)$$

Comparing the above expression for the unpaired roots found from the Seiberg-Witten curve with the result (4.3) obtained using the Cachazo-Seiberg-Witten exact solution [9], applied to the zero vacua, we observe the exact match.

Next, can use (4.11) and (4.12) to find "small" VEVs for $(\tilde{N} - r - 1)$ monopoles. They are given by (2.16), where for $P = (r + 1), ..., (\tilde{N} - 1)$,

$$\xi_P = -2\sqrt{2} \mu \sqrt{(e_P - e_N^+)(e_P - e_N^-)} = -4i \mu \Lambda_0 \sin \frac{\pi(P - r)}{\tilde{N} - r}, \qquad (4.13)$$

see (2.14). This is the famous sine formula for the monopole VEVs and the Abelian electric string tensions [34]. We reproduce it via our general expression (2.14) which we can use in particular, for pure Yang-Mills theory, see [16]. Note that r quark VEVs (2.6) are determined by the quark masses,

$$\xi_P = 2\mu m_P, \qquad P = 1, ..., r, \qquad (4.14)$$

since $e_N^{\pm}$ are very small at small m, see (2.14) and (4.1). Other $(N - \tilde{N})$ monopoles have "large" VEVs determined by $\Lambda_{\mathcal{N}=2}$, see (4.7). These monopoles decouple from low-energy physics.

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4.3 μ Dual theory in the zero vacua

The above analysis implies that the low-energy theories in the zero vacua at small μ and m have the $U(r) \times U(1)^{\tilde{N}-r}$ gauge group with r flavors of light quarks charged under the $U(r)$ subgroup and $(\tilde{N} - r - 1)$ light monopoles charged under the $(\tilde{N} - r - 1)$ $U(1)$ factors. One $U(1)$ remains unbroken. The remaining $U(1)^{N-\tilde{N}}$ gauge sector becomes heavy and decouples, along with $(N - \tilde{N})$ heavy monopoles.

The scale Λ_0 of the $U(1)^{\tilde{N}-r}$ sector is very small, as it is clearly seen from (4.9). Therefore, when we increase μ forcing VEVs (4.13) of $(\tilde{N} - r - 1)$ light monopoles to hit the scale Λ_0, the $U(1)^{\tilde{N}-r}$ monopole sector enters the strong coupling regime, and we cannot use this monopole theory to describe low-energy physics.

Nevertheless, at larger μ we can construct a dual low-energy description. Equations (2.4), (4.5), (4.6) and (4.10) show that the adjoint field in the zero vacuum has the form

$$\langle \Phi \rangle \approx -\frac{1}{\sqrt{2}} \operatorname{diag} \left[m_1, ..., m_r, 0, ..., 0, c_1 \Lambda_{\mathcal{N}=2}, ..., c_{N-\tilde{N}} \Lambda_{\mathcal{N}=2} \right], \tag{4.15}$$

where we have $(\tilde{N} - r)$ almost vanishing eigenvalues, while $(N - \tilde{N})$ "large" entries (i.e. of the order of $\Lambda_{\mathcal{N}=2}$) are associated with the decoupled $U(1)^{N-\tilde{N}}$ heavy sector. The form of the adjoint field in (4.15) signals the restoration of the $U(\tilde{N})$ gauge group (if $m \ll \Lambda_{\mathcal{N}=2}$).

The Seiberg-Witten curve takes the form

$$y^2 = \left(x + \frac{m}{\sqrt{2}} \right)^{2r} \left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}} \right)^{2(N-\tilde{N})} \left\{ \prod_{P=r+1}^{\tilde{N}} (x - \phi_P)^2 - 4 \frac{\left(x + \frac{m}{\sqrt{2}} \right)^{N_f - 2r}}{\left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}} \right)^{N-\tilde{N}}} \right\}. \tag{4.16}$$

We focus on the low-energy region, $x \ll \Lambda_{\mathcal{N}=2}$. This is the curve of the IR free $U(\tilde{N})$ gauge theory with N_f flavors. Thus, the μ-dual low-energy theory has the $U(\tilde{N})$ gauge group and N_f quark flavors. The superpotential of the theory is

$$\mathcal{W}_{\mu-\text{dual}} = \sqrt{2} \sum_{A=1}^{N_f} \left(\frac{1}{2} \tilde{q}_A \mathcal{A} q^A + \tilde{q}_A \mathcal{A}^n T^n q^A + m_A \tilde{q}_A q^A \right)$$

$$+ \mu u_2, \tag{4.17}$$

where

$$u_2 = \operatorname{Tr} \left(\frac{1}{2} \mathcal{A} + T^n \mathcal{A}^n \right)^2, \tag{4.18}$$

while the fundamental and adjoint color indices are now truncated to $l = 1, ..., \tilde{N}$ and $n = 1, ..., (\tilde{N}^2 - 1)$.

The VEVs of the adjoint field are

$$\left\langle \frac{1}{2} \mathcal{A} + T^n \mathcal{A}^n \right\rangle \approx -\frac{1}{\sqrt{2}} \operatorname{diag} \left[m_1, ..., m_r, 0, ..., 0 \right], \tag{4.19}$$

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where $(\tilde{N} - r)$ eigenvalues are quasiclassically zero. The matrix of the quark VEVs has the form

$$\langle q^{lA} \rangle = \langle \bar{\tilde{q}}^{lA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \sqrt{\xi_r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$l = 1, ..., \tilde{N}, \qquad A = 1, ..., N_f, \tag{4.20}$$

where the first r parameters ξ are given by (4.14), while all other $(\tilde{N} - r)$ ξ's are (quasi-classically) zero. Only r quark flavors develop VEVs. The U$(\tilde{N} - r)$ gauge sector remains unbroken. The U$(\tilde{N})$ theory is IR free and is weakly coupled at energies above Λ_0, see below.

Let us stress that the reason why the low-energy superpotential (4.17) is consistent with the adjoint and quark VEVs given above is a peculiar property of the zero vacuum namely, the extreme smallness of $(\tilde{N} - r - 1)$ parameters ξ which are of the order of $\Lambda_0 \ll m$, see (4.13). Generically, if only r quarks of $\tilde{N}$ (the maximal possible value allowed by the rank of the gauge group) condense, the F terms proportional to

$$\mu \frac{\partial u_2}{\partial \Phi} \sim \xi$$

are generated. To cancel these terms, additional $(\tilde{N} - r - 1)$ monopoles develop VEVs.

The reason why this does not happen in the zero vacua at $\mu \gg \Lambda_0$ is the fact that the corresponding parameters ξ are (almost) zero, see (4.13).

At energies above m the U$(\tilde{N})$ μ-dual theory at hand is IR free and weakly coupled. However, at energies below m the gauge group gets broken to U$(r) \times$U$(\tilde{N} - r)$ by adjoint VEVs (4.19). The U(r) sector with N_f light quarks is IR free and weakly coupled. However, the U$(\tilde{N} - r)$ sector becomes a pure Yang-Mills theory since the quarks charged with respect to U$(\tilde{N} - r)$ gauge group acquire masses of the order of m and decouple.[7] Thus, the U$(\tilde{N} - r)$ sector is asymptotically free and runs into strong coupling in the infrared. This happens at the scale of U$(\tilde{N} - r)$ Yang-Mills theory, which coincides with Λ_0, see (4.9).

Thus, we must admit that the μ-dual theory at hand is not exactly a weakly coupled low-energy description all the way down. It is weakly coupled only at energies above the very small scale provided by $\Lambda_0 \ll m$.

At energies below Λ_0 we have a weak coupling description in terms of the U$(r) \times$U$(1)^{\tilde{N}-r}$ gauge theory with light quarks and monopoles. As we increase μ and go to higher energy scales our system undergoes a crossover transition, and the quark-monopole description breaks down.

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[7]This is because of extreme smallness of the corresponding parameters ϕ, see (4.10).

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At energies well above Λ_0 we use weakly coupled μ-dual description in terms of the U($\tilde{N}$) gauge theory for N_f light quarks. This is quite natural because the monopoles are Abelian objects and hardly can play a role at large μ where adjoint fields decouple and no Abelianization of the theory is expected.

4.4 Superpotential

Masses of quarks and gauge bosons in the μ-dual theory are determined by the scales m and $\sqrt{\xi}$ in (4.14). Therefore, once we increase μ above m the adjoint fields decouple from low-energy physics making the theory at hand $\mathcal{N} = 1$. It is easy to integrate out adjoint fields in the superpotential (4.17). We expand u_2 in a and a^n and keep only quadratic terms (higher order terms are suppressed by powers of $m/\Lambda_{\mathcal{N}=2}$). The coefficients of this expansion are determined by using the adjoint and quark VEVs (4.19) and (4.20). In this way we get

$$\mathcal{W}_{\mu-\mathrm{dual}} = -\frac{1}{2\mu}\,(\tilde{q}_A q^B)(\tilde{q}_B q^A) + m_A \tilde{q}_A q^A \,, \qquad (4.21)$$

for further details see [6] where a similar calculation is carried out in the $r = N$ vacuum.

To summarize, at large μ the original U(N) gauge theory flows to $\mathcal{N} = 1$ SQCD. At $\mu \gg m$ low-energy physics in the zero vacua can be described by $\mathcal{N} = 1$ supersymmetric SQCD with the U($\tilde{N}$) gauge group and N_f quark flavors with the superpotential (4.21). Note, that in contrast to $\mathcal{N} = 2$ SQCD, where in each vacuum we have its own description with a distinct gauge group, in the case at hand we have one and the same superpotential for all zero vacua. The vacua differ by the number r of condensed quarks. In order to keep this IR free theory at weak coupling we assume that the ξ parameters in (4.14) are small compared to the scale of this $\mathcal{N} = 1$ μ-dual theory, determined by

$$\tilde{\Lambda}_{\mathcal{N}=1}^{N_f - 3\tilde{N}} = \frac{\Lambda_{\mathcal{N}=2}^{N_f - 2\tilde{N}}}{\mu^{\tilde{N}}}. \qquad (4.22)$$

Namely, we assume

$$\xi \sim \mu m \ll \tilde{\Lambda}_{\mathcal{N}=1}^2. \qquad (4.23)$$

Condensation of quarks leads to the formation of non-Abelian strings in the U(r) sector. These strings confine monopoles, for a review on non-Abelian strings and monopole confinement see [19]. The U($\tilde{N} - r$) sector remains unbroken. Thus, our theory is in the mixed Higgs/Coulomb phase. Quarks of the U(r) sector are screened, while monopoles are confined.

We stress that quarks are not confined at large μ, contrary to the naive duality arguments.

To conclude, let us address the question: how large should μ be to ensure the decoupling of the adjoint matter? From (4.22) we see that in order to make contact with $\mathcal{N} = 2$ theory we cannot take μ too large; in fact, it cannot exceed $\Lambda_{\mathcal{N}=2}$. This upper bound might seem too restrictive from the point of view of the original microscopic $\mathcal{N} = 1$ U(N) SQCD. Indeed

one might think that in order to decouple the adjoint matter one should take μ much larger than the scale of this theory. However, above we saw that the low-energy states in the μ-dual theory have masses determined by m or $\sqrt{\xi}$ which are way below the above-mentioned scale. Therefore, in order to decouple the adjoint matter from the low-energy sector it is sufficient to keep μ in the window $m \ll \mu \lesssim \Lambda_{\mathcal{N}=2}$.

5   Connection to Seiberg's duality

As was mentioned in Sec. 1, originally Seiberg's duality [1, 2] was formulated for $\mathcal{N} = 1$ SQCD corresponding to the limit $\mu \to \infty$ and referred to $r = 0$. A generalization of Seiberg's duality for r vacua of μ-deformed $\mathcal{N} = 2$ SQCD at large but finite μ was considered in [11, 12]. In our case of the U(N) SQCD the Seiberg's dual has the U($\tilde{N}$) gauge group, N_f flavors of Seiberg's dual quarks[8] and neutral mesonic field M_A^B defined in (3.4). The Seiberg superpotential is

$$\mathcal{W}_S = -\frac{1}{2\mu} \operatorname{Tr}\left(M^2\right) + m_A\, M_A^A + \frac{1}{\kappa}\, \tilde{h}_{Al} h^{lB}\, M_B^A \,, \tag{5.1}$$

where first two terms are obtained by integrating out the adjoint fields at the tree level in (2.1) and (2.2). Here κ is a parameter of dimension of mass needed to formulate Seiberg's duality [1, 2].

From definition (3.4) it is clear that the number of the eigenvalues of the matrix $\tilde{q}q = M$ which scales as μm at large m is r in the r vacuum. What is the vacuum structure [11, 12, 8] of the Seiberg dual theory (5.1) for the $r < N$ vacua?

If we integrate out Seiberg's dual quarks h^{lA} we end up [2, 11, 12, 8] with the Afleck-Dine-Seiberg superpotential (3.8). It correctly reproduces the total number of the $r < N$ vacua (2.3) and gives the correct values of the M condensates since Eq. (3.7) coincides with the one obtained from the Cachazo-Seiberg-Witten exact solution [9], see [8] and Sec. 3. However, the ADS superpotential (3.8) is not a superpotential of a gauge theory (gauge degrees of freedom are already integrated out). In fact this is a superpotential of the Veneziano-Yankielowicz type [35] and, as such, cannot be used to describe the spectrum of low-energy excitations, confinement or screening [8]. It is useful only for the vacuum condensates.

To describe low-energy physics we need a weakly coupled description in terms of a gauge theory. We could try to use the Seiberg dual theory (5.1) *per se*. In [12] it was noted that not all $r < N$ vacua can be seen at the classical level in the superpotential (5.1). Later it was found [8] that only the zero vacua are seen in (5.1) at the classical level, while the Λ vacua remain "missing," or quantum vacua, seen only in the ADS superpotential (3.8). Let us briefly discuss this.

Extremizing superpotential (5.1) we find the classical vacua of the generalized Seiberg

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[8]To be denoted as h^{lA} ($l = 1, ..., \tilde{N}$ and $A = 1, ..., N_f$).

dual theory. Assuming that $\langle M_A^B \rangle = \delta_A^B \, M_A$ we arrive at

$$-\frac{1}{\mu} \, M_A + \kappa \, m_A + \frac{1}{\kappa} \, \tilde{h}_{Al} h^{lA} = 0,$$

$$M_A \, h^{lA} = \tilde{h}_{Al} \, M_A = 0, \tag{5.2}$$

for all values of A. The solution of (5.2) is

$$M_A = \mu \, m_A, \qquad (\tilde{h}h)_A = 0, \qquad A = 1, ..., p \,,$$

$$(\tilde{h}h)_A = -\kappa \, m_A, \qquad M_A = 0, \qquad A = (p+1), ..., N_f \,, \tag{5.3}$$

where p should obey the constraint $p > N$, since the rank of the matrix $(\tilde{h}h)$ cannot exceed $\tilde{N}$.

This solution can describe low-energy physics if the infrared-free Seiberg dual theory is at weak coupling. To ensure that this is the case we assume the small-m limit. In this limit p does not coincide with r, the latter parameter being defined at large masses. In fact $p = N_f - r$, see [8] and (3.13). Now observe that p eigenvalues of M are given by μm, while others are classically zeros. This dependence matches the m dependence of M in the zero vacua at small m, see (3.10). Moreover, the number of classical vacua (5.3) is

$$\mathcal{N}_{0-\text{vac}} = \sum_{r=0}^{\tilde{N}-1} (\tilde{N} - r) \, C_{N_f}^r = \sum_{r=0}^{\tilde{N}-1} (\tilde{N} - r) \, \frac{N_f!}{r!(N_f - r)!} \,. \tag{5.4}$$

This is the number of choices one can pick up $r = N_f - p$ dual quarks h which develop VEVs times the Witten index in the classically unbroken by h condensation gauge group, namely $\text{SU}(\tilde{N} - r)$. This number coincides with the zero vacua number, see (3.16) and Fig. 2.

This leads us to the conclusion that vacua (5.3) classically seen in the Seiberg dual theory are in fact the zero vacua [8]. The Λ vacua are not seen classically. Our interpretation of this phenomenon is as follows (cf. [8]). In the zero vacua $\text{U}(\tilde{N})$ is the true low-energy gauge group and dual quarks h are the correct low-energy degrees of freedom. Since the Seiberg dual theory is infrared-free it is weakly coupled in the small-m limit, provided the classical vacua exist, i.e. in the zero vacua. Instead, in the Λ vacua, the dual quarks h are *not* the low-energy degrees of freedom.

This explains why the Λ vacua are not seen quasiclassically. In fact, Seiberg's dual $\text{U}(\tilde{N})$ theory (5.1) is strongly coupled in the Λ vacua. Nevertheless, integrating out dual quarks leads to the correct ADS superpotential (3.8), which can be used only to determine chiral condensates from the chiral rings, à la Veneziano-Yankielowicz.

In much the same way as in the Seiberg duality, our μ-dual theory in the zero vacua also has the $\text{U}(\tilde{N})$ gauge group (Sec. 4). Both dual theories give weakly coupled low-energy descriptions in the small-m limit. Do these two descriptions match?

The answer is positive. To see that this is the case, let us identify the quarks of the μ-dual theory with the Seiberg dual quarks. The change of variables

$$q^{lA} = \sqrt{-\frac{\mu}{\kappa}} \, h^{lA}, \quad N_A^B \equiv -\frac{1}{\mu} \, M_A^B, \quad l = 1, ..., \tilde{N}, \quad A = 1, ..., N_f \tag{5.5}$$

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brings the superpotential (5.1) to the form

$$\mathcal{W}_S = -\frac{\mu}{2} \operatorname{Tr}(N^2) - \mu m_A N_A^A + \tilde{q}_{Al} q^{lB} N_B^A. \tag{5.6}$$

The kinetic terms are not known in the Seiberg dual theory, and, hence, normalization of the h fields is unknown too, which leaves us the freedom to change the variables as in (5.5). We see that the κ parameter completely disappears from the theory and is replaced by the physical parameter μ. Equation (5.6) shows that the mesonic field N_A^B is heavy at large μ (i.e. $\mu \gg m$) and can be integrated out. The result is

$$\mathcal{W}_S = \frac{1}{2\mu} (\tilde{q}_A q^B)(\tilde{q}_B q^A) - m_A \tilde{q}_A q^A. \tag{5.7}$$

This superpotential coincides with the superpotential (4.21) up to a sign. This shows the equivalence of the Seiberg dual and μ-dual low-energy theories in the zero vacua. The identification (5.5) reveals the physical nature of Seiberg's dual quarks. They are not monopoles as naive duality suggests. Instead, they are quarks of the original theory. Remember, r quarks condense in the r vacuum, see (4.20). This leads to confinement of monopoles charged with respect to the Cartan generators of SU(r). Quarks of U($\tilde{N} - r$) sector do not condense, the dual theory is in the mixed Coulomb/Higgs phase.

6   Phases of $\mathcal{N} = 1$ QCD in the small ξ limit

Before discussing the phases of $\mathcal{N} = 1$ SQCD we briefly review the r vacua with $r > N_f/2$ at small μ.

6.1   A few words about "large"-r $(r > N_f/2)$ vacua at small μ

In the r vacua with $r > N_f/2$ physics is quite different, see [15, 7, 16]. At large μm (μ is assumed to be small so that the quark masses must be large) the low energy-theory has the gauge group U(r)×U(1)$^{N-r}$ with r condensed quarks and $(N - r - 1)$ condensed monopoles. The theory is at weak coupling because it has large condensates in the non-Abelian asymptotically free SU(r) quark sector and small condensates in the IR free monopole sector. At low ξ the theory goes through a crossover transition. At small ξ physics can be described by weakly coupled infrared-free r dual theory with the U(ν)×U(1)$^{N-\nu}$ gauge group, $\nu = N_f - r$. The r dual theory has N_f flavors of quark-like dyons. The color charges of non-Abelian quark-like dyons are identical to those of quarks.[9] However, they belong to a different representation of the global color-flavor locked group. Condensation of these dyons leads to the confinement of monopoles. Quarks from the U(ν) sector are in the "instead-of-confinement" phase: the Higgs-screened quarks decay into monopole-antimonopole pairs confined by non-Abelian

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[9]Because of monodromies quarks pick up root-like color-magnetic charges in addition to their weight-like color-electric charges at strong coupling.

strings. Singlet quarks from the $U(1)^{r-\nu}$ sector and monopoles from the $U(1)^{N-r}$ sectors are Higgs-screened. Other monopoles, charged with respect to the Cartan generators of $SU(r)$, and quarks charged with respect to the orthogonal $U(1)^{N-r}$ are confined.

6.2   Phases in the r vacua at large μ

In this section we briefly summarize the overall picture of physical phases in different r vacua in the small ξ and large μ limit. Namely, we impose

$$\mu \gg \sqrt{\xi}, \qquad \sqrt{\xi} \ll \tilde{\Lambda}_{\mathcal{N}=1}. \tag{6.1}$$

Phases of the theory in the different r vacua are shown in Fig. 3, which is the same as Fig. 2, with various physical regimes indicated.

a) *Zero vacua*:   The r parameters ξ relevant to the low-energy μ-dual theory are given by (4.14). This theory has the $U(\tilde{N})$ gauge group with N_f flavors of quarks. It is infrared-free and weakly coupled in the region (6.1) if we keep quark masses sufficiently small. The theory is in the mixed Coulomb/Higgs phase with r quarks condensed, see (4.20), while the $U(\tilde{N} - r)$ subgroup remains unbroken. Non-Abelian strings are formed in the $U(r)$ sector which entails confinement of monopoles charged with respect to the $SU(r)$ Cartan generators. The Seiberg and μ-dual descriptions are equivalent.
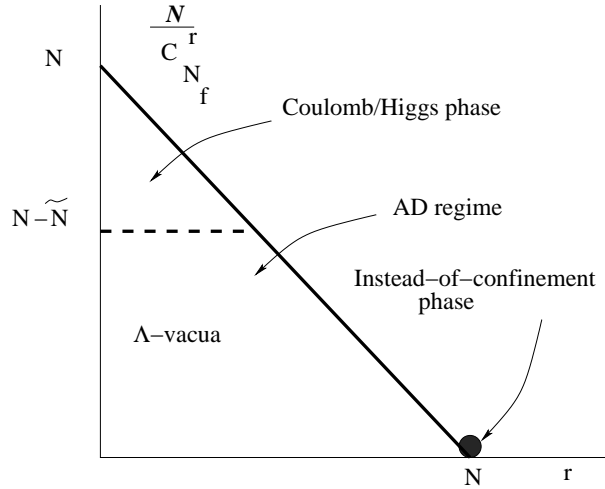

Figure 3: Phases of r vacua in $\mathcal{N} = 1$ SQCD in the region (6.1). Zero and Λ-vacua shown as in Fig. 2. Black circle denotes $r = N$ vacua.

b) *Λ vacua*:   As we increase μ, we break the condition (6.1), generally speaking. The weak-coupling description is unknown so far, and it is unclear whether or not it exists. However, we can tune the common quark mass m and approach the Argyres-Douglas (AD) point [13] where r double roots for $r < N_f/2$ vacuum and $\nu = N_f - r$ double roots for $r > N_f/2$ vacuum come close to one of the unpaired roots $e_N^{\pm}$. Then r parameters ξ for

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$r < N_f/2$ vacuum and $\nu = N_f - r$ parameters ξ for $r > N_f/2$ vacuum can be made small to satisfy the bound in (6.1), see (4.1) and [7]. This limit is a continuation of the AD conformal *strongly coupled* regime to large μ.

c) *The r = N vacuum*: The $r = N$ vacuum presents a special case. In this case the gaugino condensate vanishes and $\tilde{N}$ parameters ξ_P are proportional to μm_P. They can satisfy the bound (6.1) provided the quark masses are sufficiently small. The small-mass limit can be described by weakly coupled infrared-free r-dual theory [6, 7]. It has the U($\tilde{N}$) gauge group with N_f flavors of quark-like dyons. The quark-like dyons condense leading to the formation of non-Abelian strings which confine monopoles. The quarks and gauge bosons of the original theory are in the "instead-of-confinement" phase. Namely, the Higgs-screened quarks and gauge bosons decay into the monopole-antimonopole pairs on the curves of marginal stability (CMS) [15, 27]. The monopole-antimonopole pairs are in the confining regime. In other words, the original quarks and gauge bosons evolve at small μm into the monopole-antimonopole stringy mesons. (The latter are expected to form Regge trajectories, generally speaking). At $r = N$ the r-dual theory matches the Seiberg dual [8]. Conceptually this vacuum can be added to the zero vacua to form a class of vacua with U($\tilde{N}$) weak coupling low-energy description. Moreover, the number of condensed quark-like dyons for this vacuum is $\tilde{N}$ so it nicely adds to the set of zero vacua where the number of condensed quarks is $0 \leq r < \tilde{N}$.

To conclude this section, let us note that there is no quark confinement phase in $\mathcal{N} = 1$ SQCD in the domain (6.1). The Seiberg-Witten phase of monopole condensation and Abelian quark confinement present in the slightly deformed $\mathcal{N} = 2$ QCD at small μ [3, 4] does *not* survive in the large-μ domain where the adjoint fields decouple. This result resolves the long-standing problem of extrapolating the Seiberg-Witten scenario of quark confinement to $\mathcal{N} = 1$ SQCD. The phase most close to what we observe in the real-world QCD is the "instead-of-confinement" phase present in the $r = N$ vacuum.

7 Conclusions

We considered $\mathcal{N} = 2$ SQCD with the U(N) gauge group and N_f quark flavors ($N + 1 < N_f < \frac{3}{2}N$) perturbed by a mass term $\mu \mathcal{A}^2$. This theory has r vacua, i.e. those vacua in which r flavors of quarks condense, $r < N$ (this definition refers to large values of m_A). In this paper we analyzed the r vacua with $r < N_f/2$. Low-energy theory in these vacua at small μ is based on the U(r)×U(1)$^{N-r}$ gauge group, with r light quarks and $(N - r - 1)$ Abelian monopoles.

Among these vacua we identify a subset that we call *zero vacua*. In the zero vacua the gaugino condensate vanishes in the small quark-mass limit. We show that upon increasing μ these vacua go though a crossover into strong coupling.

At large μ the zero vacua can be described in terms of weakly coupled infrared-free μ-dual theory with the U($N_f - N$) gauge group and N_f flavors of quarks. The r quark flavors

condense triggering monopole confinement. We show that this μ-dual theory matches the Seiberg dual. This match reveals the nature of Seiberg's dual quarks which in this regime happen to be ordinary quarks of our microscopic theory flowing to $\mathcal{N}=1$ SQCD at large μ.

The above conclusions are reached on the basis of the analysis of the exact Seiberg-Witten solution of $\mathcal{N}=2$ SQCD. We focused on the μ-dual $\mathcal{N}=1$ theories in the r vacua in the regime analogous to that existing to the left of the left edge of the Seiberg conformal window. The strong-weak coupling duality is shown to exist in the zero vacua which can be found at $r < N_f - N$.

Acknowledgments

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