Roberge-Weiss transition and 't Hooft loops

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Abstract

Roberge and Weiss showed that for $SU(N)$ gauge theories, phase transitions occur in the presence of an imaginary quark chemical potential. We show that at asymptotically high temperature, where the phase transition is of first order, that even with dynamical quarks 't Hooft loops of arbitrary $Z(N)$ charge are well defined at the phase boundary. To leading order in weak coupling, the 't Hooft loop satisfies Casimir scaling in the pure glue theory, but not with quarks. Because the chemical potential is imaginary, typically the interaction measure is negative on one side of the phase transition. Using a matrix model to model the deconfining phase transition, we compute the phase diagram for heavy quarks, in the plane of temperature and imaginary chemical potential. In general we find intersecting lines of first order transitions. Using a modified Polyakov loop which is Roberge-Weiss symmetric, we suggest that the interface tension is related to the 't Hooft loop only at high temperature, where the imaginary part of this Polyakov loop, and not the real part, is discontinuous across the phase boundary.

PACS numbers: 11.30.Rd, 12.40.-y, 21.65.Qr, 25.75.Nq

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I. INTRODUCTION

Understanding the nature of the phase diagram of Quantum Chromodynamics (QCD) is one of the outstanding problems in nuclear physics. At zero chemical potential, the theory can be studied by numerical simulations on the lattice. Generally this is not possible at nonzero chemical potential because of the sign problem [1].

One way to trying to understand the theory for nonzero chemical potential is to consider a chemical potential which is purely imaginary: then there is no sign problem, and numerical simulations are possible. Although an imaginary chemical potential is not directly physical, results can be related to those for real chemical potential using Fourier and Laplace transformations [2].

The pure gauge theory is invariant under a global symmetry of $Z(N)$, but this is violated when dynamical quarks are present. Roberge and Weiss showed, however, that in the presence of ϕ, an imaginary quark chemical potential, that the theory acquires a global $Z(N)$ symmetry, under which $\phi \to \phi + 1/N$ [2]. They also showed that at high temperature, there is a first order transition at constant temperature, as ϕ is varied across $\phi_{RW} = 1/(2N)$. We generalize this to $Z(N)$ transformations of charge k, and find a first order transition when $\phi_{RW} = k/(2N)$. As the temperature is lowered, this single line of transitions can split into two lines at $\phi \neq \phi_{RW}$; these can be a true phase transition, of first or second order, or just crossover. For a transition which occurs when $\phi \neq \phi_{RW}$, the Roberge-Weiss transition mixes either with the transitions for deconfinement or for the restoration of chiral symmetry. This has been studied by numerical simulations on the lattice [3–5], in effective theories [6], and in holographic models [7].

In this paper we show that Roberge-Weiss phase transitions have an unexpected connection with pure glue theories. Consider $SU(N)$ gauge theories without dynamical quarks. Running the Wilson loop in the temporal direction gives the Polyakov loop. This is an order parameter for $Z(N)$ electric charge, whose expectation value vanishes in the confined phase and is nonzero in the deconfined phase. Similarly, the two point function of Polyakov loops exhibits an area law in the confined phase.

The response of the theory to $Z(N)$ magnetic charge is given by the 't Hooft loop [8–11]. Its behavior is converse to that of the Wilson or Polyakov loops, in that the 't Hooft loop exhibits an area law only in the deconfined phase. The string tension for the area law of

the 't Hooft loop [9] is equal to the order-order interface tension for $Z(N)$ interfaces [12]. (Throughout this paper we assume that the 't Hooft loop is purely spatial; with or without quarks, temporal 't Hooft loops do not develop an area law at any temperature [10].)

Dynamical quarks carry $Z(N)$ electric charge, and so modify the behavior of both loops. The Polyakov loop is no longer a strict order parameter, but is nonzero at any temperature $T \neq 0$. Similarly, one expects that the 't Hooft loop acts as in the deconfined phase of the pure glue theory, and so exhibits an area law. However, it is not clear how to define the 't Hooft loop in a theory with dynamical quarks [9].

In this paper we show that 't Hooft loops are well defined for a Roberge-Weiss transition at $\phi_{RW} = k/(2N)$. Establishing this result is not difficult. In a gauge theory, a global $Z(N)$ transformation is an overall rotation of the quark field by a constant phase. This can be exactly compensated by a shift in an imaginary chemical potential for the quarks. What is less obvious is how the boundary conditions of the Roberge-Weiss transition are precisely equivalent to those for a $Z(N)$ interface, and thus to the 't Hooft loop [9].

We compute the behavior of the 't Hooft loop at asymptotically high temperature, where the calculation can easily be done using semi-classical techniques. In the pure glue theory, at leading order and up to corrections $\sim g^3$, the interface tension associated with 't Hooft loops of different charges satisfies Casimir scaling [13]. We find that even to leading order, dynamical quarks do not respect the Casimir scaling found in the pure glue theory.

We also consider the thermodynamics of Roberge-Weiss phase transitions. We show that the interaction measure is negative on at least one side of the Roberge-Weiss transition. This happens because both the chemical potential and the associated quark number densities are imaginary. Thus their contribution to the energy density can be negative, and so unphysical.

It is also of interest to know how the 't Hooft loop behaves for temperatures which are not asymptotic. To study this, we consider heavy quarks, so that there are Roberge-Weiss and deconfining phase transitions in the plane of temperature and imaginary chemical potential. We use a matrix model [14, 15], which was used previously to locate the position of the deconfining critical endpoint for heavy quarks [16]. In this matrix model we find intersecting lines of first order transitions in the $T - \phi$ plane. There are lines of first order transitions both at ϕ_{RW} and $\phi \neq \phi_{RW}$. Depending upon the quark mass, these first order lines end in either critical or tri-critical points. Our results are in good agreement with recent results on the lattice [5].

The phase transitions in T and ϕ are naturally characterized by a modified Polyakov loop, which is constructed to be invariant under the Roberge-Weiss symmetry. For the transition at ϕ_{RW}, as occurs at high temperature, only the imaginary part of the Polyakov loop is discontinuous. At lower temperatures, there are first order transitions for $\phi \neq \phi_{RW}$, where both the real and imaginary parts of the modified Polyakov loop jump. Since the real part of the Polyakov loop characterizes the usual phase transition for deconfinement, the transitions when $0 < \phi \neq \phi_{RW} < 1/N$ are those where deconfinement mixes with the Roberge-Weiss transition. We find that the interface tension of a first order transition is related to a 't Hooft loop if and only if it is entirely Roberge-Weiss, occurring at ϕ_{RW}.

In this paper we do not analyze a matrix model with light quarks, where the restoration of chiral symmetry also enters [3, 4]. At high temperature there is only a Roberge-Weiss transition at ϕ_{RW}, while at lower temperatures, the order parameter for chiral symmetry breaking can mix with the imaginary part of the Polyakov loop. Nevertheless, we suggest that an analogous criterion occurs: the 't Hooft loop is related to the interface tension of a first order transition only at high temperature, for a Roberge-Weiss transition at $\phi = \phi_{RW}$.

Hence in general, 't Hooft loops can only be defined at temperatures which are above that for deconfinement or the restoration of chiral symmetry. Nevertheless, we find it noteworthy that even a limited region of an extended phase diagram, it is possible to measure 't Hooft loops in a gauge theory with dynamical quarks.

In Sec. (II) we discuss the global $Z(N)$ symmetries of the pure gauge theory, the symmetries with dynamical quarks, and the relationship to 't Hooft loops. In Sec. (III) we consider the thermodynamics of Roberge-Weiss phase transitions, and the phase diagram with heavy quarks.

II. $Z(N)$ SYMMETRY AND 'T HOOFT LOOPS

A. General analysis

We work in imaginary time τ at a temperature T, so $\tau : 0 \to 1/T$. Under a gauge transformation $U(\vec{x}, \tau)$, the gluons and quarks transform as

$$A_\mu(\vec{x}, \tau) \to \frac{1}{ig} U(\vec{x}, \tau) \, D_\mu \, U^\dagger(\vec{x}, \tau) \;\;, \;\; q(\vec{x}, \tau) \to U(\vec{x}, \tau) \, q(\vec{x}, \tau) \;, \tag{1}$$

$D_\mu = \partial_\mu + ig A_\mu$.

A special class of gauge transformations are those which are symmetric up to elements of $Z(N)$. Consider the matrix $e^{2\pi i k/N}\mathbf{1}_N$; this has determinant one, and so is an element of $SU(N)$. Since it is proportional to the unity matrix, we can consider aperiodic gauge transitions

$$U(\vec{x}, 1/T) = e^{2\pi i k/N}U(\vec{x}, 0) , \tag{2}$$

where k is an integer. Gluons must be periodic in τ, but this aperiodicity in the U's does alter the gluon boundary conditions, cancelling between U and $U^\dagger$.

Quarks are fermions, and so in the absence of a chemical potential, are anti-periodic in τ. The boundary conditions are altered by the presence of a chemical potential, μ. Consider the analytic continuation from a real chemical potential, μ, to one which is imaginary and proportional to the temperature,

$$\mu = 2\pi i \, \phi \, T . \tag{3}$$

The quarks now satisfy

$$q(\vec{x}, 1/T) = - \, e^{-2\pi i \, \phi} \, q(\vec{x}, 0) . \tag{4}$$

Because of the change in the fermion boundary conditions, when $\phi \neq 0$ the theory can exhibit unphysical behavior. For example, $\phi = \frac{1}{2}$ turns fermions into bosons. We shall see an example of this unphysical behavior later, when we find that the interaction measure is negative on one side of the Roberge-Weiss transition line.

As the $Z(N)$ transformation of Eq. (2) is proportional to the unit matrix, for the quarks it just alters them by an overall phase. Thus in all, quarks transform as

$$q(\vec{x}, 1/T) = -e^{2\pi i \, (-\phi+k/N)} \, q(\vec{x}, 0) . \tag{5}$$

It is useful to think of the boundary condition on the quarks at $\tau = 1/T$ as a type of $Z(N)$ "charge". Because the $Z(N)$ transformations of the gauge field arise from the details of the $SU(N)$ algebra, this $Z(N)$ quark charge is heuristic, meant mainly to understand the detailed computations in the next subsection. By Eq. (5) we normalize the $Z(N)$ charge of the quark as $-\phi + k/N$.

When there is an imaginary chemical potential, then, a shift in ϕ can be compensated by a $Z(N)$ transformation,

$$\phi \to \phi + \frac{k}{N} . \tag{6}$$

The action is invariant under this transformation. By construction, so are the boundary conditions. This is the Roberge-Weiss symmetry [2].

This periodicity can also be understood topologically. In the pure glue theory, the gauge group is $SU(N)/Z(N)$, and so there is a global symmetry of $Z(N)$. When $\phi = 0$, the gauge group is just $SU(N)$, and this global $Z(N)$ symmetry is lost. However, by introducing an imaginary chemical potential, ϕ, for free quarks we gain an extra global symmetry of $U(1)$. The coupling of gauge fields reduces this $U(1)$ symmetry to one of $Z(N)$.

We thus have global $Z(N)$ symmetries both in the pure glue theory and, when $\phi \neq 0$, for dynamical quarks. Thus we might expect that there are non-contractible loops which measure windings in these global $Z(N)$ symmetries. In the pure glue theory, such windings are measured by the 't Hooft loop. Since the transformation of ϕ is so intimately tied to the $Z(N)$ symmetry of the pure glue theory, it is perhaps not so surprising that the 't Hooft loop continues to measure such windings, even in the presence of dynamical quarks.

We can understand where the Roberge-Weiss transition occurs even without detailed computation. By the Roberge-Weiss symmetry, $\phi = 0$ is equivalent to $\phi = k/N$. The simplest way is to introduce a background $Z(N)$ charge k/N, so that at $\phi = k/N$, the total $Z(N)$ charge is zero. The question is then how to match these two cases, moving up from $\phi = 0$, and down from $\phi = k/N$. Since at some point the background field has to jump from zero to k/N, this might generate a transition at some ϕ.

The most natural place for the transition to occur is exactly halfway in between, at

$$\phi_{RW} = \frac{k}{2\,N} \; . \tag{7}$$

As ϕ increases from zero, the $Z(N)$ quark charge decreases, going from zero at $\phi = 0$ to $-\phi_{RW}$ just to the left of ϕ_{RW}.

Now work in the opposite direction, letting ϕ decrease from k/N. Here we have to include the background charge k/N, so that as $\phi = k/N$, the total $Z(N)$ charge vanishes, consistent with the Roberge-Weiss symmetry. As ϕ decreases, the $Z(N)$ charge increases, and equals $-\phi_{RW} + k/N = +\phi_{RW}$ just to the right of ϕ_{RW}.

Thus exactly at the Roberge-Weiss point, the quark $Z(N)$ charge is $-\phi_{RW}$ on the left, and $+\phi_{RW}$ on the right. Because of charge conjugation symmetry, various quantities transform simply as we flip the sign of the quark $Z(N)$ charge. The pressure is even in ϕ, so quarks with equal and opposite $Z(N)$ charge are degenerate, and have no forces acting between

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them. In contrast, the (imaginary) quark number density is odd in ϕ, and flips moving across ϕ_{RW}. The change in the sign of the quark number density implies that if we change ϕ at fixed temperature, that there is a phase transition at ϕ_{RW} which is of first order.

The existence of a first order transition at ϕ_{RW} is valid only at high temperature. As we show in the next section using a matrix model for heavy quarks, and in agreement with lattice results [5], at temperatures near that for deconfinement there can be transitions for $\phi \neq \phi_{RW}$, which are of either first or second order. A similar statement can be made for light quarks, near the transition for the restoration of chiral symmetry [4].

B. Semi-classical analysis

Although the above discussion of quark $Z(N)$ charge is illustrative, it is merely heuristic. This is because while the quark chemical potential only involves a $U(1)$ phase, the $Z(N)$ transformations of the gauge field involve the Lie algebra of $SU(N)$, and in particular the Cartan subalgebra of mutually commuting generators, in a detailed manner. In this subsection we perform an analysis at high temperature, where all computations can be done semiclassically.

To generate a $Z(N)$ transformation, consider the diagonal matrix [13]

$$\mathbf{Y}_k = \frac{1}{N} \, \text{diag} \, (k \ldots k, -N + k, \ldots - N + k) \, . \tag{8}$$

There are $N - k$ elements with entry k/N, and k elements equal to $-1 + k/N$. Thus Y_k has zero trace and is an element of the Lie algebra of $SU(N)$. Since

$$\mathrm{e}^{2\pi i \mathbf{Y}_k} = \mathrm{e}^{2\pi i k/N} \, \mathbf{1}_N \, , \tag{9}$$

$\mathbf{Y}_k$ generates a $Z(N)$ transformation with strength k. $\mathbf{Y}_3$ is the hypercharge matrix for three flavors.

The thermal Wilson line is given by

$$\mathbf{L}(\vec{x}) = \exp \left(ig \int_0^{1/T} A_0(\vec{x}, \tau) d\tau \right) \, . \tag{10}$$

Then the path

$$A_0 = \frac{2\pi T}{g} \, q \, \mathbf{Y}_k \, , \tag{11}$$

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takes one from the ordinary perturbative vacuum for $q = 0$, $\mathbf{L} = \mathbf{1}_N$, to the kth $Z(N)$ transform thereof, for $q = 1$, $\mathbf{L} = \mathrm{e}^{2\pi i k/N}\mathbf{1}_N$.

At the outset it is also useful to consider the matrix $\mathbf{Y}_{N-k}$, where

$$\mathbf{Y}_{N-k} = \frac{1}{N}\mathrm{diag}\left(N - k \ldots N - k, -k \ldots - k\right) ; \tag{12}$$

this has k elements with entry $1 - k/N$, and $N - k$ elements with value $-k/N$. It is clear that

$$\mathrm{e}^{2\pi i \mathbf{Y}_{N-k}} = \mathrm{e}^{2\pi i(-k)/N}\,\mathbf{1}_N . \tag{13}$$

That is, $\mathbf{Y}_{N-k}$ generates a $Z(N)$ transformation with charge $N - k$, which by the additive $Z(N)$ symmetry is equivalent to charge $-k$. This symmetry is useful, because we expect that the interface tension for charge k should be equal to that for charge $N-k$. The matrix $\mathbf{Y}_{N-k}$ makes this manifest, although our final expressions do not obviously reflect this symmetry. Because of this symmetry, we can restrict k to be less than the nearest integer $\leq N/2$.

Classically there is no potential for the q's, but one is generated at one loop order. This was first computed by Weiss [17] and Yaffe et $al.$ [18],

$$\mathcal{V}_{\mathrm{pt}}^{gl} = \frac{2\pi^2 T^4}{3}\left(-\frac{N^2 - 1}{30} + \sum_{i,j=1}^{N} q_{ij}^2(1 - |q_{ij}|)^2\right) . \tag{14}$$

The q_i are the elements of $q\,\mathbf{Y}_k$. The gluon contribution involves the adjoint covariant derivative, and so only the differences of the q_i enter, $q_{ij} = q_i - q_j$.

The q_{ij}'s arise as the gluon energies divided by $2\pi T$. Since the potential for the q_i's arises for a Matsubara sum, then it is periodic in the q_{ij}, $q_{ij} \to q_{ij}+1$. Consequently, the potential only involves the absolute value of q_{ij}, modulo one. Thus for the gluon term we can always require that $q : 0 \to 1$.

The term independent of q is the just (minus) the pressure of an ideal gas of gluons. For the q_{ij}, there are k elements $= q(k/N)$, and $N - k$ elements $= q(-1 + k/N)$. The potential arises from the (absolute value) of the $q_i - q_j$'s. For these elements, the $|q_{ij}|$'s are either zero, or q (assuming $q > 0$). There are $2k(N - k)$ such terms which give q, and so the gluon potential is

$$\mathcal{V}_{\mathrm{pt}}^{gl}(q) - \mathcal{V}_{\mathrm{pt}}^{gl}(0) = \frac{4\pi^2 T^4}{3}\,k(N - k)\,q^2(1 - q)^2 . \tag{15}$$

The potential is identical at $q = 0$ and $q = 1$, which illustrates the $Z(N)$ symmetry of the pure glue theory between the ordinary vacuum, $k = 0$, and the kth $Z(N)$ transform.

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In all, the potential is proportional to $k(N-k)$. This is known as Casimir scaling. It is satisfied in the pure glue theory up to corrections $\sim g^3$ times the leading order term [13].

For a single, massless flavor, quarks contribute to the potential for q as

$$\mathcal{V}_{\text{pt}}^{qk} = -\frac{4\pi^2 T^4}{3}\left(-\frac{N}{30} + \sum_{i=1}^{N} q_i^2(1-|q_i|)^2\right) . \tag{16}$$

For the q_i, there are $N-k$ elements

$$q_i = \frac{1}{2} - \phi + \frac{k}{N} q \, ,$$

and k elements

$$q_i = \frac{1}{2} - \phi + \left(-1 + \frac{k}{N}\right) q \, . \, , \tag{17}$$

These q's differ from the gluon case because of the change in the boundary conditions in τ, for fermions and bosons, and because the quarks carry an imaginary chemical potential $\sim \phi$. The quark contribution involves the covariant derivative in the fundamental representation, so it is the q_i, and not $q_i - q_j$, which enter. Like the gluon potential, as the quark potential arises from a sum over the Matsubara frequencies, each q_i enters only as the absolute value, modulo one. Thus it is necessary to be careful about the range of the q_i. Even so, clearly Eq. (16) is invariant under two symmetries. As only the absolute value of the q_i enter, one is $q_i \to -q_i$. The second is apparent from the form of the potential: assuming that $1 \geq q_i \geq 0$, the quark potential is also invariant under $q_i \to 1 - q_i$.

The q_i are the Euclidean energies divided by $2\pi T$. Thus the first factor of $\frac{1}{2}$ is the πT which arises because quarks are fermions, with boundary conditions which are anti-periodic in imaginary time. The second term, $-\phi$, is the contribution of the imaginary chemical potential. Lastly, the terms $\sim q$ arise from the background field.

While we explicitly compute the potential shortly, most aspects of the physics can be understood without going into such details.

First consider vanishing chemical potential, $\phi = 0$. As discussed above, for the gluons the potential is periodic in q, with $q = 1$ degenerate with $q = 0$, Eq. (15). This reflects the $Z(N)$ symmetry of the pure glue theory.

This is no longer true with dynamical quarks. Then q_i's involve factors of kq/N; for $q = 1$, this is k/N, and generates a nontrivial potential. Computation shows the quark potential has a higher value when $q = 1$ than for $q = 0$. This occurs because quarks in

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the fundamental representation do not respect the $Z(N)$ global symmetry of the pure glue theory.

Looking just at the q_i's shows how the Roberge-Weiss symmetry of Eq. (6) works. Compare the ordinary perturbative vacuum, where $\phi = q = 0$, to a state where $\phi = k/N$, and $q = 1$. When $\phi = q = 0$, all q_i's equal $\frac{1}{2}$. When $\phi = k/N$, and $q = 1$, all of the q_i's are either $\frac{1}{2}$ or $-\frac{1}{2}$. Since only the absolute value enters, both are equivalent to $\frac{1}{2}$.

Thus we see that a state with $\phi = k/N$ is equivalent to the perturbative vacuum, if we shift the background field by k/N. As we increase ϕ from 0, there is no background field, but at some point, it shifts to that with k/N. As discussed previously, this happens halfway in between, at $\phi_{RW} = k/(2N)$.

At ϕ_{RW}, when $q = 0$ all N elements equal

$$q_i = \frac{1}{2}\left(1 - \frac{k}{N}\right) . \tag{18}$$

When $q = 1$, there are $N - k$ elements equal to

$$q_i = \frac{1}{2}\left(1 + \frac{k}{N}\right) , \tag{19}$$

and k elements equal to

$$q_i = -\frac{1}{2}\left(1 - \frac{k}{N}\right) . \tag{20}$$

Even without computation one can see that the values in Eqs. (19) and (20) are equal to those in in Eq. (18). The q_i's in Eq. (19) are related to those in in Eq. (18) by $q_i \to 1 - q_i$; the q_i's in Eq. (20) are equivalent to those in Eq. (18) under $q_i \to -q_i$.

Explicitly, at $\phi_{RW} = k/(2N)$ the quark potential is given by

$$\mathcal{V}_{\mathrm{pt}}^{qk}(q) = -\frac{4\pi^2}{3}\,T^4\left((N - k)\left(\frac{1}{4} - \left(\frac{k}{N}\right)^2\left(\frac{1}{2} - q\right)^2\right)\right.$$

$$\left. + k\left(1 - \frac{k}{N}\right)^2\left(\frac{1}{2} - q\right)^2\left(1 - \left(1 - \frac{k}{N}\right)\left|\frac{1}{2} - q\right|\right)^2\right) . \tag{21}$$

The values of the potential are clearly equal when $q = 0$ and $q = 1$; note, however, that the second term does involve the absolute value of $\frac{1}{2} - q$. Also, this potential is not simply proportional to $k(N - k)$, and so does not respect Casimir scaling. As argued above, the potential will respect a transformation under $k \to N - k$, since then $\mathbf{Y}_{-k}$ enters, instead of $\mathbf{Y}_k$.

We can use these results to discuss 't Hooft loops. In the theory without dynamical quarks, Kovner, Korthals-Altes, and Stephanov [9] showed that the 't Hooft loop has a simple physical interpretation. Consider a box which is long in one spatial direction, say that in the z-direction. Put a 't Hooft loop of kth $Z(N)$ charge around the boundary of the box at one end of the box, at $z = L$. Then consider boundary conditions which are $q = 0$ at $z = 0$, and $q = 1$ at $z = L$. These boundary conditions are identical to that for a order-order $Z(N)$ interface of charge k [12, 13]. The 't Hooft loop at $z = L$ then forces q to jump from 1 back to 0, so that in all one has periodic boundary conditions. Neglecting this singularity shows that the interface tension for the order-order interface is equal to that for the 't Hooft loop.

Our analysis above shows that one has identically the same boundary conditions across the Roberge-Weiss transition point. The vacuum jumps from $q = 0$ on the left hand side, to $q = 1$ on the right hand side. Consequently, at ϕ_{RW}, we can define the 't Hooft loop in precisely the same manner as in the pure gauge theory.

We stress that the 't Hooft loop can *only* be defined at the Roberge-Weiss transition point, when ϕ_{RW}. For example, in the ordinary vacuum, $\phi = 0$, the states with $q = 0$ and $q = 1$ are not degenerate. Typically, the state with $q = 1$ is not even extremal [9]. (Depending upon the matter content, it is possible that the state with $q = 1$ is metastable [19]; see, also, [20].) In this case, one can define a 't Hooft loop, but the interface tension will have an imaginary part, reflecting this metastability. A necessary condition for the $Z(N)$ loop to be non-contractible is if the $Z(N)$ transformed states are absolutely degenerate. This only happens across a Roberge-Weiss transition.

In the pure gauge theory, the value of the 't Hooft loop depends only upon the area of the loop, but not upon its shape. Away from the Roberge-Weiss transition, the $Z(N)$ charges of the quarks are unequal, and have different pressures. This difference in pressure generates a force, which drives the 't Hooft loop to be flat.

In contrast, at the Roberge-Weiss transition the $Z(N)$ charges of the quarks are equal and opposite, and have the same pressure. Thus there is no net force which acts upon the 't Hooft loop, and its value depends only upon the area of the loop, and not its shape.

In the pure gauge theory, the 't Hooft loop measures a non-contractible loop for the global $Z(N)$ symmetry. As we argued in Sec. (II A), there is also a global $Z(N)$ symmetry if the quarks have an imaginary chemical potential. Since this global $Z(N)$ symmetry of

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dynamical quarks is intimately tied to the $Z(N)$ transformations of the pure glue theory, it is natural that the 't Hooft loop continues to measure the winding in $Z(N)$.

Our results are valid at asymptotically high temperature, where the Roberge-Weiss phase transition is manifestly of first order. As one lowers the temperature, numerical simulations on the lattice show that the single transition at $\phi = \phi_{RW}$ can split into two, for $\phi \neq \phi_{RW}$, which are of either first or second order [3–5].

In the next section we also find that for heavy quarks in a matrix model, a single line of Roberge-Weiss transitions at ϕ_{RW} splits into two transitions for $\phi \neq \phi_{RW}$, see Fig. (2). This agrees with the lattice results of Ref. [5]. We argue there that the interface tension across a first order transition is related to the 't Hooft loop if and only if the transition is entirely Roberge-Weiss. For a transition at ϕ_{RW}, the two degenerate vacua are $Z(N)$ transformations of one another, and so the interface tension is naturally related to the 't Hooft loop. In contrast, for transitions for $0 < \phi \neq \phi_{RW} < k/N$, the Roberge-Weiss and deconfining transitions mix with one another. Thus across the dotted lines in Fig. (2), when $\phi \neq \phi_{RW}$ while the vacua are degenerate across the transition, they are not $Z(N)$ transformations of one another. Thus the corresponding interface tension is not related to a 't Hooft loop.

For light quarks, numerical simulations on the lattice suggest that the single line of Roberge-Weiss transitions for ϕ_{RW} at high temperature can also split into two transitions at low temperature, with $\phi \neq \phi_{RW}$ [4]. When this happens, the Roberge-Weiss transition mixes with that for chiral symmetry restoration. We suggest that as for heavy quarks, the interface tension across such a first order transition is related to a 't Hooft loop only when $\phi = \phi_{RW}$.

C. Computing the 't Hooft loop

The explicit computation of the 't Hooft loop follows standard methods. A $Z(N)$ interface has an electric field, which contributes to the action as

$$\frac{4\pi^2 T^2}{g^2 N}\, k(N-k) \int dz\, \left(\frac{dq}{dz}\right)^2 . \tag{22}$$

The interface tension is determined as a semiclassical tunneling between $q = 0$ and $q = 1$. In the pure glue theory, this involves the sum of the kinetic term in Eq. (22) and the potential

of Eq. (15). Since each term is proportional to $k(N - k)$, the interface tension is as well, and so it respects Casimir scaling, at least to leading order in the coupling constant.

With dynamical quarks, the kinetic term remains as in Eq. (22), but now the potential is a sum of the gluon term in Eq. (15) and the quark term in Eq. (21). Since the quark potential is not proportional to $k(N-k)$, the interface tension is a more complicated function of k.

The appearance of Casimir scaling is to some extent an observation about the structure of the theory in weak coupling. Computation to corrections $\sim g^3$ beyond that of leading order shows a small violation of Casimir scaling even in the pure glue theory [13]. So there is nothing fundamental in Casimir scaling, nor in that it is violated by dynamical quarks. The only symmetry principle which must be respected is that for $Z(N)$ periodicity. This requires that the interface tension for k is the same for $N-k$; this is equivalent to $-k$, which as we have argued, is valid.

Let us assume that the path for the interface tension is along the direction $\mathbf{Y}_k$. This is a straight line in the Cartan subalgebra. If true, it is easy computing the 't Hooft loop. One has a tunneling problem in this one dimension, and it is easy to solve this by using "energy" conservation for the associated problem in quantum mechanics [12]. This involves both the potential and the kinetic term for the gluons, Eq. (22). For the pure glue theory, as both terms are $\sim k(N - k)$, and the interface tension follows immediately. With dynamical quarks, since the quark contribution to the potential is more involved, and even with the conservation of energy one is left with a single integral over q which needs to be computed numerically. We defer this exercise, and simply observe that with dynamical quarks, Casimir scaling will not be satisfied.

This assumes that the straight line path is minimal. We consider this in the next subsection.

1. *Straight line path*

While in the pure glue theory the path for the interface tension is along the direction $\mathbf{Y}_k$, in principle it can move in other directions. The general path for an $SU(N)$ gauge theory lies in the subspace of all commuting generators, which is the Cartan subalgebra, with $N-1$ dimensions.

For the pure gauge theory one can show that the straight line path is minimal [13]. This has deep geometric reasons: the $\mathbf{Y}_k$ form the boundary of the Weyl chamber, which is the smallest possible region to describe the Cartan subalgebra [14]. The endpoints of $\mathbf{Y}_k$ are the relevant endpoints for the interface tension, so then it is natural that the boundary, along $\mathbf{Y}_k$, is the minimal path which connects these two points.

With dynamical quarks, the structure of the Weyl chamber at the Roberge-Weiss transition is more involved. We have not been able to answer this question for an arbitrary numbers of colors, and so satisfy ourselves with working out the simplest possible cases, working up from $N = 2$ to $N = 4$.

The case of two colors is trivial. There is only one direction, along $\mathbf{Y}_1 \sim \sigma_3 \sim \mathrm{diag}(1, -1)$, and so the path necessarily lies along $\mathbf{Y}_1$.

The first nontrivial case arises for three colors, where there are two directions in the Cartan sub-algebra. For the quarks, the q_i's are

$$q_i = \left(\frac{1}{2} - \phi\right) \mathbf{1}_3 + \frac{q_3}{2} \, \mathrm{diag}(1, -1, 0) + \frac{q_8}{3} \, \mathrm{diag}(1, 1, -2) \,. \tag{23}$$

In the standard Gell-Mann notation, the directions are $\lambda_3 \sim (1, -1, 0)$ and $\lambda_8 \sim \mathbf{Y}_1 \sim (1, 1, -2)$, with associated coordinates q_3 and q_8. For three colors, there is only one interface tension, as $k = -2$ is equivalent to $k = 1$. The endpoints of the interface are given by $q_8 = 0$ and 1, with $q_3 = 0$. The straight line path is along $\mathbf{Y}_1$, with one transverse direction, along λ_3.

The q_i's of Eq. (23) are for quarks, but we can use them for gluons, since only the differences of the q_i's enter in the gluon potential, through $q_{ij} = q_i - q_j$. Numerically we find that for the gluon potential of Eq. (14), the straight line path along $\mathbf{Y}_1$ is minimal. That is, for the path where $q_8 \neq 0$ and $q_3 = 0$, for every value of q_8 the potential is minimal with respect to variations in the transverse direction, along q_3. As noted, this is because $\mathbf{Y}_1$ is the boundary of the Weyl chamber for three colors.

Now consider the quark potential at the Roberge-Weiss transition point, $\phi = 1/6$, using the q_i's of Eq. (23) in Eq. (16). As in the pure glue theory for three colors, numerically we have checked that the minimal path is a straight line along $\mathbf{Y}_1$.

For four colors there are two possible interfaces, $k = 1$ and $k = 2$; the associated elements of $Z(4)$ are i and -1, respectively.

For $k = 1$ we can parametrize the q_i's using the usual Cartan generators,

$$q_i = \left(\frac{1}{2} - \phi\right) \mathbf{1}_4 + \frac{q_3}{2} \operatorname{diag}(1, -1, 0, 0) + \frac{q_8}{3} \operatorname{diag}(1, 1, -2, 0) + \frac{q_{15}}{4} \operatorname{diag}(1, 1, 1, -3) \,. \quad (24)$$

The $k = 1$ interface is from $q_{15} : 0$ to 1, with $q_3 = q_8 = 0$ at either end. The straight line path is along $\mathbf{Y}_1 \sim (1, 1, 1, -3)$, with only q_{15} nonzero. Numerically we checked that the straight line path is minimal, both in the pure glue theory and with dynamical quarks at the Roberge-Weiss point for $k = 1$, where $\phi_{RW} = 1/8$.

This exercise also shows that the potential has nontrivial structure. At the Roberge-Weiss transition for $k = 1$, $\phi_{RW} = 1/8$, there is a metastable minimum in the potential when $q_{15} = 0$: it occurs for $q_8 = 1$, with $q_3 = 0$. It is metastable in all three directions, but tunnels with finite lifetime to the usual $Z(4)$ vacua, which are absolutely stable. Such metastable vacua are known to arise for these types of potentials [19].

Lastly we consider four colors with $k = 2$. For the q_i's we take

$$q_i = \left(\frac{1}{2} - \phi\right) \mathbf{1}_4 + \frac{q_2}{2} \operatorname{diag}(1, -1, 0, 0) + \frac{q_2'}{2} \operatorname{diag}(0, 0, 1, -1) + \frac{q_4}{4} \operatorname{diag}(1, 1, -1, -1) \,. \quad (25)$$

For the interface with $k = 2$, $q_4 = 0$ at one end and $q_4 = 1$ at the other, with $q_2 = q_2' = 0$ at both ends. The straight line path is along $\mathbf{Y}_2 \sim (1, 1, -1, -1)$.

To determine stability of a path it is essential to have a parameterization in three independent directions. Our path is along $\mathbf{Y}_2 \sim (1, 1, -1, -1)$. The diagonal matrix $\sim (1, -1, 0, 0)$ is a generator for (the diagonal part of) $SU(2)$ in the first two colors, and is obviously transverse to $\mathbf{Y}_2$. To determine the remaining direction, one can use brute force: one computes the linear combination of $(1, 1, -2, 0)$ and $(1, 1, 1, -3)$ which is transverse to $(1, -1, 0, 0)$ and $(1, 1, -1, -1)$. The answer, as in Eq. (25), is $\sim (0, 0, 1, -1)$. This is just the $SU(2)$ type generator for the third and fourth and fourth colors of $SU(4)$. At least after the fact, this is obvious.

By explicit computation, again one finds that the minimal path is a straight line, with $q_4 \neq 0$ and $q_2 = q_2' = 0$. This is true both for the pure glue theory, and for the theory with (massless) dynamical quarks at the Roberge-Weiss transition point, $\phi_{RW} = 1/4$. We did not find metastable minima when $k = 2$.

The examples of $k = 1$ for three colors, and $k = 1$ and $k = 2$ for four colors, suggests that for $SU(N)$ at the Roberge-Weiss transition point(s), $\phi_{RW} = k/(2N)$, that the path for the associated interface tension is *always* a straight line along $\mathbf{Y}_k$. At present, we can only

suggest this as a conjecture and have no general proof. If true, surely it is due to the nature of the Weyl chamber with dynamical quarks at the Roberge-Weiss transition point(s).

III. THERMODYNAMICS OF ROBERGE-WEISS TRANSITIONS

A. High temperature

The total pressure p and the entropy density s are a sum,

$$p = p_g + p_q, \qquad s = s_g + s_q, \tag{26}$$

where p_g and s_g are the gluon contributions, and p_q and s_q the quark contributions.

On the left side of the Roberge-Weiss transition, $\phi_{RW} = k/(2N)$, the gluon contribution is

$$p_g = (N_c^2 - 1)\frac{\pi^2 T^4}{45} \;;\; s_g = 4\,\frac{p_g}{T}, \tag{27}$$

while the quarks contribute

$$
\begin{aligned}
p_q &= \frac{\pi^2 N N_f T^4}{3}\left[\frac{7}{60} - 2\left(\frac{k}{N}q - \phi\right)^2 + 4\left(\frac{k}{N}q - \phi\right)^4\right] \;; \\
s_q &= 4\,\frac{p_q}{T} \;; \\
\mathrm{Im}(n_q) &= \frac{2\pi N N_f T^3}{3}\left[\left(\frac{k}{N}q - \phi\right) - 4\left(\frac{k}{N}q - \phi\right)^3\right],
\end{aligned}
\tag{28}
$$

Here $kq/N - \phi$ is defined between $-1/2$ and $1/2$. The quark number density is imaginary because the chemical potential is.

To define the energy density, we take the standard expression for a real chemical potential, and assume it remains valid for an imaginary chemical potential:

$$e = -p + sT + \mu n_q = -p + sT - 2\pi T\,\phi\,\mathrm{Im}(n_q). \tag{29}$$

Now consider how thermodynamic functions change on either side of the Roberge-Weiss transition, $\phi_{RW} = k/(2N)$. On the left side of the transition, $q = 0$, while on the right, $q = 1$, so $kq/N - \phi = \mp k/(2N)$ changes sign across the transition. The pressure and the entropy density are even in $kq/N - \phi$ and do not change. The quark number density is odd in $kq/N - \phi$ and so changes sign. This change in sign for the (imaginary) quark number

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density is the only reason why the energy density changes at the Roberge-Weiss point and makes the transition of first order.

If we assume that ϕ and T are both fixed, at asymptotically high temperature the contribute of the quark number density to the internal energy density is:

$$\frac{e}{T^4} \to 3\frac{p}{T^4} - 2\pi\phi\frac{\text{Im}(n_q)}{T^3} \tag{30}$$

Thus at high T the interaction measure is due entirely to the contribution from the quark number density,

$$\Delta \equiv \frac{e - 3p}{T^4} = -2\pi\phi\frac{\text{Im}(n_q)}{T^3}. \tag{31}$$

The interaction measure is nonzero even at high T because we assume that the (imaginary) chemical potential is proportional to temperature, $\mu = 2\pi i T\phi$. As discussed, it also flips sign across the transition. This holds not only at high T, but persists down to temperatures close to the transition temperature, as we see in the model calculations which follow.

B. Non-perturbative models

The perturbative potential for the q's is given by Eq. (14). It involves the function $V_2(x) = x^2(1 - |x|)^2$, and $q_{ij} = (q_i - q_j)_{\text{mod } 1}$.

The minimum of this potential is always the usual perturbative vacuum, or a $Z(N)$ transform thereof. To model the transition to deconfinement, we can add, by hand, non-perturbative terms [14–16]. The involves one new function, $V_1(x) = |x|(1 - |x|)$, where like $V_2(x)$, this function is defined to be periodic in x, modulo one.

For all values of N, a successful fit was obtained with the form

$$\mathcal{V}_g^{np} = -\frac{4\pi^2}{3} T^2 T_d^2 \sum_{i,j} \left(-\frac{c_1}{5} V_1(q_{ij}) - c_2 V_2(q_{ij}) + \frac{N^2 - 1}{60} c_3 \right) \tag{32}$$

The parameters c_1 and c_2 are assumed to be independent of temperature, while the temperature dependence of c_3 is simply

$$c_3(T) = c_3(\infty) + (c_3(T_d) - c_3(\infty)) \left(\frac{T_d}{T}\right)^2 . \tag{33}$$

Driven by the lattice data, most terms in the non-perturbative potential are $\sim T^2$. The temperature dependence of c_3 also incorporates a MIT bag constant $\sim c_3(T_d) - c_3(\infty)$.

The parameters of the model are fit by the transition in the pure glue theory, where we assume $T_d = 270$ MeV. At the outset, the model involves four parameters, c_1, c_2, $c_3(T_d)$ and $c_3(\infty)$. One then requires two conditions. First, that the transition occurs at T_d. Second, that the pressure vanishes at T_d. The first is a reasonable condition at any N. The second is an approximation, modeling that the pressure is suppressed by $1/N^2$ in the confined phase, relative to that in the deconfined phase. This leaves two parameters, which are fit by the value of the latent heat, and the fall off of the pressure with temperature at asymptotically high T.

To include quarks, we follow Ref. [16] and simply add the one loop term for quarks in the background field. This is then the only way in which the (imaginary) chemical potential enters. For constant quark mass, this contribution is

$$
\mathcal{V}_q = -2T \mathrm{tr}_{f,c} \int \frac{dp^3}{(2\pi^3)} \left[\ln\left(1 + e^{-\beta\{E_f - i2\pi T(\phi + q_c)\}}\right) + \ln\left(1 + e^{-\beta\{E_f + i2\pi T(\phi + q_c)\}}\right) \right], \quad (34)
$$

where $\mathrm{tr}_{f,c}$ represents the trace over flavor and color and $E_f(p) = \sqrt{p^2 + m_f^2}$. We assume that the up and down quarks are isospin symmetric. We thus adopt the notation that the light (up and the down) quark masses are m_l, and the strange quark mass is m_s.

For light quark masses there is a back from to the gluon potential for chiral symmetry breaking or restoration, but we can neglect this here. Such back reaction may be discussed by using the functional renormalization group or the gluon and ghost potentials from the Landau gauge gluon and ghost propagators [21]. When we consider the small quark mass region, we should also consider the meson and also the baryon contributions, but inclusion of these effects is much more involved, and will be treated later.

We shall work in the limit of heavy quark masses, where the light and strange quark masses are constant, and such effects can be neglected. Therefore, as a first step to construct the reliable model of QCD, we investigate the upper part of the Columbia plot at the Roberge-Weiss endpoint.

From the thermal Wilson line $\mathbf{L}$ in Eq. (10), we introduce a modified Polyakov loop as

$$
\Psi = e^{2\pi i \phi} \frac{1}{N} \mathrm{tr}\, \mathbf{L} , \quad \overline{\Psi} = e^{-2\pi i \phi} \frac{1}{N} \mathrm{tr}\, \mathbf{L}^\dagger . \quad (35)
$$

The Polyakov loop is the trace of the quark propagator. Multiplying by $e^{2\pi i \phi}$ is exactly like the phase which enters for the boundary conditions for a dynamical quark field, Eq. (4).

The modified loop Ψ is periodic under the Roberge-Weiss symmetry, and so henceforth, we always refer to the Polyakov loop as that of Eq. (35).

Previously it was found the the pressure and other thermodynamical quantities exhibit unphysical behavior below T_d in the matrix model [16]. This is because the behavior of color singlet quantities, such as glueballs, are not included self-consistently. To handle this, we modify the potential as

$$\mathcal{V}(q) \to \mathcal{V}(q) - \mathcal{V}(q_c) + \mathcal{V}(q_c)\overline{\Psi}\Psi \,, \tag{36}$$

where $q_c = (1/3, -1/3, 0)$. This follows a similar modification in Ref. [16] where the q-dependent part of the potential was modified in one particular direction in color space. The above form is more natural, and suppresses it in a color symmetric manner.

C. Phase diagram for heavy quarks

In this subsection we discuss the nature of the phase diagram, in the plane of temperature and quark chemical potential, as the mass of a heavy quark changes. For this purpose, we can concentrate on the behavior of the order parameter(s), which are the real and imaginary parts of the (Roberge-Weiss symmetric) Polyakov loop. After this we give results for thermodynamic quantities in the following subsection. To be definite, we assume that there are three degenerate flavors of quarks, although qualitatively our results rather insensitive to the number of flavors.

Consider first quarks in the absence of an imaginary chemical potential, $\phi = 0$. For heavy quarks, we consider how deconfinement changes as we go from the pure gauge theory, $m = \infty$, to large quark masses. For three (or more) colors, the deconfining transition is of first order. Quarks act like a background $Z(3)$ field for the transition, and so in all parts of the phase diagram, the real part of the Polyakov loop is nonzero whenever there are dynamical quarks present.

As the quark mass decreases, the latent heat of the deconfining transition decreases, until it first vanishes at a deconfining critical endpoint, when $m = m_{dce}$. Numerically, in the matrix model we find that $m_{dce}/T_d \approx 8.2 \pm .1$ [16]. When $m < m_{dce}$, there is no deconfining transition, only crossover. At the critical endpoint, $m = m_{dce}$. the critical field is the deviation of the real part of the Polyakov loop from its expectation value, with the universality class that of the Ising model in three dimensions. In the matrix model this

occurs at a temperature which is very close to that the transition temperature in the pure gauge theory, $T_{dce}/T_d \approx .991 \pm .001$ [16].

When the quarks have an imaginary chemical potential which is nonzero, we expect that the imaginary part of the Polyakov loop is also nonzero. By Roberge-Weiss periodicity, the imaginary part vanishes when $\phi = 0$ and $1/3$. In all cases, we find that the imaginary part of the Polyakov loop is positive when $\phi < 1/6$, and negative when $\phi > 1/6$.

The Roberge-Weiss transition occurs when $\phi_{RW} = 1/6$ at sufficiently high temperature for any value of the quark mass. Then the expectation value of the imaginary part of the loop is positive to the left of the Roberge-Weiss transition, and negative to the right. We illustrate the behavior of the Polyakov loops in Fig. (1). In this figure, $m = m_{dce}$, and we choose a temperature T_d. As expected, the real part of the loop is always nonzero, decreasing to a minimum at the Roberge-Weiss transition, $\phi_{RW} = 1/6$.
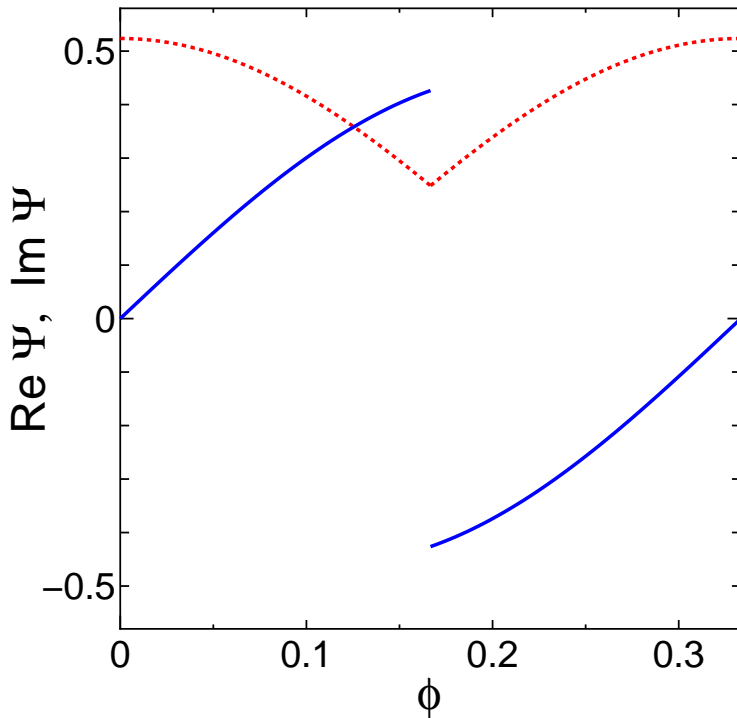
FIG. 1. Expectation value of the real (dotted line) and imaginary (solid line) parts of the Polyakov loop at the Roberge-Weiss transition, $\phi_{RW} = 1/6$. The values are for $m = m_{dce}$ and $T = T_d$, but the behavior is qualitatively similar for any Roberge-Weiss transition.

The phase diagram in the plane of temperature and imaginary chemical potential is illustrated in Fig. (2). Here we choose $m = m_{dce}$, so when $\phi = 0$ (or $1/3$), so there is a

second order transition as T is varied. The phase diagram for $m > m_{dce}$ is very similar, the only difference being that the transition at $\phi = 0$ (or $1/3$) is of first order instead of second.
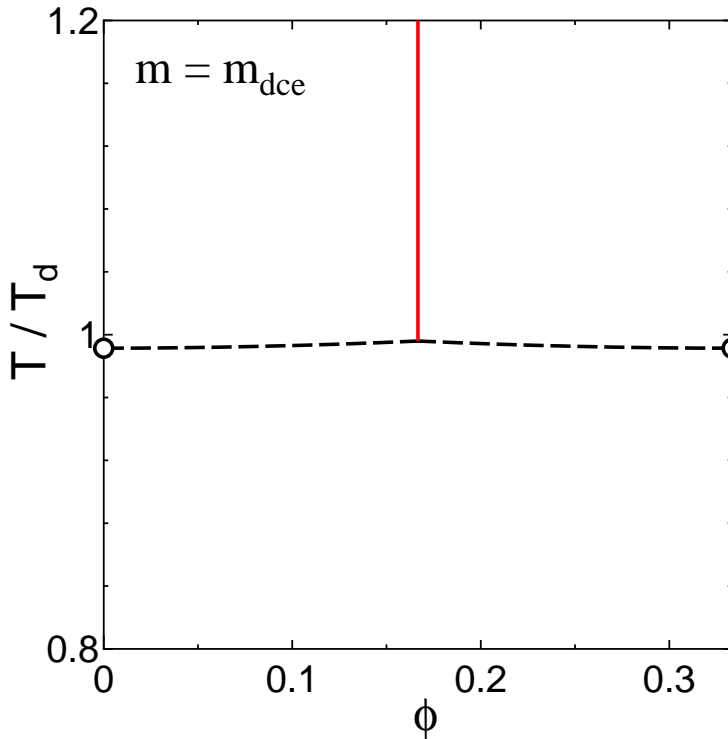

FIG. 2. Phase diagram at the deconfining critical endpoint, $m = m_{dce}$. The solid line denotes a Roberge-Weiss transition, of first order, at $\phi_{RW} = 1/6$; the dotted lines, first order transitions, which mix deconfining and Roberge-Weiss transitions. The points at $\phi = 0$ and $1/3$ are deconfining critical endpoints, and so of second order.

The solid line indicates the Roberge-Weiss transition at $\phi_{RW} = 1/6$. Across this line of first order transitions, by the arguments in the previous section the interface tension corresponds directly to the 't Hooft loop. At a temperature $\approx (1.00 \pm .01)T_d$, this line of first order transitions splits into two lines of first order transitions, drawn as dotted lines.

The first order transitions for $0 < \phi < 1/6$, and $1/6 < \phi < 1/3$, are manifestly those where the Roberge-Weiss and deconfining transitions mix. This is clear above the deconfining critical endpoint, for $m \geq m_{dce}$, since then the line of first order transitions for the Roberge-Weiss transition is directly connected by lines of first order transitions to the deconfining transition at $\phi = 0$, where the imaginary part of the Polyakov loop vanishes, and the transition is entirely one of deconfinement.

As ϕ increases from 0 to 1/6, the jump in the imaginary part of the Polyakov loop increases. In this region, transition is one where the deconfining and Roberge-Weiss transitions mix. We have checked that for $\phi \neq 1/6$, that the jump in the $Z(3)$ phase does not correspond to a $Z(3)$ transformation. Thus as argued in the previous section, the interface tension across such first order transitions is not related to a 't Hooft loop.
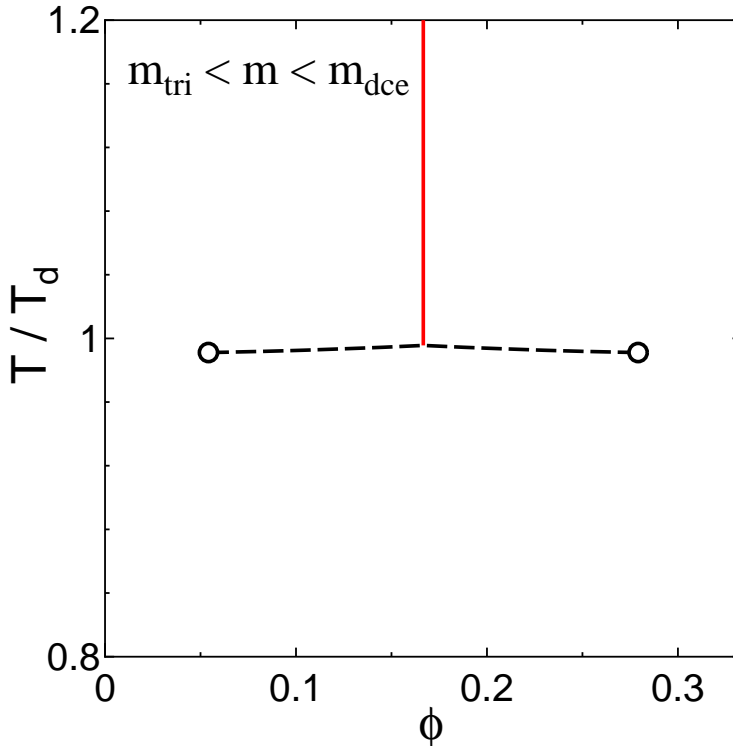

FIG. 3. Phase diagram for masses below the deconfining critical endpoint, but above the tri-critical point, $m_{dce} > m > m_{tri}$. Again, the solid line denotes a Roberge-Weiss transition, of first order, at $\phi_{RW} = 1/6$; the dotted lines, first order transitions, which mix deconfining and Roberge-Weiss transitions.

As the quark mass decreases below that for the deconfining critical endpoint, the two lines of first order transitions for $\phi \neq 1/3$ move closer to 1/6, see Fig. (3). There are two critical endpoints, at ϕ_c and $1/3 - \phi_c$. At the critical endpoints, both the real and the imaginary parts of the Polyakov loop of Eq. (35) are nonzero. The universality class is that of $Z(2)$, with the critical field some combination of real and imaginary parts of the Polyakov loop.

As $\phi_c \to 1/6$, the two critical endpoints merge, and the line of Roberge-Weiss transitions ends at tri-critical point, as illustrated in Fig. (4). In the matrix model, this occurs for a

quark mass $m_{tri}/T_d \sim 6.3 \pm .1$, at a temperature $T_{tri}/T_d \sim 0.983 \pm .001$. At this tri-critical point the effective theory is that of a single scalar field in three dimensions, where the mass squared and quartic coupling vanish at the tri-critical point. Thus the universality class is mean field, up to calculable logarithmic corrections.

For masses below m_{tri}, there is only a line of Roberge-Weiss transitions, which end in an ordinary critical endpoint, in the universality class of the Ising model in three dimensions.
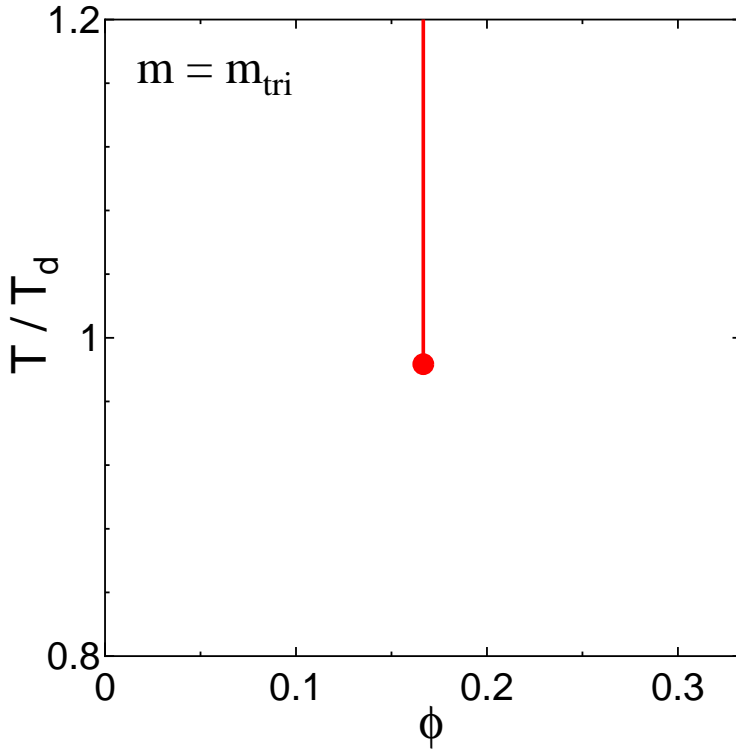
FIG. 4. Phase diagram for a quark mass at the tri-critical point, $m = m_{tri}$. In this region, there is only a Roberge-Weiss transition at $\phi_{RW} = 1/6$, which ends in a tri-critical point, denoted by a solid point. For $m < m_{tri}$, the line of Roberge-Weiss transitions ends in an ordinary critical endpoint.

We note that in general, the imaginary part of the quark number density is positive for $0 < \phi < 1/6$, and negative for $1/6 < \phi < 1/3$. For temperatures sufficiently high that there is a Roberge-Weiss transition, the sign of the imaginary part of the quark number density flips at $\phi_{RW} = 1/6$, and there is a first order transition. For temperatures below that for the Roberge-Weiss transition, the imaginary part of the quark number density is still positive for $0 < \phi < 1/6$, and negative for $1/6 < \phi < 1/3$, but because there is no phase transition,

it vanishes for $\phi_{RW} = 1/6$.

1. Thermodynamics of Roberge-Weiss transitions

We now turn to the thermodynamics of Roberge-Weiss transitions. We concentrate on the transitions for $\phi_{RW} = 1/6$, as those for $\phi \neq 1/6$ are qualitatively similar.
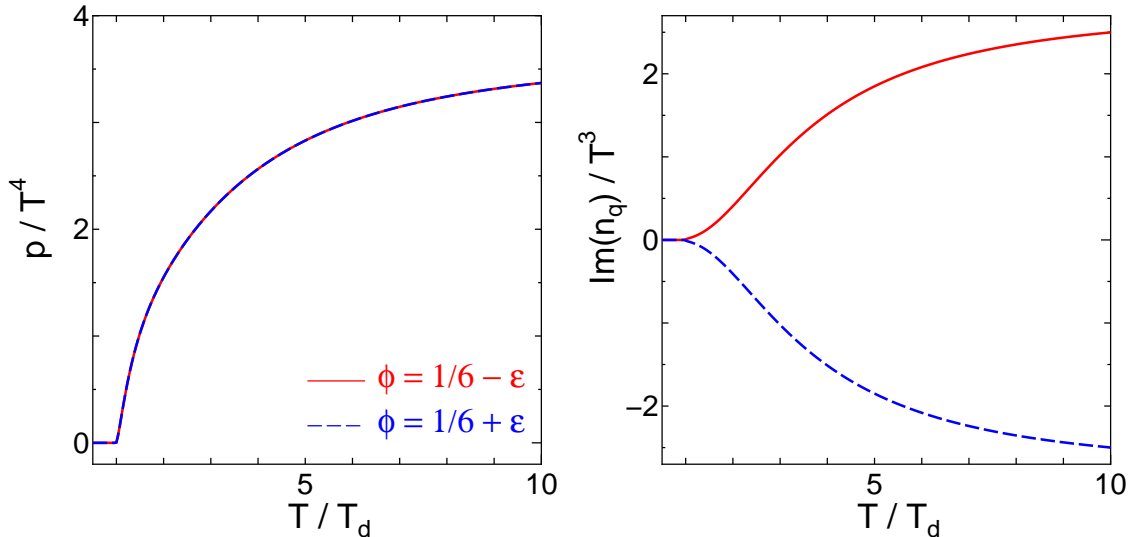

FIG. 5. The temperature dependence of the pressure and the imaginary part of the quark number density across the Roberge-Weiss transition at $\phi_{RW} = 1/6$ with $m = m_{dce}$. All quantities are scaled by appropriate powers of the temperature to be dimensionless, and shown as functions of T/T_d. The solid and dashed lines represents $\phi = 1/6 - \epsilon$ and $\phi = 1/6 + \epsilon$, respectively.

We begin with a quark mass at the deconfining critical endpoint (when $\phi = 0$). In Fig. (5) we show the pressure, $p(T)$, the imaginary part of the quark number density, $n_q(T)$. Both quantities are rescaled by powers of the temperature to be dimensionless, so we show $p(T)/T^4$ and Im $n_q(T)/T^3$. Similarly, in Fig. (6) we show the rescaled energy density $e(T)/T^4$, Eq. (29), and the interaction measure, $(e - 3p)/T^4$.

Fig. (5) shows that the pressure is like the real part of the Polyakov loop, and is continuous across the Roberge-Weiss transition. In contrast, the imaginary part of the quark number density is like the imaginary part of the Polyakov loop, and flips sign across this transition.

Because the pressure is continuous, the change in the imaginary part of the quark number density implies that the energy density is discontinuous across the transition. Thus this
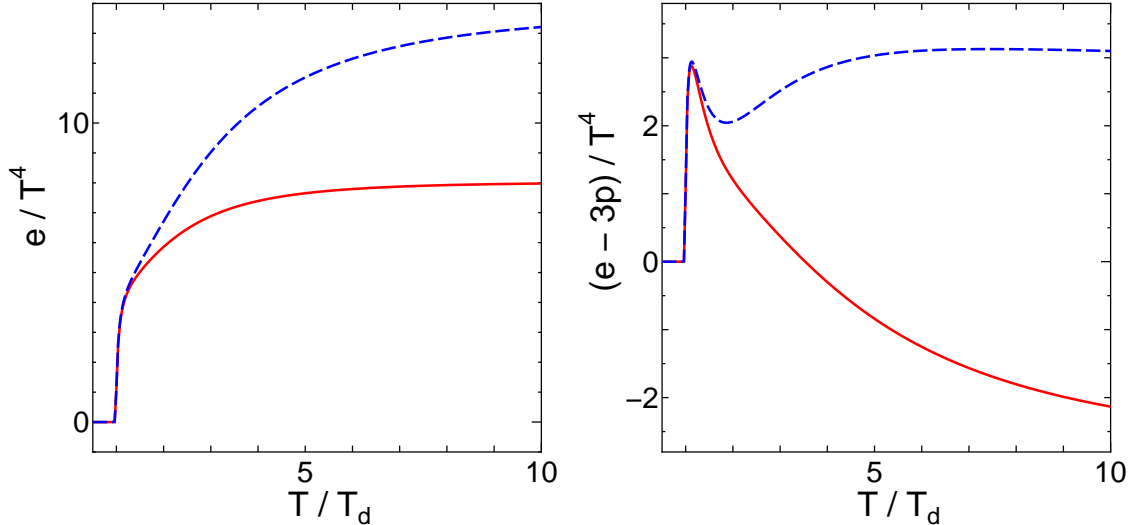
FIG. 6. The temperature dependence of the energy density, $e(T)/T^4$ and the interaction measure, $(e-3p)/T^4$, across the Roberge-Weiss transition at $\phi_{RW} = 1/6$, with $m = m_{dce}$. The solid and dashed lines represents $\phi = 1/6 - \epsilon$ and $\phi = 1/6 + \epsilon$, respectively.

change in the imaginary part of the quark number density is the only reason the transition is of first order, Fig. (6). On the right hand side of the transition, for $\phi_{RW} = 1/6^+$, the smaller value of the energy density implies that the interaction measure eventually becomes negative, Fig. (6), for temperatures above $\sim 3\,T_d$.

To the left of the Roberge-Weiss transition, the interaction measure displays a characteristic two peak structure [16]. The peak near T_d is due to the gluons, that at several times T_d is due to the quarks, because they are so heavy. Because the energy density to the right of the Roberge-Weiss transition is negative, the peak in the interaction measure persists, but that due to the quarks is completely washed out by the negative energy density for $\phi_{RW} = 1/6^+$. At high temperatures, the interaction measure has the same value for $\phi_{RW} = 1/6^\pm$, but with opposite sign, see Sec. III A.

The behavior of the thermodynamic quantities is similar across the Roberge-Weiss transition for other quark masses. In Fig. (7) we show the behavior of the imaginary part of the quark number density, and the interaction measure, for $m = m_{tri}$. This is similar to the behavior at the deconfining critical endpoint, Figs. (5) and (6), and so we do not bother to show the corresponding results for the pressure and the energy density.

One can compute the position of the tri-critical point versus the number of flavors, and
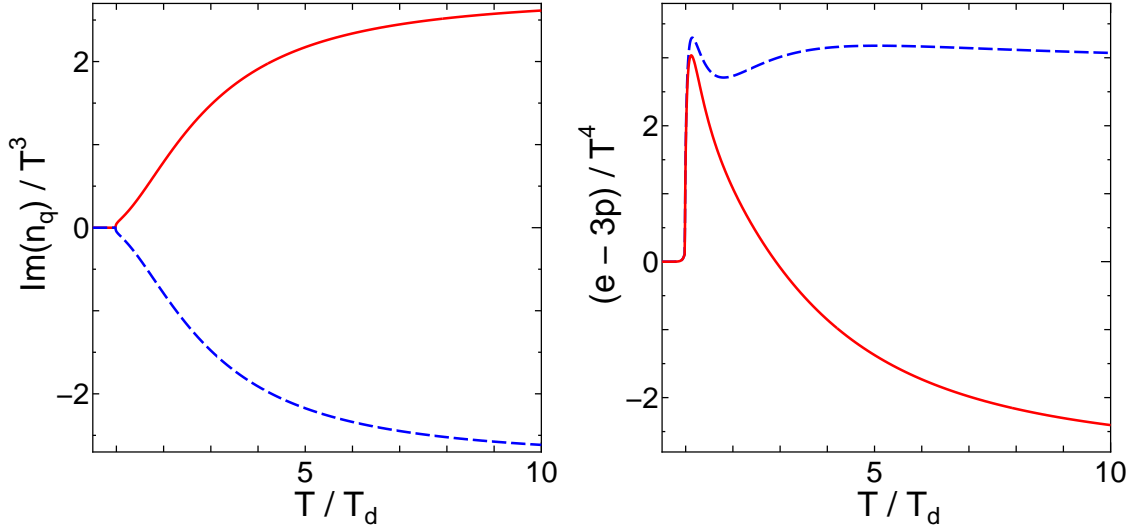
FIG. 7. The temperature dependence of the imaginary part of the quark number density, $\mathrm{Im}(n_q)/T^3$, and the interaction measure, $(e - 3p)/T^4$, across the Roberge-Weiss transition at $\phi_{RW} = 1/6$, with $m = m_{tri}$. The solid and dashed lines represents $\phi = 1/6 - \epsilon$ and $\phi = 1/6 + \epsilon$, respectively.

is illustrated in Fig. (8). Recent lattice QCD data [4] suggest that there is the tri-critical line in the heavy quark mass region at the RW endpoint. The matrix model reproduces the result and values of m_{tri}/T_{tri} are summarized in Table. I. These values well reproduce the

	$m_l = \infty$	$m_l = m_s$	$m_s = \infty$
Ref. [5]	5.56(3)	6.66(3)	6.25(3)
This model	5.0	6.4	5.9

TABLE I. Summary for m_{tri}/T_{tri} in the recent LQCD simulation with the strong coupling and hopping parameter expansions [5] and the matrix model for deconfinement. We show our result down to first decimal place.

recent LQCD simulation with the strong coupling and hopping parameter expansions [5].
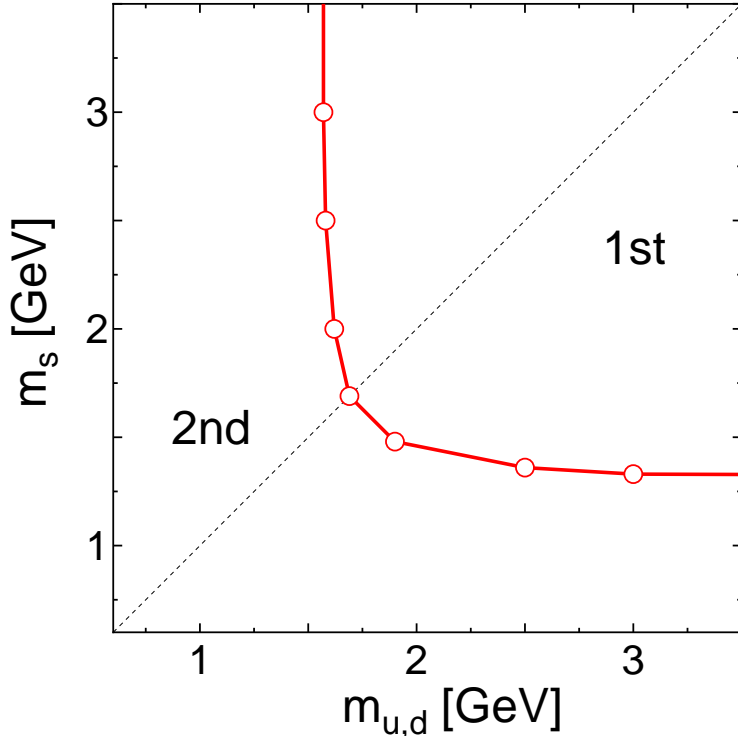
FIG. 8. The dependence of the tri-critical endpoint for $2 + 1$ flavors.

D. Summary

In this paper we showed that 't Hooft loops of arbitrary $Z(N)$ charge are well defined even with dynamical quarks at a Roberge-Weiss transition, for $\phi_{RW} = k/(2N)$. To leading order in weak coupling, the 't Hooft loop satisfies Casimir scaling in the pure glue theory, but not with quarks.

For three colors we computed thermodynamic behavior at large quark mass using an effective matrix model for deconfinement, and computed the form of the Columbia plot. We computed the interaction measure about the Roberge-Weiss transition, $\phi_{RW} = 1/6 \pm \epsilon$, and find an enhancement on one side of the transition, but not the other.

Considering an imaginary chemical potential is clearly useful to discriminate between various effective models of deconfinement. The relationship to the 't Hooft loop suggests that it probes more fundamental aspects of the dynamics in unforeseen ways.

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