



# CHORUS

This is the accepted manuscript made available via CHORUS. The article has been published as:

## Zero interface tensions at the deconfining phase transition for a matrix model of a $SU(\infty)$ gauge theory

Shu Lin, Robert D. Pisarski, and Vladimir V. Skokov

Phys. Rev. D **87**, 105002 — Published 2 May 2013

DOI: [10.1103/PhysRevD.87.105002](https://doi.org/10.1103/PhysRevD.87.105002)

# Zero interface tensions at the deconfining phase transition for a matrix model of a $SU(\infty)$ gauge theory

Shu Lin\*

*RIKEN/BNL, Brookhaven National Laboratory, Upton, NY 11973*

Robert D. Pisarski†

*Department of Physics, Brookhaven National Laboratory, Upton, NY 11973 and*

*RIKEN/BNL, Brookhaven National Laboratory, Upton, NY 11973*

Vladimir V. Skokov‡

*Department of Physics, Brookhaven National Laboratory, Upton, NY 11973*

## Abstract

Using a matrix model, we model the deconfining phase transition at nonzero temperature for a  $SU(N)$  gauge theory at large  $N$ . At infinite  $N$  the matrix model exhibits a Gross-Witten-Wadia transition. We show that at the critical temperature  $T_d$ , both the order-disorder and the order-order interface tensions vanish identically. We estimate how these quantities approach zero in the matrix model as  $T \rightarrow T_d$  and as  $N \rightarrow \infty$ . The numerical solution of the matrix model suggests possible non-monotonic behavior in  $N$  for relatively small values of  $N$ ,  $N \sim 5$ .

---

\* slin@quark.phy.bnl.gov

† pisarski@bnl.gov

‡ vskokov@quark.phy.bnl.gov

## I. INTRODUCTION

Because of dimensional transmutation, the properties of  $SU(N)$  gauge theories are of great interest. Once one mass scale is set, all other properties are determined, at least in principle. For example, if the theory exhibits a deconfining phase transition at a temperature  $T_d$ , then the nature of the phase transition is completely fixed.

Understanding the deconfining phase transition requires numerical simulations on the lattice. These have been carried out both for a small [1, 2] and a large [3] number of colors. Using these results, a matrix model was developed to model deconfinement. These models involve zero [4, 5], one [6], and two [7, 8] parameters. They are soluble analytically for two and three colors, and numerically for four or more colors.

This matrix model is also soluble analytically in the limit of infinite  $N$  [9]. The phase transition at infinite  $N$  is exceptional, and can be termed a Gross-Witten-Wadia transition [10–12]. For the deconfining transition, such a transition was first seen to occur on a femtosphere [13–17]. For such a transition, at infinite  $N$  it exhibits aspects of both first and second order phase transitions. It is of first order, in that the latent heat is nonzero and proportional to  $\sim N^2$ ; also, the Polyakov loop jumps from 0 to  $1/2$  at  $T_d$ . On the other hand, it is of second order in that several quantities, such as the specific heat, exhibit nontrivial critical exponents. Such an unusual transition only occurs at infinite  $N$ , as at finite  $N$  the transition is of first order.

Consider a phase transition which can be characterized by the change of a single field. If the transition is of first order, then at the transition temperature there are two degenerate minima, with a nonzero barrier between them. The order-disorder interface tension is nonzero, given by the probability to tunnel between the two minima. Conversely, if the transition is of second order, at the transition temperature the two minima coincide. There is no barrier to tunnel between the two minima, and the order-disorder interface tension vanishes.

In a gauge theory asking about the potential is more subtle. In a matrix model of an  $SU(N)$  gauge theory, the potential exists in  $N - 1$  dimensions (the number of commuting diagonal generators). A physical question is to ask how the interface tensions behave, especially at the phase transition. There are two such interface tensions. There is the usual order-disorder interface tension, which exists only at  $T_d$ . There are also order-order interface

tensions [18–24], which are directly related to the 't Hooft loops for  $Z(N)$  charges [25, 26]. These exist for all  $T \geq T_d$ .

In this paper we consider the interface tensions in the matrix model near  $T_d$ . We find that in the matrix model, at infinite  $N$  both the order-order interface tension and the order-disorder interface tensions vanish identically at the transition temperature,  $T_d$ .

There is a simple heuristic explanation for our results. Consider the potential for the simplest Polyakov loop, that in the fundamental representation,  $\ell_1 = (1/N)\text{tr } \mathbf{L}$ , where  $\mathbf{L}$  is the thermal Wilson line. Then the Gross-Witten-Wadia transition occurs because at infinite  $N$ , the potential for  $\ell_1$  is completely flat between 0 and 1/2. That is, at  $T_d$  there are two distinct minima, as expected for a first order transition. Nevertheless, because the potential is flat at  $N = \infty$  between the two minima, there is no barrier to tunnel between them, and so the interface tensions vanish. Such a flat potential was first found for the Gross-Witten-Wadia transition on a femto-sphere [13–17].

As noted, in a gauge theory there are other degrees of freedom. For example, one can consider higher powers of  $\mathbf{L}$ ,  $\ell_j = (1/N)\text{tr } \mathbf{L}^j$ , for  $j = 2 \dots N - 1$ . For the Gross-Witten-Wadia transition, though, all  $\ell_j$  vanish at  $T_d$  when  $j \geq 2$ , which allows us to consider the potential as a function of a single variable,  $\ell_1$ . This is true only at  $T_d$ , and not for  $T \neq T_d$ . Indeed, while we estimate the behavior of the order-order interface tensions for  $T > T_d$ , we cannot solve the problem in full generality. This is because away from  $T_d$ , *all*  $\ell_j$  contribute, and it is a much more difficult problem than at  $T_d$ .

The order-disorder interface tension,  $\alpha^{od}$ , has been measured through numerical simulations on the lattice at  $N = 3$  by Beinlich, Karsch, and Peikert [27], and for  $N = 3, 4$ , and 6 by Lucini, Teper, and Wenger [3, 28]. These results were used to extrapolate to infinite  $N$ , giving

$$\frac{\alpha^{od}}{N^2 T_d^3} = 0.0138(3) - \frac{0.104(3)}{N^2}. \quad (1)$$

We find it striking that the coefficient of the leading term,  $\sim N^2$ , is an order of magnitude smaller than that of the term at next to leading order.

In the matrix model the coefficient  $\sim N^2$  vanishes identically, and the true behavior is only  $\sim N$ . This is because in the matrix model, the height of the barrier between the two distinct minima is small,  $\sim 1/N$ . Thus perhaps the small value of the order-order interface tension found on the lattice is because the coefficient  $\sim N^2$  vanishes. We discuss this further

in Sec. (IV) and in the Conclusions, Sec. (V).

## II. REVIEW OF LARGE $N$ THERMODYNAMICS

We are interested in the thermodynamics of pure  $SU(N)$  gauge theory for temperatures a few times that for the deconfining phase transition at  $T_d$ . The order parameter of the thermodynamics is taken to be the Wilson line:

$$\mathbf{L}(\vec{x}) = \text{P exp} \left( ig \int_0^\beta A_0(\vec{x}, \tau) d\tau \right). \quad (2)$$

By a gauge transformation, we can diagonalize the field  $A_0$  as

$$A_0^{ij} = \frac{2\pi T}{g} q_i \delta^{ij}, \quad (3)$$

where  $i, j = 1 \cdots N$  and the eigenvalues  $q_i$  are subject to the  $SU(N)$  constraint  $\sum_i^N q_i = 0$ . In our model  $q_i$  are the fundamental variables to characterize the transition. We assume that after integrating out the other components of gluon field  $A_i$ , that we obtain an effective potential for  $q_i$  [4–9].

$$\tilde{V}_{eff}(q) = -d_1(T)\tilde{V}_1(q) + d_2(T)\tilde{V}_2(q), \quad (4)$$

$$\tilde{V}_n(q) = \sum_{i,j=1}^N |q_i - q_j|^n (1 - |q_i - q_j|)^n. \quad (5)$$

The potential includes both perturbative (proportional to  $T^4$ ) and non-perturbative (proportional to  $T^2 T_d^2$ ) contributions. The temperature dependent functions  $d_1$  and  $d_2$  are given by

$$d_1(T) = \frac{2\pi}{15} c_1 T^2 T_d^2, \quad d_2(T) = \frac{2\pi}{3} (T^4 - c_2 T^2 T_d^2). \quad (6)$$

At tree level, the kinetic term is

$$\tilde{K}(q) = \frac{1}{2} \text{tr} F_{\mu\nu}^2 = \left( \frac{2\pi T}{g} \right)^2 \sum_{i=1}^N (\nabla q_i)^2. \quad (7)$$

In a mean-field approximation, the kinetic term does not contribute for a spatially homogeneous state, and so it can be ignored for thermodynamic quantities. The kinetic term does enter in computation of interface tension in the following sections.

In the infinite  $N$  limit we introduce a continuous variable  $x = i/N$ . Labeling the eigenvalue  $q_i \rightarrow q(x)$ , we introduce the eigenvalue density

$$\rho(q) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i^N \delta(q - q_i) = \int_0^1 dx \delta(q - q(x)) = \frac{dx}{dq}. \quad (8)$$

At finite  $N$  there are two identities,

$$\sum_i^N 1 = N, \quad \sum_i^N q_i = 0 \quad (9)$$

which become

$$\int dq \rho(q) = 1, \quad \int dq \rho(q) q = 0, \quad (10)$$

at infinite  $N$ . The potential is proportional to  $N^2$ ,

$$\begin{aligned} \tilde{V}_n(q) &= N^2 V_n(q) = N^2 \int dx dy |q(x) - q(y)|^n (1 - |q(x) - q(y)|)^n \\ &= N^2 \int dq dq' \rho(q) \rho(q') |q - q'|^n (1 - |q - q'|)^n. \end{aligned} \quad (11)$$

The minimum of Eq. (4) was found in Ref. [9]. The solution is

$$\rho(q) = 1 + b \cos dq, \quad -q_0 < q < q_0, \quad (12a)$$

$$d = \sqrt{\frac{12d_2}{d_1}}. \quad (12b)$$

$$\cot(dq_0) = \frac{d}{3} \left( \frac{1}{2} - q_0 \right) - \frac{1}{d(1/2 - q_0)}, \quad (12c)$$

$$b^2 = \frac{d^4}{9} \left( \frac{1}{2} - q_0 \right)^4 + \frac{d^2}{3} \left( \frac{1}{2} - q_0 \right)^2 + 1. \quad (12d)$$

For this solution, the eigenvalues span the region between  $-q_0$  and  $+q_0$ . Both  $q_0$  and  $b$  are implicitly functions of temperature, determined through the above equations from the dependence of the parameter  $d$  as a function of temperature, Eqs. (6) and (12b).

Above the transition temperature,  $q_0 < 1/2$ , and the eigenvalues do not span the full range in  $q$ . The density is discontinuous at the end points:  $\rho(\pm q_0) \neq 0$ , but  $\rho(q) = 0$  when  $|q| > q_0$ . Also,  $d > 2\pi$  for  $T > T_d$ .

At the transition temperature, the eigenvalue density is continuous for all values of  $q$  in the range between  $-1/2$  to  $+1/2$ , and vanishes at the endpoint,  $\rho(\pm 1/2) = 0$ . Also,  $d = 2\pi$ .

Below the transition temperature, the theory is in confined phase, with a uniform distribution of eigenvalues over the unit circle, with  $q_0 = \frac{1}{2}$  and  $b = 0$ . This is the usual confined phase expected in matrix models.

For the potential in Eq. (4) and the eigenvalue distribution of Eq. (12a), at  $T_d$ ,  $q_0 = \frac{1}{2}$ , and the potential is independent of  $b$ . Changing  $b$  from 0 to 1 interpolates between the confined and deconfined phases, but does not change the potential. Hence  $b$  is a zero mode of the potential corresponding to changing the overall shape of the distribution. This will play an important role in the construction of the interface. For short we call the change of shape related to the parameter  $b$  as the “ $b$ -mode”.

It is also worth emphasizing that in the derivation of Ref. [9], we have assumed that the eigenvalue density is symmetric in  $q$ . We can obtain different distributions by applying an arbitrary  $Z_N$  transform to a given solution. A  $Z_N$  transform of charge  $k$ ,  $k = 1 \dots (N - 1)$ , is given by

$$q_1, q_2, \dots, q_N \rightarrow q_1 + \frac{k}{N}, \dots, q_{N-k} + \frac{k}{N}, q_{N-k+1} + \frac{k-N}{N}, \dots, q_N + \frac{k-N}{N}. \quad (13)$$

Assuming  $q_1 \leq q_2 \leq \dots \leq q_N$  and  $|q_i - q_j| < 1$ , we can relabel the eigenvalues such that they are in an increasing order:

$$q_1, q_2, \dots, q_N \rightarrow q_{N-k+1} + \frac{k-N}{N}, \dots, q_N + \frac{k-N}{N}, q_1 + \frac{k}{N}, \dots, q_{N-k} + \frac{k}{N}. \quad (14)$$

In the infinite  $N$  limit, the  $Z_N$  transform takes the following form. Define the inverse function of  $x(q) = \int_{-q_0}^q dq' \rho(q')$  as  $Q(x)$ ,

$$q = Q(x) \rightarrow q = \begin{cases} Q(x + 1 - \Delta) - 1 + \Delta, & 0 < x < \Delta, \\ Q(x - \Delta) + \Delta, & \Delta < x < 1. \end{cases} \quad (15)$$

Since the potential is invariant under  $Z_N$  transformations, a smooth change in  $\Delta$  is another zero mode of the potential, which we call the “shift mode”. This is also relevant for the construction of the interface.

We stress that both the  $b$  and shift modes are zero modes only at infinite  $N$ . The former is because of the flatness of the potential. The latter is because the  $Z_N$  symmetry becomes a continuous symmetry of  $U(1)$  at infinite  $N$ . We comment on what happens at finite  $N$  later.

### III. INTERFACE TENSION

An interface is a topological object interpolating between two vacua of the theory. Suppose the two vacua are separated in the  $z$  direction and extended in the transverse plane. The effective action has an area law when the transverse size  $L_{tr}$  is large:  $S_{eff} = \alpha\beta L_{tr}$ . The proportionality constant defines the interface tension. Up to cubic order in the perturbative expansion, the order-order interface tension exhibits Casimir scaling [20, 21]

$$\alpha \propto k(N - k). \quad (16)$$

An important question to address is whether the order-order interface tension satisfies Casimir scaling in the matrix model.

An interface tension necessarily involves a spatial gradient along the  $z$  direction. Consequently, the kinetic term must be included. Since the potential is simple when written in terms of the eigenvalue density  $\rho(q)$ , it is useful to write the kinetic term in terms of the same variable as well. Assuming that there is a spatial gradient only along the  $z$  direction,

$$\tilde{K}(q) = N^2 K(q) = N^2 \frac{(2\pi T)^2}{g^2 N} \int dz dx \left( \frac{\partial q(x, z)}{\partial z} \right)^2, \quad (17)$$

where the partial derivative is taken at fixed  $x$ . Notice that in this expression and henceforth, that  $z$  is a spatial coordinate, while  $x$  labels the eigenvalue.

We start with the eigenvalue density  $\rho(q, z) = \partial x(q, z)/\partial q$ , and assume that the range of the eigenvalue distribution does not change over the interface,

$$x(q, z) = \int_{-q_0}^q dq' \rho(q', z), \quad (18)$$

modulo a  $Z(N)$  transform. Using the chain rule, we have

$$\frac{\partial q(x, z)}{\partial z} = - \frac{\partial q(x, z)}{\partial x} \frac{\partial x(q, z)}{\partial z} = - \frac{1}{\rho(q, z)} \frac{\partial}{\partial z} \int^q dq' \rho(q', z). \quad (19)$$

The kinetic term becomes

$$K(q) = \frac{(2\pi T)^2}{g^2 N} \int dz dq \frac{1}{\rho(q, z)} \left( \frac{\partial}{\partial z} \int^q dq' \rho(q', z) \right)^2. \quad (20)$$

This form was derived previously by Polchinski [29].



### A. Interface tension at $T_d$

We begin with the interface tension at the transition temperature. At  $T_d$  there are both order-order and order-disorder interfaces. Due to the complicated form of the kinetic energy in Eq. (20), an explicit analytic solution for the interface appears difficult. However, the presence of the zero modes allows us to show that both interface tensions vanish. It is straightforward to construct an order-order interface using the shift mode:

$$q = \begin{cases} Q(x + 1 - \Delta f(z)) - 1 + \Delta f(z), & 0 < x < \Delta f(z); \\ Q(x - \Delta f(z)) + \Delta f(z), & \Delta f(z) < x < 1, \end{cases} \quad (21)$$

with  $f(-L) = 0$  and  $f(L) = 1$  at the two boundaries of the interface. A  $k$ -wall interpolating two vacua related by a  $Z(N)$  transformation of charge  $k$  corresponds to  $\Delta = k/N$ .

The idea is to take  $f(z) = z/(2L) + 1/2$  such that  $\partial q/\partial z \sim 1/L$ . As  $f$  is a zero mode, this not change the potential energy. There is a contribution from the kinetic energy, but at each point in the  $z$  direction, this is  $\sim (1/L)^2$ , and so integrating over  $z$ , it is  $\sim L(1/L)^2 \sim 1/L$ , and so negligible as  $L \rightarrow \infty$ . Thus the interface tension vanishes at  $T_d$  at infinite  $L$ .

Actually this construction using the shift mode does not work. To see that, we need to take a closer look at the kinetic term. On the interface, the density of eigenvalues is given by

$$\rho = \begin{cases} 1 + \cos 2\pi(q + 1 - \Delta f(z)), & Q(1 - \Delta f(z)) - 1 + \Delta f < q < -1/2 + \Delta f(z), \\ 1 + \cos 2\pi(q - \Delta f(z)), & -1/2 + \Delta f(z) < q < Q(1 - \Delta f(z)) + \Delta f(z). \end{cases} \quad (22)$$

Integrating with respect to  $q$ , we obtain

$$x = q + \frac{1}{2} + \frac{1}{2\pi} \sin[2\pi(q - \Delta f)], \quad Q(1 - \Delta f) - 1 + \Delta f < q < Q(1 - \Delta f) + \Delta f, \quad (23)$$

from which it follows

$$\frac{\partial q}{\partial z} = -\frac{\cos[2\pi(q - \Delta f)]\Delta f'}{1 + \cos[2\pi(q - \Delta f)]}. \quad (24)$$

Plugging Eq. (24) into Eq. (20), we identify a non-integrable singularity at  $q = 1/2 + \Delta f$ . Therefore, we conclude that the interface built by the shift mode is ruled out by a divergence in the kinetic energy.

A second possibility is to build an interface with the  $b$ -mode. In this case, we build an order-order interface by joining two order-disorder interfaces, so the two vacua at the ends

of the interface are joined through a confining phase in the middle. Define  $Q_b(x)$  as the inverse function of

$$x = q + \frac{1}{2} + \frac{b}{2\pi} \sin(2\pi q). \quad (25)$$

At  $T = T_d$ ,  $Q_1(x)$  reduces to the distribution for the deconfined phase, and  $Q_0(x)$  for the confined phase. The interface is constructed as

$$\begin{aligned} \text{part I: } & q = Q_{b(z)}(x), \quad -L < z < 0 \quad \leftrightarrow \\ \text{part II: } & q = \begin{cases} Q_{b(z)}(x + 1 - \Delta) - 1 + \Delta, & 0 < x < \Delta; \\ Q_{b(z)}(x - \Delta) + \Delta, & \Delta < x < 1. \end{cases} \quad 0 < z < L, \end{aligned} \quad (26)$$

with the boundary conditions  $b(\pm L) = 1$  and  $b(0) = 0$ . The distributions from part I and part II join at  $z = 0$ , and given explicitly by  $q = x - 1/2$ . We note that part II is a  $Z(N)$  transform of part I, flipped in  $z$ .

We show below that the kinetic energy for this configuration does not diverge, and that its kinetic energy is as before,  $\sim 1/L$  after integrating over  $L$ . Since the path given by the first and second lines of Eq. (26) are related by  $Z_N$  transformations, they necessarily have the same kinetic and potential energies. It is sufficient to restrict ourselves to the first line. The gradient is

$$\frac{\partial q}{\partial z} = -\frac{b'}{2\pi} \frac{\sin 2\pi q}{1 + b \cos 2\pi q}, \quad (27)$$

where here and in the following primes denote the derivatives with respect to the argument. We see possible singularities at  $q = \pm 1/2$  from the denominator are cancelled by the numerator, giving rise to a finite result for the kinetic energy. For a given  $b$ , it is not difficult to compute the integral in  $q$  by contour integration,

$$\int_{-1/2}^{1/2} dq \rho(q) \left( \frac{\partial q}{\partial z} \right)^2 = \left( \frac{b'}{2\pi} \right)^2 \left( \frac{-b^4 + b^2(5 - 3\sqrt{1 - b^2}) + 4(-1 + \sqrt{1 - b^2})}{b^2 \sqrt{1 - b^2} (-1 + \sqrt{1 - b^2})^2} \right). \quad (28)$$

We checked that the kinetic term is finite in the limit  $b \rightarrow 1$ . Consequently both the order-order and the order-disorder interface tensions vanish identically at the transition temperature.

## B. Order-order interface tension above $T_d$

The order-disorder interface tension is only defined at the transition temperature. Above  $T_d$ , the order-order interface tension, which is equivalent to the 't Hooft loop, is nonzero.

Because of the complexity of the kinetic term, we are not able to compute the order-order interface tension in full generality. We can compute in two limiting cases, which we consider in turn.

1. *Small  $Z(N)$  rotations*

A  $Z(N)$  rotation has charge  $k$ , where the parameter  $\Delta = k/N$ . In this subsection we consider the case of a small  $Z(N)$  rotation, where  $\Delta \ll 1$ . At infinite  $N$  we consider an infinitesimal rotation,  $\Delta \rightarrow 0$ .

In the case  $\Delta \rightarrow 0$ , the end points of the interface are given by

$$q = Q(x), \quad q = \begin{cases} Q(x + 1 - \Delta) - 1 + \Delta, & 0 < x < \Delta; \\ Q(x - \Delta) + \Delta, & \Delta < x < 1. \end{cases} \quad (29)$$

The eigenvalues for  $x : \Delta \rightarrow 1$  shift infinitesimally, while those between  $x : 0 \rightarrow \Delta$  have a finite jump. The latter is suppressed when  $\Delta$  is small, though. We then take the following path:

$$q = (Q(x + 1 - \Delta) - 1 + \Delta - Q(x))g(z) + Q(x), \quad 0 < x < \Delta; \quad (30)$$

$$q = (Q(x - \Delta) + \Delta - Q(x))f(z) + Q(x), \quad \Delta < x < 1. \quad (31)$$

The unknown functions  $g(z)$  and  $f(z)$  interpolate between 0 and 1. They could in principle depend on  $x$ , which characterizes the change of shape of the eigenvalue density. To leading order in  $\Delta$  we can ignore this dependence. The potential energy along this path is

$$V = \int_0^\Delta dx \int_0^\Delta dy V(q(x) - q(y)) + \int_\Delta^1 dx \int_\Delta^1 dy V(q(x) - q(y)) \\ + 2 \int_0^\Delta dx \int_\Delta^1 dy V(q(y) - q(x)). \quad (32)$$

The first term is of order  $O(\Delta^3)$  and may be ignored. The second term starts with the vacuum potential energy at leading order. The third term is of order  $O(\Delta)$ . We need to know the  $O(\Delta)$  correction to the potential:

$$\delta V = \int_0^1 dx \int_0^1 dy \frac{\partial V}{\partial |q(x) - q(y)|} \Delta |Q'(x) - Q'(y)| f \\ + 2 \int_0^\Delta dx \int_0^1 dy [V(q(y) + q_0) - V(q(y) + q_0 + (1 - 2q_0)g)]. \quad (33)$$

Evaluating Eq. (33) for the vacuum solution Eq. (12a), we find the first term vanishes identically, while the second term gives

$$\delta V = d_2(1 - 2q_0)^4 g^2(1 - g)^2. \quad (34)$$

The kinetic energy reads

$$\frac{(2\pi T)^2}{g^2 N} \int dz dx \left( \frac{\partial q}{\partial z} \right)^2 \quad (35)$$

$$\text{with } \frac{\partial q}{\partial z} = \begin{cases} (2q_0 - 1)g', & 0 < x < \Delta; \\ \Delta(1 - Q'(x))f', & \Delta < x < 1. \end{cases} \quad (36)$$

We ignore the contribution from  $\Delta < x < 1$  because it is of order  $O(\Delta^2)$ . Combining the kinetic and potential terms, we have

$$\int dz (K + \delta V) = \int dz \left( \frac{(2\pi T)^2}{g^2 N} \Delta (1 - 2q_0)^2 g'^2 + \Delta d_2 (1 - 2q_0)^4 g^2 (1 - g)^2 \right). \quad (37)$$

We need to minimize the above action with the boundary condition  $g(-\infty) = 0$  and  $g(\infty) = 1$ . Using the identity  $x^2 + y^2 \geq 2xy$ , we can obtain the minimum without solving for  $g$ :

$$\begin{aligned} \int dz (K + \delta V) &\geq \Delta \frac{2\pi T \sqrt{d_2}}{\sqrt{g^2 N}} \int dz 2 \sqrt{(1 - 2q_0)^2 g'^2 (1 - 2q_0)^4 g^2 (1 - g)^2} \\ &= \Delta \frac{4\pi T \sqrt{d_2}}{\sqrt{g^2 N}} \int dg g(1 - g) \frac{d(1 - 2q_0)^3}{\sqrt{3}} = \Delta \frac{2\pi T \sqrt{d_2}}{3\sqrt{g^2 N}} (1 - 2q_0)^3. \end{aligned} \quad (38)$$

After rescaling the overall factor of  $N^2$ ,

$$\alpha = \Delta \frac{2\pi T \sqrt{d_2}}{3\sqrt{g^2 N}} N^2 (1 - 2q_0)^3. \quad (39)$$

## 2. Near the deconfining transition

Now we consider temperatures close to the deconfining transition temperature,  $T \rightarrow T_d$ . We expect the interface to mimic the  $T = T_d$  case, given by two vacua joined through a confined distribution. This suggests an interface composed of three parts. In the spatial direction  $z$ , the three parts extend over

$$\text{I : } -L < z < -\frac{L}{2} \quad ; \quad \text{II : } -\frac{L}{2} < z < \frac{L}{2} \quad ; \quad \text{III : } \frac{L}{2} < z < L. \quad (40)$$

In the first part of the interface, I, we use of the  $b$ -mode to deform the eigenvalue distribution. For temperatures near  $T_d$ , the  $b$  mode is nearly a zero mode, and so this should increase the

potential energy by a small amount. The contribution from the second part, II, is suppressed by  $(1 - 2q_0)$  because the eigenvalues do not move by a large amount. The third part, III is just a  $Z_N$  transform of part I, flipped in  $z$ . Explicitly,

$$\begin{aligned}
\text{I: } & \rho = a(z) + b(z) b_0 \cos d q , \quad -q_0 < q < q_0 , \\
\text{II: } & \rho = \begin{cases} \frac{1}{2q_0}, & -q_0 - (1 - \Delta)(1 - 2q_0)f(z) < q < -(1 - 2\Delta)q_0 - (1 - \Delta)(1 - 2q_0)f(z) ; \\ \frac{1}{2q_0}, & -(1 - 2\Delta)q_0 + \Delta(1 - 2q_0)g(z) < q < q_0 + \Delta(1 - 2q_0)g(z) , \end{cases} \\
\text{III: } & \rho = \begin{cases} a(z) + b(z) \cos d(q + 1 - \Delta), & Q_b(1 - \Delta) - 1 + \Delta < q < q_0 - 1 + \Delta ; \\ a(z) + b(z) \cos d(q + 1 - \Delta), & -q_0 + \Delta < q < Q_b(1 - \Delta) + \Delta . \end{cases} \quad (41)
\end{aligned}$$

The boundary conditions are

$$b(\pm L) = 1 \quad , \quad b(\pm L/2) = 0 \quad , \quad f(-L/2) = 0 \quad , \quad f(L/2) = 1 . \quad (42)$$

In Eq. (41)  $q_0$  and  $b_0$  are determined by Eqs. (12c) and Eq. (12d), respectively. The normalization condition  $\int dq \rho(q) = 1$  forces  $a(z)q_0 + (b(z)/d) \sin dq_0 = 1/2$ , while the tracelessness condition  $\int dq \rho(q) q = 0$  sets  $f(z) = g(z)$ .

We start with part I. We can expand in  $1/2 - q_0$ , which is small when  $T \sim T_d$ :

$$\delta V = d_1 \frac{60 - d^2}{90} (1 - b)^2 \left( \frac{1}{2} - q_0 \right)^2 + O \left( \frac{1}{2} - q_0 \right)^3 . \quad (43)$$

The kinetic energy can be taken as at  $T = T_d$ , ignoring higher order corrections Eq. (28). With these ingredients, we can already work out the contribution to the interface tension from part I. It is worth noting prior doing any computations that  $\delta V \sim (\frac{1}{2} - q_0)^2$ ,  $K \sim O(1)$ .

Next we consider part II. This case is particular easy because of the constant eigenvalue density in this part. To leading order, the potential energy and kinetic energy are

$$\begin{aligned}
\delta V &= \frac{d_1}{90} \left[ 60 - d^2 - 60f(1 - f)\Delta(1 - \Delta) (12 - d^2\Delta(1 - \Delta)) \right] \left( \frac{1}{2} - q_0 \right)^2 , \\
T &= \frac{(2\pi T)^2}{g^2 N} f^2 \Delta(1 - \Delta) \left( \frac{1}{2} - q_0 \right)^2 . \quad (44)
\end{aligned}$$

We note in part II both potential and kinetic terms are suppressed by  $(\frac{1}{2} - q_0)^2$ , leading only to contributions of higher order in  $(T - T_d)$  for the interface tension.

Adding up contributions from all three parts, we obtain a contribution to leading order which is twice that of part I. It is not difficult to convince ourselves that  $\alpha \sim (\frac{1}{2} - q_0)$  and it is independent of  $\Delta$ .

We will improve the result by considering a more sophisticated ansatz. Note that in Eq. (41), we have chosen to turn on the  $b$ -mode and  $f$ -modes separately. Here we consider more general ansatz by turning them on simultaneously. Furthermore, we neglect contributions from part II as subleading, and only modify part I, taking the ansatz

$$\begin{aligned} \rho &= a(z) + b(z) \cos d(q + (1 - \Delta)f_L(z)) , & -q_0 - (1 - \Delta)f_L < q < Q_b(\Delta) - (1 - \Delta)f_L , \\ \rho &= a(z) + b(z) \cos d(q - \Delta f_R(z)) , & Q_b(\Delta) + \Delta f_R < q < q_0 + \Delta f_R . \end{aligned} \quad (45)$$

The tracelessness condition forces  $f_L = f_R \equiv f$ . It is also natural to require  $f \sim 1 - 2q_0$  such that the eigenvalues remain close together. The introduction of the  $f$ -mode induces corrections to the potential in Eq. (44). Defining  $f = \bar{f} \left(\frac{1}{2} - q_0\right) + O\left(\frac{1}{2} - q_0\right)^2$ , we can expand the potential as

$$\delta V = d_1 \left( R(\Delta, b) + P(\Delta, b)\bar{f} + Q(\Delta, b)\bar{f}^2 \right) \left( \frac{1}{2} - q_0 \right)^2 , \quad (46)$$

where

$$R = \frac{1}{90} \left[ 60 - d^2 - 60f(1 - f)\Delta(1 - \Delta) (12 - d^2\Delta(1 - \Delta)) \right] , \quad (47)$$

$$\begin{aligned} P &= -\frac{1}{48}(b - 1) (32b + (1 - 4Q_b(\Delta)^2)(-48 + d^2(1 + 8Q_b(\Delta) + 12Q_b(\Delta)^2 - 16\Delta Q_b(\Delta)))) \\ &\quad - 16b(-1 + 12Q_b(\Delta)^2) \cos dQ_b(\Delta) , \end{aligned} \quad (48)$$

$$\begin{aligned} Q &= 384b^2 - 48bd^2 (1 + 4Q_b(\Delta)^2 - Q_b(\Delta)(4 - 8\Delta)) + d^2 \left[ 48 - 192Q_b(\Delta)^2 \right. \\ &\quad \left. + d^2 (-5 - 24Q_b(\Delta)^2 + 48Q_b(\Delta)^4 + Q_b(\Delta)^3(32 - 64\Delta) + 16\Delta(1 - \Delta) + 24Q_b(\Delta)(-1 + 2\Delta)) \right] \\ &\quad + 48b (8b + d^2(1 - 4Q_b(\Delta)^2)) \cos dQ_b(\Delta) . \end{aligned} \quad (49)$$

We have also factored out the overall  $\left(\frac{1}{2} - q_0\right)^2$  dependence. One important property we confirmed numerically is that  $Q > 0$ . To lower the potential energy, we choose  $\bar{f} = \frac{P}{2Q}$ . As a result,

$$\delta V = \left( R - \frac{P^2}{4Q} \right) \left( \frac{1}{2} - q_0 \right)^2 . \quad (50)$$

At the same time, the correction also makes the interface tension  $\Delta$  dependent. With the same kinetic energy as at leading order, we obtain the interface tension

$$\alpha = 2 \int dz \sqrt{\delta V K} \sim \frac{1}{2} - q_0 . \quad (51)$$

### 3. Summary

The final result is obtained numerically and shown in Fig. (1). Both scenarios are included. Comparing the dependence on  $(1 - 2q)$  in Eq. (39) and Eq. (51) we conclude that the two scenarios do not have a common region of validity: with can either take an infinitesimal  $Z(N)$  rotation,  $\Delta \ll 1$ , or work close to  $T_d$ .

We also note that despite the limited coverage in  $\Delta$ , that the temperature dependence is nontrivial. This is in contrast to the  $T^3$  dependence in the perturbative regime [18, 23] and  $T^4/M_{KK}$  from holographic models [24]. Of course this nontrivial dependence upon temperature arises because we are close to the transition temperature.

One explanation for the differences in the two limits of  $\Delta \rightarrow 0$  and  $T \rightarrow T_d$  is that at infinite  $N$ , they do not commute. Given the unusual nature of the Gross-Witten-Wadia transition, this would not be that remarkable. Thus it is important to evaluate corrections at finite  $N$  near  $T_d$ , where both results should match smoothly onto one another.

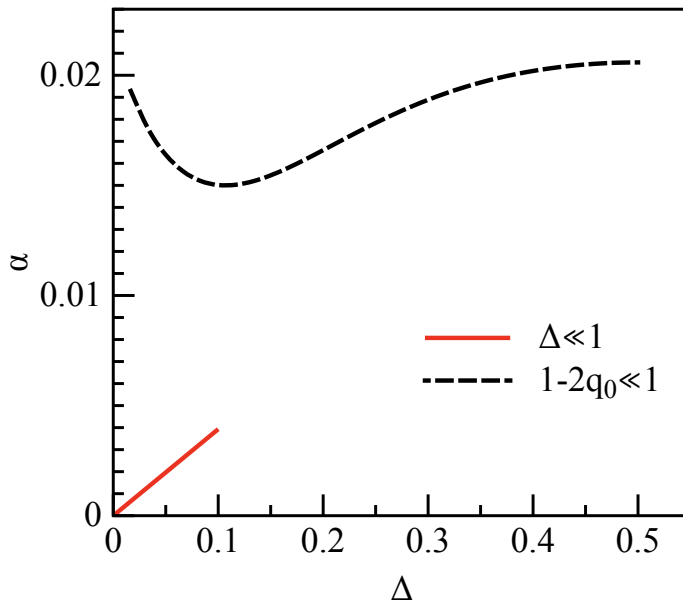


FIG. 1. Interface tension, in units of  $2\pi T\sqrt{d_1}N^2/(\sqrt{g^2N})$  at  $q_0 = 0.3$ . Due to the symmetry  $\Delta \leftrightarrow 1 - \Delta$ , we have only shown half the range in  $\Delta$ . The red linear line is for  $\Delta \ll 1$ , while the black dashed line is obtained assuming  $T \rightarrow T_d$ , or  $1/2 - q_0 \ll 1$ .

#### IV. FINITE $N$ CORRECTION NEAR $T_d$

At finite  $N$ , corrections arise when the continuous integrals of infinite  $N$  are replaced by discrete sums. This can be evaluated with Euler-MacLaurin formula [30]

$$\sum_{i=1}^N F(i) = N \int_{\frac{1}{N}}^1 dx f(x) + \frac{f(1) + f(\frac{1}{N})}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)! N^{2k-1}} \left( f^{(2k-1)'}(1) - f^{(2k-1)'}\left(\frac{1}{N}\right) \right), \quad (52)$$

where  $f(i/N) = F(i)$  and  $B_{2k}$  are Bernoulli numbers. In the integral on the right hand side, we wish to shift the lower bound of integration from  $1/N$  to 0. By Taylor expansion, we find

$$\sum_{i=1}^N F(i) = N \int_0^1 dx f(x) + \frac{f(1) - f(0)}{2} + \dots \quad (53)$$

Applying Eq. (53) to the sums, we have

$$\begin{aligned} N &= \sum_i^N 1 = N \int_0^1 dx \\ 0 &= \sum_i^N q = N \int_0^1 dx q + \frac{q(1) - q(0)}{2} + \dots \\ V &= \sum_{i,j=1}^N V(q_i, q_j) = N^2 \int_0^1 dx \int_0^1 dy V(q(x) - q(y)) \\ &\quad + 2N \int_0^1 dx \frac{V(q(x) - q(1)) - V(q(x) - q(0))}{2} + \dots \end{aligned} \quad (54)$$

where  $\dots$  denote terms of higher order in the  $1/N$  expansion. As before, we define the eigenvalue density  $\rho(q) = dx/dq$ , so that Eq. (54) becomes

$$\begin{aligned} 1 &= \int_{q_-}^{q_+} dq \rho(q) \\ 0 &= \int_{q_-}^{q_+} dq \rho(q) q + \frac{q_+ - q_-}{2N} + \dots \\ V &= N^2 \int_{q_-}^{q_+} dq \int_{q_-}^{q_+} dq' \rho(q) \rho(q') V(q - q') + N \int_{q_-}^{q_+} dq \rho(q) (V(q_+ - q) - V(q - q_-)) + \dots, \end{aligned} \quad (55)$$

with  $q_- = q(0)$  and  $q_+ = q(1)$ . For the infinite  $N$  eigenvalue distribution,  $\rho(q) = \rho(-q)$  and  $q(0) + q(1) = 0$ . The correction of  $\sim N$ , evaluated using the solution at infinite  $N$ , vanishes.



However, the eigenvalue distribution at finite  $N$  changes from that at infinite  $N$ . To determine this, we vary  $V$  with respect to  $\rho$ , subject to the usual normalization and tracelessness constraints in Eq. (55). This gives

$$2 \int_{q_-}^{q_+} dq' \rho(q') V(q - q') + \frac{1}{N} (V(q_+ - q) - V(q - q_-)) + \lambda_1 + \lambda_2 q = 0, \quad (56a)$$

$$\int_{q_-}^{q_+} dq \rho(q) = 1, \quad (56b)$$

$$\int_{q_-}^{q_+} dq \rho(q) q + \frac{q_+ - q_-}{2N} = 0, \quad (56c)$$

where  $\lambda_1$  and  $\lambda_2$  are Lagrange multipliers. We follow the method of [9] in solving for  $\rho$ . Taking the derivative with respect to the equation of motion,  $q$ , four times, we arrive at

$$\rho''(q) + d^2(\rho(q) - 1) = 0. \quad (57)$$

The general solution is given by

$$\rho(q) = 1 + b \cos(dq) + c \sin(dq). \quad (58)$$

Plugging this into Eq. (56a), we find the result is organized as a fourth order polynomial in  $q$ . The coefficients of  $q$  and the constant term can always be set to zero by choosing  $\lambda_1$  and  $\lambda_2$  properly. The remaining coefficients of  $q^4$ ,  $q^3$  and  $q^2$  give three independent equations. Together with Eq. (56b) and Eq. (56c), we have in total five equations, to be satisfied by four constants  $b$ ,  $c$  and  $q_{\pm}$ . It turns out if four of the equations are satisfied, then the fifth automatically holds. In practice, we solve for  $\cos(dq_{\pm})$  and  $\sin(dq_{\pm})$  in terms of the constants  $b$ ,  $c$  and  $q_{\pm}$ . Defining  $q_s = q_- + q_+$  and  $q_d = q_+ - q_-$ , we first find an equation for  $q_d$ ,

$$\cot\left(\frac{dq_d}{2}\right) = -\frac{12 - d^2(1 - q_d)^2}{6d(1 - q_d)}. \quad (59)$$

It is easy to see  $q_d = 2q_0$  as defined in Eq. (12c) is free of corrections in  $1/N$ , and that

$$q_s = \frac{1}{N} \frac{-12 - 2d^2q_0(1 - 2q_0)^2}{d^2(1 - 2q_0)^2}, \quad (60)$$

$$\frac{(144 + 12d^2(1 - 2q_0)^2 + d^4(1 - 2q_0)^4)(36 + d^2N^2(1 - 2q_0)^4)}{d^2(1 - 2q_0)^4} = 144(b^2 + c^2)N^2 \quad (61)$$

The first shows that  $q_s$  is suppressed in  $1/N$ . The second combined with

$$\tan\left(\frac{dq_s}{2}\right) = \frac{1 - \cos(dq_s)}{\sin(dq_s)}, \quad (62)$$

can be used to determine  $b$  and  $c$ . To have a consistent expansion in  $1/N$ , we need to have

$$b = b_0 + \frac{b_2}{N^2} + \dots, \quad c = \frac{c_1}{N} + \dots. \quad (63)$$

At leading order  $b_0$  agrees with Eq. (12d).  $c_1$  is

$$c_1 = -b_0 d q_0. \quad (64)$$

We have obtained Eqs. (58), (59), (60), (63) and (64) as the new eigenvalue distribution, valid up to corrections  $\sim 1/N$ .

The appearance of  $\sin(dq)$  in Eq. (58) is, at first sight, worrisome. It appears to imply that the Polyakov loop has a nonvanishing imaginary part, which would violate charge conjugation symmetry. This does not occur, as one has to expand consistently, taking all factors of  $1/N$  into account.

Using our experience from before, we expect that to order  $1/N$ , the constants have the following form:

$$b = b_0, \quad c = \frac{c_1}{N}, \quad q_d = 2q_0, \quad q_s = \frac{q_{s1}}{N}. \quad (65)$$

Plugging Eq. (65) into Eqs. (56a), (56b), (56c) and keeping terms up to order  $1/N$ , we find

$$c_1 = 0, \quad q_{s1} = -\frac{12}{d^2(1-2q_0)^2}. \quad (66)$$

The term  $\sin(dq)$  naturally drops out, and we find explicitly that the imaginary part of all Polyakov loops vanish to order  $1/N$ .

To  $\sim 1/N$ , the eigenvalue distribution is then

$$\begin{aligned} \rho &= 1 + b \cos(dq), \quad q_- < q < q_+; \\ q_- + q_+ &= -\frac{12}{d^2 N (1-2q_0)^2}, \quad q_+ - q_- = 2q_0. \end{aligned} \quad (67)$$

Thus we have

$$(q_0 - q_+) \frac{d^2 N (1-2q_0)^2}{6} = 1. \quad (68)$$

In Fig. (2) we compare this combination with the numerical results for the maximal eigenvalue,  $q_{\max}$ , which at high  $N$  is approximated by  $q_+$ , for different number of colors and different  $d$ , that is temperature. There is good agreement between the analytic expression

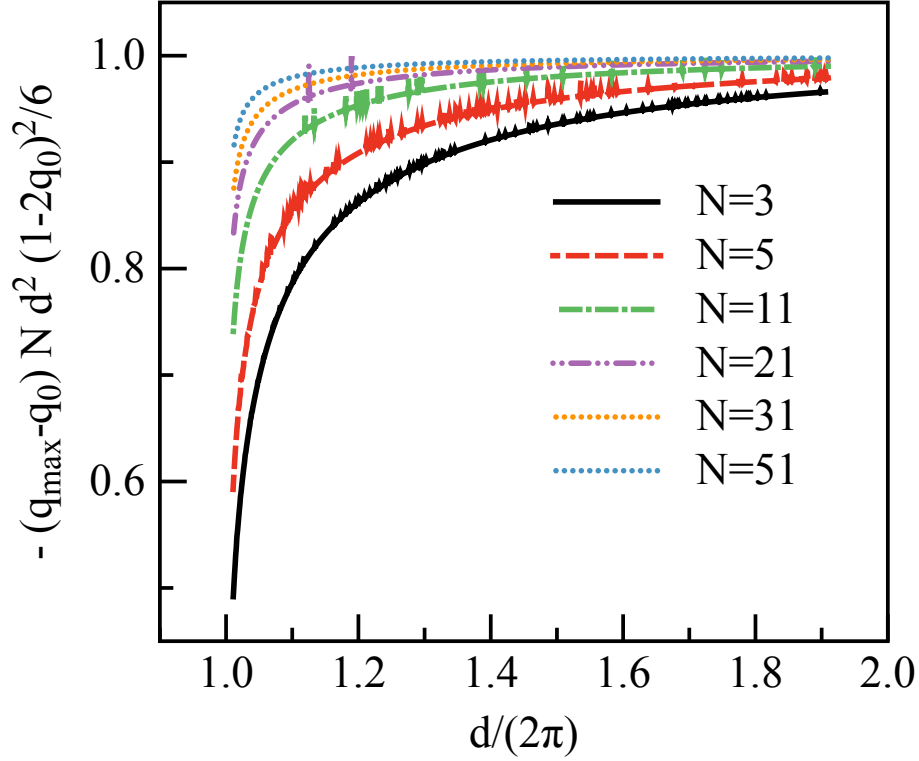


FIG. 2. The maximal eigenvalue,  $q_{\max}$ , as a function of  $d$  for different number of colors,  $N$ .

in Eq. (68) and the numerical results. From Eq. (67) and our numerical simulations we also see that when  $N(1 - 2q_0)^2 \sim 1$ , that the large  $N$  expansion breaks down. We also find that in contrast to the behavior of the maximal eigenvalue in the confined and the deconfined phase,  $q_{\max} - q_0 \sim 1/N$ , that at the critical temperature,  $q_{\max} - q_0 \sim 1/\sqrt{N}$ .

With the correct eigenvalue distribution, we can proceed to evaluate the correction to the potential. We find the correction vanishes to order  $1/N$ . To obtain the correction to the next order, we need to find  $\rho$  at the corresponding order. This is a lengthy exercise, which we defer for now. We can estimate this size of the correction by using Eq. (67). We use

$$\begin{aligned}
V = & \int_{q_-}^{q_+} dq \int_{q_-}^{q_+} dq' \rho(q) \rho(q') V(q - q') + \frac{1}{N} \int_{q_-}^{q_+} dq \rho(q) (V(q_+ - q) - V(q - q_-)) \\
& - \frac{V(q_+ - q_-)}{2N^2} + \frac{B_2}{N^2} \int_{q_-}^{q_+} dq \rho(q) \left( \frac{V'(q_+ - q)}{\rho(q_+)} + \frac{V'(q - q_-)}{\rho(q_-)} \right) + \dots . \quad (69)
\end{aligned}$$

Taking into account the correction  $\sim 1/N$  to  $q_s$ , we have contributions  $\sim 1/N^2$ . After some

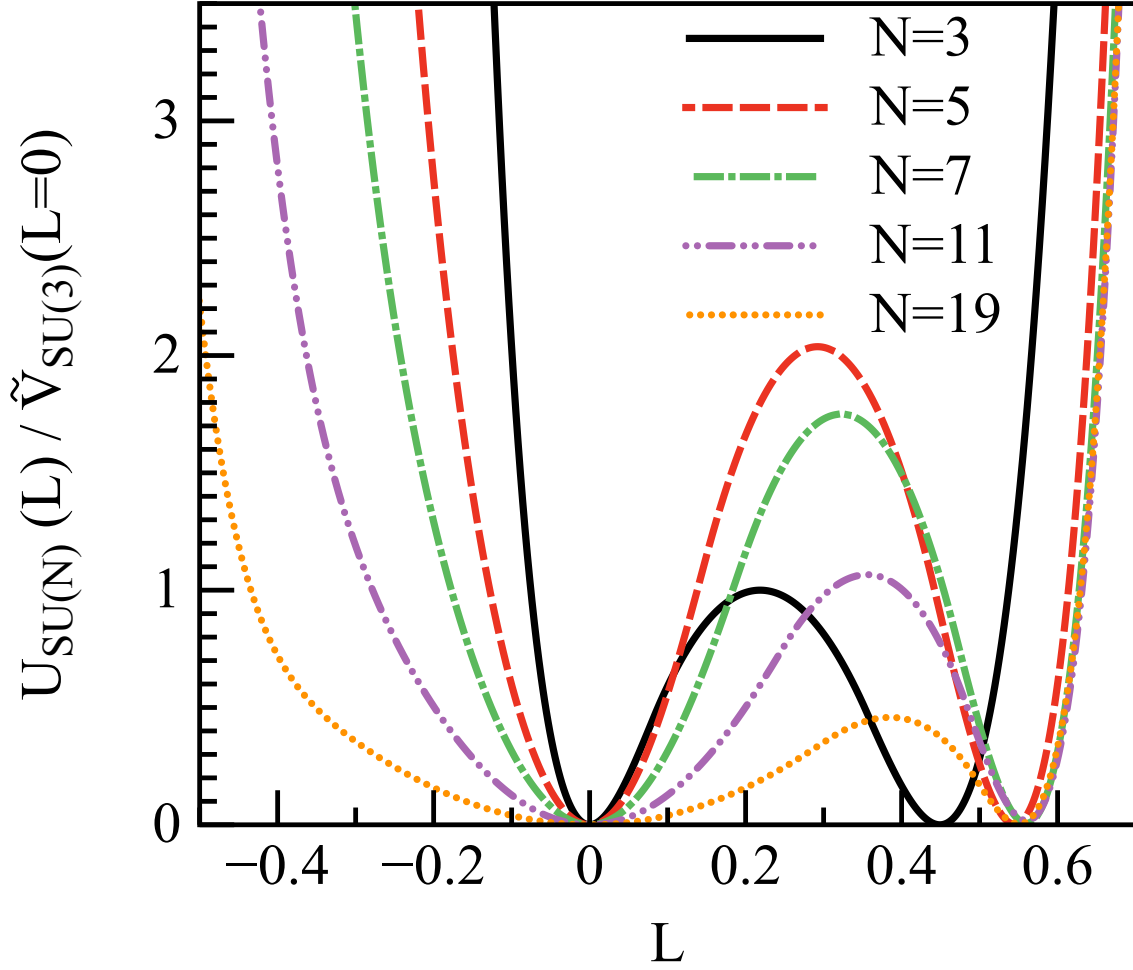


FIG. 3. The non-equilibrium potential  $U(\ell) = \tilde{V}(\ell) - \tilde{V}(\ell = 0)$  at the transition temperature.

computation, we find

$$V_1 = d_1 \frac{180 - 5 d^4 q_0^2 (1 - 2q_0)^5 + 24 d^2 q_0^2 (-5 + 20 q_0 - 30 q_0^2 + 16 q_0^3)}{30 d^2 N^2 (1 - 2q_0)^3} \quad (70)$$

$$V_2 = -d_1 \frac{q_0 (1 - 2q_0) (6 - d^2 q_0 (1 - 2q_0))}{3 N^2} \quad (71)$$

$$V_3 = d_1 \frac{q_0 (1 - 2q_0) (6 - d^2 q_0 (1 - 2q_0))}{6 N^2}, \quad (72)$$

where  $V_1$ ,  $V_2$  and  $V_3$  denote contributions from the orders  $\sim 1$ ,  $\sim 1/N$  and  $\sim 1/N^2$  in Eq. (69), respectively. We note the large  $N$  expansion breaks down when  $N^2(1 - 2q_0)^3 \sim 1$ . In this, bulk thermodynamic quantities have a less stringent criterion for the convergence of the  $1/N$  expansion than the eigenvalue distribution.

We were able to analytically estimate corrections to the full potential only at the mini-

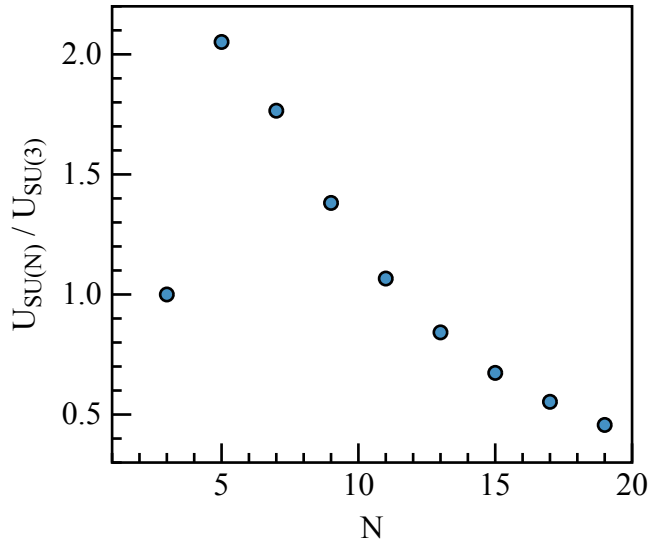


FIG. 4. The maximum of the potential  $U_{\text{max}}$  for different  $N$  at  $T = T_d$ .

mum. Numerically it is possible to go beyond this and compute the potential as a function of the Polyakov loop  $\ell = 1/N \text{tr } \mathbf{L}$  at the transition temperature. In Fig. (3), we show the potential, divided by  $N^2 - 1$ , as a function of the Polyakov loop for different  $N$ . This figure shows a surprising feature. The confining phase is  $\ell = 0$ , and the deconfined phase,  $\ell \approx 1/2$ . For three colors, the potential looks like a standard potential for a transition of first order, with significant curvature about each minimum. As  $N$  increases, though, the curvature in the deconfined phase remains significant, while that in the confined phase decreases significantly.

This suggests that in order to look for evidence of the Gross-Witten-Wadia transition, that it is better looking at the decrease of the mass in the confined phase, and not that in the deconfined phase. That is, to look for the decrease in the string tension below  $T_d$ , and not that of the Debye mass, above  $T_d$ .

In Fig. (4) we concentrate on one particular feature of the potential, its maximum. This shows that while the height of the potential increases from  $N = 3$  to  $N = 5$ , it then decreases as  $N$  increases further. For the sake of argument, let us assume that the interface tensions at  $T_d$  can be characterized entirely by the potential in  $\ell$ . At  $T_d$ , the order-disorder interface tension is given by the integral of the potential. The width is very nearly constant, from 0 to  $\approx 1/2$  for all  $N \geq 3$ . Consequently, the area under the barrier is approximately proportional to its height. This suggests that the order-disorder interface tension, scaled by

$N^2 - 1$ , increases from  $N = 3$  to  $N = 5$ , and then decreases. We stress that this argument is only qualitative: the order-disorder interface tension is a tunneling problem not in one dimension, but in  $N - 1$  dimensions. Even numerically, though, determining the path for tunneling in such a high dimensional space is rather nontrivial.

## V. CONCLUSIONS

The matrix model studied here is clearly only one of many possible matrix models. Its advantage is that it can be solved analytically at infinite  $N$ , and numerically at finite  $N$ . In this model all interface tensions vanish at the deconfining phase transition. Since the transition is of first order, this would be striking evidence that it is an unusual transition, perhaps of the Gross-Witten-Wadia type.

Presently, numerical simulations of  $SU(N)$  gauge theories can only be carried out at relatively small  $N$ ,  $N < 10$ . For two colors the order-disorder interface tension vanishes, as the transition is of second order. For three colors the order-disorder interface tension is relatively small [27]. This presumably reflects that the transition for three colors is weakly first order, because of its proximity to the second order transition for two colors. This leads one to expect that as  $N$  increases, that the order-disorder interface tension, divided by  $N^2 - 1$ , increases monotonically from  $N = 3$ , and becomes constant at infinite  $N$ .

As discussed at the end of the last section, the numerical solution of the matrix model indicates the contrary: that the order-disorder interface tension, divided by  $N^2 - 1$ , behaves non-monotonically with  $N$ . Fig. (4) suggests this quantity increases from  $N = 3$  to  $N = 5$ , and then decreases, slowly, as  $N$  increases further.

Such non-monotonic behavior in  $N$  is unexpected, and could well just be an artifact of the model. This could be settled by numerical studies on the lattice of the order-disorder interface tension for moderate values of  $N \sim 5$ . It might even provide hints of a Gross-Witten-Wadia transition at infinite  $N$ .

## ACKNOWLEDGMENTS

The research of R.D.P. and V.S. is supported by the U.S. Department of Energy under contract #DE-AC02-98CH10886. S.L. is supported by the RIKEN Special Postdoctoral

- [1] C. DeTar and U. Heller, *Eur.Phys.J.* **A41**, 405 (2009), arXiv:0905.2949 [hep-lat].
- [2] P. Petreczky, *J.Phys.* **G39**, 093002 (2012), arXiv:1203.5320 [hep-lat].
- [3] B. Lucini and M. Panero, (2012), arXiv:1210.4997 [hep-th].
- [4] P. N. Meisinger, T. R. Miller, and M. C. Ogilvie, *Phys.Rev.* **D65**, 034009 (2002), arXiv:hep-ph/0108009 [hep-ph].
- [5] P. N. Meisinger and M. C. Ogilvie, *Phys.Rev.* **D65**, 056013 (2002), arXiv:hep-ph/0108026 [hep-ph].
- [6] A. Dumitru, Y. Guo, Y. Hidaka, C. P. Korthals Altes, and R. D. Pisarski, *Phys.Rev.* **D83**, 034022 (2011), arXiv:1011.3820 [hep-ph].
- [7] A. Dumitru, Y. Guo, Y. Hidaka, C. P. K. Altes, and R. D. Pisarski, (2012), arXiv:1205.0137 [hep-ph].
- [8] K. Kashiwa, R. D. Pisarski, and V. V. Skokov, *Phys.Rev.* **D85**, 114029 (2012), arXiv:1205.0545 [hep-ph].
- [9] R. D. Pisarski and V. V. Skokov, *Phys.Rev.* **D86**, 081701 (2012), arXiv:1206.1329 [hep-th].
- [10] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, *Rev.Mod.Phys.* **53**, 43 (1981).
- [11] S. R. Wadia, (Univ. of Chicago preprint EFI-79/44).
- [12] S. R. Wadia, *Phys.Lett.* **B93**, 403 (1980).
- [13] B. Sundborg, *Nucl.Phys.* **B573**, 349 (2000), arXiv:hep-th/9908001 [hep-th].
- [14] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, *Adv.Theor.Math.Phys.* **8**, 603 (2004), arXiv:hep-th/0310285 [hep-th].
- [15] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, *Phys.Rev.* **D71**, 125018 (2005), arXiv:hep-th/0502149 [hep-th].
- [16] H. J. Schnitzer, *Nucl.Phys.* **B695**, 267 (2004), arXiv:hep-th/0402219 [hep-th]; L. Alvarez-Gaume, C. Gomez, H. Liu, and S. R. Wadia, *Phys.Rev.* **D71**, 124023 (2005), arXiv:hep-th/0502227 [hep-th]; L. Alvarez-Gaume, P. Basu, M. Marino, and S. R. Wadia, *Eur.Phys.J.* **C48**, 647 (2006), arXiv:hep-th/0605041 [hep-th]; T. J. Hollowood and J. C. Myers, *JHEP* **0911**, 008 (2009), arXiv:0907.3665 [hep-th]; S. Hands, T. J. Hollowood, and J. C. Myers, *ibid.* **1007**, 086 (2010), arXiv:1003.5813 [hep-th]; T. J. Hollowood, S. P. Kumar, and J. C.

- Myers, *ibid.* **1111**, 138 (2011), arXiv:1110.0696 [hep-th]; T. J. Hollowood and J. C. Myers, (2012), arXiv:1207.4605 [hep-th].
- [17] A. Dumitru, J. Lenaghan, and R. D. Pisarski, Phys.Rev. **D71**, 074004 (2005), arXiv:hep-ph/0410294 [hep-ph].
- [18] T. Bhattacharya, A. Gocksch, C. P. Korthals Altes, and R. D. Pisarski, Phys.Rev.Lett. **66**, 998 (1991).
- [19] T. Bhattacharya, A. Gocksch, C. P. Korthals Altes, and R. D. Pisarski, Nucl.Phys. **B383**, 497 (1992), arXiv:hep-ph/9205231 [hep-ph].
- [20] P. Giovannangeli and C. P. Korthals Altes, Nucl.Phys. **B721**, 1 (2005), arXiv:hep-ph/0212298 [hep-ph].
- [21] P. Giovannangeli and C. P. Korthals Altes, Nucl.Phys. **B721**, 25 (2005), arXiv:hep-ph/0412322 [hep-ph].
- [22] S. A. Hartnoll and S. P. Kumar, Phys.Rev. **D74**, 026001 (2006), arXiv:hep-th/0603190 [hep-th].
- [23] A. Armoni, S. P. Kumar, and J. M. Ridgway, JHEP **0901**, 076 (2009), arXiv:0812.0773 [hep-th].
- [24] H.-U. Yee, JHEP **0904**, 029 (2009), arXiv:0901.0705 [hep-th].
- [25] C. P. Korthals Altes, A. Kovner, and M. A. Stephanov, Phys.Lett. **B469**, 205 (1999), arXiv:hep-ph/9909516 [hep-ph].
- [26] C. P. Korthals Altes and A. Kovner, Phys.Rev. **D62**, 096008 (2000), arXiv:hep-ph/0004052 [hep-ph].
- [27] B. Beinlich, F. Karsch, and A. Peikert, Phys.Lett. **B390**, 268 (1997), arXiv:hep-lat/9608141 [hep-lat].
- [28] B. Lucini, M. Teper, and U. Wenger, JHEP **0502**, 033 (2005), arXiv:hep-lat/0502003 [hep-lat].
- [29] J. Polchinski, Phys.Rev.Lett. **68**, 1267 (1992), arXiv:hep-th/9109007 [hep-th].
- [30] Y. Y. Goldschmidt, J.Math.Phys. **21**, 1842 (1980).