Two-loop hard thermal loop pressure at finite temperature and chemical potential

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Two-loop HTL pressure at finite temperature and chemical potential

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We calculate the two-loop pressure of a plasma of quarks and gluons at finite temperature and chemical potential using the hard thermal loop perturbation theory (HTLpt) reorganization of finite temperature/density quantum chromodynamics. The computation utilizes a high temperature expansion through fourth order in the ratio of the chemical potential to temperature. This allows us to reliably access the region of high temperature and small chemical potential. We compare our final result for the leading- and next-to-leading-order HTLpt pressure at finite temperature and chemical potential with perturbative quantum chromodynamics (QCD) calculations and available lattice QCD results.

I. INTRODUCTION

Quantum chromodynamics (QCD) exhibits a rich phase structure and the equation of state (EOS) which describes the matter can be characterized by different degrees of freedom depending upon the temperature and the chemical potential. Hadrons are the relevant degrees of freedom at low temperature and chemical potential where chiral symmetry is spontaneously broken and quarks and gluons are confined but the matter is approximately $SU(3)_c$ center-symmetric. At high temperatures the system is expected to make a phase transition to a quasifree state known as quark-gluon plasma (QGP). In the QGP chiral symmetry is restored and the expectation value of the Polyakov Loop becomes close to one, signaling deconfinement.[1] At high temperatures and moderate chemical potentials one therefore expects the system to be in the QGP phase. Such conditions are generated in relativistic heavy ion collisions at Brookhaven National Laboratory's Relativistic Heavy Ion Collider (RHIC) [1], the European Organization for Nuclear Research's Large Hadron Collider (LHC) [2], and are expected to be generated at the Gesellschaft fur Schwerionenforschung's Facility for Antiproton and Ion Research (FAIR) [3].

The determination of the equation of state (EOS) of QCD matter is extremely important to QGP phenomenology. There are various effective models (see e.g. [7–11]) to describe the EOS of strongly interacting matter; however, one would prefer to utilize systematic first-principles QCD methods. The currently most reliable method for determining the EOS is lattice QCD [4]. At this point in time lattice calculations can be performed at arbitrary temperature, however, they are restricted to relatively small chemical potentials [5, 6]. Alternatively, perturbative QCD (pQCD) [12–15] can be applied at high temperature and/or chemical potentials where the strong coupling ($g^2 = 4\pi\alpha_s$) is small in magnitude and non-perturbative effects are expected to be small. However, due to infrared singularities in the gauge sector, the perturbative expansion of the finite-temperature and density QCD partition function breaks down at order g^6 requiring non-perturbative input albeit through a single numerically computable number [15, 16]. Up to order $g^6 \ln(1/g)$ it possible to calculate the necessary coefficients using analytic (resummed) perturbation theory.

Since the advent of pQCD there has been a tremendous effort to compute the pressure order by order in the weak coupling expansion [14, 15, 17, 18]. The pressure has been calculated to order of $g^6 \ln(1/g)$ at zero chemical potential ($\mu = 0$) and finite temperature T [15] and finite chemical potential/temperature ($\mu \geq 0$ and $T \geq 0$) [17]. In addition, the pressure is known to order g^4 for large μ and arbitrary T [18]. Unfortunately, one finds that as successive perturbative orders are included, the series converges poorly and the dependence on the renormalization scale increases rather than decreases. The resulting perturbative series only becomes convergent at very high temperature ($T \sim 10^5\, T_c$). One could be tempted to say that this is due to the largeness of the QCD coupling constant at realistic temperatures; however, in practice one finds that the relevant small quantity is, in fact, α_s/π which for phenomenologically relevant temperatures is on the order of one-tenth. Instead, one finds that the coefficients of α_s/π are large. This can be seen by examining the weak coupling expansion of the free energy $\mathcal{F}(T,\mu)$ of QGP calculated [17] up to order $\alpha_s^3 \ln(\alpha_s)$

$$\mathcal{F} = -\frac{8\pi^2}{45}T^4\left[\mathcal{F}_0 + \mathcal{F}_2\frac{\alpha_s}{\pi} + \mathcal{F}_3\left(\frac{\alpha_s}{\pi}\right)^{3/2} + \mathcal{F}_4\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{F}_5\left(\frac{\alpha_s}{\pi}\right)^{5/2} + \mathcal{F}_6\left(\frac{\alpha_s}{\pi}\right)^3 + \cdots\right], \tag{1}$$

where we have specialized to the case $N_c = 3$ and

$$\mathcal{F}_0 = 1 + \frac{21}{32}N_f\left(1 + \frac{120}{7}\hat{\mu}^2 + \frac{240}{7}\hat{\mu}^4\right), \tag{2}$$

$$\mathcal{F}_2 = -\frac{15}{4}\left[1 + \frac{5N_f}{12}\left(1 + \frac{72}{5}\hat{\mu}^2 + \frac{144}{5}\hat{\mu}^4\right)\right], \tag{3}$$

$$\mathcal{F}_3 = 30\left[1 + \tfrac{1}{6}\left(1 + 12\hat{\mu}^2\right)N_f\right]^{3/2} \tag{4}$$

$$\mathcal{F}_4 = 237.223 + \left(15.963 + 124.773\,\hat{\mu}^2 - 319.849\hat{\mu}^4\right)N_f$$

[1] With dynamical quarks the center symmetry Z(3) in SU(3) is explicitly broken yet it can be regarded as an approximate symmetry and the expectation value of the Polyakov Loop is still useful as an order parameter.

$$
\begin{aligned}
& -\left(0.415+15.926\,\hat\mu^2+106.719\,\hat\mu^4\right)N_f^2 \\
& +\frac{135}{2}\left[1+\tfrac{1}{6}\left(1+12\hat\mu^2\right)N_f\right]\log\left[\frac{\alpha_s}{\pi}\left(1+\tfrac{1}{6}\left(1+12\hat\mu^2\right)N_f\right)\right] \\
& -\frac{165}{8}\left[1+\frac{5}{12}\left(1+\frac{72}{5}\hat\mu^2+\frac{144}{5}\hat\mu^4\right)N_f\right]\left(1-\frac{2}{33}N_f\right)\log\hat\Lambda\,, \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_5 = & -\left(1+\frac{1+12\hat\mu^2}{6}N_f\right)^{1/2}\Bigg[799.149+\left(21.963-136.33\,\hat\mu^2+482.171\,\hat\mu^4\right)N_f \\
& +\left(1.926+2.0749\,\hat\mu^2-172.07\,\hat\mu^4\right)N_f^2\Bigg] \\
& +\frac{495}{2}\left(1+\frac{1+12\hat\mu^2}{6}N_f\right)\left(1-\frac{2}{33}N_f\right)\log\hat\Lambda\,, \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_6 = & -\Bigg[659.175+\left(65.888-341.489\,\hat\mu^2+1446.514\,\hat\mu^4\right)N_f \\
& +\left(7.653+16.225\,\hat\mu^2-516.210\,\hat\mu^4\right)N_f^2 \\
& -\frac{1485}{2}\left(1+\frac{1+12\hat\mu^2}{6}N_f\right)\left(1-\frac{2}{33}N_f\right)\log\hat\Lambda\Bigg]\log\left[\frac{\alpha_s}{\pi}\left(1+\frac{1+12\hat\mu^2}{6}N_f\right)4\pi^2\right] \\
& -475.587\log\left[\frac{\alpha_s}{\pi}\,4\pi^2 C_A\right], \qquad (7)
\end{aligned}
$$

where here and throughout all hatted quantities are scaled by $2\pi T$, e.g. $\hat\mu = \mu/(2\pi T)$, Λ is the modified minimum subtraction ($\overline{\text{MS}}$) renormalisation scale, and $\alpha_s = \alpha_s(\hat\Lambda)$ is the running coupling. At finite T the central value of the renormalisation scale is usually chosen to be $2\pi T$. However, at finite T and μ we use the central scale $\Lambda = 2\pi\sqrt{T^2+(\mu/\pi)^2}$, which is the geometric mean between $2\pi T$ and 2μ [17, 19]. In Fig. 1 we plot the ratio of the pressure to an ideal gas of quarks and gluons. The figure clearly demonstrates the poor convergence of the naive perturbative series and the increasing sensitivity of the result to the renormalisation scale as successive orders in the weak coupling expansion are included.

In this context one should note that one can explicitly separate the contributions coming from the soft sector (momenta on the order of g_sT where $g_s^2 = 4\pi\alpha_s$) and the hard sector (momenta on the order of T) using effective field theory/dimensional reduction methods [21–27]. After doing this one finds that the hard-sector contributions, which form a power series in even powers of g_s, converge reasonably well; however, the soft sector perturbative series, which contains odd powers of g_s, is poorly convergent. This suggests that in order to improve the convergence of the resulting perturbative approximants one should treat the soft sector non-perturbatively, or at least resum soft corrections to the pressure. There have been works in the framework of dimensional reduction which effectively perform such soft-sector resummations by not truncating the soft-scale contributions in a power series in g_s, see e.g. [23, 24, 27]. This method seems to improve the convergence of the perturbation series and provides motivation to find additional analytic methods to accomplish soft-sector resummations.

In order to better describe the soft-scale contributions there have been various resummation schemes developed which attempt to improve the convergence of the successive approximations by reorganizing the calculation in terms of quasiparticle degrees of freedom [28–44]. These resummation methods include some relevant physical ingredients, e.g. screening masses and Landau damping. These reorganizations of perturbation theory canonically include quasiparticle degrees of freedom from the outset, as opposed to naive perturbation theory. In the naive perturbative treatment an expansion around the vacuum is made and one only includes quasiparticle effects in order to regulate infrared divergences. Based on Hard Thermal Loop (HTL) resummation [29, 30], a manifestly gauge-invariant reorganization of finite temperature/density QCD called HTL perturbation theory (HTLpt) has been developed [36]. HTLpt has so far been applied primarily to the case of finite temperature and zero chemical potential. In HTLpt [36] the

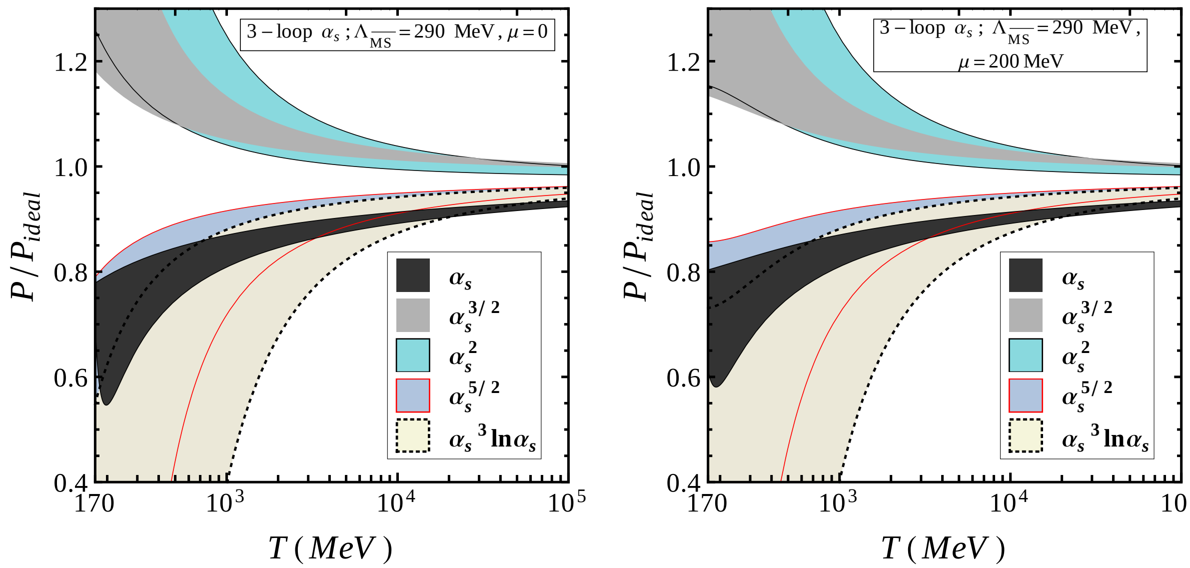

FIG. 1. The $N_f = 3$ pQCD pressure specified in Eq. (1) as a function of the temperature. Successive perturbative approximations are shown through order $\alpha_s^3 \ln \alpha_s$ for vanishing μ (left) and for non-vanishing μ (right). The shaded bands indicate the variation of the pressure as the $\overline{\rm MS}$ renormalisation scale is varied around a central value of $\Lambda = 2\pi\sqrt{T^2 + \mu^2/\pi^2}$ [17, 19] by a factor of two. We use $\Lambda_{\overline{\rm MS}} = 290$ MeV based on recent lattice calculations [20] of the three-loop running of α_s.

next-to-leading order (NLO) [37] and next-to-next-to-leading order (NNLO) [38] thermodynamic functions have been calculated at finite T but $\mu = 0$. Recently [40, 43] the leading order (LO) HTL pressure for finite T and μ has been calculated and approximately a decade ago it was applied at LO for finite μ but $T = 0$ [44].

In view of the ongoing RHIC beam energy scan and planned FAIR experiments, one is motivated to reliably determine the thermodynamic functions at finite chemical potential. In this article we compute the NLO pressure of quarks and gluons at finite T and μ. The computation utilizes a high temperature expansion through fourth order in the ratio of the chemical potential to temperature. This allows us to reliably access the region of high temperature and small chemical potential. We compare our final result for the NLO HTLpt pressure at finite temperature and chemical potential with state-of-the-art perturbative quantum chromodynamics (QCD) calculations and available lattice QCD results.

The paper is organized as follows. In Sec. II, we will briefly review HTLpt. In Sec. III we discuss various quantities required to be calculated at finite chemical potential based on prior calculations of the NLO thermodynamic at zero chemical potential [37]. In Sec. IV we reduce the sum of various diagrams to scalar sum-integrals. A high temperature expansion is made in Sec. V to obtain analytic expressions for both the LO and NLO thermodynamic potential. We then use this to compute the pressure in Sec.VI. We conclude in Sec. VII. Finally, in Appendices A and B we collect the various integrals and sum-integrals necessary to obtain the results presented in the main body of the text.

II. HARD THERMAL LOOP PERTURBATION THEORY

HTL perturbation theory [36–38] is a reorganization of the perturbation series for hot and dense QCD which has the following Lagrangian density

$$\mathcal{L} = \left(\mathcal{L}_{\rm QCD} + \mathcal{L}_{\rm HTL}\right)\Big|_{g\to\sqrt{\delta}g} + \Delta\mathcal{L}_{\rm HTL}\,, \tag{8}$$

where $\Delta\mathcal{L}_{\rm HTL}$ collects all necessary renormalization counterterms and $\mathcal{L}_{\rm HTL}$ is the HTL effective Lagrangian [29, 30]. It can be written compactly as

$$\mathcal{L}_{\rm HTL} = -\frac{1}{2}(1-\delta)m_D^2 {\rm Tr}\left(G_{\mu\alpha}\left\langle \frac{y^\alpha y^\beta}{(y\cdot D)^2}\right\rangle_y G^\mu{}_\beta\right) + (1-\delta)\, i m_q^2 \bar\psi\gamma^\mu \left\langle \frac{y^\mu}{y\cdot D}\right\rangle_y \psi\,, \tag{9}$$

where D is a covariant derivative operator, $y = (1, \mathbf{y})$ is a light like vector and $\langle\cdots\rangle$ is the average over all possible directions, $\hat{y}$, of the loop momenta. The HTL effective action is gauge invariant, nonlocal, and can generate all of the HTL n-point functions [29, 30], which are interrelated through Ward identities. The mass parameters m_D and m_q are the Debye screening and quark masses in a hot and dense medium, respectively, which depend on the strong coupling g, temperature T, and the chemical potential μ. In the high temperature limit the leading-order expressions for m_D and m_q are

$$m_D^2 = \frac{g^2}{3}\left[\left(N_c + \frac{N_f}{2}\right)T^2 + \frac{3N_f}{2\pi^2}\mu^2\right], \tag{10}$$

$$m_q^2 = \frac{g^2}{4}\frac{N_c^2-1}{4N_c}\left(T^2 + \frac{\mu^2}{\pi^2}\right). \tag{11}$$

We will not assume these expressions a priori, but instead treat m_D and m_q as free parameters to be fixed at the end of the calculation. In order to make the calculation tractable we make expansions in m_D and m_q in (9) treating the masses as order g [36–38]. The $n^{\rm th}$ loop order in the HTLpt loop expansion is obtained by expanding the partition function through order δ^{n-1} and then taking $\delta \to 1$ [36–42]. In this work, we will fix the parameters m_D and m_q by employing a variational prescription which requires that the first derivative of the thermodynamic potential with respect to both m_D and m_q vanishes, such that the free energy is minimized. In the following, we generalize the NLO thermodynamic potential calculation from the case of zero chemical potential [37] to finite chemical potential.

III. INGREDIENTS FOR THE NLO THERMODYNAMIC POTENTIAL IN HTLPT

The LO HTLpt thermodynamic potential, $\Omega_{\rm LO}$, for an $SU(N_c)$ gauge theory with N_f massless quarks in the fundamental representation can be written as [36, 37]

$$\Omega_{\rm LO} = d_A\mathcal{F}_g + d_F\mathcal{F}_q + \Delta_0\mathcal{E}_0\,, \tag{12}$$

where $d_F = N_f N_c$ and $d_A = N_c^2 - 1$ with N_c is the number of colors. $\mathcal{F}_q$ and $\mathcal{F}_g$ are the one loop contributions to quark and gluon free energies, respectively. The LO counterterm is the same as in the case of zero chemical potential [36]

$$\Delta_0\mathcal{E}_0 = \frac{d_A}{128\pi^2\epsilon}m_D^4\,. \tag{13}$$

At NLO one must consider the diagrams shown in Fig. 2. The resulting NLO HTLpt thermodynamic potential can be written in the following general form [37]

$$\begin{aligned}\Omega_{\rm NLO} = \Omega_{\rm LO} + d_A\left[\mathcal{F}_{3g} + \mathcal{F}_{4g} + \mathcal{F}_{gh} + \mathcal{F}_{gct}\right] + d_A s_F\left[\mathcal{F}_{3qg} + \mathcal{F}_{4qg}\right] \\ + d_F\mathcal{F}_{qct} + \Delta_1\mathcal{E}_0 + \Delta_1 m_D^2\frac{\partial}{\partial m_D^2}\Omega_{\rm LO} + \Delta_1 m_q^2\frac{\partial}{\partial m_q^2}\Omega_{\rm LO}\,,\end{aligned} \tag{14}$$

where $s_F = N_F/2$. At NLO the terms that depend on the chemical potential are $\mathcal{F}_q$, $\mathcal{F}_{3qg}$, $\mathcal{F}_{4qg}$, $\mathcal{F}_{qt}$, $\Delta_1 m_q^2$, and $\Delta_1 m_D^2$ as displayed in Fig. 2. The other terms, e.g. $\mathcal{F}_g$, $\mathcal{F}_{3g}$, $\mathcal{F}_{4g}$, $\mathcal{F}_{gh}$ and $\mathcal{F}_{gct}$ coming from gluon and ghost loops remain the same as the $\mu = 0$ case [37]. We also add that the vacuum energy counterterm, $\Delta_1\mathcal{E}_0$, remains the same as

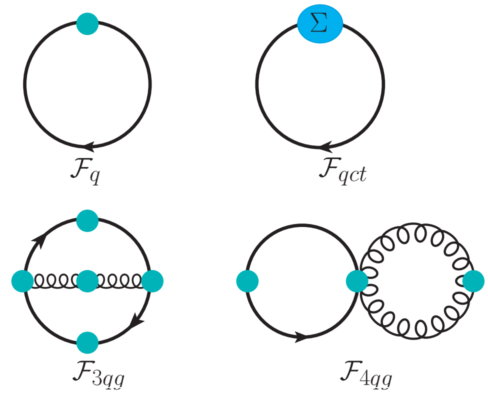

FIG. 2. Diagrams containing fermionic lines relevant for NLO thermodynamics potential in HTLpt with finite chemical potential. Shaded circles indicate HTL n-point functions.

the $\mu = 0$ case whereas the mass counterterms, $\Delta_1 m_D^2$ and $\Delta_1 m_q^2$, have to be computed for $\mu \neq 0$. These counterterms are of order δ. This completes a general description of contributions one needs to compute in order to determine NLO HTLpt thermodynamic potential at finite chemical potential. We now proceed to the scalarization of the necessary diagrams.

IV. SCALARIZATION OF THE FERMIONIC DIAGRAMS

The one-loop quark contribution coming from the first diagram in Fig. 2 can be written as

$$\mathcal{F}_q = -\sum\!\!\!\!\!\!\!\int_{\{P\}} \log\det\left[\not{P} - \Sigma(P)\right] = -2\sum\!\!\!\!\!\!\!\int_{\{P\}} \log P^2 - 2\sum\!\!\!\!\!\!\!\int_{\{P\}} \log\left[\frac{A_S^2 - A_0^2}{P^2}\right] , \tag{15}$$

where

$$A_0(P) = iP_0 - \frac{m_q^2}{iP_0}\mathcal{T}_P , \tag{16}$$

$$A_S(P) = |\mathbf{p}| + \frac{m_q^2}{|\mathbf{p}|}\left[1 - \mathcal{T}_P\right] , \tag{17}$$

and $\mathcal{T}_P$ is defined by the following integral [37]

$$\mathcal{T}_P = \left\langle \frac{P_0^2}{P_0^2 + p^2 c^2} \right\rangle_c = \frac{\omega(\epsilon)}{2}\int_{-1}^{1} dc\,(1-c^2)^{-\epsilon}\frac{iP_0}{iP_0 - |\mathbf{p}|c} , \tag{18}$$

with $w(\epsilon) = 2^{2\epsilon}\,\Gamma(2-2\epsilon)/\Gamma^2(1-\epsilon)$. In three dimensions $\epsilon \to 0$ and (18) reduces to

$$\mathcal{T}_P = \frac{iP_0}{2|\mathbf{p}|}\log\frac{iP_0 + |\mathbf{p}|}{iP_0 - |\mathbf{p}|} , \tag{19}$$

with $P \equiv (P_0, \mathbf{p})$. In practice, one must use the general form and only take the limit $\epsilon \to 0$ after regularization/renormalization.

The HTL quark counterterm at one-loop order can be rewritten from the second diagram in Fig. 2 as

$$\mathcal{F}_{qct} = -4\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{P^2 + m_q^2}{A_S^2 - A_0^2} . \tag{20}$$

The two-loop contributions coming from the third and fourth diagrams in Fig. 2 are given, respectively, by

$$\mathcal{F}_{3qg} = \frac{1}{2}g^2 \sum\!\!\!\!\!\!\!\int_{\{PQ\}} \mathrm{Tr}\left[\Gamma^\mu(P,Q,R)S(Q) \times \Gamma^\nu(P,Q,R)S(R)\right]\Delta_{\mu\nu}(P)\,, \tag{21}$$

$$\mathcal{F}_{4qg} = \frac{1}{2}g^2 \sum\!\!\!\!\!\!\!\int_{\{PQ\}} \mathrm{Tr}\left[\Gamma^{\mu\nu}(P,-P,Q,Q)S(Q)\right]\Delta_{\mu\nu}(P)\,, \tag{22}$$

where S is the quark propagator which is given by $S = (\gamma^\mu \mathcal{A}_\mu)^{-1}$ with $\mathcal{A}_\mu = (A_0(P), A_S(P)\hat{\mathbf{p}})$ and $\Delta^{\mu\nu}$ is the gluon propagator. The general covariant gauge $\Delta^{\mu\nu}$ can be expressed most conveniently in Minkowski space

$$\Delta^{\mu\nu}(p) = \left[-\Delta_T(p)g^{\mu\nu} + \Delta_X(p)n^\mu n^\nu\right] - \frac{n\!\cdot\! p}{p^2}\Delta_X(p)\left(p^\mu n^\nu + n^\mu p^\nu\right) + \left[\Delta_T(p) + \frac{(n\!\cdot\! p)^2}{p^2}\Delta_X(p) - \frac{\xi}{p^2}\right]\frac{p^\mu p^\nu}{p^2}\,, \tag{23}$$

where n^μ is thermal rest frame four-vector and Δ_T and Δ_L are the transverse and longitudinal propagators

$$\Delta_T(p) = \frac{1}{p^2 - \Pi_T(p)}\,, \tag{24}$$

$$\Delta_L(p) = \frac{1}{-n_p^2 p^2 + \Pi_L(p)}\,, \tag{25}$$

with $n_p^\mu = n^\mu - (n_\mu p^\mu/p^2)p^\mu$. It is convenient to introduce the following linear combination of transverse and longitudinal propagators which turn out to make the calculations easier to manage in practice

$$\Delta_X(p) = \Delta_L(p) + \frac{1}{n_p^2}\Delta_T(p)\,. \tag{26}$$

Also above Γ^μ and $\Gamma^{\mu\nu}$ are HTL-resummed 3- and 4-point functions. Many more details concerning the HTL n-point functions including the general Coulomb gauge propagator etc. can be found in appendices of Refs. [36, 37].

In general covariant gauge, the sum of (21) and (22) reduces to

$$\mathcal{F}_{3qg+4qg} = \frac{1}{2}g^2 \sum\!\!\!\!\!\!\!\int_{\{PQ\}} \Bigg\{\Delta_X(P)\mathrm{Tr}\left[\Gamma^{00}S(Q)\right] - \Delta_T(P)\mathrm{Tr}\left[\Gamma^\mu S(Q)\Gamma^\mu S(R')\right] + \Delta_X(P)\mathrm{Tr}\left[\Gamma^0 S(Q)\Gamma^0 S(R')\right]\Bigg\}\,, \tag{27}$$

where Δ_T is the transverse gluon propagator, Δ_X is a combination of the longitudinal and transverse gluon propagators [37], and $R' = Q - P$. After performing the traces of the γ-matrices one obtains [37]

$$\mathcal{F}_{3qg+4qg} = -g^2 \sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{A_S^2(Q) - A_0^2(Q)}\Bigg\{2(d-1)\Delta_T(P)\frac{\hat{\mathbf{q}}\!\cdot\!\hat{\mathbf{r}}A_S(Q)A_S(R) - A_0(Q)A_0(R)}{A_S^2(R) - A_0^2(R)}$$
$$-2\Delta_X(P)\frac{A_0(Q)A_0(R) + A_S(Q)A_S(R)\hat{\mathbf{q}}\!\cdot\!\hat{\mathbf{r}}}{A_S^2(R) - A_0^2(R)}$$
$$-4m_q^2\Delta_X(P)\left\langle \frac{A_0(Q) - A_s(Q)\hat{\mathbf{q}}\!\cdot\!\hat{\mathbf{y}}}{(P\!\cdot\! Y)^2 - (Q\!\cdot\! Y)^2}\frac{1}{(Q\!\cdot\! Y)}\right\rangle_{\hat{\mathbf{y}}}$$

$$+\frac{8m_q^2\Delta_T(P)}{A_S^2(R)-A_0^2(R)}\left\langle\frac{(A_0(Q)-A_S(Q)\hat{\mathbf{q}}\cdot\hat{\mathbf{y}})(A_0(R)-A_S(R)\hat{\mathbf{r}}\cdot\hat{\mathbf{y}})}{(Q\cdot Y)(R\cdot Y)}\right\rangle_{\hat{\mathbf{y}}}$$
$$+\frac{4m_q^2\Delta_X(P)}{A_S^2(R)-A_0^2(R)}\left\langle\frac{2A_0(R)A_S(Q)\hat{\mathbf{q}}\cdot\hat{\mathbf{y}}-A_0(Q)A_0(R)-A_S(Q)A_S(R)\hat{\mathbf{q}}\cdot\hat{\mathbf{r}}}{(Q\cdot Y)(R\cdot Y)}\right\rangle_{\hat{\mathbf{y}}}\Bigg\}$$
$$+O(g^2m_q^4)\,, \tag{28}$$

where A_0 and A_S are defined in (16) and (17), respectively. We add that the exact evaluation of two-loop free energy could be performed numerically and would involve 5-dimensional integrations; however, one would need to be able to identify all divergences and regulate the numerical integration appropriately. Short of this, one can calculate the sum-integrals by expanding in a power series in m_D/T, m_q/T, and μ/T in order to obtain semi-analytic expressions.

V. HIGH TEMPERATURE EXPANSION

As discussed above, we make an expansion of two-loop free energies in a power series of m_D/T and m_q/T to obtain a series which is nominally accurate to order g^5. By "nominally accurate" we mean that we expand the scalar integrals treating m_D and m_q as $\mathcal{O}(g)$ keeping all terms which contribute through $\mathcal{O}(g^5)$; however, the resulting series is accurate to order g^5 in name only. At each order in HTLpt the result is an infinite series in g. Using the mass expansion we keep terms through order g^5 at all loop-orders of HTLpt in order to make the calculation tractable. At LO one obtains only the correct perturbative coefficients for the g^0 and g^3 terms when one expands in a strict power series in g. At NLO one obtains the correct g^0, g^2, and g^3 coefficients and at NNLO one obtains the correct g^0, g^2, g^3, g^4, and g^5 coefficients. The resulting approximants obtained when going from LO to NLO to NNLO are expected to show improved convergence since the loop expansion is now explicitly expanded in terms of the relevant high-temperature degrees of freedom (quark and gluon high-temperature quasiparticles).

In practice, the HTL n-point functions can have both hard and soft momenta scales on each leg. At one-loop order the contributions can be classified "hard" or "soft" depending on whether the loop momenta are order T or gT, respectively; however, since the lowest fermionic Matsubara mode corresponds to $P_0 = \pi T$, fermion loops are always hard. The two-loop contributions to the thermodynamic potential can be grouped into hard-hard (hh), hard-soft (hs), and soft-soft (ss) contributions. However, we note that one of the momenta contributing is always hard since it corresponds to a fermionic loop and therefore there will be no two-loop soft-soft contribution. Below we calculate the various contributions to the sum-integrals presented in Sec. IV.

A. One-loop sum-integrals

The one-loop sum-integrals (15) and (20) correspond to the first two diagrams in Fig. 2. They represent the leading-order quark contribution and order-δ HTL counterterm. We will expand the sum-integrals through order m_q^4 taking m_q to be of (leading) order g. This gives a result which is nominally accurate (at one-loop) through order g^5. [2]

[2] Of course, this won't reproduce the full g^5 pQCD result in the limit $g \to 0$. In order to reproduce all known coefficients through $\mathcal{O}(g^5)$, one would need to perform a NNLO HTLpt calculation.

1. Hard Contribution

The hard contribution to the one-loop quark self-energy in (15) can be expanded in powers of m_q^2 as

$$\mathcal{F}_q^{(h)} = -2\sum\!\!\!\!\!\!\!\int_{\{P\}} \log P^2 - 4m_q^2 \sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2} + 2m_q^4 \sum\!\!\!\!\!\!\!\int_{\{P\}} \left[\frac{2}{P^4} - \frac{1}{p^2P^2} + \frac{2\mathcal{T}_P}{p^2P^2} - \frac{(\mathcal{T}_P)^2}{p^2P_0^2}\right]. \tag{29}$$

Note that the function $\mathcal{T}_P$ does not appear in m_q^2 term. The expressions for the sum-integrals in (29) are listed in Appendix A. Using those expressions, the hard contribution to the quark free energy becomes

$$\begin{aligned}\mathcal{F}_q^{(h)} = &-\frac{7\pi^2}{180}T^4\left(1+\frac{120}{7}\hat{\mu}^2+\frac{240}{7}\hat{\mu}^4\right) + \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\frac{m_q^2T^2}{6}\Bigg[\left(1+12\hat{\mu}^2\right) \\ &+ \epsilon\left(2-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+24(\gamma+2\ln 2)\,\hat{\mu}^2 - 28\,\zeta(3)\,\hat{\mu}^4 + \mathcal{O}\left(\hat{\mu}^6\right)\right)\Bigg] \\ &+ \frac{m_q^4}{12\pi^2}(\pi^2-6)\,. \end{aligned} \tag{30}$$

Expanding the HTL quark counterterm in (20) one can write

$$\mathcal{F}_{\rm qct}^{(h)} = 4m_q^2 \sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2} - 4m_q^4 \sum\!\!\!\!\!\!\!\int_{\{P\}} \left[\frac{2}{P^4} - \frac{1}{p^2P^2} + \frac{2}{p^2P^2}\mathcal{T}_P - \frac{1}{p^2P_0^2}\left(\mathcal{T}_P\right)^2\right], \tag{31}$$

where the expressions for various sum-integrals in (31) are listed in Appendix A. Using those expressions, the hard contribution to the HTL quark counterterm becomes

$$\mathcal{F}_{qct}^{(h)} = -\frac{m_q^2T^2}{6}\left(1+12\hat{\mu}^2\right) - \frac{m_q^4}{6\pi^2}(\pi^2-6)\,. \tag{32}$$

We note that the first term in (32) cancels the order-ϵ^0 term in the coefficient of m_q^2 in (30). There are no soft contributions either from the leading-order quark term in (15) or from the HTL quark counterterm in (20).

B. Two-loop sum-integrals

Since the two-loop sum-integrals given in (27) contain an explicit factor of g^2, we only require an expansion to order $m_q^2 m_D/T^3$ and m_D^3/T^3 in order to determine all terms contributing through order g^5. We note that the soft scales are given by m_q and m_D whereas the hard scale is given by T, which leads to two different phase-space regions as discussed in Sec. V A. In the hard-hard region, all three momenta P, Q, and R are hard whereas in the hard-soft region, two of the three momenta are hard and the other one is soft.

1. The hh contribution

The self-energies for hard momenta are suppressed [29, 30, 37] by m_D^2/T^2 or m_q^2/T^2 relative to the propagators. For hard momenta, one just needs to expand in powers of gluon self-energies Π_T, Π_L, and quark self-energy Σ. So, the hard-hard contribution of $\mathcal{F}_{3qg}$ and $\mathcal{F}_{4qg}$ in (27) can be written as

$$\mathcal{F}_{3qg+4qg}^{(hh)} = (d-1)g^2\left[\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2} - 2\sum\!\!\!\!\!\!\!\int_{P\{Q\}} \frac{1}{P^2Q^2}\right] + 2m_D^2 g^2 \sum\!\!\!\!\!\!\!\int_{P\{Q\}} \left[\frac{1}{p^2P^2Q^2}\mathcal{T}_P + \frac{1}{P^4Q^2} - \frac{d-2}{d-1}\frac{1}{p^2P^2Q^2}\right]$$

$$+ m_D^2 g^2 \sum\!\!\!\!\!\!\int_{\{PQ\}} \left[\frac{d+1}{d-1}\frac{1}{P^2Q^2r^2} - \frac{4d}{d-1}\frac{q^2}{P^2Q^2r^4} - \frac{2d}{d-1}\frac{P\cdot Q}{P^2Q^2r^4}\right]\mathcal{T}_R$$

$$+ m_D^2 g^2 \sum\!\!\!\!\!\!\int_{\{PQ\}} \left[\frac{3-d}{d-1}\frac{1}{P^2Q^2R^2} + \frac{2d}{d-1}\frac{P\cdot Q}{P^2Q^2r^4} - \frac{d+2}{d-1}\frac{1}{P^2Q^2r^2} + \frac{4d}{d-1}\frac{q^2}{P^2Q^2r^4} - \frac{4}{d-1}\frac{q^2}{P^2Q^2r^2R^2}\right]$$

$$+ 2m_q^2 g^2(d-1) \sum\!\!\!\!\!\!\int_{\{PQ\}} \left[\frac{1}{P^2Q_0^2Q^2} + \frac{p^2-r^2}{q^2P^2Q_0^2R^2}\right]\mathcal{T}_Q + 2m_q^2 g^2(d-1) \sum\!\!\!\!\!\!\int_{P\{Q\}} \left[\frac{2}{P^2Q^4} - \frac{1}{P^2Q_0^2Q^2}\mathcal{T}_Q\right]$$

$$+ 2m_q^2 g^2(d-1) \sum\!\!\!\!\!\!\int_{\{PQ\}} \left[\frac{d+3}{d-1}\frac{1}{P^2Q^2R^2} - \frac{2}{P^2Q^4} - \frac{p^2-r^2}{q^2P^2Q^2R^2}\right], \tag{33}$$

where the various sum-integrals are evaluated in Appendices A and B. Using those sum-integral expressions, the hh contribution becomes

$$\mathcal{F}_{3qg+4qg}^{(hh)} = \frac{5\pi^2}{72}\frac{\alpha_s}{\pi}T^4\left[1+\frac{72}{5}\,\hat\mu^2+\frac{144}{5}\,\hat\mu^4\right]$$
$$- \frac{1}{72}\frac{\alpha_s}{\pi}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\left[\frac{1+6(4-3\zeta(3))\,\hat\mu^2 - 120(\zeta(3)-\zeta(5))\,\hat\mu^4+\mathcal{O}\left(\hat\mu^6\right)}{\epsilon}\right.$$
$$\left. + 1.3035 - 59.9055\,\hat\mu^2 - 75.4564\,\hat\mu^4 + \mathcal{O}\left(\hat\mu^6\right)\right]m_D^2T^2$$
$$+ \frac{1}{8}\frac{\alpha_s}{\pi}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\left[\frac{1-12\,\hat\mu^2}{\epsilon} + 8.9807 - 152.793\,\hat\mu^2 + 115.826\,\hat\mu^4 + \mathcal{O}\left(\hat\mu^6\right)\right]m_q^2T^2\,. \tag{34}$$

2. The hs contribution

Following Ref. [37] one can extract the hard-soft contribution from (27) as the momentum P is soft whereas momenta Q and R are always hard. The function associated with the soft propagator $\Delta_T(0,\mathbf{p})$ or $\Delta_X(0,\mathbf{p})$ can be expanded in powers of the soft momentum $\mathbf{p}$. For $\Delta_T(0,\mathbf{p})$, the resulting integrals over $\mathbf{p}$ are not associated with any scale and they vanish in dimensional regularization. The integration measure $\int_{\mathbf{p}}$ scales like m_D^3, the soft propagator $\Delta_X(0,\mathbf{p})$ scales like $1/m_D^2$, and every power of p in the numerator scales like m_D.

The contributions that survive only through order $g^2m_D^3T$ and $m_q^2m_Dg^3T$ from $\mathcal{F}_{3qg}$ and $\mathcal{F}_{4qg}$ in (27) are

$$\mathcal{F}_{3qg+4qg}^{(hs)} = g^2T\int_{\mathbf{p}}\frac{1}{p^2+m_D^2}\sum\!\!\!\!\!\!\int_{\{Q\}}\left[\frac{2}{Q^2}-\frac{4q^2}{Q^4}\right] + 2m_D^2g^2T\int_{\mathbf{p}}\frac{1}{p^2+m_D^2}\sum\!\!\!\!\!\!\int_{\{Q\}}\left[\frac{1}{Q^4}-\frac{2(3+d)}{d}\frac{q^2}{Q^6}+\frac{8}{d}\frac{q^4}{Q^8}\right]$$
$$-4m_q^2g^2T\int_{\mathbf{p}}\frac{1}{p^2+m_D^2}\sum\!\!\!\!\!\!\int_{\{Q\}}\left[\frac{3}{Q^4}-\frac{4q^2}{Q^6}-\frac{4}{Q^4}\mathcal{T}_Q-\frac{2}{Q^2}\left\langle\frac{1}{(Q\cdot Y)^2}\right\rangle_{\hat{\mathbf{y}}}\right]. \tag{35}$$

Using the sum-integrals contained in Appendices A and B, the hard-soft contribution becomes

$$\mathcal{F}_{3qg+4qg}^{(hs)} = -\frac{1}{6}\alpha_s m_D T^3(1+12\,\hat\mu^2) + \frac{\alpha_s}{24\pi^2}\left[\frac{1}{\epsilon}+1+2\gamma+4\ln 2-14\zeta(3)\,\hat\mu^2+62\zeta(5)\,\hat\mu^4+\mathcal{O}\left(\hat\mu^6\right)\right]$$
$$\times\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\left(\frac{\Lambda}{2m_D}\right)^{2\epsilon}m_D^3T - \frac{\alpha_s}{2\pi^2}m_q^2m_DT\,. \tag{36}$$

3. The ss contribution

As discussed earlier in Sec. V there is no soft-soft contribution from the diagrams in Fig. 2 since at least one of the loops is fermionic.

C. Thermodynamic potential

Now we can obtain the HTLpt thermodynamic potential $\Omega(T,\mu,\alpha_s,m_D,m_q,\delta)$ through two-loop order for which the contributions involving quark lines are computed here whereas the ghost and gluon contributions are computed in Ref. [37]. We also follow the same prescription as in Ref. [37] to determine the mass parameter m_D and m_q from respective gap equations but with finite quark chemical potential, μ.

1. Leading order thermodynamic potential

Using the expressions of $\mathcal{F}_q$ with finite quark chemical potential in (30) and $\mathcal{F}_g$ from Ref. [37], the total contributions from the one-loop diagrams including all terms through order g^5 becomes

$$\begin{aligned}\Omega_{\rm one\ loop} = -d_A\frac{\pi^2T^4}{45}\Bigg\{ & 1+\frac{7}{4}\frac{d_F}{d_A}\left(1+\frac{120}{7}\hat\mu^2+\frac{240}{7}\hat\mu^4\right)-\frac{15}{2}\left[1+\epsilon\left(2+2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\hat\Lambda}{2}\right)\right]\hat m_D^2 \\ & -30\frac{d_F}{d_A}\left[\left(1+12\hat\mu^2\right)+\epsilon\left(2-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\hat\Lambda}{2}+24(\gamma+2\ln 2)\hat\mu^2-28\zeta(3)\hat\mu^4+\mathcal{O}\left(\hat\mu^6\right)\right)\right]\hat m_q^2 \\ & +\ 30\left(\frac{\Lambda}{2m_D}\right)^{2\epsilon}\left[1+\frac{8}{3}\epsilon\right]\hat m_D^3+\frac{45}{8}\left(\frac{1}{\epsilon}+2\ln\frac{\hat\Lambda}{2}-7+2\gamma+\frac{2\pi^2}{3}\right)\hat m_D^4-60\frac{d_F}{d_A}(\pi^2-6)\hat m_q^4\Bigg\}\ , \end{aligned} \tag{37}$$

where $\hat m_D$, $\hat m_q$, $\hat\Lambda$, and $\hat\mu$ are dimensionless variables:

$$\hat m_D = \frac{m_D}{2\pi T}\ , \tag{38}$$

$$\hat m_q = \frac{m_q}{2\pi T}\ , \tag{39}$$

$$\hat\Lambda = \frac{\Lambda}{2\pi T}\ , \tag{40}$$

$$\hat\mu = \frac{\mu}{2\pi T}\ . \tag{41}$$

Adding the counterterm in (13), we obtain the thermodynamic potential at leading order in the δ-expansion:

$$\begin{aligned}\Omega_{\rm LO} = -d_A\frac{\pi^2T^4}{45}\Bigg\{ & 1+\frac{7}{4}\frac{d_F}{d_A}\left(1+\frac{120}{7}\hat\mu^2+\frac{240}{7}\hat\mu^4\right)-\frac{15}{2}\left[1+\epsilon\left(2+2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\hat\Lambda}{2}\right)\right]\hat m_D^2 \\ & -30\frac{d_F}{d_A}\left[\left(1+12\hat\mu^2\right)+\epsilon\left(2-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\hat\Lambda}{2}+24(\gamma+2\ln 2)\hat\mu^2-28\zeta(3)\hat\mu^4+\mathcal{O}\left(\hat\mu^6\right)\right)\right]\hat m_q^2 \\ & +30\left(\frac{\Lambda}{2m_D}\right)^{2\epsilon}\left[1+\frac{8}{3}\epsilon\right]\ \hat m_D^3+\frac{45}{8}\left(2\ln\frac{\hat\Lambda}{2}-7+2\gamma+\frac{2\pi^2}{3}\right)\hat m_D^4-60\frac{d_F}{d_A}(\pi^2-6)\hat m_q^4\Bigg\}\ , \end{aligned} \tag{42}$$

where we have kept terms of $\mathcal{O}(\epsilon)$ since they will be needed for the two-loop renormalization.

The complete expression for the next-to-leading order correction to the thermodynamic potential is the sum of the contributions from all two-loop diagrams, the quark and gluon counterterms, and renormalization counterterms. Adding the contributions of the two-loop diagrams, $\mathcal{F}_{3qg+4qg}$, involving a quark line in (34) and (36) and the contributions of $\mathcal{F}_{3g+4g+gh}$ from Ref. [37], one obtains

$$\begin{aligned}\Omega_{\text{two loop}} = -d_A\frac{\pi^2T^4}{45}\frac{\alpha_s}{\pi}\Bigg\{&-\frac{5}{4}\left[c_A+\frac{5}{2}s_F\left(1+\frac{72}{5}\,\hat{\mu}^2+\frac{144}{5}\,\hat{\mu}^4\right)\right]+15\left(c_A+s_F\left(1+12\,\hat{\mu}^2\right)\right)\hat{m}_D\\&-\frac{55}{8}\Bigg[\left(c_A-\frac{4}{11}s_F\left[1+6(4-3\zeta(3))\,\hat{\mu}^2-120(\zeta(3)-\zeta(5))\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]\right)\left(\frac{1}{\epsilon}+4\ln\frac{\hat{\Lambda}}{2}\right)\\&-s_F\left(0.4712-34.8761\,\hat{\mu}^2-21.0214\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)-c_A\left(\frac{72}{11}\ln\hat{m}_D-1.96869\right)\Bigg]\hat{m}_D^2\\&-\frac{45}{2}s_F\left[\left(1-12\,\hat{\mu}^2\right)\left(\frac{1}{\epsilon}+4\,\ln\frac{\hat{\Lambda}}{2}\right)+8.9807-152.793\,\hat{\mu}^2+115.826\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]\hat{m}_q^2\\&+180s_F\,\hat{m}_D\hat{m}_q^2+\frac{165}{4}\Bigg[\left(c_A-\frac{4}{11}s_F\right)\left(\frac{1}{\epsilon}+4\,\ln\frac{\hat{\Lambda}}{2}-2\ln\hat{m}_D\right)+c_A\left(\frac{27}{11}+2\gamma\right)\\&-\frac{4}{11}s_F\left(1+2\gamma+4\ln2-14\zeta(3)\,\hat{\mu}^2+62\zeta(5)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\Bigg]\hat{m}_D^3\Bigg\}\,,\end{aligned} \tag{43}$$

where $c_A = N_c$ and $s_F = N_f/2$.

The HTL gluon counterterm is the same as obtained at zero chemical potential [37]

$$\Omega_{\text{gct}} = -d_A\frac{\pi^2T^4}{45}\left[\frac{15}{2}\,\hat{m}_D^2-45\hat{m}_D^3-\frac{45}{4}\left(\frac{1}{\epsilon}+2\ln\frac{\hat{\Lambda}}{2}-7+2\gamma+\frac{2\pi^2}{3}\right)\hat{m}_D^4\right], \tag{44}$$

The HTL quark counterterm as given by (32) is

$$\Omega_{qct} = -d_F\frac{\pi^2T^4}{45}\left[30(1+12\,\hat{\mu}^2)\,\hat{m}_q^2+120(\pi^2-6)\,\hat{m}_q^4\right]. \tag{45}$$

The ultraviolet divergences that remain after adding (43), (44), and (45) can be removed by renormalization of the vacuum energy density $\mathcal{E}_0$ and the HTL mass parameter m_D and m_q. The renormalization contributions [37] at first order in δ are

$$\Delta\Omega = \Delta_1\mathcal{E}_0+\Delta_1 m_D^2\frac{\partial}{\partial m_D^2}\Omega_{LO}+\Delta_1 m_q^2\frac{\partial}{\partial m_q^2}\Omega_{LO}\,. \tag{46}$$

The counterterm $\Delta_1\mathcal{E}_0$ at first order in δ will be same as the zero chemical potential counterterm

$$\Delta_1\mathcal{E}_0 = -\frac{d_A}{64\pi^2\epsilon}m_D^4\,. \tag{47}$$

The mass counterterms necessary at first order in δ are found to be

$$\Delta_1\hat{m}_D^2 = -\frac{\alpha_s}{3\pi\epsilon}\left[\frac{11}{4}c_A-s_F-s_F\left(1+6\hat{m}_D\right)\left[(24-18\zeta(3))\hat{\mu}^2+120(\zeta(5)-\zeta(3))\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]\right]\,\hat{m}_D^2 \tag{48}$$

and

$$\Delta_1 \hat{m}_q^2 \;=-\frac{\alpha_s}{3\pi\epsilon}\left[\frac{9}{8}\frac{d_A}{c_A}\right]\frac{1-12\,\hat{\mu}^2}{1+12\,\hat{\mu}^2}\;\hat{m}_q^2\;. \tag{49}$$

Using the above counterterms, the complete contribution from the counterterms in (46) at first order in δ at finite chemical potential becomes

$$\begin{aligned}
\Delta\Omega = -d_A\frac{\pi^2T^4}{45}\Bigg\{&\frac{45}{4\epsilon}\hat{m}_D^4+\frac{\alpha_s}{\pi}\Bigg[\frac{55}{8}\left(c_A-\frac{4}{11}s_F\left[1+(24-18\zeta(3))\hat{\mu}^2+120(\zeta(5)-\zeta(3))\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]\right)\\
&\left(\frac{1}{\epsilon}+2+2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\hat{\Lambda}}{2}\right)\hat{m}_D^2-\frac{165}{4}\left(c_A-\frac{4}{11}s_F\right)\left(\frac{1}{\epsilon}+2+2\ln\frac{\hat{\Lambda}}{2}-2\ln\hat{m}_D\right)\hat{m}_D^3\\
-&\ \frac{165}{4}\frac{4}{11}s_F\left[(24-18\zeta(3))\hat{\mu}^2+120(\zeta(5)-\zeta(3))\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]\left(2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\hat{m}_D\right)\hat{m}_D^3\\
+&\ \frac{45}{2}s_F\frac{1-12\,\hat{\mu}^2}{1+12\,\hat{\mu}^2}\Bigg(\frac{1+12\,\hat{\mu}^2}{\epsilon}+2+2\ln\frac{\hat{\Lambda}}{2}-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+24(\gamma+2\ln 2)\;\hat{\mu}^2\\
-&\ 28\zeta(3)\;\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\Bigg)\hat{m}_q^2\Bigg]\Bigg\}\,.
\end{aligned} \tag{50}$$

Adding the contributions from the two-loop diagrams in (43), the HTL gluon and quark counterterms in (44) and (45), the contribution from vacuum and mass renormalizations in (50), and the leading-order thermodynamic potential in (42) we obtain the complete expression for the QCD thermodynamic potential at next-to-leading order in HTLpt:

$$\begin{aligned}
\Omega_{\rm NLO} = -d_A\frac{\pi^2T^4}{45}\Bigg\{&1+\frac{7}{4}\frac{d_F}{d_A}\left(1+\frac{120}{7}\hat{\mu}^2+\frac{240}{7}\hat{\mu}^4\right)-15\hat{m}_D^3-\frac{45}{4}\left(\log\frac{\hat{\Lambda}}{2}-\frac{7}{2}+\gamma+\frac{\pi^2}{3}\right)\hat{m}_D^4\\
&+60\frac{d_F}{d_A}\left(\pi^2-6\right)\hat{m}_q^4+\frac{\alpha_s}{\pi}\Bigg[-\frac{5}{4}\left(c_A+\frac{5}{2}s_F\left(1+\frac{72}{5}\,\hat{\mu}^2+\frac{144}{5}\,\hat{\mu}^4\right)\right)+15\left(c_A+s_F(1+12\hat{\mu}^2)\right)\hat{m}_D\\
&-\frac{55}{4}\Bigg\{c_A\left(\log\frac{\hat{\Lambda}}{2}-\frac{36}{11}\log\hat{m}_D-2.001\right)-\frac{4}{11}s_F\Bigg[\left(\log\frac{\hat{\Lambda}}{2}-2.337\right)\\
&+\ (24-18\zeta(3))\left(\log\frac{\hat{\Lambda}}{2}-15.662\right)\hat{\mu}^2+120\left(\zeta(5)-\zeta(3)\right)\left(\log\frac{\hat{\Lambda}}{2}-1.0811\right)\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\Bigg]\Bigg\}\hat{m}_D^2\\
&-45\,s_F\left\{\log\frac{\hat{\Lambda}}{2}+2.198-44.953\hat{\mu}^2-\left(288\ln\frac{\hat{\Lambda}}{2}+19.836\right)\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right\}\hat{m}_q^2\\
&+\frac{165}{2}\left\{c_A\left(\log\frac{\hat{\Lambda}}{2}+\frac{5}{22}+\gamma\right)-\frac{4}{11}s_F\left(\log\frac{\hat{\Lambda}}{2}-\frac{1}{2}+\gamma+2\ln 2-7\zeta(3)\hat{\mu}^2+31\zeta(5)\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right\}\hat{m}_D^3\\
&+15s_F\left(2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\hat{m}_D\right)\left[(24-18\zeta(3))\hat{\mu}^2+120(\zeta(5)-\zeta(3))\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]\hat{m}_D^3+180\,s_F\hat{m}_D\hat{m}_q^2\Bigg]\Bigg\}.
\end{aligned} \tag{51}$$

For convenience and comparison with lattice data [6], we define the pressure difference

$$\Delta P(T,\mu)=P(T,\mu)-P(T,0)\,. \tag{52}$$

VI. PRESSURE

In the previous section we have computed both LO and NLO thermodynamic potential in presence of quark chemical potential and temperature. All other thermodynamic quantities can be calculated using standard thermodynamic relations. The pressure is defined as

$$P = -\Omega(T, \mu, m_q, m_D)\,, \tag{53}$$

where m_D and m_q are determined by requiring

$$\begin{aligned}\frac{\partial \Omega_{\rm NLO}}{\partial \hat{m}_D} &= 0\,,\\ \frac{\partial \Omega_{\rm NLO}}{\partial \hat{m}_q} &= 0\,.\end{aligned} \tag{54}$$

This leads to the following two gap equations which will be solved numerically

$$\begin{aligned}45\hat{m}_D^2\left[1+\left(\ln\frac{\hat{\Lambda}}{2}-\frac{7}{2}+\gamma+\frac{\pi^2}{3}\right)\hat{m}_D\right] &= \frac{\alpha_s}{\pi}\Bigg\{15(c_A+s_F(1+12\hat{\mu}^2))-\frac{55}{2}\left[c_A\left(\ln\frac{\hat{\Lambda}}{2}-\frac{36}{11}\ln\hat{m}_D-3.637\right)\right.\\ &\quad -\frac{4}{11}s_F\left\{\ln\frac{\hat{\Lambda}}{2}-2.333+(24-18\zeta(3))\left(\ln\frac{\hat{\Lambda}}{2}-15.662\right)\hat{\mu}^2\right.\\ &\quad \left.\left.+120(\zeta(5)-\zeta(3))\left(\ln\frac{\hat{\Lambda}}{2}-1.0811\right)\hat{\mu}^4\right\}\right]\hat{m}_D+\frac{495}{2}\left[c_A\left(\ln\frac{\hat{\Lambda}}{2}+\frac{5}{22}+\gamma\right)\right.\\ &\quad -\frac{4}{11}s_F\left\{\ln\frac{\hat{\Lambda}}{2}-\frac{1}{2}+\gamma+2\ln 2-7\zeta(3)\hat{\mu}^2+31\zeta(5)\hat{\mu}^4\right.\\ &\quad \left.\left.-\left(\frac{\zeta'(-1)}{\zeta(-1)}+\ln m_D+\frac{1}{3}\right)\left((24-18\zeta(3))\hat{\mu}^2+120(\zeta(5)-\zeta(3))\hat{\mu}^4\right)\right\}\right]m_D^2+180s_F\hat{m}_q^2\Bigg\},\end{aligned} \tag{55}$$

and

$$\hat{m}_q^2 = \frac{d_A}{8d_F\left(\pi^2-6\right)}\frac{\alpha_s s_F}{\pi}\left[3\left(\ln\frac{\hat{\Lambda}}{2}+2.198-44.953\,\hat{\mu}^2-\left(288\ln\frac{\hat{\Lambda}}{2}+19.836\right)\hat{\mu}^4\right)-12\hat{m}_D\right]. \tag{56}$$

In Figs. 3 and 4 we present a comparison of NLO HTLpt pressure with that of four-loop pQCD [17] as a function of the temperature for two and three loop running of α_s. The only difference between Figs. 3 and 4 is the choice of order of the running coupling used. As can be seen from these figures, the dependence on the order of the running coupling is quite small. However, we note that in both figures even at extremely large temperatures there is a sizable correction when going from LO to NLO. This was already seen in the $T = 0$ results of Ref. [37] where it was found that due the logarithmic running of the coupling, it was necessary to go to very large temperatures in order for the LO and NLO predictions to overlap. This is due to over-counting problems at LO which lead to an order-g^2 perturbative coefficient which is twice as large as it should be [37, 45]. This problem is corrected at NLO, but the end result is that there is a reasonably large correction ($\sim 5\%$) at the temperatures shown.

The NLO HTLpt result differs from the pQCD result through order $\alpha_s^3 \ln \alpha_s$ at low temperatures. A NNLO HTLpt calculation at finite μ would agree better with pQCD $\alpha_s^3 \ln \alpha_s$ as found in $\mu = 0$ case [38]. The HTLpt result clearly indicates a modest improvement over pQCD in respect of convergence and sensitivity of the renormalisation scale. In Fig. 5 the pressure difference, ΔP, is also compared with the same quantity computed using pQCD [17] and lattice

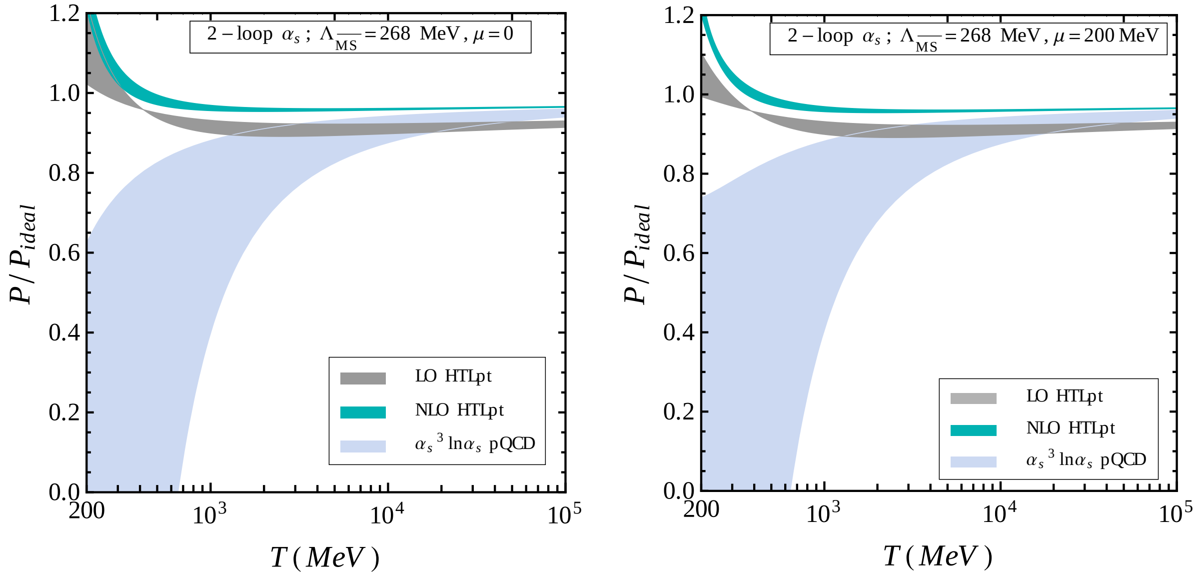

FIG. 3. The NLO HTLpt pressure scaled with ideal gas pressure plotted along with four-loop pQCD pressure [17] for two different values of chemical potential with $N_f = 3$ and 2-loop running coupling constant α_s. The bands are obtained by varying the renormalisation scale by a factor of 2 around its central value $\Lambda = 2\pi\sqrt{T^2 + \mu^2/\pi^2}$ [17, 19]. We use $\Lambda_{\overline{\rm MS}} = 290$ MeV based on recent lattice calculations [20] of the three-loop running of α_s.

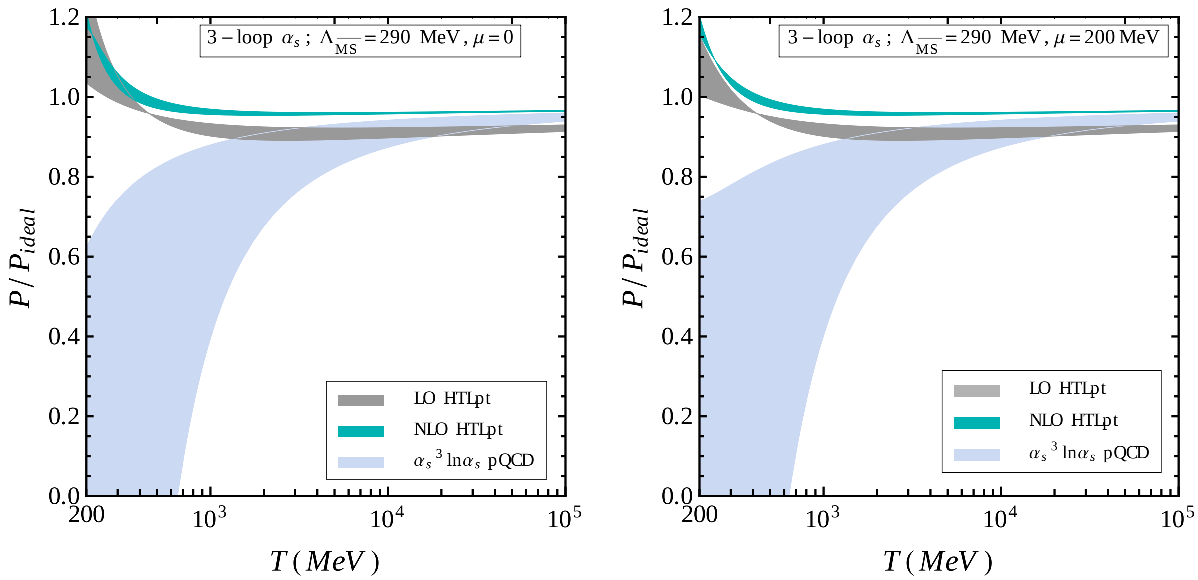

FIG. 4. Same as Fig. 3 but for 3-loop α_s.

QCD [6]. Both LO and NLO HTLpt results are less sensitive to the choice of the renormalisation scale than the weak coupling results with the inclusion of successive orders of approximation. Comparison with available lattice QCD data [6] suggests that HTLpt and pQCD cannot accurately account for the lattice QCD results below approximately $3\,T_c$; however, the results are in very good qualitative agreement with the lattice QCD results without any fine tuning.

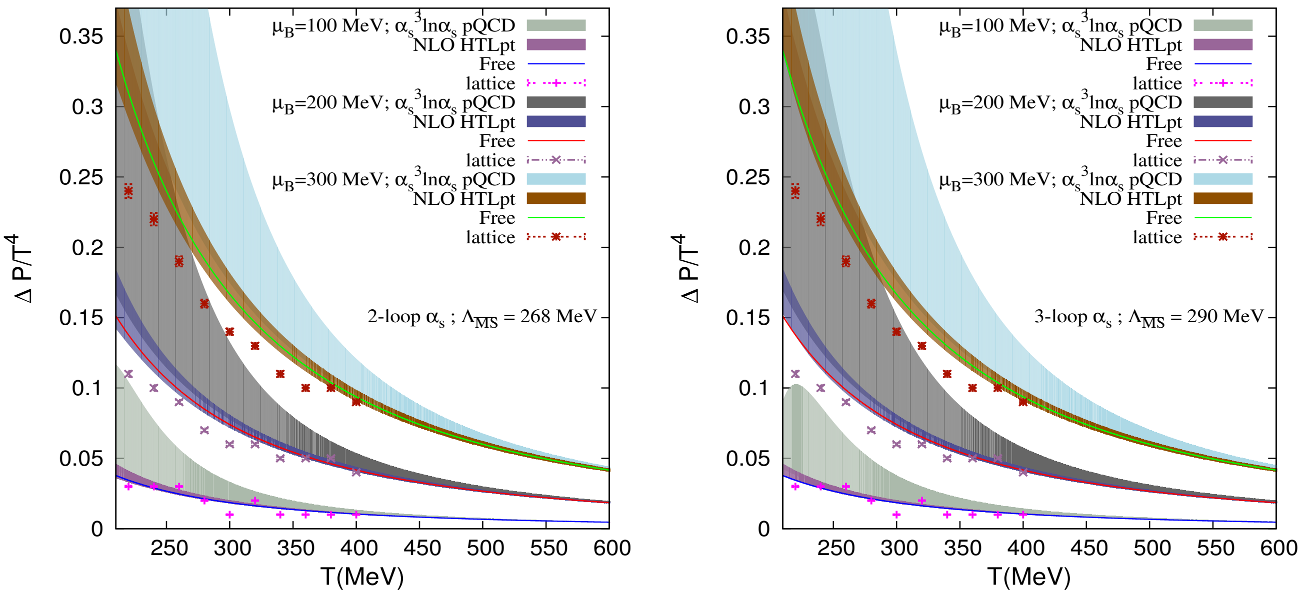

FIG. 5. (Left panel) ΔP for $N_f = 3$ is plotted as a function of T for two-loop HTLpt result along with those of four-loop pQCD up to $\alpha_s^3 \ln \alpha_s$ [17] and lattice QCD [6] up to $\mathcal{O}\left(\mu^2\right)$ using 2-loop running coupling constant α_s. (Right panel) Same as left panel but using 3-loop running coupling. In both cases three different values of μ are shown as specified in the legend. The bands in both HTLpt and pQCD are obtained by varying the renormalisation scale by a factor of 2 around its central value $\Lambda = 2\pi\sqrt{T^2 + \mu^2/\pi^2}$ [17, 19].

VII. CONCLUSIONS AND OUTLOOK

In this paper we have generalized the zero chemical potential NLO HTLpt calculation of the QCD thermodynamic potential [37] to finite chemical potential. We have obtained (semi-)analytic expressions for the thermodynamic potential at both LO and NLO in HTLpt. The results obtained are trustworthy at high temperatures and small chemical potential since we performed an expansion in the ratio of the chemical potential over the temperature.

This calculation will be useful for the study of finite temperature and chemical potential QCD matter. This is important in view of the ongoing RHIC beam energy scan and proposed heavy-ion experiments at FAIR. Using the NLO HTLpt thermodynamic potential, we have obtained a variational solution for both mass parameters, m_q and m_D, and we have used this to obtain the pressure at finite temperature and chemical potential. When compared with the weak coupling expansion of QCD, the HTLpt pressure helps somewhat with the problem of oscillation of successive approximations found in pQCD. Furthermore, the scale variation of the NLO HTLpt result for pressure is smaller than that obtained with the weak coupling result. The HTLpt pressure shows some deviations from the lattice data below 3 T_c which suggests that the calculation should be extended to NNLO. In addition, getting better agreement with pQCD at low temperature will require going to NNLO. This is indeed a very challenging job which represents work in progress.

We also note that, based on the results obtained herein, one can straightforwardly compute quark susceptibilities. In a forthcoming paper we will compare the NLO HTLpt results for quark susceptibilities with lattice data and other theoretical models of QCD matter.

ACKNOWLEDGMENTS

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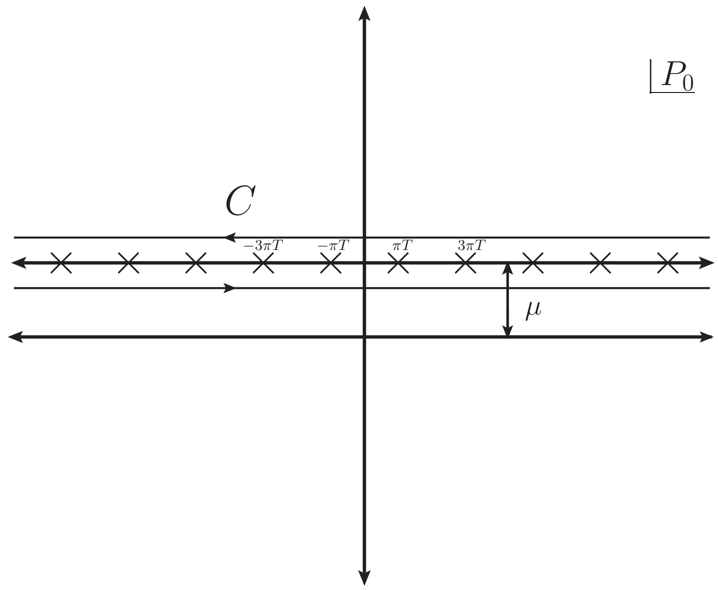

FIG. 6. The contour corresponding to (A.3) in complex P_0 plane. The crosses are poles of the thermal weight factor which are shifted by an amount μ from the Re$[P_0]$ axis.

Appendix A: Sum-Integrals

In the imaginary-time (Euclidean time) formalism for the field theory of a hot and dense medium, the 4-momentum $P = (P_0, \mathbf{p})$ is Euclidean with $P^2 = P_0^2 + \mathbf{p}^2$. The Euclidean energy P_0 has discrete values: $P_0 = 2n\pi T$ for bosons and $P_0 = (2n+1)\pi T - i\mu$ for fermions, where n is an integer running from $-\infty$ to ∞, μ is the quark chemical potential, and $T = 1/\beta$ is the temperature of the medium. Loop diagrams usually then involve sums over P_0 and integrals over $\mathbf{p}$. In dimensional regularization, the integral over spatial momentum is generalized to $d = 3 - 2\epsilon$ spatial dimensions. We define the dimensionally regularized sum-integral as

$$\sum\!\!\!\!\!\!\int_P \equiv \left(\frac{e^\gamma \Lambda^2}{4\pi}\right)^\epsilon T \sum_{P_0 = 2n\pi T} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \,, \tag{A.1}$$

$$\sum\!\!\!\!\!\!\int_{\{P\}} \equiv \left(\frac{e^\gamma \Lambda^2}{4\pi}\right)^\epsilon T \sum_{P_0 = (2n+1)\pi T - i\mu} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \,, \tag{A.2}$$

where $3-2\epsilon$ is the dimension of space, Λ is an arbitrary momentum scale, P is the bosonic loop momentum, and $\{P\}$ is the fermionic loop momentum. The factor $(e^\gamma/4\pi)^\epsilon$ is introduced so that, after minimal subtraction of the poles in ϵ due to ultraviolet divergences, Λ coincides with the renormalization scale of the $\overline{\text{MS}}$ renormalization scheme.

We describe below the technique of contour integration [12, 13] in the complex plane to evaluate the frequency sum over P_0. Consider a meromorphic function $f(P_0)$ that originates from a loop diagram, then one can write

$$T \sum_{P_0 = (2n+1)\pi T - i\mu} f(P_0) = T \oint_C \frac{dP_0}{2\pi i} f(P_0) \frac{\beta}{2} \tanh \frac{\beta(iP_0 - \mu)}{2} = -\frac{T}{2\pi i} \times \frac{\beta}{2} \times (2\pi i) \sum \text{Residues} \,, \tag{A.3}$$

provided $f(P_0)$ is regular in $\text{Re}(iP_0) = \mu$ line as shown in Fig. 6. Below we demonstrate two examples, a simpler one involving only loop momentum and a complicated one involving fourth power of loop momentum and the HTL angular function, which would be relevant for evaluating sum-integrals:

(i) Simpler one:

$$\sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2} = \left(\frac{e^{\gamma}\Lambda^2}{4\pi}\right)^{\epsilon} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}\, T \sum_{P_0=(2n+1)\pi T - i\mu} \frac{1}{2p}\left[\frac{1}{iP_0+p} - \frac{1}{iP_0-p}\right]$$

$$= -\left(\frac{e^{\gamma}\Lambda^2}{4\pi}\right)^{\epsilon} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \left(\frac{n_F(p)}{2p}\right), \tag{A.4}$$

where $n_F(p) = [e^{\beta(p-\mu)}+1]^{-1} + [e^{\beta(p+\mu)}+1]^{-1} = [n_F^-(p) + n_F^+(p)]$.

(ii) Involving HTL term:

$$\sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^4}\mathcal{T}_P = \sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^4}\left\langle \frac{P_0^2}{P_0^2+p^2c^2}\right\rangle_c$$

$$= \left\langle \frac{1}{1-c^2}\right\rangle_c \sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^4} + \left\langle \frac{c^2}{1-c^2} \sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2(P_0^2+p^2c^2)}\right\rangle_c$$

$$= \left\langle \frac{1}{1-c^2}\right\rangle_c \sum\!\!\!\!\!\!\int_{\{P\}} \left(-\frac{1}{2p}\right)\frac{d}{dp}\frac{1}{P^2} + \left\langle \frac{c^2-c^{1+2\epsilon}}{(1-c^2)^2}\right\rangle_c \sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{p^2P^2}$$

$$= \frac{1}{2}\left(\frac{e^{\gamma}\Lambda^2}{4\pi}\right)^{\epsilon} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}\, \frac{1}{p}\frac{d}{dp}\left[\frac{n_F(p)}{2p}\right] - \left\langle \frac{c^2-c^{1+2\epsilon}}{(1-c^2)^2}\right\rangle_c \left(\frac{e^{\gamma}\Lambda^2}{4\pi}\right)^{\epsilon} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \left[\frac{n_F(p)}{2p^3}\right]. \tag{A.5}$$

After performing the frequency sum, one is left with dimensionally regularized spatial momentum integration, which are also discussed in Appendix B. However, all other frequency sums can be evaluated in similar way as discussed above.

1. Simple one loop sum-integrals

The specific fermionic one-loop sum-integrals needed are

$$\sum\!\!\!\!\!\!\int_{\{P\}} \ln P^2 = \frac{7\pi^2}{360}T^4\left(1+\frac{120\,\hat\mu^2}{7}+\frac{240\,\hat\mu^4}{7}\right). \tag{A.6}$$

$$\sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2} = -\frac{T^2}{24}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\Bigg[1+12\,\hat\mu^2+\epsilon\left(2-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+24(\gamma+2\ln 2)\,\hat\mu^2-28\zeta(3)\,\hat\mu^4+\mathcal{O}\left(\hat\mu^6\right)\right)$$
$$+\epsilon^2\left(4+\frac{\pi^2}{4}-4\ln 2-2\ln^2 2+4(1-\ln 2)\frac{\zeta'(-1)}{\zeta(-1)}+2\frac{\zeta''(-1)}{\zeta(-1)}+94.5749\,\hat\mu^2-143.203\,\hat\mu^4+\mathcal{O}\left(\hat\mu^6\right)\right)\Bigg]. \tag{A.7}$$

$$\sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^4} = \frac{1}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\Bigg[\frac{1}{\epsilon}+\left(2\gamma+4\ln 2-14\,\zeta(3)\,\hat\mu^2+62\,\zeta(5)\,\hat\mu^4+\mathcal{O}\left(\hat\mu^6\right)\right)$$
$$+\epsilon\left(4(2\gamma+\ln 2)\ln 2-4\gamma_1+\frac{\pi^2}{4}-71.6013\,\hat\mu^2+356.329\,\hat\mu^4+\mathcal{O}\left(\hat\mu^6\right)\right)\Bigg]. \tag{A.8}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{p^2}{P^4} = -\frac{T^2}{16}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\Bigg[1+12\,\hat{\mu}^2+\epsilon\left(\frac{4}{3}-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+8(3\gamma+6\ln 2-1)\,\hat{\mu}^2\right.$$
$$\left.-\,28\,\zeta(3)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\Bigg]. \tag{A.9}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{p^2}{P^6} = \frac{1}{(4\pi)^2}\frac{3}{4}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\left[\frac{1}{\epsilon}+\left(2\gamma-\frac{2}{3}+4\ln 2-14\,\zeta(3)\,\hat{\mu}^2+62\,\zeta(5)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right]. \tag{A.10}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{p^4}{P^6} = -\frac{5T^2}{64}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\Bigg[1+12\,\hat{\mu}^2+\epsilon\left(\frac{14}{15}-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+8\left(-\frac{8}{5}+3\gamma+6\ln 2\right)\,\hat{\mu}^2\right.$$
$$\left.-\,28\,\zeta(3)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\Bigg]. \tag{A.11}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{p^4}{P^8} = \frac{1}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\frac{5}{8}\left[\frac{1}{\epsilon}+\left(2\gamma-\frac{16}{15}+4\ln 2-14\,\zeta(3)\,\hat{\mu}^2+62\,\zeta(5)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right]. \tag{A.12}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{p^2P^2} = \frac{1}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}2\Bigg[\frac{1}{\epsilon}+\left(2+2\gamma+4\ln 2-14\,\zeta(3)\,\hat{\mu}^2+62\,\zeta(5)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)$$
$$+\epsilon\left(4+8\ln 2+4\ln^2 2+4\gamma+8\gamma\ln 2+\frac{\pi^2}{4}-4\gamma_1-105.259\,\hat{\mu}^2+484.908\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\Bigg]. \tag{A.13}$$

2. HTL one loop sum-integrals

We also need some more difficult one-loop sum-integrals that involve the HTL function defined in (18). The specific fermionic sum-integrals needed are

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^4}\mathcal{T}_P = \frac{1}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\frac{1}{2}\left[\frac{1}{\epsilon}+\left(1+2\gamma+4\ln 2-14\,\zeta(3)\,\hat{\mu}^2+62\,\zeta(5)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right]. \tag{A.14}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{p^2P^2}\mathcal{T}_P = \frac{2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\Bigg[\frac{\ln 2}{\epsilon}+\left(\frac{\pi^2}{6}+\ln 2\left(2\gamma+5\ln 2-14\,\zeta(3)\,\hat{\mu}^2+62\,\zeta(5)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right)$$
$$+\,\epsilon\left(17.5137-85.398\,\hat{\mu}^2+383.629\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\Bigg]. \tag{A.15}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2P_0^2}\mathcal{T}_P = \frac{1}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon}\Bigg[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}2\left(\gamma+2\ln 2-7\,\zeta(3)\,\hat{\mu}^2+31\,\zeta(5)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)$$
$$+\left(\frac{\pi^2}{4}+4\ln^2 2+8\gamma\ln 2-4\gamma_1-71.6014\,\hat{\mu}^2+356.329\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\Bigg]. \tag{A.16}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{p^2 P_0^2} \left(\mathcal{T}_P\right)^2 = \frac{4}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \ln 2 \left[\frac{1}{\epsilon} + (2\gamma + 5\ln 2) - 14\ \zeta(3)\ \hat{\mu}^2 + 62\ \zeta(5)\ \hat{\mu}^4 + \mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{A.17}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^2} \left\langle \frac{1}{(P\cdot Y)^2} \right\rangle_{\hat{\mathbf{y}}} = -\frac{1}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[\frac{1}{\epsilon} - 1 + 2\gamma + 4\ln 2 - 14\ \zeta(3)\ \hat{\mu}^2 + 62\ \zeta(5)\ \hat{\mu}^4 + \mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{A.18}$$

3. Simple two loop sum-integrals

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2 Q^2 R^2} = \frac{T^2}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon} \left[\frac{\hat{\mu}^2}{\epsilon} + 2(4\ln 2 + 2\gamma + 1)\ \hat{\mu}^2 - \frac{28}{3}\zeta(3)\ \hat{\mu}^4 + \mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{A.19}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2 Q^2 r^2} = \frac{T^2}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon} \left(-\frac{1}{6}\right) \left[\frac{1}{\epsilon}\left(1 + 12\hat{\mu}^2\right) + 4 - 2\ln 2 + 4\frac{\zeta'(-1)}{\zeta(-1)} + 48\left(1 + \gamma + \ln 2\right)\ \hat{\mu}^2 - 76\zeta(3)\ \hat{\mu}^4 + \mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{A.20}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{P^2 Q^2 r^4} = \frac{T^2}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon} \left(-\frac{1}{12}\right) \left[\frac{1}{\epsilon}\left(1 + 12\hat{\mu}^2\right) + \left(\frac{11}{3} + 2\gamma - 2\ln 2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right) + 4\left(7 + 12\gamma + 12\ln 2 - 3\zeta(3)\right)\ \hat{\mu}^2 - 4\left(27\zeta(3) - 20\zeta(5)\right)\ \hat{\mu}^4 + \mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{A.21}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{P\cdot Q}{P^2 Q^2 r^4} = \frac{T^2}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon} \left(-\frac{1}{36}\right) \left[1 - 6\gamma + 6\frac{\zeta'(-1)}{\zeta(-1)} + 24\left\{2 + 3\zeta(3)\right\}\ \hat{\mu}^2 + 48(7\zeta(3) - 10\zeta(5))\ \hat{\mu}^4 + \mathcal{O}(\hat{\mu}^6)\right]. \tag{A.22}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{r^2 P^2 Q^2 R^2} = -\frac{T^2}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon} \frac{1}{72} \left[\frac{1}{\epsilon}\left[1 - 12(1 - 3\zeta(3))\ \hat{\mu}^2 + 240(\zeta(3) - \zeta(5))\ \hat{\mu}^4 + \mathcal{O}\left(\hat{\mu}^6\right)\right] - \left(7.001 - 108.218\ \hat{\mu}^2 - 304.034\ \hat{\mu}^4 + \mathcal{O}(\hat{\mu}^6)\right)\right]. \tag{A.23}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{q^2 P^2 Q^2 R^2} = \frac{T^2}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon} \frac{5}{72} \left[\frac{1}{\epsilon}\left(1 - \frac{12}{5}\left(1 + 7\zeta(3)\right)\ \hat{\mu}^2 - \frac{24}{5}\left(14\zeta(3) - 31\zeta(5)\right)\ \hat{\mu}^4 + \mathcal{O}\left(\hat{\mu}^6\right)\right) + \left(9.5424 - 185.706\ \hat{\mu}^2 + 916.268\ \hat{\mu}^4 + \mathcal{O}(\hat{\mu}^6)\right)\right]. \tag{A.24}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{r^2}{q^2 P^2 Q^2 R^2} = -\frac{T^2}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon} \frac{1}{18} \left[\frac{1}{\epsilon}\left(1 + 3(-2 + 7\zeta(3))\ \hat{\mu}^2 + 6(14\zeta(3) - 31\zeta(5))\ \hat{\mu}^4 + \mathcal{O}(\hat{\mu}^6)\right) + \left(8.1428 + 96.9345\ \hat{\mu}^2 - 974.609\ \hat{\mu}^4 + \mathcal{O}(\hat{\mu}^6)\right)\right]. \tag{A.25}$$

The generalized two loop sum-integrals can be written from [37] as

$$\sum\!\!\!\!\!\!\int_{\{PQ\}} F(P)G(Q)H(R) = \int_{PQ} F(P)G(Q)H(R) - \int_{p_0,\mathbf{p}} \epsilon(p_0)n_F(|p_0|)\, 2\,\mathrm{Im}F(-ip_0+\varepsilon,\mathbf{p})\,\mathrm{Re}\int_Q G(Q)H(R)\Big|_{P_0=-ip_0+\varepsilon}$$

$$- \int_{p_0,\mathbf{p}} \epsilon(p_0)n_F(|p_0|)\, 2\,\mathrm{Im}G(-ip_0+\varepsilon,\mathbf{p})\,\mathrm{Re}\int_Q H(Q)F(R)\Big|_{P_0=-ip_0+\varepsilon} \tag{A.26}$$

$$+ \int_{p_0,\mathbf{p}} \epsilon(p_0)n_B(|p_0|)\, 2\,\mathrm{Im}H(-ip_0+\varepsilon,\mathbf{p})\,\mathrm{Re}\int_Q F(Q)G(R)\Big|_{P_0=-ip_0+\varepsilon}$$

$$+ \int_{p_0,\mathbf{p}} \epsilon(p_0)n_F(|p_0|)\, 2\,\mathrm{Im}F(-ip_0+\varepsilon,\mathbf{p}) \int_{q_0,\mathbf{q}} \epsilon(q_0)n_F(|q_0|)\, 2\,\mathrm{Im}G(-iq_0+\varepsilon,\mathbf{q})\,\mathrm{Re}H(R)\Big|_{R_0=i(p_0+q_0)+\varepsilon}$$

$$- \int_{p_0,\mathbf{p}} \epsilon(p_0)n_F(|p_0|)\, 2\,\mathrm{Im}G(-ip_0+\varepsilon,\mathbf{p}) \int_{q_0,\mathbf{q}} \epsilon(q_0)n_B(|q_0|)\, 2\,\mathrm{Im}H(-iq_0+\varepsilon,\mathbf{q})\,\mathrm{Re}F(R)\Big|_{R_0=i(p_0+q_0)+\varepsilon}$$

$$- \int_{p_0,\mathbf{p}} \epsilon(p_0)n_B(|p_0|)\, 2\,\mathrm{Im}H(-ip_0+\varepsilon,\mathbf{p}) \int_{q_0,\mathbf{q}} \epsilon(q_0)n_F(|q_0|)\, 2\,\mathrm{Im}F(-iq_0+\varepsilon,\mathbf{q})\,\mathrm{Re}G(R)\Big|_{R_0=i(p_0+q_0)+\varepsilon}. \tag{A.27}$$

After applying Eq. (A.27) and using the delta function to calculate the P_0 and Q_0 integrations, the sum-integral (A.23) reduces to

$$\sum\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2R^2} = \int_{\mathbf{pq}} \frac{n_F^-(p) - n_F^+(p)}{2p}\,\frac{n_F^-(q) - n_F^+(q)}{2q}\,\frac{2p\;q}{\Delta(p,q,r)}\,, \tag{A.28}$$

where

$$n_F^{\pm}(p) = \frac{1}{e^{\beta(p\pm\mu)}+1} \quad \text{and} \quad \Delta(p,q,r) = p^4+q^4+r^4-2(p^2q^2+q^2r^2+p^2r^2) = -4p^2q^2(1-x^2)\,, \tag{A.29}$$

and using the result of Eq. (B.10), we get sum-integral (A.19) and agree with [17].

After applying Eq. (A.27), the sum-integral (A.20) reduces to

$$\sum\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2r^2} = -2\int_{\mathbf{p}} \frac{n_F(p)}{2p}\int_Q \frac{1}{Q^2r^2} + \int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{4pq}\,\frac{1}{r^2}\,, \tag{A.30}$$

where $n_F(p) = n_F^-(p) + n_F^+(p)$. Now using the result of 4-dimensional integrals from [37] and applying Eq. (B.3) and Eq. (B.5), we can calculate sum-integral Eq. (A.20). The sum-integrals (A.21) can be calculated in same way:

$$\sum\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{P^2Q^2r^4} = -2\int_{\mathbf{p}} \frac{n_F(p)}{2p}\int_Q \frac{p^2}{Q^2r^4} + \int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{4pq}\,\frac{p^2}{r^4}. \tag{A.31}$$

The sum-integral (A.22) can be written as

$$\sum\!\!\!\!\!\!\int_{\{PQ\}} \frac{P\cdot Q}{P^2Q^2r^4} = \sum\!\!\!\!\!\!\int_{\{PQ\}} \frac{P_0Q_0}{P^2Q^2r^4} + \frac{1}{2}\sum\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2r^2} - \sum\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{P^2Q^2r^4} \tag{A.32}$$

Using Eq. (A.27) and after doing P_0 and Q_0 integrations, first sum-integral above reduces to

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{P_0 Q_0}{P^2 Q^2 r^4} = \int_{\mathbf{pq}} \frac{n_F^-(p) - n_F^+(p)}{2\ p}\ \frac{n_F^-(q) - n_F^+(q)}{2\ q}\ \frac{p\ q}{r^4}\,, \tag{A.33}$$

and the result is given in Eq. (B.9). The second term and third terms sum-integrals above are linear combinations of Eq. (A.20) and Eq. (A.21). Adding all of them, we get required sum-integral.

Similarly after applying Eq. (A.27), the sum-integral (A.23) reduces to

$$\begin{aligned}\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{r^2 P^2 Q^2 R^2} &= \int_{\mathbf{p}} \frac{n_B(p)}{p} \int_Q \frac{r^2}{p^2 Q^2 R^2}\bigg|_{P_0=-ip} - \int_{\mathbf{p}} \frac{n_F(p)}{2p} \int_Q \frac{1}{Q^2 R^2}\left(\frac{q^2}{r^2} + \frac{p^2}{q^2}\right)\bigg|_{P_0=-ip} \\ &+ \int_{\mathbf{pq}} \frac{n_F(p) n_F(q)}{4pq}\ \frac{q^2}{r^2}\ \frac{r^2-p^2-q^2}{\Delta(p,q,r)} - \int_{\mathbf{pq}} \frac{n_F(p) n_B(q)}{4pq}\ \frac{p^2+r^2}{q^2}\ \frac{r^2-p^2-q^2}{\Delta(p,q,r)}\,,\end{aligned} \tag{A.34}$$

So

$$\left\langle \frac{p^2+r^2}{q^2}\ \frac{r^2-p^2-q^2}{\Delta(p,q,r)} \right\rangle_{\hat{p}\cdot\hat{q}} = \frac{1}{2q^2\ \epsilon}\,, \tag{A.35}$$

and

$$\begin{aligned}\left\langle \frac{q^2}{r^2}\ \frac{r^2-p^2-q^2}{\Delta(p,q,r)} \right\rangle_{\hat{p}\cdot\hat{q}} &= \left\langle \frac{q^2}{\Delta(p,q,r)} \right\rangle_x - \left\langle \frac{q^2(p^2+q^2)}{\Delta(p,q,r)} \right\rangle_x\,, \\ &= \frac{1-2\epsilon}{8\epsilon}\frac{1}{p^2} - \frac{1}{2\epsilon}\left\langle \frac{q^2}{r^4} \right\rangle_x - \frac{1-2\epsilon}{8\epsilon}\frac{1}{p^2} \\ &= -\frac{1}{2\epsilon}\left\langle \frac{q^2}{r^4} \right\rangle_x\,.\end{aligned} \tag{A.36}$$

Using the above angular integration, Eq. (A.34) becomes

$$\begin{aligned}\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{r^2 P^2 Q^2 R^2} &= \int_{\mathbf{p}} \frac{n_B(p)}{p} \int_Q \frac{r^2}{p^2 Q^2 R^2}\bigg|_{P_0=-ip} - \int_{\mathbf{p}} \frac{n_F(p)}{2p} \int_Q \frac{1}{Q^2 R^2}\left(\frac{q^2}{r^2} + \frac{p^2}{q^2}\right)\bigg|_{P_0=-ip} \\ &- \frac{1}{2\epsilon}\int_{\mathbf{pq}} \frac{n_F(p) n_F(q)}{4pq}\ \frac{p^2}{r^4} - \frac{1}{2\epsilon}\int_{\mathbf{pq}} \frac{n_F(p) n_B(q)}{4pq}\ \frac{1}{q^2}\,.\end{aligned} \tag{A.37}$$

Using the 4-dimensional integrals from [37] and Eqs. (B.2), (B.3), (B.4) and (B.6), we obtain the sum-integral (A.23).

Similarly after applying Eq. (A.27), the sum-integral (A.24) reduces to

$$\begin{aligned}\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{q^2 P^2 Q^2 R^2} &= \int_{\mathbf{p}} \frac{n_B(p)}{p} \int_Q \frac{q^2}{Q^2 r^2 R^2}\bigg|_{P_0=-ip} - \int_{\mathbf{p}} \frac{n_F(p)}{2p} \int_Q \frac{1}{Q^2 R^2}\left(\frac{p^2}{q^2} + \frac{q^2}{p^2}\right)\bigg|_{P_0=-ip} \\ &+ \int_{\mathbf{pq}} \frac{n_F(p) n_F(q)}{4pq}\ \frac{p^2}{q^2}\ \frac{r^2-p^2-q^2}{\Delta(p,q,r)} - \int_{\mathbf{pq}} \frac{n_F(p) n_B(q)}{4pq}\ \left(\frac{p^2}{r^2} + \frac{r^2}{p^2}\right)\ \frac{r^2-p^2-q^2}{\Delta(p,q,r)}\,.\end{aligned} \tag{A.38}$$

Now

$$\left\langle \frac{p^2}{q^2}\ \frac{r^2-p^2-q^2}{\Delta(p,q,r)} \right\rangle_{\hat{p}\cdot\hat{q}} = 0\,, \tag{A.39}$$

and

$$\left\langle \left(\frac{p^2}{r^2}+\frac{r^2}{p^2}\right) \frac{r^2-p^2-q^2}{\Delta(p,q,r)} \right\rangle_{\hat{p}\cdot\hat{q}} = \frac{1}{2\epsilon}\frac{1}{p^2} - \frac{1}{2\epsilon}\left\langle \frac{p^2}{r^4} \right\rangle_x . \tag{A.40}$$

Using the above angular average, we find

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{q^2P^2Q^2R^2} = \int_{\mathbf{p}} \frac{n_B(p)}{p} \int_Q \frac{q^2}{Q^2r^2R^2}\bigg|_{P_0=-ip} - \int_{\mathbf{p}} \frac{n_F(p)}{2p} \int_Q \frac{1}{Q^2R^2}\left(\frac{p^2}{q^2}+\frac{q^2}{p^2}\right)\bigg|_{P_0=-ip}$$
$$-\frac{1}{2\epsilon}\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{2pq}\,\frac{1}{p^2} + \frac{1}{2\epsilon}\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{2pq}\,\frac{p^2}{r^4} \tag{A.41}$$

Using the 4-dimensional integrals from [37] and Eqs. (B.2), (B.3), (B.4) and (B.7), we obtain the sum-integral (A.24).

Similarly after applying Eq. (A.27), the sum-integral (A.25) reduces to

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{r^2}{p^2P^2Q^2R^2} = \int_{\mathbf{p}} \frac{n_B(p)}{p} \int_Q \frac{p^2}{Q^2r^2R^2}\bigg|_{P_0=-ip} - \int_{\mathbf{p}} \frac{n_F(p)}{2p} \int_Q \frac{1}{Q^2R^2}\left(\frac{r^2}{p^2}+\frac{r^2}{q^2}\right)\bigg|_{P_0=-ip}$$
$$+\int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{4pq}\,\frac{r^2}{p^2}\,\frac{r^2-p^2-q^2}{\Delta(p,q,r)} - \int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{4pq}\left(\frac{q^2}{r^2}+\frac{q^2}{p^2}\right)\frac{r^2-p^2-q^2}{\Delta(p,q,r)} . \tag{A.42}$$

Now

$$\left\langle \frac{r^2}{p^2}\,\frac{r^2-p^2-q^2}{\Delta(p,q,r)} \right\rangle_{\hat{p}\cdot\hat{q}} = \frac{1}{2p^2\epsilon} , \tag{A.43}$$

and

$$\left\langle \left(\frac{q^2}{r^2}+\frac{q^2}{p^2}\right) \frac{r^2-p^2-q^2}{\Delta(p,q,r)} \right\rangle_{\hat{p}\cdot\hat{q}} = -\frac{1}{2\epsilon}\left\langle \frac{q^2}{r^4} \right\rangle_x . \tag{A.44}$$

Using the above angular average, we have

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{q^2P^2Q^2R^2} = \int_{\mathbf{p}} \frac{n_B(p)}{p} \int_Q \frac{q^2}{Q^2r^2R^2}\bigg|_{P_0=-ip} - \int_{\mathbf{p}} \frac{n_F(p)}{2p} \int_Q \frac{1}{Q^2R^2}\left(\frac{p^2}{q^2}+\frac{q^2}{p^2}\right)\bigg|_{P_0=-ip}$$
$$+\frac{1}{2\epsilon}\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{2pq}\,\frac{1}{p^2} + \frac{1}{2\epsilon}\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{2pq}\,\frac{q^2}{r^4} . \tag{A.45}$$

Using the 4-dimensional integrals from [37] and Eqs. (B.2), (B.3), (B.4) and (B.8), we obtain the sum-integral (A.24).

4. HTL two loop sum-integrals

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{1}{P^2Q^2r^2}\mathcal{T}_R = \frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{48}\right)\left[\frac{1}{\epsilon^2} + \left(2+12(1+8\,\hat{\mu}^2)\ln 2 + 4\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon}\right.$$
$$\left. + \left(136.3618 + 460.23\,\hat{\mu}^2 - 273.046\,\hat{\mu}^4 + \mathcal{O}\left(\hat{\mu}^6\right)\right)\right] . \tag{A.46}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{p^2}{P^2Q^2r^4}\mathcal{T}_R = \frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{576}\right)\left[\frac{1}{\epsilon^2}+\left(\frac{26}{3}+4(13+144\,\hat{\mu}^2)\ln 2+4\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon}\right.$$
$$\left.+\left(446.397+2717.86\,\hat{\mu}^2-1735.61\,\hat{\mu}^4+\mathcal{O}(\hat{\mu}^6)\right)\right]. \tag{A.47}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{P\cdot Q}{P^2Q^2r^4}\mathcal{T}_R = \frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\left(-\frac{1}{96}\right)\left[\frac{1}{\epsilon^2}+\left(4\ln 2+4\frac{\zeta'(-1)}{\zeta'(-1)}\right)\frac{1}{\epsilon}+\left(69.1737+118.244\,\hat{\mu}^2\right.\right.$$
$$\left.\left.+136.688\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right]. \tag{A.48}$$

$$\sum\!\!\!\!\!\!\!\int_{\{PQ\}} \frac{r^2-p^2}{P^2q^2Q_0^2R^2}\mathcal{T}_Q = -\frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\frac{1}{8}\left[\frac{1}{\epsilon^2}\left(1+4\,\hat{\mu}^2\right)+\frac{1}{\epsilon}\left(2+2\gamma+\frac{10}{3}\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}\right.\right.$$
$$\left.+\;2\left(8\gamma+16\ln 2-7\zeta(3)\right)\hat{\mu}^2-\frac{2}{3}\left(98\zeta(3)-93\zeta(5)\right)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)$$
$$\left.+\;\left(46.8757-41.1192\,\hat{\mu}^2+64.0841\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right]. \tag{A.49}$$

Appendix B: Integrals

1. Three dimensional integrals

We require one integral that does not involve the Bose-Einstein distribution function. The momentum scale in these integrals is set by the mass $m = m_D$. The one-loop integral is

$$\int_{\mathbf{p}} \frac{1}{p^2+m^2} = -\frac{m}{4\pi}\left(\frac{\Lambda}{2m}\right)^{2\epsilon}\left[1+2\epsilon\right]. \tag{B.1}$$

2. Thermal Integrals

$$\frac{\Lambda^{2\epsilon}}{(4\pi)^2}\int_{\mathbf{p}} \frac{n_B(p)}{p}p^{-2\epsilon} = \frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\left(\frac{1}{12}\right)\left\{1+\epsilon\left[2-2\ln 2+4\frac{\zeta'(-1)}{\zeta(-1)}\right]\right.$$
$$\left.+\;2\epsilon^2\left[\frac{7\pi^2}{8}-2+\ln^2 2-2\ln 2+4(1+\ln 2)\left(1+\frac{\zeta'(-1)}{\zeta(-1)}\right)+4\frac{\zeta''(-1)}{\zeta(-1)}\right]\right\}. \tag{B.2}$$

$$\frac{\Lambda^{2\epsilon}}{(4\pi)^2}\int_{\mathbf{p}} \frac{n_F(p)}{2p}p^{-2\epsilon} = \frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\left(\frac{1}{24}\right)\left[\left(1+12\hat{\mu}^2\right)+\epsilon\left\{2-2\ln 2+4\frac{\zeta'(-1)}{\zeta(-1)}\right.\right.$$
$$\left.\left.+\;24\left(2\gamma+5\ln 2-1\right)\,\hat{\mu}^2-56\zeta(3)\,\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right\}\right]. \tag{B.3}$$

$$\frac{\Lambda^{2\epsilon}}{(4\pi)^2}\int_{\mathbf{p}}\frac{n_F(p)}{2p}\frac{1}{p^2}p^{-2\epsilon}=-\frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\left[\frac{1}{\epsilon}+2+2\gamma+10\ln 2-28\zeta(3)\ \hat{\mu}^2+124\zeta(5)\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{B.4}$$

$$\int_{\mathbf{pq}}\frac{n_F(p)n_F(q)}{4pq}\frac{1}{r^2}=\frac{T^2}{(4\pi)^2}\left[\frac{1}{3}(1-\ln 2)+4(2\ln 2-1)\hat{\mu}^2+\frac{10}{3}\zeta(3)\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{B.5}$$

$$\int_{\mathbf{pq}}\frac{n_F(p)n_F(q)}{4pq}\frac{p^2}{r^4}=\frac{T^2}{(4\pi)^2}\left(-\frac{1}{36}\right)\Bigg[\Bigg(5+6\gamma+6\ln 2-6\frac{\zeta'(-1)}{\zeta(-1)}-12(-13+12\ln 2+3\zeta(3))\hat{\mu}^2$$
$$+12\left(-13\zeta(3)+20\zeta(5)\right)\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\Bigg)+\epsilon\left(3.0747+31.2624\ \hat{\mu}^2+262.387\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\Bigg]. \tag{B.6}$$

$$\int_{\mathbf{pq}}\frac{n_B(p)n_F(q)}{2pq}\frac{p^2}{r^4}=\frac{T^2}{(4\pi)^2}\left(-\frac{1}{36}\right)\Bigg[\Bigg\{7-6\gamma-18\ln 2+6\frac{\zeta'(-1)}{\zeta(-1)}+6(-22+21\zeta(3))\ \hat{\mu}^2$$
$$+6\left(126\zeta(3)-155\zeta(5)\right)\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\Bigg\}+\epsilon\left(29.5113+158.176\ \hat{\mu}^2-557.189\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\Bigg]. \tag{B.7}$$

$$\int_{\mathbf{pq}}\frac{n_B(p)n_F(q)}{2pq}\frac{q^2}{r^4}=\frac{T^2}{(4\pi)^2}\left(\frac{1}{18}\right)\Bigg[\Bigg\{1-6\gamma-12\ln 2+6\frac{\zeta'(-1)}{\zeta(-1)}+12\hat{\mu}^2-6\left(28\zeta(3)-31\zeta(5)\right)\ \hat{\mu}^4$$
$$+\mathcal{O}\left(\hat{\mu}^6\right)\Bigg\}+\epsilon\left(31.0735+222.294\ \hat{\mu}^2-\ 416.474\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{B.8}$$

$$\int_{\mathbf{pq}}\frac{n_F^-(p)-n_F^+(p)}{2\ p}\ \frac{n_F^-(q)-n_F^+(q)}{2\ q}\ \frac{p\ q}{r^4}=\frac{T^2}{(4\pi)^2}\frac{1}{3}\left[(1-3\zeta(3))\ \hat{\mu}^2-20(\zeta(3)-\zeta(5))\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{B.9}$$

Thermal integrals containing the triangle function:

$$\int_{\mathbf{pq}}\frac{n_F^-(p)-n_F^+(p)}{2p}\frac{n_F^-(q)-n_F^+(q)}{2q}\frac{2p\ q}{\Delta(p,q,r)}=\frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\left[\frac{\hat{\mu}^2}{\epsilon}+2(4\ln 2+2\gamma+1)\ \hat{\mu}^2-\frac{28}{3}\zeta(3)\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{B.10}$$

Thermal integrals containing both the triangle function and HTL average are listed below:

$$\int_{\mathbf{pq}}\frac{n_F(p)n_F(q)}{4pq}\mathrm{Re}\left\langle c^2\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c=\frac{T^2}{(4\pi)^2}\left[0.014576+0.238069\ \hat{\mu}^2+0.825164\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{B.11}$$

$$\int_{\mathbf{pq}}\frac{n_F(p)n_F(q)}{4pq}\mathrm{Re}\left\langle c^4\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c=\frac{T^2}{(4\pi)^2}\left[0.017715+0.28015\ \hat{\mu}^2+0.87321\ \hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]. \tag{B.12}$$

$$\int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{4pq}\mathrm{Re}\left\langle \frac{q^2}{r^2}c^2\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c = -\frac{T^2}{(4\pi)^2}\left[0.01158+0.17449\;\hat{\mu}^2+0.45566\;\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]. \quad \text{(B.13)}$$

$$\int_{\mathbf{pq}} \frac{n_B(p)n_F(q)}{2pq}\mathrm{Re}\left\langle \frac{p^2-q^2}{r^2}\frac{r^2c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,rc)}\right\rangle_c = \frac{T^2}{(4\pi)^2}\left[0.17811+1.43775\;\hat{\mu}^2-2.45413\;\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right]. \quad \text{(B.14)}$$

Second set of integrals involve the variables $r_c = |\mathbf{p}+\mathbf{q}/c|$:

$$\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{2pq}\mathrm{Re}\left\langle c^{-1+2\epsilon}\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}\left[0.19678+1.07745\;\hat{\mu}^2-2.63486\;\hat{\mu}^4+\mathcal{O}\left(\;\hat{\mu}^6\right)\right]. \quad \text{(B.15)}$$

$$\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{2pq}\mathrm{Re}\left\langle c^{1+2\epsilon}\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c = \frac{T^2}{(4\pi)^2}\left[0.048368+0.23298\;\hat{\mu}^2-0.65074\;\hat{\mu}^4+\mathcal{O}\left(\;\hat{\mu}^6\right)\right]. \quad \text{(B.16)}$$

$$\begin{aligned}\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{2pq}\frac{p^2}{q^2}\mathrm{Re}\left\langle c^{1+2\epsilon}\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c &= \frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\frac{1}{96}\left[\left(1+12\;\hat{\mu}^2\right)\frac{1}{\epsilon}\right.\\ &\left.+\left(7.77236+81.1057\;\hat{\mu}^2-48.5858\;\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right]. \quad \text{(B.17)}\end{aligned}$$

$$\begin{aligned}\int_{\mathbf{pq}} \frac{n_F(p)n_B(q)}{2pq}\mathrm{Re}\left\langle c^{1+2\epsilon}\frac{r_c^2}{q^2}\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c &= \frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\frac{11-8\ln 2}{288}\left[\frac{1}{\epsilon}\left(1+12\;\hat{\mu}^2\right)\right.\\ &\left.+\left(7.7995+70.5162\;\hat{\mu}^2-57.9278\;\hat{\mu}^4+\mathcal{O}\left(\;\hat{\mu}^6\right)\right)\right]. \quad \text{(B.18)}\end{aligned}$$

$$\begin{aligned}\int_{\mathbf{pq}} \frac{n_F(p)n_F(q)}{4pq}\mathrm{Re}\left\langle c^{-1+2\epsilon}\frac{r_c^2-p^2}{q^2}\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c &= -\frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\frac{1}{24}\left[\left(1+12\;\hat{\mu}^2\right)\frac{1}{\epsilon^2}\right.\\ +\frac{2}{\epsilon}\left(1+\gamma+\ln 2+\frac{\zeta'(-1)}{\zeta(-1)}+(24\gamma+48\ln 2-7\zeta(3))\;\hat{\mu}^2+(31\zeta(5)-98\zeta(3))\;\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)&\\ &\left.+\left(40.3158+261.822\;\hat{\mu}^2-1310.69\;\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right]. \quad \text{(B.19)}\end{aligned}$$

$$\begin{aligned}\int_{\mathbf{pq}} \frac{n_B(p)n_F(q)}{2pq}\quad\mathrm{Re}\left\langle c^{-1+2\epsilon}\frac{r_c^2-p^2}{q^2}\frac{r_c^2-p^2-q^2}{\Delta(p+i\varepsilon,q,r_c)}\right\rangle_c &= -\frac{T^2}{(4\pi)^2}\left(\frac{\Lambda}{4\pi T}\right)^{4\epsilon}\frac{1}{12}\left[\frac{1}{\epsilon^2}\right.\\ +\frac{1}{\epsilon}\left(2+2\gamma+4\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}-14\zeta(3)\;\hat{\mu}^2+62\zeta(5)\;\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)&\\ \left.+\left(52.953-190.103\;\hat{\mu}^2+780.921\;\hat{\mu}^4+\mathcal{O}\left(\hat{\mu}^6\right)\right)\right]. &\quad \text{(B.20)}\end{aligned}$$

The integral (B.12) can be evaluated directly in three dimensions at finite chemical potential. The other integrals Eqs. (B.13)–(B.20) can be evaluated following the same procedure as discussed in [37] at finite chemical potential.

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