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Systematic biases in parameter estimation of binary black-hole mergers

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Parameter estimation of binary-black-hole merger events in gravitational-wave data relies on matched-filtering techniques which, in turn, depend on accurate model waveforms. Here we characterize the systematic biases introduced in measuring astrophysical parameters of binary black holes by applying the currently most accurate effective-one-body templates to simulated data containing non-spinning numerical-relativity waveforms. We quantify the systematic bias by using a Markov chain Monte Carlo algorithm to sample the posterior distribution function of noise-free data, and compare the offset of the maximum a priori waveform parameters (the bias) to the width of the distribution, which we refer to as the statistical error. For advanced ground-based detectors, we find that the systematic biases are well within the statistical error for realistic signal-to-noise ratios (SNR). These biases grow to be comparable to the statistical errors at high ground-based-instrument SNRs (SNR ~ 50), but never dominate the error budget. At the much larger signal-to-noise ratios expected for space-based detectors, these biases will become large compared to the statistical errors, but for astrophysical black hole mass estimates the absolute biases (of at most a few percent) are still fairly small.

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I. INTRODUCTION

Binary-black-hole (BBH) coalescences are cornerstone sources for gravitational-wave (GW) detectors, be they existing ground-based detectors like LIGO [1] and Virgo [2], planned space-based detectors such as classic LISA [3] or eLISA [4], or a pulsar timing array [5]. The analysis of GW data to detect and characterize binary-black-hole merger events and to test the predictions of general relativity requires some family of efficiently computable signal models, representing all the possible waveforms consistent with general relativity.

Modeling BBH waveforms has historically been separated into three regimes, distinguished by the different computational procedures suitable for each. The “inspiral” where the individual black holes are sufficiently separated for post-Newtonian (PN) theory to be valid [6–9], the “merger” of the binary where numerical relativity (NR) is needed [10–12], and the “ringdown” phase of a single, post-merger, perturbed object relaxing to a Kerr black hole [13, 14].

During recent years, work at the interface between analytical and numerical relativity has provided the community with a variety of semi-analytical inspiral-merger-ringdown waveform sets [15–26] of varying scope in parameter range and accuracy. These waveforms have already been used to search for GWs from high-mass [27, 28] and intermediate-mass [29] binary black holes in LIGO and Virgo data, and also to carry out preliminary parameter-estimation studies for ground-based

detectors [30] and space-based detectors, such as classic LISA [31, 32]. In this paper we shall study a set of inspiral-merger-ringdown waveforms based on the Effective-One-Body (EOB) framework [33–37].

Template waveforms will always be an approximation to the true signals and the difference, if large enough, can bias inferences made from the GW data about the astrophysical parameters of the system or the validity of general relativity. Estimates of the systematic errors introduced by waveform approximants in the literature [25, 38–42] have focused only on the inspiral, or used general conservative criteria to determine when the waveform has a bias, never using the parameter-estimation techniques employed in actual data analysis.

In this work, we will carry out the first *measurement* of systematic biases introduced when determining the physical parameters of a BBH merger by using EOB waveforms as templates. We do so by simulating data that contain NR waveforms as the “signal” to be detected. Then, using the Markov Chain Monte Carlo (MCMC) method [43, 44], we sample the posterior distribution function for the binary parameters using EOB waveforms as the templates. The characteristic width of the posterior as determined by the MCMC is taken as the *statistical* error, while the distance in parameter space between the dominant mode of the posterior and the true, or “injected” waveform parameters is the *systematic* error, or bias, introduced by these waveforms. We study several different BBH systems, sampling the total mass, mass-ratio, and signal-to-noise ratio (SNR) space for both a LIGO/Virgo network in the advanced-

detector era [45, 46] and a LISA-like configuration.

Before proceeding further, we wish to make clear what we mean by “statistical error.” In the following examples, we typically do not include any noise in our data simulations. The resulting posterior distribution function is, in a strict sense, representative of the statistical error which would be realized by repeated measurement of the same system, averaged over all possible noise realizations. This *is not* representative of the expected measurement accuracy achieved for a single detection, the uncertainty of which will be significantly impacted by the particular noise realization with which the signal competes [47]. Therefore, we make no claims about what sort of statistical uncertainty should be expected for future GW detections – what we call “statistical error” is just used to normalize the bias introduced by approximate waveforms. We make this choice because any attempt to both characterize statistical error as well as systematic error will be hamstrung by the inaccuracy of our data simulations, as simulated noise would be gaussian distributed with a known power spectrum and, thus far, actual detector data is not so easily characterized [48, 49]. Ultimately, including noise in the data simulations would only introduce uncontrolled variables in our experiment, without providing any useful insight.

For this first analysis we employ non-spinning waveforms for quasi-circular orbits. In particular, for the injected signals, we consider the NR waveforms produced by the Caltech-Cornell-CITA collaboration in Ref. [50]. For the templates we use the EOB waveforms that were calibrated in Ref. [25] to those NR waveforms [50]. Because our emphasis is on BBHs, and the merger waveforms in particular, it is certainly the case that spin magnitude and orientation play an important role in the waveform, and thus in parameter estimation [51–55]. NR waveforms with spins aligned or anti-aligned with the orbital angular momentum are available, and EOB waveforms that include spins have been developed in Refs. [22, 26, 37]. However, the spinning EOB waveforms are currently restricted to the dominant mode and additional code development is needed before they can be employed in stochastic sampling methods like the MCMC. Thus, we leave to the future the extension of this study to spinning BBHs.

Within these limitations, we show that the EOB waveforms developed in Ref. [25] and tested here are accurate enough to introduce little to no significant biases when the data contain NR waveforms at SNRs consistent with expectations for likely LIGO/Virgo detections ($\text{SNR} \lesssim 50$). For LISA-like detections, where the expected SNRs are much higher than for ground-based detectors, statistically significant biases do emerge. Nonetheless, we find that the discrepancies between the true and measured parameters, at a few percent for the black-hole masses, are small enough to not impact key astrophysical conclusions that may be drawn from the data (e.g., black-hole seed models, etc.) [4, 56, 57]. However, when very high accuracies are required, as when testing

the validity of general relativity [58, 59], best-fit EOB waveforms from the existing model will leave behind significant residual power, making them ill suited for these applications without further development.

The remainder of the paper is organized as follows. In Sec. II we describe the numerical and analytic waveforms used in this work. In Sec. III we lay out how the study will proceed, describing in particular the MCMC sampler that we use. We then discuss in detail the results for stellar-mass BBHs in ground-based detectors (Sec. IV) and supermassive BBHs in space-based observatories (Sec. V). In Sec. VI we summarize the findings from this work, address limitations, and discuss future directions to be pursued.

II. INSPIRAL-MERGER-RINGDOWN WAVEFORMS USED IN THE ANALYSIS

Our study involves comparisons between two sets of waveforms. We primarily seek to *evaluate* a continuously parameterizable family of model waveforms based on the EOB framework against a discrete set of highly accurate numerical relativity (NR) waveforms. In analogy with observational algorithm tests, we can think of the numerical waveforms as “injected” signals, which we challenge the “template” EOB waveforms to match.

We employ as injected signals the non-spinning NR waveforms produced by the Caltech-Cornell-CITA collaboration [50], using the spectral Einstein code. The NR polarizations have mass-ratio $q \equiv m_1/m_2 = 1, 2, 3, 4, 6$ and contain -2 spin-weighted spherical harmonics $(\ell, m) = (2, \pm 2), (2, \pm 1), (2, 0), (3, \pm 3), (3, \pm 2), (4, \pm 4), (5, \pm 5),$ and $(6, \pm 6)$. These waveforms provide 30–40 (quadrupole) GW cycles before merger, depending on the mass-ratio.

The phase and amplitude errors of the NR waveforms vary with mass-ratio and gravitational mode. The numerical errors grow toward merger and ringdown, and typically at merger, for the dominant (2, 2) mode, the phase error ranges between 0.05 and 0.25 rad, while the fractional amplitude error is at most 1%. The subdominant modes can have somewhat larger errors, especially the (3, 3) and (4, 4) modes.

Applying the numerical waveforms to generate mock signal observations, we will test the ability of a previously published family of template waveforms to characterize these signals [25]. These template waveforms are based on the EOB framework, founded on the very accurate results of PN theory, an expansion of general-relativity dynamics in powers of v/c , where v is the characteristic velocity of the binary. In the EOB approach, however, the PN expansions are applied in a resummed form that maps the dynamics of two compact objects into the dynamics of a reduced-mass test particle moving in a deformed Schwarzschild geometry [33–37]. Waveforms in the EOB formalism are derived from such particle dynamics up to the light-ring (unstable photon orbit) radius. The subse-

quent ringdown portion of the waveforms is a superposition of quasinormal modes matched continuously to the inspiral. Tunable parameters, effectively standing in for currently unknown higher-order PN terms, are fixed by matching to numerical-relativity simulation results.

The comparable-mass NR waveforms in Ref. [50] were used, together with the small-mass-ratio waveforms produced by the Teukolsky code in Ref. [60], to calibrate a non-spinning EOB model in Ref. [25]. More specifically, the numerical waveforms available at discrete points in the parameter space were employed to fix a handful of EOB adjustable parameters entering the EOB conservative dynamics and gravitational modes. These adjustable parameters were then interpolated over the entire mass-ratio space. The EOB model in Ref. [25] contains four subdominant gravitational modes, $(2, \pm 1)$, $(3, \pm 3)$, $(4, \pm 4)$ and $(5, \pm 5)$, beyond the dominant mode $(2, \pm 2)$.

The EOB model in Ref. [25] has been coded in the (public) LIGO Algorithm Library (LAL) [61] (under the name EOBNRv2). We carry out our study using LAL to generate template waveforms. Henceforth, we denote the EOB model with only the dominant $(2, \pm 2)$ mode as EOB₂₂, and the model that includes the four subdominant modes $(2, \pm 1)$, $(3, \pm 3)$, $(4, \pm 4)$ and $(5, \pm 5)$ as EOB_{HH}. We will also omit the \pm in mode labels.

The phase difference of the $(2, 2)$ mode between the calibrated EOB model and numerical simulation remains below ~ 0.1 rad throughout the evolution for all mass-ratios considered; the fractional amplitude difference at merger (i.e., at the waveform’s peak) of the $(2, 2)$ mode is 2%, growing to 12% during the ringdown. Around merger and ringdown, the phase and amplitude differences of the subdominant modes between the EOB and NR waveforms are somewhat larger than those of the $(2, 2)$ mode. [The numerical errors, and phase and amplitude differences between the EOB and NR waveforms can be read off from Figs. 6–10 in Ref. [25].]

To quantify how these differences between template and signal would affect GW searches in Advanced LIGO, Ref. [25] studied the effectualness and measurement accuracy of the EOB model. When investigating the effectualness for detection purposes, they found that the NR polarizations containing the strongest seven modes¹ have a maximum mismatch of 7% for stellar-mass BBHs, and 10% for intermediate-mass BBHs, when using only EOB₂₂ for $q = 1, 2, 3, 4, 6$ and binary total masses 20–200 M_\odot . However, the mismatches decrease when using the full EOB_{HH} model, reaching an upper bound of 0.5% for stellar-mass BBHs, and 0.8% for intermediate-mass BBHs. Thus, the EOB model developed in Ref. [25] is accurate enough for detection, which generally requires a mismatch not larger than 7%.

To understand whether this EOB model is pre-

cise enough for measurement purposes, the authors of Ref. [25] carried out a preliminary study, adopting as accuracy requirement for measurement the one proposed in Refs. [41, 42]. Using a single Advanced LIGO detector, Ref. [25] computed the SNRs below which the EOB polarizations are accurate enough that systematic biases are smaller than statistical errors. Since subdominant modes have non-negligible contribution for large mass-ratios, and those modes have the largest amplitude errors, they found that the upper-bound SNRs are lower for the most asymmetric systems, such as $q = 6$. However, as stressed in Ref. [25], the accuracy requirement in Ref. [41, 42] may be too conservative, and by itself does not say which of the binary parameters will be biased and how large the bias will be. It could turn out that the biased parameters have little relevance in astrophysics or tests of general relativity. It is the main goal of this paper to measure the actual biases of the EOB model with and without the subdominant modes.

Our study is restricted to binary systems moving along quasi-circular orbits where the spin of each constituent black hole is negligible, thus reducing the model parameters θ from a space of 17 dimensions to 9 dimensions:

$$\theta = \{\ln M, \ln \mathcal{M}, \ln D_L, t_p, \sin \delta, \alpha, \cos \iota, \psi, \varphi_p\}. \quad (1)$$

In the above parameter list, $M \equiv (1+z)(m_1+m_2)$ is the redshifted total mass of the binary, and $\mathcal{M} \equiv \nu^{3/5} M$ is its chirp mass, where $\nu \equiv m_1 m_2 / (m_1 + m_2)^2 = q / (1+q)^2$ is the symmetric mass-ratio. We denote by D_L the luminosity distance, which, along with the right ascension α and declination δ , describes the location of the binary. The orientation of the binary’s orbital angular-momentum vector $\hat{\mathbf{L}}$ with respect to the line of sight $\hat{\mathbf{k}}$ from the observer is encoded in the model using the inclination ι , polarization angle ψ , and phase φ_p — the Euler angles that describe the rotation from $\hat{\mathbf{k}}$ to $\hat{\mathbf{L}}$. The parameter t_p is the time of the $(2, 2)$ mode’s maximum amplitude, a proxy for the binary merger time, and φ_p is the GW phase at t_p .

Our comparisons between the EOB waveforms and the NR data are restricted to the late-inspiral, merger and ringdown, that is roughly 30–40 GW cycles before merger, depending on the mass-ratio. The injected signals contain only the NR waveforms available. We do not match the NR waveforms to EOB or PN waveforms at low frequency to increase the number of cycles, because we do not know how well the EOB or PN waveforms would approximate the NR waveforms outside the region of calibration and we do not want to introduce unknown errors when estimating the systematic biases.

There is no guarantee that a template that is in phase with the NR waveform during the last 30–40 GW cycles will remain so throughout the entire inspiral. The mass parameters are strongly encoded in the GW phase, so any additional de-phasing at times earlier than covered by the NR simulations can potentially increase the systematic biases. Therefore, in order for our study to be meaningful, we consider binary systems with a total

¹ Note that in Ref. [25] the $(2, 0)$ mode was not included.

mass M such that the majority of the SNR is accumulated during the last 30–40 GW cycles before merger.

III. STATISTICAL VERSUS SYSTEMATIC ERRORS USING MCMC TECHNIQUES

We use the MCMC algorithm [43, 44] to produce samples from the posterior distribution function for the waveform parameters. The MCMC sampler is built on the foundation of Bayes’ Theorem, which, in the context of parameter inference, defines the posterior distribution function for parameter vector θ and data d as

$$p(\theta|d, \mathcal{I}) \equiv \frac{p(d|\theta, \mathcal{I})p(\theta|\mathcal{I})}{p(d|\mathcal{I})}. \quad (2)$$

Here $p(\cdot|\cdot)$ are conditional probability densities with arguments on the right-hand side of the bar assumed to be true, $p(d|\theta, \mathcal{I})$ is the likelihood, $p(\theta|\mathcal{I})$ is the prior distribution, and $p(d|\mathcal{I})$ is the model evidence, which, in parameter-estimation applications, serves only as a normalization constant. The information \mathcal{I} denotes all of the assumptions that are built into the analysis, particularly that the NR waveforms represent reality (see Sec. II for associated discussion). Henceforth, to simplify notation, we shall not include \mathcal{I} when writing the conditional probability density $p(\cdot|\cdot)$. When comparing different model combinations, we adopt the notation for conditional probabilities with arguments to the right of a vertical bar representing the data and arguments to the left signifying which model was used as templates. For example, results labeled $p(\text{EOB}_{22}|\text{NR})$ come from the posterior distribution functions for models using EOB_{22} as the templates and an NR waveform as the data.

With the posteriors, we compare the statistical error (the characteristic width of the posterior) to the systematic error (the displacement from the injected parameter values of the posterior’s mode). We do not include a noise realization in the simulated data, as that introduces additional biases — each noise realization pushing the best-fit solution away from the injected value in a different way — that are not easily quantified [48]. As the control in this experiment, we simulated (noise-free) data with EOB waveforms and use EOB templates for parameter estimation, giving us one set of results with no systematic bias, apart from the sampling error in the Markov chains due to their finite length. We use these controlled results as code verification, and as the standard against which the other models’ performance is compared. We then test the EOB waveform models by injecting NR waveforms (summed over all available modes) and using the two EOB models discussed in Sec. II as templates. The EOB_{22} model is used as a baseline, as it has been employed in LIGO/Virgo search pipelines to analyze data collected in recent science runs [29, 62]. The EOB_{HH} model is the most complete waveform at our disposal, and is used to measure how well the EOB model could perform on NR data.

In Eq. (2) we use the standard gaussian logarithmic likelihood $\ln p(d|\theta) \equiv -(d - h(\theta)|d - h(\theta))/2 + C$, where C is a normalization constant that does not depend on model parameters and is henceforth neglected, h is the template, and

$$(a|b) \equiv 2 \sum_{i=1}^I \int_{f_{\min}}^{f_{\text{Nyq}}} \frac{\tilde{a}_i^*(f)\tilde{b}_i(f) + \tilde{a}_i(f)\tilde{b}_i^*(f)}{S_n^i(f)} df, \quad (3)$$

denotes a noise-weighted inner product with the sum on i over I (independent) interferometer channels and $S_n^i(f)$ is the one-sided noise power spectral density (PSD) for detector i . The bounds of integration, f_{\min} & f_{Nyq} , are the minimum frequency of the NR waveform and the Nyquist frequency of the data, respectively. The Nyquist frequency is chosen to ensure that the highest-frequency portion of the waveform is well below the instrument sensitivity curve, while f_{\min} is set by the duration of the NR waveform and the total mass of the system, such that we only integrate over frequencies where numerical data exists.

For each case, the Markov chains are run for $\sim 10^6$ iterations, taking about 10^3 CPU hours to complete. The chains rely on parallel tempering [63], differential evolution [64], and jumps along eigenvectors of the Fisher Information Matrix (FIM) (e.g., see Ref. [65]) computed from PN waveforms to efficiently explore the posterior distribution function. We use burn-in times of 10^4 samples, and run several chains with different initial locations to check for convergence. Prior distributions for all parameters are chosen to be uniform. Azimuthal angular parameters (α , ψ , and φ_p) have support over $[0, 2\pi)$ with periodic boundary conditions, while declination-like angle parameters ($\sin \delta$, $\cos \iota$) range from $[-1, 1]$ with reflecting boundary conditions. The ranges for $\ln M$ and $\ln D_L$ are chosen to be large enough so as to not influence the posteriors. The prior range on $\ln \mathcal{M}$ is coupled to $\ln M$, as the maximum value of the chirp mass occurs for the $q = 1$ ($\nu = 1/4$) case, and depends on the total mass M of the system. Because of this, the prior boundary on chirp mass does affect the posteriors for the equal-mass systems considered in this work.

The products of our analysis procedure are samples from the posterior distribution function $p(\theta|d)$ — an oddly shaped, sometimes multimodal, blob living in a 9D space. There is no perfect way of distilling this information into a simple, robust, statistic to assess parameter-estimation accuracy. We will make do with the “fractional systematic error” $\delta\beta_\theta$: For parameter θ , we first define the systematic error $\beta_\theta \equiv |\theta_{\text{MAP}} - \theta_0|$ where θ_{MAP} is the *maximum a posteriori* (MAP) value and θ_0 is the injected value, while the statistical error is quantified by the standard deviation σ_θ of the 1D marginalized posterior distribution function. We then define the fractional systematic error as the ratio between β_θ and the statistical error:

$$\delta\beta_\theta \equiv \frac{\beta_\theta}{\sigma_\theta}. \quad (4)$$

We consider templates that consistently yield $\delta\beta_\theta \lesssim 1$ as introducing negligible bias, assuming the NR waveforms are exact, which, as seen in Sec. II, is not the case².

The fractional systematic error (4) can be interpreted as the number of standard deviations away from the injected value at which we find the MAP waveform. This choice of statistic is not perfect — low-SNR systems have very non-gaussian posteriors making the standard deviation a poor choice for characterizing the statistical error. Furthermore, the MAP parameters are a single point and tell us nothing about how large a region in parameter space had similar posterior support to the current best estimate. Additionally, the MAP value is a feature of the full 9D posterior, while the variances are computed from the *marginalized* posterior distribution functions. This introduces complications for some special cases, as we shall discuss in detail below.

IV. RESULTS FOR ADVANCED LIGO DETECTORS

The first test of the EOB waveforms uses simulated data from the network of advanced ground-based detectors expected to come on line in the middle of this decade: the two LIGO detectors in the USA and the Virgo detector in Italy. We use the same noise PSD for each interferometer, the “zero-detuned high-power” curve from Ref. [66], which is the sensitivity curve for the fully completed Advanced LIGO detector. The GW response in each interferometer is modeled by convolving the GW signal with the beam-pattern function for that detector and applying the appropriate time-delays between interferometers [67].

A. Choice of binary configurations

We study several binary configurations using different mass-ratios, total masses, and SNRs. The SNR of the system is computed via $\text{SNR} = \sqrt{\langle d|d \rangle}$, and its value is controlled by adjusting the luminosity distance D_L . Because our simulated data d contain no simulated noise, the SNR is simply the inner product of the injected waveform with itself. Also, our definition of the inner product in Eq. (3) includes a summation over all interferometer channels, thus we quote the *network* SNRs for the ground-based studies.

The first three panels of Fig. 1 show the time-domain EOB_{HH} waveforms (blue, dotted) and whitened by the noise spectral density (red, solid) for three representa-

tive cases studied here³. The vertical lines indicate intervals in which contribute 10% of the signal power, starting from f_{\min} for each system, with the rightmost line indicating where 99.9% of the power has accumulated. We focus on the late-inspiral, merger, and ringdown portions of the waveform, beginning 1000 M before the peak of the (2, 2) mode’s amplitude. The power intervals are included as a guide to see which portions of the waveform contribute most to the parameter estimation. For instance, 50% of the $q = 1$, $M = 50 M_\odot$ waveform’s integrated power is contained in the late-inspiral, merger, and ringdown, while the $q = 2$, $M = 23 M_\odot$ signal is much less dominated by this interval, accounting for only 30% of the power. The $q = 6$, $M = 120 M_\odot$ examples are most influenced by the end of the waveform, which makes up over 70% of the power.

The bottom-right panel of Fig. 1 shows the strain spectral densities $|\hat{h}(f)|$ for the same three systems, now using the NR waveforms used in this study. Also included is the Advanced LIGO power spectral density.

We focus on moderately high-mass black-hole mergers with $M \sim 50 M_\odot$ (e.g., the equal-mass case shown in Fig. 1: top-left panel, red solid curve in bottom-right panel). Beyond their potential as Advanced LIGO sources⁴, high-mass systems serve an important role in testing the waveform models for two reasons. First, as explained in Sec. II, the NR signals are short in duration and we do not supplement the waveform by hybridizing the numerical data with analytic inspiral models at low frequency. We therefore require higher-mass systems, merging at lower frequency, to ensure that most of the inspiral missing from the NR data will fall outside the sensitive measurement band of the detector. With $M \sim 50 M_\odot$, NR waveforms start at ~ 30 Hz, setting f_{\min} in the inner product defined in Eq. (3). Comparing these data to EOB waveforms with the same parameters but $f_{\min} = 10$ Hz (below which the Advanced LIGO sensitivity is very poor), we find the NR waveforms contain $\sim 85\%$ of the total signal power, or $\sim 90\%$ of the total SNR.

A second reason for focusing on $M \sim 50 M_\odot$ for the binary is that these systems are “centrally located” in

³ “Whitened” waveforms are ones that have been Fourier-transformed to the frequency domain, rescaled by $1/\sqrt{S_n(f)}$, and finally re-transformed to the time domain [68].

⁴ A binary with $M \sim 50 M_\odot$ is astrophysically relevant, as the largest black-hole mass ever observed is in the range of 23–34 M_\odot [69, 70]. Recent results from population-synthesis studies suggest that massive, low-metallicity stars are capable of producing black holes as large as $M \sim 80 M_\odot$ [71], although these findings are for single stars only, and binary evolution could either increase or decrease the maximum black-hole mass. Additionally, there exists at least one example of a massive Wolf-Rayet star, R136a1 [72], with $M \sim 250 M_\odot$ at a distance of ~ 0.1 Mpc and with sufficiently low metallicity to produce a massive black hole. However, it cannot be excluded that the star goes instead through a pair-instability supernova, leaving no remnant.

² Currently we have no way of incorporating the numerical error of the NR waveforms into our estimate and so we neglect it.

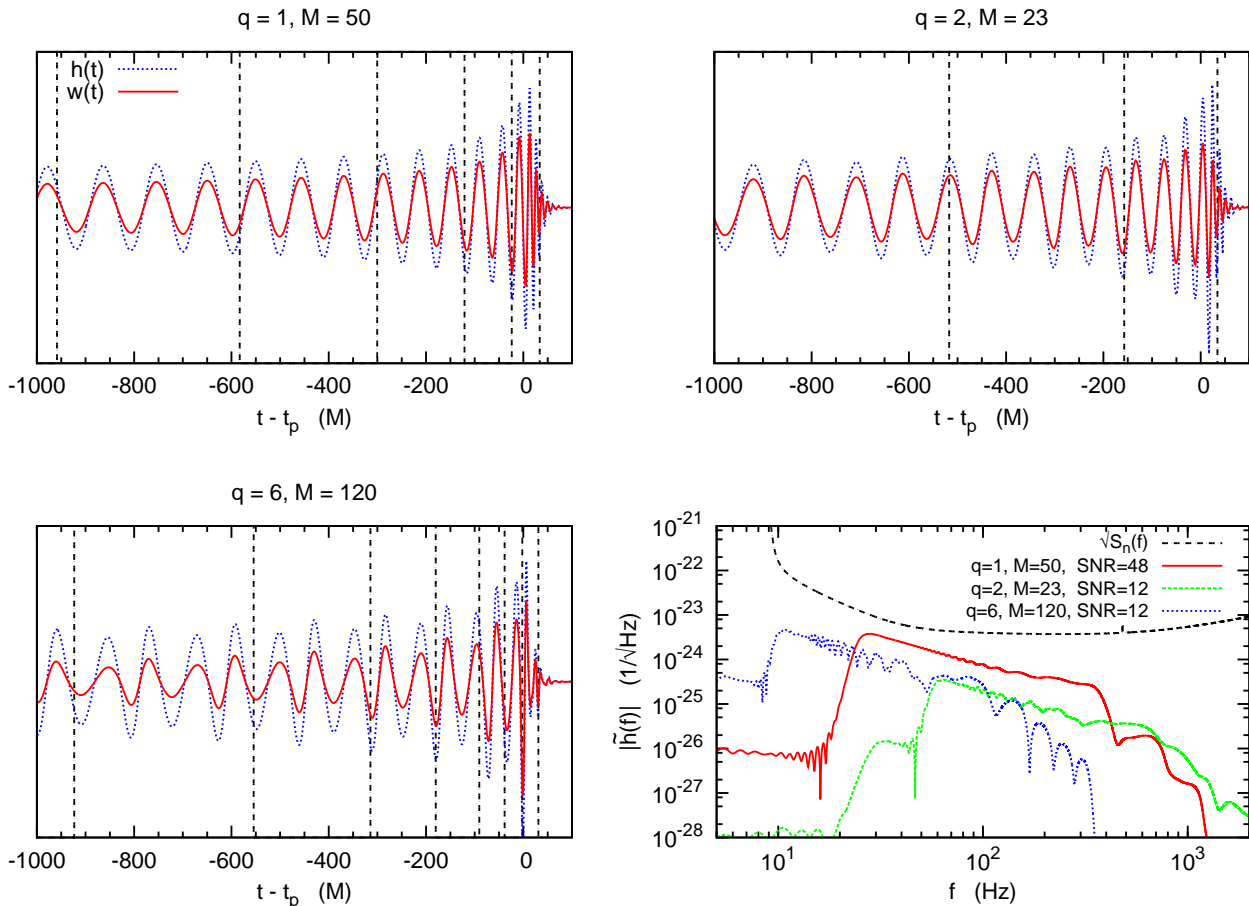


FIG. 1: [Top row, and bottom-left] Time-domain EOB waveforms (blue dotted) and the same signal whitened by the noise spectral density $\sqrt{S_n(f)}$ (red solid) representing different test cases studied in this work. The time-axis is scaled by the total mass M and shifted by the merger time t_p . The vertical dotted lines demarcate 10% intervals for accumulated signal power, with the rightmost line at the 99.9% mark. [Bottom right] Strain spectral densities $|h(f)|$ showing an estimate for the Advanced LIGO noise curve (gray dotted line) and the NR waveforms for the same representative examples as the other panels.

frequency over the most sensitive band of the advanced detectors (~ 30 to $\sim 10^3$ Hz, e.g., see the bottom-right panel in Fig. 1), such that inspiral, merger, ringdown, and additional modes all contribute to the overall signal power and, accordingly, the parameter-estimation capabilities. Generally speaking, we expect such systems to make the greatest demands on complete inspiral-merger-ringdown waveform model accuracy.

While the $M \sim 50 M_\odot$ systems serve as the basis for our comparisons, we include additional examples to probe regions of signal space that are of particular interest. These include a $q = 6, 120 M_\odot$ system (Fig. 1: bottom-left panel, blue dotted curve in bottom-right panel) chosen such that the subdominant modes contribute the most, as they will be most pronounced at high mass-ratios, and signal power from the higher frequency modes is still in the sensitive band of the detector. At this mass, $f_{\min} = 10$ Hz, so our analysis is not missing any signal power due to the length of the NR data.

We also go to lower masses, using a $q = 2, M = 23 M_\odot$ binary (Fig. 1: top-right panel, green dashed curve in bottom-right panel) as a more likely LIGO/Virgo detection, to demonstrate the EOB models' parameter-estimation accuracy not at the extremes of a potential binary signal, but within reasonable expectations of what the coming data may hold (apart from including the black-hole spins). It is worth noting that for these low-mass systems, we are missing a large portion of the inspiral, as $f_{\min} = 60$ Hz and $\sim 30\%$ of the full SNR will be accumulated below that frequency. Therefore, these results might change in the future, when longer EOB and NR waveforms become available.

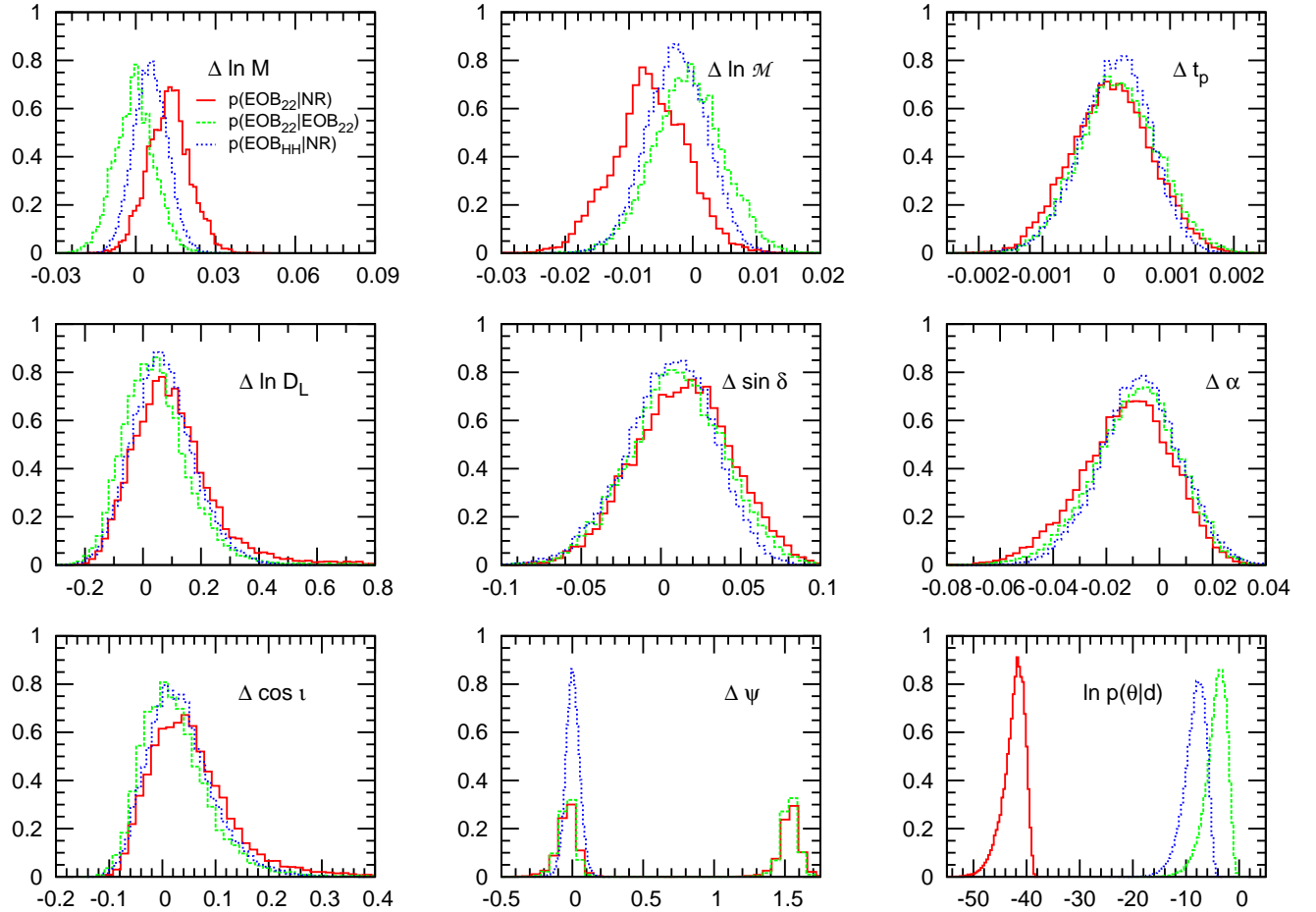


FIG. 2: Example marginalized posterior distribution functions for a binary system with mass-ratio $q = 2$, total mass $M = 51M_\odot$, and network SNR = 48 produced using an MCMC sampler. The green (dashed) histograms arise from injecting the EOB₂₂ waveforms and using the same model as the template. The red (solid) distributions use the EOB₂₂ waveform while the blue (dotted) curves use the EOB_{HH} waveform to recover an NR injection. The x -axis shows the distance away from the injected parameter value, so distributions that peak at zero show no bias. The bottom-right panel shows the logarithmic likelihood distributions, which are used to quantify the amount of residual power left behind by the waveform model. The y -axes have been normalized to range between $[0, 1]$.

B. Results on systematic biases at fixed inclination angle

In Fig. 2 we plot the 1D marginalized posterior distribution functions for each parameter for the case of a binary with mass-ratio $q = 2$, total mass $M = 51 M_\odot$, and network SNR = 48, produced using an MCMC sampler. The inclination angle is chosen to be $\iota = \pi/3$. The independent variables in these plots are $\Delta\theta = \theta - \theta_{\text{inj}}$, where θ_{inj} are the injected parameter values. The $p(\text{EOB}_{22}|\text{EOB}_{22})$ histograms (green, dashed lines) are the posteriors using the EOB₂₂ waveforms for both the signal and the templates. These confirm that the MCMC sampler is working properly, as the posteriors all show strong support for the injected waveform parameters (peaking at or near 0), and statistical errors consistent with results in Ref. [30] obtained using FIM estimates and phenomenological inspiral-merger-ringdown

waveforms.

The red (solid) lines and blue (dotted) lines are for data containing an NR signal and the EOB₂₂ and EOB_{HH} waveforms as templates, respectively. The bottom-right panel shows the logarithmic likelihood distributions for each chain. We can see from these posteriors that the EOB₂₂ waveform is significantly biased away from the NR injected value, by $\sim 1\%$ in both M and \mathcal{M} . This bias is substantially reduced when using the EOB_{HH} template, to the point where the systematic error is well within the statistical error of the posterior. For the extrinsic parameters such as distance and sky location, the posteriors for the approximate templates are nearly identical to those produced by using the exact same waveform for both data simulation and parameter estimation. We also see the role that subdominant modes play in breaking the $\pi/2$ degeneracy in the polarization angle ψ , which can aid in distance and sky-location determination for

some systems.

The SNR of the residual $d - h$, given by $\text{SNR}_{\text{res}} = \sqrt{-2 \ln p(d|h)}$ can be inferred from the bottom-right panel. Viewed in this way, there is a distinct excess in the residual for even the EOB_{HH} case (blue, dotted) in comparison with the idealized control MCMC residuals (green, dashed). To understand the significance of this, consider applying a detection threshold of $\text{SNR} = 6$ [73] for the residual waveform. This corresponds to a maximum logarithmic likelihood of $\lesssim -18$, below which the residual could potentially contain enough power to be detected after the best-fit waveform is regressed from the data. Suppose in the near future we have model waveforms at our disposal containing all of the details of black-hole mergers (i.e. spins and eccentricity) that generally produce $\text{SNR}_{\text{res}} \lesssim 6$ for equally detailed NR simulations, and yet coherent residuals are consistently found in the data. Such an event could suggest a possible departure from general relativity.

The results from our MCMC studies for different systems are displayed in Tables I and II, which show the fractional systematic error $\delta\beta$ (see Eq. (4)) for the mass, sky location, and distance parameters. The injected waveforms again had an inclination angle of $\iota = \pi/3$. We include our estimate of the statistical error σ_θ to quantify the precision of the advanced detectors for the systems considered here. The standard deviations should be interpreted with caution; we do not include noise in the simulated data, so the deviations are not representative of the “error bars” on a *particular* detection, but instead represent an ensemble average over idealized Gaussian stationary noise realizations of the statistical error for these particular systems.

Table I contains the *intrinsic* parameters — those that affect the shape of the waveform. Because we consider non-spinning black holes, the only intrinsic parameters are the masses. The *extrinsic*, or observer-dependent, parameters (i.e., distance and sky location) are given in Table II. They are encoded in the instrument response to the GW, instead of being imprinted in the phase and amplitude evolution of the waveform itself. We do not report on the orientation parameters ι and ψ or reference time t_p and phase φ_p parameters in this fashion, but note that results for these other parameters are consistent with the extrinsic variables in Table II.

From Table I we can see that generically, the EOB₂₂ waveforms are not as accurate as the EOB_{HH} waveforms, which include the subdominant modes. This is true even for comparable-mass systems, where the subdominant modes only minimally contribute to the overall waveform power. The bias introduced by neglecting additional harmonics is not due to missing waveform power as much as it is caused by phase differences between a quadrupole-only template and the full NR data, as coherent matched-filtering analyses are typically more sensitive to phase than amplitude.

The parameter estimation accuracy of the EOB_{HH} model up to $\text{SNR} \sim 50$ exceeds expectations from

Ref. [25], as can be seen by focusing on rows 2 and 7 in Table I. Here we find systems chosen specifically to compare with Fig. 15 in Ref. [25] where, based on the accuracy requirement proposed in Refs. [41, 42], they predicted that systematic error could exceed statistical error at *single-detector* $\text{SNR} \sim 35$ ($q = 1$, $M = 50M_\odot$) and ~ 11 ($q = 6$, $M = 56M_\odot$).⁵

Our analysis uses the LIGO/Virgo network of detectors, as opposed to the single-detector studies from Ref. [25]. This difference will not heavily impact the results, as it is the measurement of *intrinsic* parameters that is most affected by differences in the waveform model due to both the accuracy with which they are measured, and the way they are encoded in the phase evolution of the signal. Measurement of the intrinsic parameters is not greatly influenced by the inclusion of additional detectors in the network (at a fixed SNR). Our findings show that, even at SNRs that are rather high for an expected LIGO detection, the EOB_{HH} model introduces systematic errors that differ by $\lesssim 1\sigma$ from the injected parameters.

The extrinsic parameters, on the other hand, are inferred mostly from the overall amplitude of the waveform, which is not as well-measured as phase, and the time of arrival of the signal at each detector. We thus expect that the extrinsic parameters, determined with lower fidelity than the masses, will be better able to tolerate small differences between template waveform models within the statistical error. Adding additional detectors to the network dramatically improves the statistical error for extrinsic parameters, mostly due to the increased baseline [74], but not to the point of becoming influenced by the waveform systematics.

Indeed, we find that the relative systematic biases for extrinsic parameters are generally smaller than those of the intrinsic parameters. For systems with $q \geq 2$, regardless of the SNR or the EOB model, the systematic errors are consistently smaller than the statistical errors, even when the (2, 2)-only waveform is used as the template. This is evident in Fig. 2, where the D_L , $\sin \delta$, and α posteriors are nearly indistinguishable, despite the significant difference in the residual left behind by the waveform model, as shown in the bottom-right panel containing the logarithmic likelihood distributions. The same can not be said for the equal-mass cases (top two rows in Table II), where we encounter a subtle effect from our choice of statistic, $\delta\beta$.

For the $\text{SNR} = 12$, equal-mass case, the $\delta\beta$ statistic breaks down and results are omitted from the table. At such a low signal strength, the orientation parameters are very poorly measured, with the polarization angle ψ

⁵ The accuracy criterion used in Refs. [25, 41, 42] is a “sufficient” but “not necessary” requirement for parameter estimation, and it does not say which of the binary parameters will be biased and how large the bias will be. Thus, the authors of Ref. [25] were making conservative judgments about the waveform accuracy.

q	$M (M_\odot)$	$f_{\text{low}} \text{ (Hz)}$	SNR	$p(\text{EOB} \text{EOB})$				$p(\text{EOB}_{22} \text{NR})$				$p(\text{EOB}_{\text{HH}} \text{NR})$			
				$\sigma_{\ln M}$	$\sigma_{\ln \mathcal{M}}$	$\delta\beta_{\ln M}$	$\delta\beta_{\ln \mathcal{M}}$	$\sigma_{\ln M}$	$\sigma_{\ln \mathcal{M}}$	$\delta\beta_{\ln M}$	$\delta\beta_{\ln \mathcal{M}}$	$\sigma_{\ln M}$	$\sigma_{\ln \mathcal{M}}$	$\delta\beta_{\ln M}$	$\delta\beta_{\ln \mathcal{M}}$
1	50	30	12	0.02	0.02	0.11	0.05	0.02	0.02	0.34	0.17	0.02	0.02	0.29	0.10
1	50	30	48	4×10^{-3}	3×10^{-3}	1×10^{-3}	0.14	0.01	0.01	1.76	0.84	2×10^{-3}	2×10^{-3}	0.79	0.87
2*	23	60	12	0.02	0.01	0.01	0.02	0.03	0.02	0.27	0.07	0.03	0.02	0.18	0.26
2	51	30	12	0.03	0.02	3×10^{-3}	4×10^{-3}	0.03	0.02	0.47	0.28	0.03	0.02	0.01	0.01
2	51	30	48	0.01	0.01	0.01	0.01	0.01	0.01	1.92	1.39	0.01	0.01	0.93	0.32
6	56	30	12	0.03	0.03	0.03	0.08	0.03	0.03	0.94	0.66	0.02	0.02	0.58	0.39
6	56	30	48	0.01	5×10^{-3}	0.23	0.04	0.01	0.01	3.47	2.51	0.01	4×10^{-3}	1.43	0.84
6*	120	10	12	0.03	0.03	0.05	0.14	0.03	0.03	1.67	0.60	0.02	0.02	0.29	0.06

TABLE I: Fractional systematic biases $\delta\beta$ [see Eq. (4)] and statistical errors σ for intrinsic parameters as determined by the MCMC sampler. An asterisk in the mass-ratio column indicates examples where EOB_{HH} was used for the $p(\text{EOB}|\text{EOB})$ study. All other examples used the EOB_{22} waveform.

q	$M (M_\odot)$	$f_{\text{low}} \text{ (Hz)}$	SNR	$p(\text{EOB} \text{EOB})$		$p(\text{EOB}_{22} \text{NR})$				$p(\text{EOB}_{\text{HH}} \text{NR})$			
				$\sigma_{\sin \delta}$	$\sigma_\alpha \text{ (rad)}$	$\sigma_{\ln D_L}$	$\delta\beta_{\ln D_L}$	$\delta\beta_{\sin \delta}$	$\delta\beta_\alpha$	$\sigma_{\ln D_L}$	$\delta\beta_{\ln D_L}$	$\delta\beta_{\sin \delta}$	$\delta\beta_\alpha$
1	50	30	12	0.06	0.04	—	—	—	—	—	—	—	—
1	50	30	48	0.03	0.01	0.10	0.11	0.01	0.09	0.12	0.12	0.17	0.03
2*	23	60	12	0.06	0.03	0.24	0.02	0.04	0.02	0.27	0.28	0.02	0.06
2	51	30	12	0.06	0.04	0.28	0.03	0.08	0.06	0.29	0.21	0.28	0.07
2	51	30	48	0.03	0.02	0.12	0.10	0.10	0.08	0.14	0.11	0.19	0.11
6	56	30	12	0.07	0.05	0.29	0.07	0.04	0.21	0.29	0.28	0.14	0.09
6	56	30	48	0.03	0.02	0.14	0.29	0.10	0.07	0.19	0.28	0.14	0.09
6*	120	10	12	0.08	0.05	0.21	0.23	0.28	0.13	0.30	0.86	0.37	0.54

TABLE II: Same as Table I, except here we show a subset of the extrinsic parameters corresponding to the binary's location. Because of their similarity between each run, the statistical errors are displayed once but apply to each example. Results for the $q = 1$, SNR = 12 example are omitted due to the failure of our $\delta\beta$ statistic.

effectively unconstrained. These large measurement uncertainties cascade through the 1D posteriors via strong $\psi - \iota$ and $\iota - D_L$ covariances. We are left with an unconstrained D_L distribution that is poorly characterized by the variance, and large stochastic variation from one Markov chain run to the next as to where the MAP parameters lie. This degeneracy is evident in Fig. 3, where we show the 2D marginalized posterior distribution function of the $\psi - \cos \iota$ plane (left panel) from a $p(\text{EOB}_{\text{HH}}|\text{NR})$ run, and the maximum logarithmic likelihood found in the Markov chain for different bins in D_L space (right panel) from both $p(\text{EOB}_{22}|\text{EOB}_{22})$ and $p(\text{EOB}_{\text{HH}}|\text{NR})$. We see a virtually flat distribution of the maximum logarithmic-likelihood values between ~ 0.5 to ~ 2.25 Gpc, with well over half of the allowed parameter space in the $\psi - \cos \iota$ plane receiving significant posterior support. The injected value of D_L was near 1 Gpc.

The over-density at $\{\psi, \cos \iota\} \simeq \{\pi/3, 0.5\}$ in the left panel corresponds to the injected parameter values, with the $\pi/2 - \text{shift}$ degenerate mode still appearing despite the inclusion of subdominant modes. Recall that this is an equal-mass system, where the subdominant modes are the least noticeable. The over-density at $\cos \iota \sim 1$ is due to the Markov chain preferring template waveforms with minimal contribution from subdominant modes (to match the strictly equal-mass injection) as the sampler explores higher mass-ratios, up to $q \sim 3$ in this case. Sys-

tems with $\cos \iota \equiv \hat{\mathbf{k}} \cdot \hat{\mathbf{L}} = 1$ are face-on and it is this configuration where the subdominant modes are least prominent.

C. Dependence of the results on the inclination angle

Due to the high computational cost of each MCMC run, we are not able to Monte Carlo over a large population of binary systems. We instead have chosen extrinsic parameters away from the extremes of parameter space. This means sky locations that are away from nulls in any detector's response, and inclinations ($\iota = \pi/3$) that were not edge- or face-on with respect to the observer's line of sight.

One of the more interesting results from this study is the impact of the subdominant modes on the parameter-estimation capabilities of ground-based detectors. The role that the additional modes play in the waveform depends heavily on both the mass-ratio and the orientation of the binary — edge-on systems have the largest contribution from the additional modes, while face-on systems are most dominated by the (2,2) mode. It is therefore possible that, for some more extreme orientations, systematic biases could become large because of the increased importance of the additional modes.

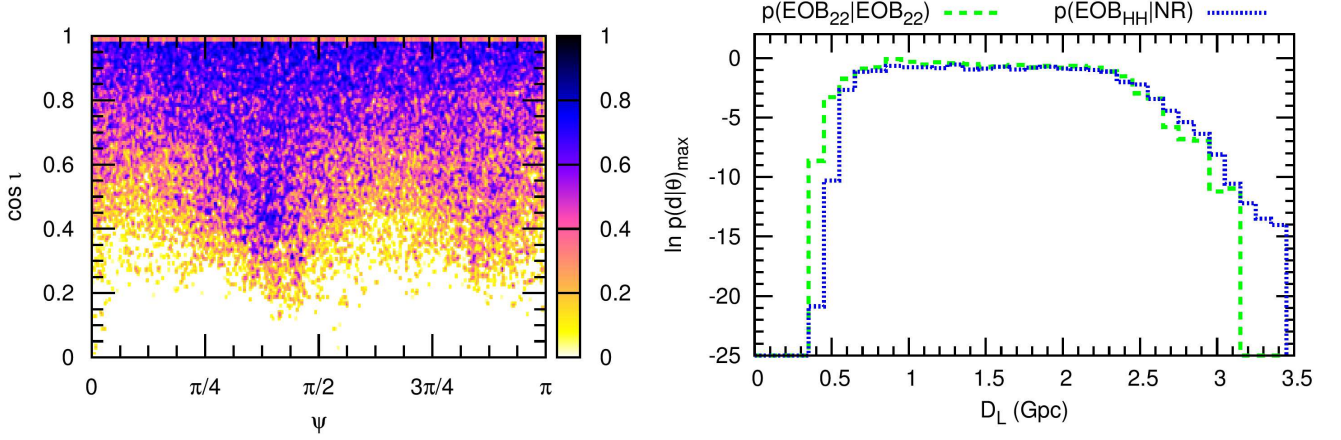


FIG. 3: Selected results from the $q = 1$, $\text{SNR} = 12$ run to exhibit the breakdown of $\delta\beta$ as a useful statistic. The left-hand panel shows the 2D marginalized posterior for the orientation angles ψ and $\cos i$, with darker colors corresponding to higher probability density. The right-hand panel displays the maximum logarithmic likelihood as a function of D_L , which is effectively uniform between ~ 0.5 and ~ 2.25 Gpc.

To allay this concern we performed a series of MCMC runs on a system where the subdominant modes would play an important role, exploring the edges of orientation space for each run. We chose the $M = 120M_\odot$, $q = 6$ system (row 8 in Tables I and II) and analyzed three different orientations: edge-on ($i = \pi/2$), face-on ($i = 0$), and moderate tilt ($i = \pi/3$). We compare the $\Delta \ln M$ and $\Delta \ln \mathcal{M}$ posteriors for each of these systems using the EOB_{HH} model as a template to study data containing an NR waveform injected at $\text{SNR} = 12$. The results in

i	$\sigma_{\ln M}$	$\delta\beta_{\ln M}$	$\sigma_{\ln \mathcal{M}}$	$\delta\beta_{\ln \mathcal{M}}$	%HH
0	0.03	0.10	0.04	0.12	~ 0
$\pi/3$	0.02	0.29	0.03	0.06	7.5
$\pi/2$	0.02	0.18	0.03	0.01	10

TABLE III: Fractional systematic errors and statistical errors for $\ln M$ and $\ln \mathcal{M}$ when $M = 120M_\odot$, $q = 6$, $\text{SNR} = 12$ and for three different inclinations: Edge-on ($i = \pi/2$), face-on ($i = 0$), and an intermediate orientation ($i = \pi/3$). We also include the percentage that the additional modes (HH) contribute to the total SNR.

Table III show the fractional systematic error well below unity for each orientation regardless of the inclination angle. We also include the percentage of the total SNR that comes from the subdominant harmonics (HH). This result confirms that the parameter-estimation accuracy of the EOB model is robust to different orientations, and thus different strengths of the additional modes.

D. Simulating a detection

All of the above results have been performed on simulated data that do not contain any noise, but do include

the noise PSD in the inner product defined in Eq. (3). Thus the posteriors that we generate are not representative of a probability density function for an actual GW measurement, but instead are the hypothetical averaged measurements of the same system in an ensemble of noise realizations [47, 75, 76]. To more realistically demonstrate the parameter-estimation capabilities of advanced ground-based interferometers, we want now to simulate a single LIGO/Virgo detection.

To that end, we use again the binary configuration with $q = 2$, $M = 51M_\odot$ and network SNR of 12, but now add stationary, gaussian noise to the NR waveform using the same PSD as in the noise-free study. The resultant posteriors are then representative parameter-estimation products, subject to the following important caveats:

- We use the same PSD for each detector when, in practice, each interferometer will have different sensitivity at any given time. Furthermore, the Virgo design sensitivity is not identical to LIGO (although it is qualitatively similar). We also effectively introduce a noise “wall” at 30 Hz to account for the limited duration of the NR data.
- We do not include any calibration errors in the waveform injections, which could prove to be a significant contribution to the overall parameter-estimation error budget [49]. Furthermore, we do not account for intrinsic error in the NR waveforms.
- We recognize that simulated additive gaussian noise is different from injecting waveforms into real LIGO/Virgo noise [48].

For this study we find the 2D marginalized posterior distribution functions to be of the most interest. We show results for the sky location in Fig. 4 and mass parameters in Figs. 5 and 6.

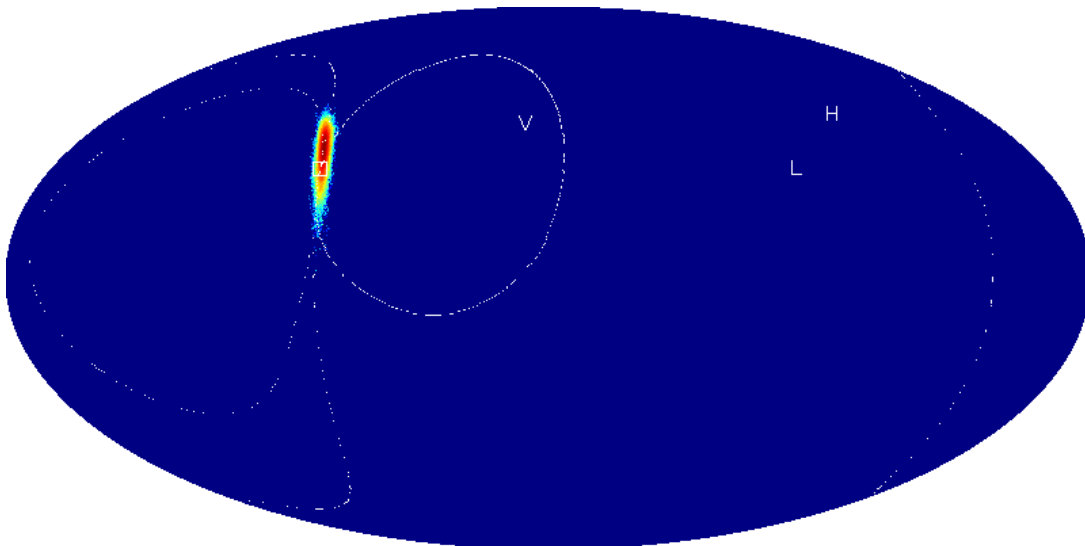


FIG. 4: 2D marginalized posterior distribution function for the sky location of a $q = 2$, $M = 50M_{\odot}$, binary with $\text{SNR} = 12$ injected into simulated stationary gaussian noise colored by the Advanced LIGO noise power spectral density. The red, yellow, and cyan regions roughly correspond to 1, 2, and 3σ confidence regions. The white box represents the injected sky location of this source. The position of each interferometer in the network is projected onto the sky, labeled with H (LIGO Hanford), L (LIGO Livingston), and V (Virgo). The white, dashed lines show the locations that yield the same time-delay between each pair of detectors for the injected sky position.

In Fig. 4, the sky-location posterior is shown in a Mollweide projection with the detector locations projected onto the celestial sphere. The white, dotted lines show the circles of constant time delay between each pair of detectors. The posterior should sit at intersections of these lines, and the principal axis should lie along a line. A small white square is included, centered on the injected position. The injected values for the sky location are contained within the $\sim 63\%$ confidence interval of the posterior (the red region of the error ellipse). The injected sky location was chosen to be a region where the SNR in each detector was roughly equivalent.

Of more pertinence to this study are the mass posteriors. While $\ln M$ and $\ln q$ are the most convenient parameters for the MCMC sampler, being the most orthogonal, they are not of the most interest to the wider astrophysical community. A better data product would be posteriors on either the individual masses m_1 and m_2 , or the total mass M and mass-ratio q . In post-processing we take the MCMC chains and compute the relevant mass parameters at each step in the chain. The injected component black holes have masses $m_1 = 34M_{\odot}$ and $m_2 = 17M_{\odot}$. We show the 2D marginalized posterior distribution functions for the m_1 - m_2 (Fig. 5), and M - q plane (Fig. 6), where the color corresponds to the posterior density.

These figures give a good depiction of just how correlated the mass parameters are with one another, and how much of parameter space is supported by the chain in part due to that strong correlation. For example, the M - q plane has significant support for mass-ratios between 1 and 3, compatible with previous LIGO MCMC

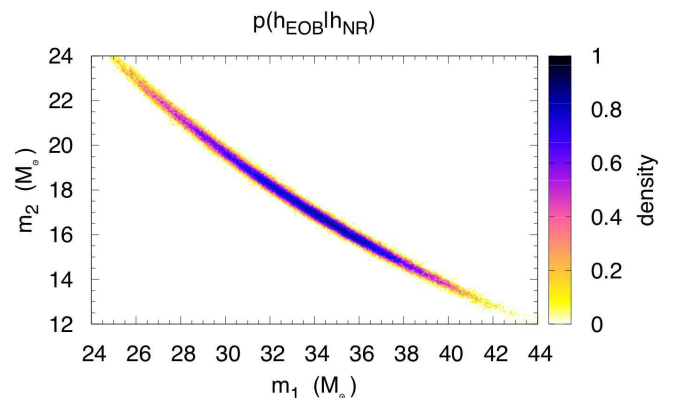


FIG. 5: 2D marginalized posterior distribution function for the individual masses m_1 and m_2 of a $q = 2$, $M = 51M_{\odot}$, binary with $\text{SNR} = 12$, injected into simulated stationary gaussian noise colored by the Advanced LIGO noise power spectral density. The injected values for $\{m_1, m_2\}$ were $\{34, 17\}M_{\odot}$.

studies using PN waveforms (e.g., see Ref. [54]). These are the type of parameter-estimation products that the astrophysics community can anticipate as the advanced detectors come on line in the coming years.

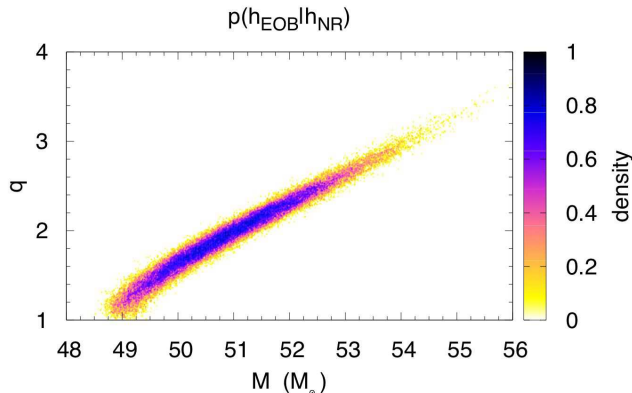


FIG. 6: Same as Fig. 5, but now depicting total mass M and mass-ratio q .

V. RESULTS FOR SPACE-BASED DETECTORS

For EOB waveforms that include subdominant modes, we have found relatively small systematic errors in parameter-estimation results for ground-based observations with $\text{SNR} < 50$. Because ground-based GW instrument rates are limited by sensitivity, higher-SNR events are exceedingly unlikely in the first generation of detections. Proposed space-based instruments will be sensitive to supermassive black-hole (SMBH) mergers out to cosmological scales, such that a significant fraction of detected events may have $\text{SNR} > 100$. Space-based instruments are typically sensitive to these events over a broad bandwidth covering a large number of cycles leading up to merger [4, 77]. Such observations will make much greater demands on the accuracy and efficiency of inspiral-merger-ringdown waveform templates. Though considerable effort has gone into estimating the ability of space-based instruments to measure astrophysical parameters assuming accurate waveforms, very little has been done to assess the template requirements for these future observations. Here we make a limited exploration of this capability with current numerical-relativity and EOB waveforms.

Of several proposed space-based GW interferometer instruments [3, 4, 78, 79], the best-studied concept is the classic LISA mission [3]. While acknowledging that there is currently considerable uncertainty about when and how the first space-based GW instrument will be developed, we choose to study the classic LISA configuration to make contact with the large body of work that has already been dedicated to black-hole merger parameter estimation (e.g., see Refs. [31, 32, 80–82]). To compare with other concepts, the most relevant alteration from the classic LISA design is the arm length (e.g., from 5 Gm for classic LISA down to 1 Gm for eLISA), which sets the overall scale for parameter-estimation capabilities; our results for a more modest detector configura-

tion would be very similar to those for classic LISA, after appropriately rescaling the total mass of the black-hole system.

We follow here the same procedure outlined for the LIGO/Virgo studies in Sec. IV, where NR waveforms are injected into simulated noise-free data, and the signals are analyzed using the EOB model as a template. Because we saw significant bias in the EOB₂₂ model at $\text{SNR} \sim 50$ it is safe to assume that those errors will only grow with SNR, and so we focus these runs only on the EOB_{HH} model.

The duration of the available NR data restricts us to brief LISA observations, for which we can apply the static limit for the detector; thus we neglect LISA's orbital motion during the observation time. Consistent with this, we focus on systems of mass $3 \times 10^7 M_\odot$ at the high end of LISA's sensitive range. For such observations the maximum frequency attained by the merger signal is well below the transfer frequency of the detector (when the wavelength of the GW signal is comparable to the size of the detector). In this low-frequency, static regime, the instrument response is equivalent to two 60° Michelson interferometers, co-located, and misaligned by $\pi/3$ radians. The antenna patterns for this configuration, and the discussion of the two limits applied here, can be found in Ref. [83].

We consider mass-ratios in the range $2 \leq q \leq 6$, observed at $\text{SNR}=100$, which would make these unusually distant for LISA observations. As shown in Fig. 7, the power-spectral density is similar over this mass-ratio range with more structure at $q = 6$. The noise-weighted waveforms for these cases are most comparable to the largest-mass $120 M_\odot$ LIGO case that we studied. Space-based instruments are not expected to have a strong power-law slope like the seismic noise wall in the LIGO sensitivity curve, meaning that even for large masses there is a softer degradation of sensitivity going back to the early portions of the signal, making our f_{min} cut somewhat more artificial here.

Generally the higher SNRs of our nominal LISA observations would predict larger bias from systematic errors in the template waveforms. Because of the differences between the sensitivity curves and response for LISA and LIGO, however, it is not straightforward to scale up such expectations. Table IV shows our results for the parameter biases for mass M and chirp mass \mathcal{M} . Unlike the LIGO results, the biases here are already statistically significant in most cases for LISA observations at $\text{SNR}=100$, reaching a few times the statistical error level. In Table IV we also provide the SNR of the residuals after the MAP waveforms are removed from the data. These residuals with $\text{SNR} > 6$ would be detectable and could therefore lead to biases in estimates of overlapping signals. Residuals at this level would also limit the utility of the current waveform templates for studies aimed at testing general relativity [58, 59].

For LISA, however, such a system would have to be exceedingly distant, at redshift $z > 20$ or more, in order

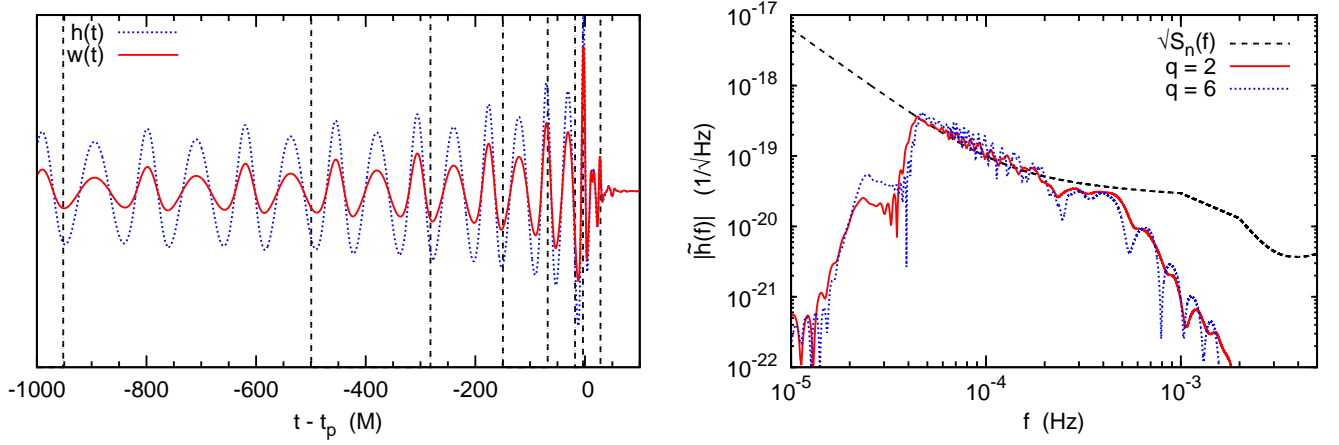


FIG. 7: Same as Fig. 1 now showing the LISA examples using $M = 3 \times 10^7 M_\odot$ at $\text{SNR} = 100$. The time-domain waveform in the left-hand panel is for the $q = 6$ case.

q	$M (M_\odot)$	$f_{\text{low}} (\text{Hz})$	SNR	$p(\text{EOB}_{\text{HH}} \text{EOB}_{\text{HH}})$					$p(\text{EOB}_{\text{HH}} \text{NR})$				
				$\sigma_{\ln M}$	$\sigma_{\ln \mathcal{M}}$	$\delta\beta_{\ln M}$	$\delta\beta_{\ln \mathcal{M}}$	SNR_{res}	$\sigma_{\ln M}$	$\sigma_{\ln \mathcal{M}}$	$\delta\beta_{\ln M}$	$\delta\beta_{\ln \mathcal{M}}$	SNR_{res}
2	3×10^7	5×10^{-5}	100	2×10^{-3}	2×10^{-3}	0.09	0.08	0.5	2×10^{-3}	2×10^{-3}	1.19	1.27	6.5
3	3×10^7	5×10^{-5}	100	2×10^{-3}	3×10^{-3}	0.02	0.04	0.7	3×10^{-3}	2×10^{-3}	0.59	2.82	7.7
4	3×10^7	5×10^{-5}	100	3×10^{-3}	2×10^{-3}	0.10	0.09	0.5	3×10^{-3}	3×10^{-3}	1.31	3.02	9.6
6	3×10^7	5×10^{-5}	100	2×10^{-3}	2×10^{-3}	0.16	0.27	0.5	3×10^{-3}	2×10^{-3}	0.62	0.92	11.5

TABLE IV: Same as Table I but for SMBH mergers as seen by LISA. Here we also include the residual SNR after the MAP waveform is regressed from the data.

to expect an SNR as small as 100 [82]. Actual SNRs could be as much as 100 times larger, and we would expect correspondingly larger relative biases and residuals. Full interpretation of LISA data would thus require higher levels of template accuracy. Even the level of errors in the numerical simulations used here in place of the exact predictions of general relativity are far too large to avoid biasing such high-SNR measurements.

That said, while the statistical error should decrease linearly with increasing SNR (causing $\delta\beta$ to grow), the systematic error should remain approximately the same, and the absolute biases in the mass parameters are small. Reading from Table IV, we see that the systematic errors in M and \mathcal{M} are $\lesssim 1\%$. Converting these into the individual masses, the biases on m_1 and m_2 are at most a few percent. While the MAP mass parameters may end up many σ away from the true values and would thus fall short of an optimal analysis of LISA data, such $\sim 1\%$ errors in the masses are still small in astrophysical terms and may have little impact on key inferences made about the supermassive black-hole population. For example, when trying to constrain black-hole formation scenarios using simulated eLISA BBH catalogs, Amaro-Seoane *et al.* [4] (based on the procedure in Ref. [57]) endured statistical errors as high as $\sim 1\%$ and were still able to easily discriminate between black hole seed models.

The limitations of our current analysis prevent us from

providing any detailed indication of how far template accuracy must be improved for space-based observations. For the detector configuration used here, the *extrinsic* parameter estimation at $\text{SNR} = 100$ is actually worse than that for the ground-based detectors. Space-based detectors rely on the long duration of the signals, the amplitude and Doppler modulations caused by the orbital motion, and finite-arm-length effects, to break degeneracies among the system parameters. We are limited here to working in a regime where all of those features are missing from our instrument model, so our classic LISA response is more like a two-detector co-located LIGO network operating at significantly lower frequencies than the existing ground-based detectors.

More complete analysis will require either dramatically longer-duration numerical simulations or a suitably well-controlled way of matching analytical and NR waveforms at low frequency that does not add as much systematic error as the template models being tested. We would also require more computationally efficient signal generation to successfully study template biases with long-duration waveforms.

VI. CONCLUSIONS AND DISCUSSION

In this paper, we have produced the first measurement of systematic errors introduced by using EOB templates to analyze NR waveforms. Our study's main focus was on stellar-mass BBHs observed by Advanced LIGO and Virgo. We also considered supermassive black-hole mergers detectable by space-based interferometers like LISA. We have injected NR waveforms into simulated data and have used an MCMC sampler to characterize the posterior distribution function for the astrophysical parameters. Our metric for assessing the size of the systematic error is to compare the offset between the injected and best-fit parameters to the statistical error characterized by the standard deviation of the 1D marginalized posteriors.

For the stellar-mass systems we have investigated, when including the subdominant modes, we find systematic biases consistently comparable to, or smaller than, the statistical errors for mass-ratios up to $q = 6$ and SNRs $\lesssim 50$. We have tested these waveforms in the most stringent way possible, simulating high-SNR events where the merger (the least reliable part of the waveform calculations) is peaking in the most sensitive band of the detector, and the higher-frequency modes contribute significantly to the overall signal power. For the $q = 6$ waveforms weighted by the detector PSD, the fraction of power contained in the subdominant modes is 11% and 16% for the $M = 56M_\odot$ and $M = 120M_\odot$ systems, respectively. We also tested low-mass systems ($M \sim 20 M_\odot$) to better represent likely Advanced LIGO/Virgo detections.

In all of these examples, the bias introduced by the EOB waveform in Ref. [25] was *at worst* comparable to the statistical errors. For several of our examples chosen specifically to compare with that paper, we find that the EOB waveforms accurately recover binary parameters at SNRs higher than were predicted there using the (deliberately) conservative accuracy requirements of Refs. [41, 42].

Matched-filtering analyses are most sensitive to the phase of the signal, and it is the phase of the waveform that is most influenced by different choices of the model. It is therefore predictable that the largest biases appear in the mass parameters. On the other hand, the extrinsic parameters have comparatively less impact on the shape of the signal — the distance comes in as an overall amplitude scaling, and the sky location is (for ground-based interferometers) predominantly determined through triangulation based on time delays between detectors. Therefore a model waveform used for parameter estimation has much more room for error if, for instance, the location of the binary is the primary interest (say, for optical counterpart searches) and the requirements on the phase-matching are not as severe.

While the results here are undoubtedly positive, there is still work to be done in waveform modeling. We are only testing the EOB waveforms over the last 30–40 GW

cycles before merger and there is no guarantee that longer waveforms will not accumulate larger phase errors during the early portion of the inspiral. Furthermore, this study neglects significant aspects of the waveform structure related to black-hole spins and orbital eccentricity. A similar study will need to be performed with long-duration, spinning systems once both the NR and EOB waveforms are prepared for that test. It will also be valuable to test the EOB waveforms over a broader class of NR simulations, including those that were not used to calibrate the template model. Moreover, we have assumed in this paper that the NR waveforms were exact but, as discussed in Sec. II, this is not the case. One possibility for taking the numerical error into account is to inject NR waveforms computed at different resolutions and/or extracted at different radii and measure the EOB systematic biases in each case. The difference between these biases can provide us with an estimate of the intrinsic error caused by the NR waveforms deviating from the exact solution in general relativity. Finally, we do not consider the effects of real detector noise or calibration errors in the data, both of which could prove to be a significant contribution to the overall error budget [48, 49].

While EOB waveforms that include subdominant modes were found to have relatively small systematic errors in parameter-estimation results for ground-based observations with $\text{SNR} < 50$, proposed space-based instruments are sensitive to SMBH mergers with $\text{SNR} > 100$. For these scenarios, we find statistically significant biases in the mass parameters for mass-ratios in the range $2 \leq q \leq 6$ observed at $\text{SNR} = 100$, on the order of $\sim 1\%$ for the component masses of the system. However, as discussed in Sec. V, systematic errors introduced by the EOB templates are small enough to still place strong constraints on the population of supermassive black holes in the Universe.

In the LISA examples, the residual power SNR is > 6 , sufficient to compromise both parameter estimation of overlapping signals and studies aimed at testing general relativity. Space-based GW data analysis will require more accurate templates, grounded in even more accurate numerical simulations.

A more complete analysis is needed, but will require either dramatically longer-duration numerical simulations or a suitably well-controlled way of matching analytical and NR waveforms at low frequency. We would also require more computationally efficient signal generation to successfully study template biases with long-duration waveforms.

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