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CFT representation of interacting bulk gauge fields in AdS

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We develop the representation of interacting bulk gauge fields and charged scalar matter in AdS in terms of non-local observables in the dual CFT. We work in holographic gauge in the bulk, $A_z = 0$. The correct statement of micro-causality in holographic gauge is somewhat subtle, so we first discuss it from the bulk point of view. We then show that in the $1/N$ expansion CFT correlators can be lifted to obtain bulk correlation functions which satisfy micro-causality. This requires adding an infinite tower of higher-dimension multi-trace operators to the CFT definition of a bulk observable. For conserved currents the Ward identities in the CFT prevent the construction of truly local bulk operators (i.e. operators that commute at spacelike separation with everything), however the resulting non-local commutators are exactly those required by the bulk Gauss constraint. In contrast a CFT which only has non-conserved currents can be lifted to a bulk theory which is truly local. Although our explicit calculations are for gauge theory, similar statements should hold for gravity.

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1 Introduction

The question of observables in quantum gravity has a long history; for reviews see [1, 2, 3, 4]. The problem is that, as emphasized by Dirac [5], only gauge-invariant quantities can be assigned a physical meaning. In gravity this rules out the existence of local observables. Indeed in the AdS/CFT context a complete set of observables lives at the boundary, so one must be able to express any definition of a bulk observable in terms of CFT data. In the limit of free scalar fields in the bulk (i.e. zero Planck length, $N \to \infty$) the construction was made in the early days of AdS/CFT [6, 7, 8, 9]. It has been recast in the form of a smearing function [10, 11]

$$\phi(z,x) = \int dx' K_\Delta(z,x|x')\mathcal{O}_\Delta(x') \tag{1}$$

where the kernel K has support only on boundary points x' which are space-like separated from the bulk point (z,x). The dimension of the boundary operator Δ is determined by the mass of the bulk field. It turns out that using complex boundary coordinates is a very convenient computational tool [11, 12]. These constructions were carried out in the free field limit, and it was shown that the CFT expectation value of two such operators reproduces the free bulk two point function.

Building on these works, the construction of interacting bulk observables in terms of smeared CFT operators has been developed. Two approaches have been worked out, both relying on perturbation theory in $1/N$. One approach is based on the bulk equations of motion, while the other uses bulk micro-causality as a guiding principle.

The first approach was introduced in [13] and further developed in [14]. The basic idea is to solve the bulk equations of motion perturbatively. This can be done in a fixed gauge (holographic gauge), using the radial supergravity Hamiltonian on a fixed background. This procedure gives a bulk operator written in terms of smeared CFT operators, whose correlation functions in the CFT reproduce bulk correlators. This construction can be carried out independently of holography. It is just a rewriting of bulk correlation functions in terms of boundary correlators, in the same way that one could have computed a bulk correlator in terms of correlation functions on some initial time slice by solving time evolution equations. The only difference is that in AdS/CFT it is convenient to evolve in a spacelike direction, using a spacelike Green's function [13, 14]. It is an extra condition, that the boundary correlation functions are those of a unitary CFT, which makes the relationship holographic. But in the approach of solving bulk equations of motion, the role played by holography is not so clear.

In the second approach, more intrinsic to the CFT, one tries to build up the bulk operator by requiring that it satisfy bulk micro-causality [13].[1] This program was carried out for

[1]For a discussion of micro-causality in curved space see [15].

scalar fields in [13], where the requirement of micro-causality is just that bulk operators commute at space-like separation. The basic point is that, if one inserts the smeared CFT operator (1) inside a CFT three point function, there are in general singularities at bulk space like separation. These singularities lead to a non-zero commutator which spoils micro-causality. However these singularities can be suppressed (in a precise sense) by redefining the bulk operator to include an infinite tower of appropriately smeared higher dimension scalar primaries (these are the multi-trace operators also discussed in [16]),

$$\phi(z,x) = \int dx' K_\Delta(z,x|x')\mathcal{O}_\Delta(x') + \sum_l a_l \int dx' K_{\Delta_l}(z,x|x')\mathcal{O}_{\Delta_l}(x'). \tag{2}$$

Order-by-order in $1/N$ one has the required spectrum of higher dimension operators, and one can choose the coefficients a_l in such a way that the bulk operator satisfies micro-causality. The resulting bulk observable agrees with what one would construct in the $1/N$ expansion by solving the bulk equations of motion perturbatively.

The CFT approach gives a different perspective from the approach based on bulk equations of motion, and gives a glimpse of the non-locality which is expected when the boundary theory is a finite-N unitary CFT. More specifically, the CFT construction requires the existence of an infinite tower of higher-dimension primary operators with prescribed properties. Such operators can be constructed in $1/N$ perturbation theory as multi-trace operators, but these operators do not actually exist in a unitary CFT at finite N. Thus at finite N we can see how bulk locality breaks down in a non-perturbative way.

In extending the CFT construction of interacting bulk observables to include gravity we face the difficulty mentioned at the start of the introduction, that in gravity there are no local physical observables (see [14, 17, 18] for discussions of this in the context of AdS/CFT). There is, however, clearly some sense in which local observables are available even in a theory of gravity. To address this we need to deal with the underlying gauge symmetry. There are two approaches we could take: either construct a set of gauge-invariant observables, or carry out the construction in a fixed gauge. We will adopt the gauge-fixed approach, which may not seem so elegant but is in fact natural in AdS/CFT.

Of course the two approaches are related. To be concrete, consider a charged scalar field in the bulk $\phi(x,z)$, coupled to an abelian gauge field A. Here $x = (t,\vec{x})$ are coordinates in the CFT and z is a radial coordinate. We work in holographic gauge which sets $A_z = 0$. In holographic gauge $\phi(x,z)$ is (by definition) a gauge-invariant observable. It can be identified with the manifestly gauge-invariant quantity

$$\exp\left[i\int_{(x,z)}^{(x,0)} A_z dz\right]\phi(x,z)$$

where we have attached a Wilson line running from the bulk point to the boundary of AdS in the z direction. But now one sees the difficulty, that although a bulk scalar field in holographic gauge is an observable quantity, it is secretly non-local, with a Wilson line extending in the z direction. So there is no reason to expect our gauge-fixed operators to commute at spacelike separation, and indeed in axial gauge there are non-local commutators [19].

In gauge theory it's tempting to avoid this issue by working in terms of local gauge-invariant quantities, such as $\operatorname{Tr} F^2$ or $\phi^\dagger \phi$, but in gravity this is not an option. So let's work directly with the scalar field in holographic gauge, and see if there is a useful sense in which we can discuss bulk locality.[2]

A key observation is that in holographic gauge non-local commutators are indeed present, but *only to the extent required by the constraints.* For example consider a charged scalar field $\phi(x, z)$ and the electric flux observable

$$\Phi_E = \oint {}^*F$$

Since charge can be measured by a surface integral arbitrarily far away, it's clear that $\phi(x, z)$ and Φ_E will in general not commute at equal times. But there's no obstacle to having ϕ commute with itself at equal times. We will make this more precise in section 2, where we consider scalar electrodynamics in holographic gauge and show that the scalar field indeed commutes with itself at spacelike separation. There are some non-local commutators in holographic gauge. However at equal times the only non-local commutators involve either the time component of the gauge field A_0 or the z component of the electric field E_z. This behavior is exactly what's required by the Gauss constraint, and it can be understood as due to the Wilson lines extending in the z direction.[3]

Our conclusion is that we can construct bulk observables in holographic gauge by demanding that, for example, charged bulk scalar fields commute at spacelike separation. Of course gauge-invariant combinations such as a field strength in the bulk and a scalar field on the boundary will also commute at spacelike separation. The remainder of this paper is devoted to showing that these requirements, which we view as encoding bulk micro-causality, suffice to uniquely determine the way in which boundary CFT correlators can be lifted into the bulk.

[2] An alternative approach would be to work in some type of covariant gauge in the bulk, where locality is manifest but there are additional unphysical degrees of freedom. It's not clear to us how this could be represented in the CFT.

[3] The extension to gravity seems clear: scalar fields will commute with each other at spacelike separation, but they will have non-zero commutators with h_{00} and with certain components of the curvature.

2 Bulk micro-causality

In this section we consider scalar electrodynamics in holographic gauge and show that the scalar field commutes with itself at spacelike separation. Our treatment of the canonical formalism for scalar electrodynamics in holographic gauge closely follows section 5.C of [19].

We work in AdS_{d+1} with metric

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + |d\vec{x}|^2 + dz^2\right)$$

and consider scalar electrodynamics with action ($D_M = \partial_M + iqA_M$)

$$S = \int d^{d+1}x\sqrt{-g}\left(-D_M\phi^* D^M\phi - \frac{1}{4}F_{MN}F^{MN}\right) \tag{3}$$

The canonical momenta are

$$\begin{aligned}
\pi_0 &= 0 \\
\pi_i &= \left(\frac{R}{z}\right)^{d-3}\left(\partial_0 A_i - \partial_i A_0\right) \qquad i = 1,\ldots,d \\
\pi_\phi &= \left(\frac{R}{z}\right)^{d-1} D_0\phi^* \\
\pi_\phi^* &= \left(\frac{R}{z}\right)^{d-1} D_0\phi
\end{aligned}$$

Thus we have the primary constraint

$$\chi_1 \equiv \pi_0 = 0$$

and the secondary constraint (Gauss' law)

$$\chi_2 \equiv \partial_i\pi_i + iq\left(\pi_\phi\phi - \pi_\phi^*\phi^*\right) = 0$$

Conjugate to these we impose the two gauge-fixing conditions

$$\begin{aligned}
\chi_3 &\equiv A_z = 0 \\
\chi_4 &\equiv \pi_z + \left(\frac{R}{z}\right)^{d-3}\partial_z A_0 = 0
\end{aligned}$$

The first condition fixes holographic gauge, while the second condition enforces the usual relation between the z component of the electric field and the gauge field.

The matrix of Poisson brackets is (setting $\mathbf{x} = (\vec{x}, z)$)

$$C_{ab} = \{\chi_a(\mathbf{x}), \chi_b(\mathbf{x}')\} = \begin{pmatrix} 0 & 0 & 0 & -(\frac{R}{z'})^{d-3}\partial_z\delta^d(\mathbf{x}-\mathbf{x}') \\ 0 & 0 & \partial_z\delta^d(\mathbf{x}-\mathbf{x}') & 0 \\ 0 & \partial_z\delta^d(\mathbf{x}-\mathbf{x}') & 0 & -\delta^d(\mathbf{x}-\mathbf{x}') \\ -(\frac{R}{z'})^{d-3}\partial_z\delta^d(\mathbf{x}-\mathbf{x}') & 0 & \delta^d(\mathbf{x}-\mathbf{x}') & 0 \end{pmatrix}$$

This has an inverse

$$C^{-1}_{ab} = \begin{pmatrix} 0 & g(\mathbf{x},\mathbf{x}') & 0 & (\frac{z'}{R})^{d-3}f(\mathbf{x},\mathbf{x}') \\ -g(\mathbf{x},\mathbf{x}') & 0 & -f(\mathbf{x},\mathbf{x}') & 0 \\ 0 & -f(\mathbf{x},\mathbf{x}') & 0 & 0 \\ (\frac{z}{R})^{d-3}f(\mathbf{x},\mathbf{x}') & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

where

$$\begin{aligned} f(\mathbf{x},\mathbf{x}') &= \delta^{d-1}(x-x')\theta(z'-z) \\ g(\mathbf{x},\mathbf{x}') &= \delta^{d-1}(x-x')\theta(z'-z)\frac{(z')^{d-2}-z^{d-2}}{(d-2)R^{d-3}} \end{aligned} \tag{5}$$

The inverse is not unique; our explicit choice for f and g corresponds to introducing a Wilson line towards the boundary of AdS, as opposed to towards the Poincaré horizon. Also note that the case $d = 2$ is special as it corresponds to Chern-Simons theory in the bulk [20]. For the canonical formalism in this case see appendix B of [18].

Given the structure of the constraint algebra – in particular the fact that $C^{-1}_{22} = 0$ – it follows that the physical degrees of freedom have canonical Dirac brackets at equal times.

$$\begin{aligned} &\{\pi_{\hat{\imath}}(\mathbf{x}), A_{\hat{\jmath}}(\mathbf{x}')\} = \delta_{\hat{\imath}\hat{\jmath}}\,\delta^d(\mathbf{x}-\mathbf{x}') \qquad \hat{\imath},\hat{\jmath} = 1,\ldots,d-1 \\ &\{\pi_\phi(\mathbf{x}), \phi(\mathbf{x}')\} = \delta^d(\mathbf{x}-\mathbf{x}') \\ &\{\pi^*_\phi(\mathbf{x}), \phi^*(\mathbf{x}')\} = \delta^d(\mathbf{x}-\mathbf{x}') \end{aligned}$$

However these fields have non-local Dirac brackets with A_0 and π_z, namely

$$\begin{aligned} &\{A_0(\mathbf{x}), A_{\hat{\imath}}(\mathbf{x}')\} = \partial_{\hat{\imath}}g(\mathbf{x},\mathbf{x}') \\ &\{A_0(\mathbf{x}), \phi(\mathbf{x}')\} = iqg(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}') \\ &\{A_0(\mathbf{x}), \pi_\phi(\mathbf{x}')\} = -iqg(\mathbf{x},\mathbf{x}')\pi_\phi(\mathbf{x}') \end{aligned}$$

$$\begin{aligned} &\{\pi_z(\mathbf{x}), A_{\hat{\imath}}(\mathbf{x}')\} = \partial_{\hat{\imath}}f(\mathbf{x},\mathbf{x}') \\ &\{\pi_z(\mathbf{x}), \phi(\mathbf{x}')\} = iqf(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}') \\ &\{\pi_z(\mathbf{x}), \pi_\phi(\mathbf{x}')\} = -iqf(\mathbf{x},\mathbf{x}')\pi_\phi(\mathbf{x}') \end{aligned}$$

along with the complex conjugates. These brackets reflect the fact that the field $\phi(x,z)$ produces a tube of electric flux extending towards $z = 0$.

This shows that, as promised, the scalar field commutes with itself at equal times. However we'd like to make a stronger statement, that the scalar field commutes with itself at spacelike separation. This can be argued based on the results obtained above. Imagine inserting the scalar field at two spacelike separated points (x, z) and (x', z'), with Wilson lines secretly extending off in the z direction. By acting with an AdS isometry the two bulk points can be brought to equal times. However the isometry will act on the Wilson lines, so they will no longer extend in the z direction. We could perform a compensating gauge transformation to restore holographic gauge $A_z = 0$, but it's simpler to leave the Wilson lines pointing in whatever direction is implied by the isometry. How would this affect the above calculation? The brackets with A_0 and π_i will clearly be different, because the electric flux tubes now go in a different direction, but the bracket of ϕ with itself will still be zero. This means that in holographic gauge the scalar field commutes with itself at arbitrary spacelike separation.

3 Bulk construction of local operators

Although it won't be the main emphasis of this paper, one can construct local bulk observables from the bulk point of view, by solving the bulk equations of motion perturbatively [13, 14]. Here we sketch the construction for scalar electrodynamics.

The equations of motion which follow from (3) are

$$\frac{1}{\sqrt{-g}} D_M \sqrt{-g} D^M \phi = 0 \tag{6}$$

$$\frac{1}{\sqrt{-g}} \partial_M \sqrt{-g} F^{MN} = J^N \tag{7}$$

where $D_M = \partial_M + iqA_M$ and $J^M = iq(D^M\phi^*\phi - \phi^* D^M\phi)$. We wish to solve these equations perturbatively in q (which we identify as being $\mathcal{O}(1/N)$), in the gauge $A_z = 0$.

We begin with the scalar equation of motion (6), which in terms of a Christoffel connection ∇ on AdS reads

$$\nabla_M \nabla^M \phi + iq(\nabla_M A^M)\phi + 2iqA^M \partial_M \phi - q^2 A^2 \phi = 0$$

We can solve this perturbatively, setting

$$\begin{aligned} \phi &= \phi^{(0)} + \phi^{(1)} + \cdots \\ A_M &= A_M^{(0)} + A_M^{(1)} + \cdots \end{aligned}$$

where

$$\begin{aligned}
\nabla_M \nabla^M \phi^{(0)} &= 0 && (8)\\
\nabla_M \nabla^M \phi^{(1)} &= -iq(\nabla_M A^{M\,(0)})\phi^{(0)} - 2iqA^{M\,(0)}\partial_M \phi^{(0)} && (9)\\
&\vdots
\end{aligned}$$

The first equation can be solved – with suitable boundary conditions that match on to the CFT as $z \to 0$ – using the scalar smearing function constructed in [11, 12]. The second equation can be solved using a spacelike Green's function as in [13, 14].

Next we look at the z component of the gauge field equation of motion (7), which reduces to[4]

$$\partial_z(\partial_\mu A^\mu) = -\frac{R^2}{z^2} J_z$$

This fixes

$$\partial_\mu A^\mu(x,z) = -\int_0^z dz'\, \frac{R^2}{z'^2} J_z(x,z') \qquad (10)$$

In the absence of a source note that $\partial_\mu A^\mu = 0$ as in [18], so that in fact $\nabla_M A^{M\,(0)} = 0$ in (9).

The remaining components of the gauge field equations of motion reduce to

$$\frac{1}{\sqrt{-g}}\partial_M \sqrt{-g} g^{MN} \partial_N \phi_\lambda + \frac{d-1}{R^2}\phi_\lambda = z\left(J_\lambda + \frac{z^2}{R^2}\partial_\lambda(\partial_\mu A^\mu)\right) \qquad (11)$$

where we've introduced $\phi_\lambda = zA_\lambda$. This is convenient because the left hand side of (11) is the wave equation for a scalar field of mass $m^2R^2 = 1-d$. Expanding in powers of the coupling, we have the tower of equations

$$\begin{aligned}
&\frac{1}{\sqrt{-g}}\partial_M \sqrt{-g} g^{MN} \partial_N \phi_\lambda^{(0)} + \frac{d-1}{R^2}\phi_\lambda^{(0)} = 0 && (12)\\
&\frac{1}{\sqrt{-g}}\partial_M \sqrt{-g} g^{MN} \partial_N \phi_\lambda^{(1)} + \frac{d-1}{R^2}\phi_\lambda^{(1)} = zJ_\lambda^{(1)} - z^3\partial_\lambda \int_0^z dz'\, \frac{1}{z'^2} J_z^{(1)}(x,z')\\
&\qquad\vdots
\end{aligned}$$

where the first-order current $J^{(1)}$ is expressed in terms of the lowest-order field $\phi^{(0)}$, and where we've used (10) to express $\partial_\mu A^\mu$ in terms of the bulk current. Just as for the scalar field, the first equation can be solved using an appropriate scalar smearing function, while the second equation can be solved using a spacelike Green's function.

In the rest of this paper we will see how this structure emerges directly from the CFT, without using bulk equations of motion.

[4]Indices tangent to the boundary are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$.

4 CFT construction: bulk scalars

In this section, as a warm-up illustrative example, we will see how things work for an interacting scalar field. This extends the construction of [13] to $d+1$ dimensions and gives results which will be useful later. Similarly to what was done in the AdS_3 case one expects the 3-point function of a bulk scalar and two boundary scalars to have the form

$$\begin{aligned} < \phi_i(x,z)\mathcal{O}_j(y_1)\mathcal{O}_k(y_2) > &= c_{ijk}\frac{1}{(y_1-y_2)^{\Delta_j+\Delta_k-\Delta_i}}\left[\frac{z}{z^2+(x-y_1)^2}\right]^{(\Delta_j+\Delta_i-\Delta_k)/2} \\ &\times \left[\frac{z}{z^2+(x-y_2)^2}\right]^{(\Delta_k+\Delta_i-\Delta_j)/2} f(\chi) \end{aligned} \tag{13}$$

where

$$\chi = \frac{[(x-y_1)^2+z^2][(x-y_2)^2+z^2]}{z^2(y_2-y_1)^2} \tag{14}$$

To compute $f(\chi)$ we look at the limit of large y_2, where the CFT 3-point function reduces to

$$< \mathcal{O}_i(x)\mathcal{O}_j(0)\mathcal{O}_k(y_2) > \rightarrow c_{ijk}\frac{1}{(y_2)^{2\Delta_k}}\frac{1}{x^{2\Delta_0}} \tag{15}$$

Here $x^2 = |\vec{x}|^2 - t^2$ and $\Delta_0 = (\Delta_i + \Delta_j - \Delta_k)/2$. We now define $\phi_i(z,x)$ by smearing $\mathcal{O}_i$ into the bulk using the appropriate scalar smearing function

$$\phi(z,t,\vec{x}) = \frac{\Gamma(\Delta - d/2+1)}{\pi^{d/2}\Gamma(\Delta-d+1)}\int_{t'^2+|\vec{y}'|^2\le z^2} dt' d^{d-1}y' \left(\frac{z^2-t'^2-|\vec{y}'|^2}{z}\right)^{\Delta-d} \mathcal{O}(t+t',\vec{x}+i\vec{y}')\,. \tag{16}$$

This gives

$$< \phi_i(x,z)\mathcal{O}_j(0)\mathcal{O}_k(y_2) > \rightarrow c_{ijk}\frac{1}{(y_2)^{2\Delta_k}} g(x,z) \tag{17}$$

where

$$g(x,z) = \frac{\Gamma(\Delta_i - \frac{d}{2}+1)}{\pi^{\frac{d}{2}}\Gamma(\Delta_i-d+1)}\int_{t'^2+y^2\le z^2} d^{d-1}y dt' \left(\frac{z^2-t'^2-y^2}{z}\right)^{\Delta_i-d} \frac{1}{((\vec{x}+i\vec{y})^2-(t+t')^2)^{\Delta_0}} \tag{18}$$

The integral can be done by setting all but one of the x components to zero, doing the integral, and then restoring Lorentz invariance. The result is

$$g(x,z) = \frac{z^{\Delta_i}}{x^{2\Delta_0}} F(\Delta_0, \Delta_0 - d/2+1, \Delta_i - d/2+1, -\frac{z^2}{x^2}) \tag{19}$$

Comparing to the large-y_2 behavior of (13) one finds

$$f(\chi) = \left(\frac{\chi}{\chi-1}\right)^{\Delta_0} F(\Delta_0, \Delta_0 - \frac{d}{2}+1, \Delta_i - \frac{d}{2}+1, \frac{1}{1-\chi}) \tag{20}$$

$\chi = 0$ corresponds to the bulk point being lightlike to one of the boundary points. The dangerous region is $0 < \chi < 1$ where all points are spacelike to each other. We will see that the 3-point function is not analytic in this region so the operators will not commute.

The analytic structure depends on whether $d/2$ is integer or half-integer. If $d/2$ is half-integer one can use the transformation formula

$$\begin{aligned} F(a,b,c,z) &= (-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}F(a,a-c+1,a-b+1,\frac{1}{z}) \\ &+ (-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}F(b,b-c+1,b-a+1,\frac{1}{z}) \end{aligned} \tag{21}$$

This gives the scalar 3-point function a non-analytic term (near $\chi = 1$) of the form

$$\frac{1}{(\chi-1)^{\frac{d}{2}}}\sum_{n=0} a_n(1-\chi)^{n+1} \tag{22}$$

Thus we have square root singularities, which will give a non-zero commutator for $0 < \chi < 1$.

If $d/2$ is an integer then the above formula is not correct since either $(a-b)$ or $(b-a)$ are negative integers. Instead the correct formula to use is

$$\begin{aligned} F(a,a+n,c,z) &= \frac{\Gamma(c)(-z)^{-a}}{\Gamma(c-a)\Gamma(a+n)}\sum_{k=0}^{n-1}\frac{(n-k-1)!(a)_k(1-c+a)_k}{k!}(-z)^{-k} \\ &+ \frac{\Gamma(c)(-z)^{-a}}{\Gamma(a)\Gamma(c-a-n)}\sum_{k=0}^{\infty}\frac{(a+n)_k(1-c+a+n)_k}{(n+k)!k!}[\psi(k+1)+\psi(n+k+1) \\ &- \psi(a+n+k)-\psi(c-a-n-k)+\ln(-z)]z^{-n-k} \end{aligned} \tag{23}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ and $(n)_k = \frac{\Gamma(n+k)}{\Gamma(n)}$. We see that for the scalar 3-point function the terms contributing to the commutator are of the form

$$\sum_{k=0}^{d/2} b_k(1-\chi)^{-\frac{d}{2}+1+k} + \ln(\chi-1)\sum_{k=0}^{\infty} a_k(1-\chi)^k \tag{24}$$

To summarize, for even d one gets logarithmic singularities in regions where the points are space-like separated and for odd d one gets square root singularities.

The region where there is a non-vanishing commutator between the bulk scalar and one of the boundary scalars, while still having all three points at bulk spacelike separation, is $0 < \chi < 1$. From the above formulas we see that the non-zero commutator in this region has the form of a power series in $(\chi - 1)$. We also see that the singularity structure is the same regardless of the dimension of the operators involved. We wish to define a bulk operator $\phi_i(x,z)$ in such a way as to have the smallest possible commutator with the boundary operators at spacelike separation, transform as a bulk scalar under AdS isometries, and have the correct boundary behavior

$$\phi_i(x,z) \overset{z\to 0}{\to} z^{\Delta_i}\mathcal{O}_i. \tag{25}$$

If we have higher dimension primary scalar operators with dimensions Δ_l, whose 3-point functions with $\mathcal{O}_j$ and $\mathcal{O}_k$ are non-zero, we can redefine the bulk operator $\phi_i(x,z)$ to have the form

$$\phi_i(z,x) = \int dx' K_{\Delta_i}(z,x|x')\mathcal{O}_i(x') + \sum_l a_l \int dx' K_{\Delta_l}(z,x|x')\mathcal{O}_l(x') \tag{26}$$

Since the singularity structure is the same for any $\mathcal{O}_l$, we can choose the coefficients a_l in such a way as to make the commutator of order $(\chi-1)^{\Delta_{\max}}$ where $\Delta_{\max}$ is as large as we wish. If we have an infinite number of suitable higher dimension operators, with conformal dimensions that are unbounded above, we can make the bulk scalar commute at bulk spacelike separation. This is how we define the bulk scalar field. Clearly for any two different $\mathcal{O}_j$ and $\mathcal{O}_k$, we will need a different tower of higher dimension primaries. Fortunately in the large N limit the required operators can be built up from operator products of $\mathcal{O}_j$ and $\mathcal{O}_k$ with derivatives. If $\mathcal{O}_j$ and $\mathcal{O}_k$ are single trace operators this procedure begins with a double trace operator and thus $a_l \sim 1/N$.[5]

5 CFT construction: bulk scalars coupled to vectors

In this section we consider charged scalar fields in the bulk and study the corrections we need to add to the definition of a bulk observable to take into account interactions with currents in the CFT. There are two cases we consider. First, in section 5.1, we consider corrections due to interactions with a non-conserved current in the CFT (dual to a massive vector field in the bulk). Then in section 5.2 we consider interactions with a conserved current in the CFT (dual to a bulk gauge field). We carry out the construction from the CFT perspective, by adding an infinite tower of higher-dimension operators and requiring bulk micro-causality. Thus we extend the program of [13] to include scalars which couple to boundary currents, conserved or not.

[5]For a general discussion of large-N counting in this context see p. 26 of [13].

5.1 Coupling to non-conserved currents

Following the approach of [13, 18] we look at the three point function of a non-conserved current of dimension Δ and two primary scalars of dimension Δ_1 and Δ_2. Up to an overall normalization factor

$$\begin{aligned} & < j_\mu(x)\mathcal{O}_1(y_1)\mathcal{O}_2(y_2) > \\ = \quad & \frac{1}{(y_1-y_2)^{\Delta_1+\Delta_2-\Delta+1}(y_1-x)^{\Delta+\Delta_1-\Delta_2-1}(y_2-x)^{\Delta+\Delta_2-\Delta_1-1}}\left(\frac{(y_1-x)_\mu}{(y_1-x)^2}-\frac{(y_2-x)_\mu}{(y_2-x)^2}\right) \end{aligned} \tag{27}$$

This can be written as

$$\begin{aligned} & \left(\frac{1}{\Delta_2+1-\Delta_1-\Delta}\frac{\partial}{\partial(y_1-x)_\mu}-\frac{1}{\Delta_1+1-\Delta_2-\Delta}\frac{\partial}{\partial(y_2-x)_\mu}\right) \\ & \left[\frac{1}{(y_1-y_2)^{\Delta_1+\Delta_2-\Delta+1}(y_1-x)^{\Delta+\Delta_1-\Delta_2-1}(y_2-x)^{\Delta+\Delta_2-\Delta_1-1}}\right] \end{aligned} \tag{28}$$

where the partial derivative with respect to $(y_1-x)_\mu$ means we are keeping $|y_1-y_2|$ and $|y_2-x|$ fixed (and similarly with $y_1 \leftrightarrow y_2$). The term in square brackets in (28) can be written as

$$(y_2-x)^2 < \tilde{\mathcal{O}}_1(x)\tilde{\mathcal{O}}_2(y_1)\tilde{\mathcal{O}}_3(y_2) > \tag{29}$$

where the expectation value is that of three primary scalars of dimensions $\Delta, \Delta_1, \Delta_2+1$ respectively. Thus one gets

$$\begin{aligned} < j_\mu(x)\phi_1(z,y_1)\mathcal{O}_2(y_2) >= & \left(\frac{1}{\Delta_2+1-\Delta_1-\Delta}\frac{\partial}{\partial(y_1-x)_\mu}-\frac{1}{\Delta_1+1-\Delta_2-\Delta}\frac{\partial}{\partial(y_2-x)_\mu}\right) \\ & \frac{1}{(x-y_2)^{\Delta+\Delta_2-1-\Delta_1}}\left[\frac{z}{z^2+(y_1-x)^2}\right]^{(\Delta+\Delta_1-\Delta_2-1)/2}\left[\frac{z}{z^2+(y_1-y_2)^2}\right]^{(\Delta_2+1+\Delta_1-\Delta)/2} \\ & \left(\frac{\chi}{\chi-1}\right)^{\Delta_0} F(\Delta_0,\Delta_0-\frac{d}{2}+1,\Delta_1-\frac{d}{2}+1,\frac{1}{1-\chi}) \end{aligned} \tag{30}$$

where

$$\chi = \frac{[(x-y_1)^2+z^2][(y_2-y_1)^2+z^2]}{z^2(y_2-x)^2} \tag{31}$$

and $\Delta_0 = \frac{1}{2}(\Delta_1+\Delta-\Delta_2-1)$.

We know the singularity structure of the scalar three point function (13), and we know we can cancel the non-causal singularities in it by adding higher dimension smeared scalar primaries. Thus when smearing $\mathcal{O}_1$ into the bulk we can cancel the non-causal singularities in (27) by adding a tower of higher dimension smeared scalar primaries to our definition of a bulk scalar. This should come as no surprise since from the bulk point of view a theory of a massive vector coupled to scalars is a conventional local theory. Thus there should be no obstacle to constructing a local bulk scalar field.

However the question remains, where do these higher dimension scalars come from? In the large N limit we can construct them as double trace operators. For example the first higher dimension operator we can construct (starting from the case $\Delta_1 = \Delta_2$) is

$$\alpha \partial_\mu j^\mu \mathcal{O}_2 + \beta j^\mu \partial_\mu \mathcal{O}_2 \tag{32}$$

With the choice $\alpha = \frac{1}{\Delta - d + 1}$ and $\beta = -\frac{1}{\Delta_2}$ this is a primary scalar. This reproduces the bulk result, where for a massive vector field the first correction is sourced by terms proportional to $A^M \partial_M \phi$, since the near boundary behavior of this bulk quantity is exactly the operator above.[6] Additional higher dimension operators can be constructed by inserting derivatives in various fashions.

5.2 Coupling to conserved currents

Let's see how things change when we have a conserved current in the CFT, dual to a gauge field in the bulk. From the CFT point of view the only difference is that now there is a Ward identity which restricts correlation functions involving the current, e.g.

$$\partial_\mu \langle j^\mu(x) \mathcal{O}(y_1) \bar{\mathcal{O}}(y_2) \rangle = -iq \langle \mathcal{O}(y_1) \bar{\mathcal{O}}(y_2) \rangle \left(\delta(x - y_1) - \delta(x - y_2) \right) \tag{33}$$

Here q is the charge of the scalar operator and an overbar denotes complex conjugation.

We start with the three point function of a conserved current (which necessarily has dimension $d-1$) and two primary scalars having the same dimension Δ_1.

$$< j_\mu(x) \mathcal{O}(y_1) \bar{\mathcal{O}}(y_2) >= \frac{1}{(y_1 - y_2)^{2\Delta_1 - d + 2} (y_2 - x)^{d-2} (y_1 - x)^{d-2}} \left(\frac{(y_1 - x)_\mu}{(y_1 - x)^2} - \frac{(y_2 - x)_\mu}{(y_2 - x)^2} \right) \tag{34}$$

As we saw before in (28) this can be written as

$$\frac{1}{2-d} \left(\frac{\partial}{\partial_{(y_1 - x)_\mu}} - \frac{\partial}{\partial_{(y_2 - x)_\mu}} \right) \left[\frac{1}{(y_1 - y_2)^{2\Delta_1 - d + 2} (y_1 - x)^{d-2} (y_2 - x)^{d-2}} \right] \tag{35}$$

and the term in the square bracket is just

$$(y_2 - x)^2 < \tilde{\mathcal{O}}_1(x) \tilde{\mathcal{O}}_2(y_1) \tilde{\mathcal{O}}_3(y_2) > \tag{36}$$

[6] The z component of $A^M \partial_M \phi$ gives rise to the $\partial_\mu j^\mu \mathcal{O}_2$ term, while the μ components give rise to $j^\mu \partial_\mu \mathcal{O}_2$. This follows from the massive vector smearing function given below in (73), (74).

where the three primary scalars are of dimension

$$\tilde{\Delta}_1 = d - 1 \qquad \tilde{\Delta}_2 = \Delta_1 \qquad \tilde{\Delta}_3 = \Delta_1 + 1 \tag{37}$$

One can use the result (13) to find

$$< j_\mu(x)\phi(z, y_1)\bar{\mathcal{O}}(y_2) >= \frac{1}{2-d}\left(\frac{\partial}{\partial_{(y_1-x)_\mu}} - \frac{\partial}{\partial_{(y_2-x)_\mu}}\right) \times$$
$$\frac{1}{(y_2-x)^{d-2}}\left[\frac{z}{z^2+(x-y_1)^2}\right]^{(d-2)/2}\left[\frac{z}{z^2+(y_1-y_2)^2}\right]^{(\Delta_1-d/2+1)}\left(\frac{\chi}{\chi-1}\right)^{(d-2)/2} \tag{38}$$

where now

$$\chi = \frac{[(x-y_1)^2+z^2][(y_1-y_2)^2+z^2]}{z^2(y_2-x)^2} \tag{39}$$

However at this point we cannot proceed as we did for a non-conserved current: the Ward identity (33) forbids a non-zero three point function of a conserved current with two scalars of unequal dimension, which means we cannot correct our definition of a bulk scalar by adding higher-dimension primaries. This can also be seen from the results of the previous section, where the higher-dimension primary we had to add in the non-conserved case (32) involved the divergence of the current.[7]

More specifically in the limit of large y_2 the leading term in (34) is

$$\frac{1}{y_2^{2\Delta_1}}\frac{(y_1-x)_\mu}{(y_1-x)^d} \tag{40}$$

Upon smearing $\mathcal{O}(y_1)$ into the bulk this generates some non-causal terms we would like to cancel. We could try to correct our definition of the bulk scalar by adding a smeared $j_\rho\partial^\rho\mathcal{O}(y_1)$, but this won't work since by large-N factorization

$$\langle j_\mu(x)\, j_\rho\partial^\rho\mathcal{O}(y_1)\,\bar{\mathcal{O}}(y_2)\rangle = \langle j_\mu(x)\, j_\rho(y_1)\rangle\,\langle\partial^\rho\mathcal{O}(y_1)\bar{\mathcal{O}}(y_2)\rangle$$

and thus as $y_2 \to \infty$ the leading dependence on y_2 that such a correction would produce is

$$\frac{y_{2\rho}}{y_2^{2\Delta_1+2}} \tag{41}$$

This means there is no way to cancel the unwanted terms. A term like (40) could be canceled in the non-conserved case, by adding a correction of the form $\partial^\rho j_\rho\mathcal{O}(y_1)$; with no derivatives acting on $\mathcal{O}$ the leading $y_2 \to \infty$ dependence would be $1/y_2^{2\Delta_1}$. But for conserved currents

[7] One could try to take a limit where the current is conserved, with $\Delta \to d-1$, but then in (32) we'd have $\alpha \to \infty$ while the divergence of the current goes to zero. It does not seem to make sense to take such a limit at the operator level.

this operator is not available. From this perspective this is the only difference (though a crucial one) between conserved and non-conserved currents.

This failure to restore bulk locality is actually desirable, since as we will show in section 5.4 the resulting non-commutativity is exactly what one needs in order to satisfy the bulk Gauss constraint. But it raises a question, how should we go about determining the appropriate higher-dimension operators to add to our bulk scalar? To find the answer we make a strategic retreat, and study causality for correlators involving the field strength of a massive vector.

Causality for massive vector field strengths
Consider the 3-point function of two scalars and one field strength $F_{\mu\nu} = \partial_\mu j_\nu - \partial_\nu j_\mu$ built from a non-conserved current. From (27) this is

$$< F_{\mu\nu}(x)\mathcal{O}_1(y_1)\mathcal{O}_2(y_2) >\sim \frac{1}{(y_1-y_2)^{\Delta_1+\Delta_2-\Delta+1}(y_1-x)^{\Delta+\Delta_1-\Delta_2+1}(y_2-x)^{\Delta+\Delta_2-\Delta_1+1}}\left[(y_1-x)_\mu(y_2-x)_\nu - \mu\leftrightarrow\nu\right] \tag{42}$$

As shown in section 5.1, when lifted into the bulk this correlator has non-causal singularities which can be canceled by adding higher-dimension operators.

For example, when the higher dimension operator (32) is inserted in place of $\mathcal{O}_1$ in the original 3-point function, the resulting CFT correlator can be computed by large-N factorization as a product of two-point functions. From the current - current correlator[8]

$$< j_\mu(x) j_\nu(0) >= \left(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\right)\frac{1}{(x^2)^\Delta} \tag{43}$$

it follows that

$$< F_{\mu\nu}(x)\partial^\rho j_\rho >= 0 \tag{44}$$

and therefore at leading order in $1/N$

$$< F_{\mu\nu}(x)\,\partial^\rho j_\rho \mathcal{O}_2(y_1)\,\mathcal{O}_2(y_2) >\simeq\, 0 \tag{45}$$

To leading order in $1/N$ the terms that are missing for conserved currents do not contribute in the non-conserved case, at least for a 3-point function involving $F_{\mu\nu}$. Since we know the massive vector can be made local in the bulk, this means the cancellation must come from the $j_\rho\partial^\rho\mathcal{O}_2$ term. To leading order in $1/N$ this term gives

$$< F_{\mu\nu}(x) j_\rho(y_1) >< \partial^\rho \mathcal{O}_2(y_1)\mathcal{O}_2(y_2) > \tag{46}$$

and indeed after some algebra this has the same form as (42), with $\Delta_1 = \Delta + \Delta_2 + 1$ as appropriate for this operator. This result is valid even in the limit $\Delta \to d-1$ where the

[8]Some useful formulas are recorded in section 4.1 of [18].

current is conserved, since no property of the non-conserved current was used (i.e. having a divergence of the current $\partial^\mu j_\mu \neq 0$ played no role).

Lessons for massless vectors

For massive vectors we found that the operator which is absent in the conserved case, namely $\partial^\rho j_\rho \mathcal{O}$, played no role in restoring causality for correlators involving the boundary field strength $F_{\mu\nu}$. Thus we expect that in a three point function of the type $\langle F_{\mu\nu}\mathcal{O}\bar{\mathcal{O}}\rangle$ locality can be respected even for conserved currents, just by adding smeared operators which are scalars but not primary. For example, given the operator (32), we would correct the bulk scalar by just smearing the $j_\rho\partial^\rho\mathcal{O}$ term. As shown below (46) this suffices to restore causality for correlators involving a massless boundary field strength.

So for conserved currents, even though one cannot build a primary scalar out of the available ingredients (the current, other primary scalars, derivative operators), one can still build an operator which can be treated as though it were a primary scalar, at least when inserted in three point functions involving $F_{\mu\nu}$ rather than j_μ. By taking this non-primary operator and smearing it as though it were primary we can cancel the non-local terms in $\langle F_{\mu\nu}\mathcal{O}\bar{\mathcal{O}}\rangle$, just as we did for non-conserved currents. In this way the bulk scalar can be corrected so that it is local with respect to the field strength $F_{\mu\nu}$ on the boundary.[9] This fits with the bulk perspective developed in section 2, where at equal times the bulk scalar commuted with the field strengths $\pi_{\hat{\imath}}$, $F_{\hat{\imath}\hat{\jmath}}$ on the boundary. This requirement is what captures the appropriate notion of micro-causality when there are conserved currents.

Presumably all of the required higher-dimension non-primary scalar operators can be constructed in the $1/N$ expansion. For example, let's look at the next correction to (32), involving an operator of dimension $d+\Delta_2+2$. To determine its form we first build a scalar primary out of a non-conserved current of dimension Δ, then we drop terms involving the divergence of the current and take the limit $\Delta \to d-1$ with $\Delta_1 = \Delta_2$. Denoting the scalar operators $\mathcal{O}$ and $\bar{\mathcal{O}}$, this leads to

$$\alpha(\nabla^2 j_\rho)\partial^\rho\mathcal{O} + \beta j_\rho\partial^\rho\nabla^2\mathcal{O} + \gamma\partial_\delta j_\rho\partial^\rho\partial^\delta\mathcal{O} \tag{47}$$

with

$$\alpha = \frac{1}{2d^2} \qquad \beta = \frac{1}{2(\Delta_1+1)(2\Delta_1+2-d)} \qquad \gamma = -\frac{1}{2d(\Delta_1+1)} \tag{48}$$

Adding these higher-dimension non-primary operators cancels the unwanted non-analyticity and restores locality in correlators involving $F_{\mu\nu}$. This means the corrected bulk scalar will commute at spacelike separation with $F_{\mu\nu}$ on the boundary. It also means that two scalar fields will commute at spacelike separation, even in the presence of a spectator $F_{\mu\nu}$.

[9] We do not expect it to be local with respect to $F_{\mu z} \sim j_\mu$ near the boundary.

5.3 AdS covariance

The procedure we have outlined restores bulk locality, at least in correlators involving $F_{\mu\nu}$, but it seems dangerous. We have added to the original scalar field an operator which is smeared like a primary scalar but is not actually a primary scalar. This means the resulting bulk field will not transform as a bulk scalar under AdS isometries. At first this sounds problematic, but we now show that it's the expected result: in holographic gauge charged scalar fields acquire an anomalous transformation rule under AdS isometries which do not preserve the gauge-fixing condition.

First let's study this from the bulk point of view. We are working in holographic gauge, $A_z = 0$. This completely fixes the gauge, so all our bulk fields are physical. In this gauge a charged scalar $\phi(z,x)$ can be identified with the manifestly gauge-invariant observable

$$\phi_{\text{phys}}(z,x) = e^{i\int dz A_z}\phi(z,x) \tag{49}$$

As such, under an isometry which does not preserve the condition $A_z = 0$ the field will not transform like a scalar: rather a compensating gauge transformation will be required. Indeed under a special conformal transformation

$$\phi'_{\text{phys}}(z',x') = e^{i\lambda(z,x)}\phi_{\text{phys}}(z,x) \tag{50}$$

where $\lambda(z,x)$ is given to first order in the parameter of the special conformal transformation b_μ by [18]

$$\lambda = -\frac{1}{\text{vol}(S^{d-1})}\int d^d x' \theta(\sigma z') 2b\cdot j \tag{51}$$

We will be interested in the boundary behavior

$$\lambda(z,x) \overset{z\to 0}{\to} z^d 2b\cdot j \tag{52}$$

from which we find that to first order in b_μ and as $z\to 0$

$$\phi'_{\text{phys}}(z',x') = \phi_{\text{phys}}(z,x) + 2iz^{\Delta+d} b\cdot j\mathcal{O}(x) \tag{53}$$

Now let's see how the CFT reproduces this behavior. We saw that to leading order in $1/N$ and as $z\to 0$ the correction to the bulk scalar field in the CFT has the form

$$\phi(z,x) = \int K_\Delta(z,x|y)\mathcal{O}(y) + \int K_{\Delta+d}(z,x|y) j^\mu \partial_\mu \mathcal{O}(y) \tag{54}$$

We use the behavior under infinitesimal special conformal transformations acting on y

$$\begin{aligned} d^d y' &= (1+2db\cdot y)d^d y \\ K'_\Delta &= K_\Delta(1+2(\Delta-d)b\cdot y) \\ \mathcal{O}'(y') &= (1-2\Delta b\cdot y) \\ j'^\mu(y') &= (1-2db\cdot y)\frac{\partial y'^\mu}{\partial y^\nu} j^\nu(y) \end{aligned} \tag{55}$$

The right hand side of (54) transforms to

$$\int K_{\Delta}(z,x|y)\mathcal{O}(y) + \int K_{\Delta+d}(z,x|y)j^{\mu}\partial_{\mu}\mathcal{O}(y) - \Delta \int K_{\Delta+d}(z,x|y)2b\cdot j\mathcal{O}(y) \tag{56}$$

The last term as $z \to 0$ behaves like

$$z^{\Delta+d}b\cdot j\mathcal{O}(x) \tag{57}$$

which matches what one expects from the bulk perspective. So the inability to construct a higher-dimension primary scalar from a conserved current in the CFT, translates in a nice way to the anomalous behavior under AdS isometries of a charged scalar field in the bulk.

5.4 Gauss constraint

Although we have been able to correct our definition of a bulk scalar field so as to achieve locality in correlators involving the boundary field strength $F_{\mu\nu}$, correlators involving the conserved current j_{μ} itself will still be non-local. We now show that this was to be expected, since the non-locality which is present is exactly the bulk non-locality required by the Gauss constraint.

We start with the 3-point function (38) of a bulk scalar, a boundary conserved current and an additional boundary scalar

$$< \phi(z,y_1)j_{\mu}(x)\bar{\mathcal{O}}(y_2) >= \frac{1}{2-d}\left(\frac{d}{d(y_1-x)_{\mu}} - \frac{d}{d(y_2-x)_{\mu}}\right)\times \tag{58}$$
$$\left[\frac{z^{\Delta_1}}{(y_2-x)^{d-2}}\left(\frac{1}{z^2+(y_1-y_2)^2}\right)^{\Delta_1-\frac{d-2}{2}}\left(z^2+(y_1-x)^2-\frac{z^2(y_2-x)^2}{z^2+(y_1-y_2)^2}\right)^{\frac{2-d}{2}}\right]$$

We assume the points x and y_2 are spacelike to each other. We compute the commutator of the current and the bulk operator inside the 3-point function as the difference of two $i\epsilon$ prescriptions, one where the time component of x has a $+i\epsilon$ and one where it has a $-i\epsilon$. The only singularities that can contribute to the commutator arise when the derivatives act on the third factor in (58). These derivatives give

$$2\left(z^2+(y_1-x)^2-\frac{z^2(y_2-x)^2}{z^2+(y_1-y_2)^2}\right)^{\frac{-d}{2}}((y_1-x)_{\mu}-\frac{(y_2-x)_{\mu}z^2}{z^2+(y_1-y_2)^2}) \tag{59}$$

Note that

$$z^2+(y_1-x)^2-\frac{z^2(y_2-x)^2}{z^2+(y_1-y_2)^2} = \frac{(y_1-y_2)^2}{z^2+(y_1-y_2)^2}\left(x-y_1-\frac{z^2(y_1-y_2)}{(y_1-y_2)^2}\right)^2 \tag{60}$$

and that the delta function in $d-1$ dimensions can be written as

$$\delta(\vec{x}) = \lim_{\epsilon\to 0} \frac{\Gamma(d/2)}{\pi^{d/2}} \frac{\epsilon}{(|\vec{x}|^2+\epsilon^2)^{d/2}}. \tag{61}$$

In the simple case (one can generalize this) that the time components of x, y_1, y_2 are equal, then taking the difference of (59) with $+i\epsilon$ and $-i\epsilon$ prescriptions gives zero for $\mu \neq 0$, while for $\mu = 0$ we get

$$\frac{2i\pi^{d/2}}{\Gamma(d/2)} \left[\frac{(y_1-y_2)^2}{z^2+(y_1-y_2)^2}\right]^{-\frac{d}{2}+1} \delta(x - y_1 - \frac{z^2(y_1-y_2)}{(y_1-y_2)^2}) \tag{62}$$

To find the commutator with the charge operator Q we restore the first two factors in (58) and integrate over the spatial coordinates $\vec{x}$. This gives

$$< [\phi(y_1,z), Q]\bar{\mathcal{O}}(y_2) > \sim \left[\frac{z}{z^2+(y_1-y_2)^2}\right]^{\Delta_1} \sim < \phi(z,y_1)\bar{\mathcal{O}}(y_2) > \tag{63}$$

which is the expected commutator of the charge operator with a charged scalar field. This shows that the bulk Gauss constraint is obeyed, at least when the lowest-order smearing function is used. It would be interesting to show that (63) continues to hold when higher-dimension operators are added to the definition of the bulk scalar field.

5.5 Scalar commutator

Adding a higher dimension non-primary operator to our definition of a bulk scalar field had some desirable properties: it made correlators with $F_{\mu\nu}$ local, and it gave the scalar field the correct behavior under AdS isometries. However one might wonder: does the resulting scalar field commute with its complex conjugate at bulk spacelike separation? From the bulk perspective developed in section 2 we would expect this to happen even in the presence of interactions. Here we give some evidence for this from the CFT point of view.

As shown in section 5.2, our bulk scalars still commute inside a 3-point function with a boundary field strength $F_{\mu\nu}$, so let's examine what happens in a 3-point function with a gauge field. This was given in (34). The leading $y_2 \to \infty$ singularity of this expression cannot be canceled, but in order to isolate the commutator between the bulk scalar and the boundary scalar, we instead look at the limit $x \to \infty$. In this limit

$$< j_\mu(x)\mathcal{O}(y_1)\bar{\mathcal{O}}(y_2) > \overset{x\to\infty}{\sim} \frac{1}{x^{2d-2}}(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2})\partial^\nu_{y_1} \frac{1}{(y_1-y_2)^{2\Delta_1-d}} \tag{64}$$

Smearing the first scalar operator into the bulk we find

$$< j_\mu(x)\phi(z,y_1)\bar{\mathcal{O}}(y_2) > \overset{x\to\infty}{\sim} \frac{1}{x^{2d-2}}(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}) \frac{z^{\Delta_1}(y_1-y_2)^\nu}{(y_1-y_2)^d((y_1-y_2)^2+z^2)^{\Delta_1-d+1}} \tag{65}$$

The leading singularity of this expression as $(y_1 - y_2)^2 \to 0$ is

$$\frac{1}{x^{2d-2}}(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2})\frac{(y_1 - y_2)^\nu z^{2d-2-\Delta_1}}{(y_1 - y_2)^d} \tag{66}$$

Now let's examine the leading behavior as $x \to \infty$ when we insert a non-primary operator of the type we talked about. We consider operators of the form[10]

$$j_\nu \partial^\nu (\nabla^2)^n \mathcal{O}(y_1) \tag{67}$$

Inserting this operator instead of $\mathcal{O}(y_1)$ in the 3-point function gives the leading $x \to \infty$ behavior of the CFT correlator

$$\frac{1}{x^{2d-2}}(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2})\partial^\nu_{y_1}\frac{1}{(y_1 - y_2)^{2\Delta_1+2n}} \tag{68}$$

Smearing y_1 into the bulk with a scalar smearing function of dimension $\Delta_1 + d + 2n$ gives the large-x behavior

$$\frac{1}{x^{2d-2}}(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2})\partial^\nu_{y_1}\frac{z^{\Delta_1+d+2n}}{(y_1 - y_2)^{2\Delta_1+2n}}F(\Delta_1+n, \Delta_1+n-\frac{d}{2}+1, \Delta_1+\frac{d}{2}+2n+1, -\frac{z^2}{(y_1 - y_2)^2}) \tag{69}$$

Using the identity

$$x\frac{d}{dx}F(a, b, c, x) = a(F(a+1, b, c, x) - F(a, b, c, x)) \tag{70}$$

this can be rewritten as

$$\frac{1}{x^{2d-2}}(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2})\frac{(y_1 - y_2)_\mu z^{\Delta_1+d+2n}}{(y_1 - y_2)^{2\Delta_1+2n+2}}F(\Delta_1+n+1, \Delta_1+n-\frac{d}{2}+1, \Delta_1+\frac{d}{2}+2n+1, -\frac{z^2}{(y_1 - y_2)^2}) \tag{71}$$

The leading singularity of this expression as $(y_1 - y_2)^2 \to 0$ matches (66). With enough such higher dimension operators one can cancel the non-analyticity to any order in $\frac{(y_1-y_2)^2}{z^2}$. (Note that in this limit the problematic regime is $-1 < \frac{(y_1-y_2)^2}{z^2} < 0$.) While this is not a complete proof that adding higher dimension non-primary operators makes the bulk scalar field commute with itself inside a 3-point function with a gauge field, it is a strong indication of it.

[10]These are not the only corrections, but these are the operators which contribute to the leading behavior as $x \to \infty$.

6 CFT construction: bulk vectors

In this section we look at the correction, from the CFT perspective, that one needs to add to lift a boundary current to an interacting local vector field in the bulk. We first consider non-conserved currents in the CFT, dual to massive vectors in the bulk, then treat conserved currents.

6.1 Bulk massive vectors

We begin by computing the three-point function of a massive vector in the bulk with two scalars on the boundary. Then we look at the higher-dimension operators we need to add to cancel the unwanted singularities in this expression.

Our starting point is the three-point function of a boundary current of dimension Δ with two scalar operators of dimension Δ_1 in a d-dimensional CFT.

$$\begin{aligned} &< j_\mu(x)\mathcal{O}(y_1)\bar{\mathcal{O}}(y_2) > \\ &= \frac{1}{(y_1-y_2)^{2\Delta_1-\Delta+1}(y_1-x)^{\Delta-1}(y_2-x)^{\Delta-1}}\left(\frac{(y_1-x)_\mu}{(y_1-x)^2}-\frac{(y_2-x)_\mu}{(y_2-x)^2}\right) \end{aligned} \tag{72}$$

For $A_z^0(x)=\frac{1}{d-\Delta-1}\partial^\mu j_\mu(x)$ the correlator is

$$\begin{aligned} < A_z^0(x)\mathcal{O}(y_1)\bar{\mathcal{O}}(y_2) > \; &= \; -\frac{1}{(y_1-y_2)^{2\Delta_1-\Delta+1}(y_1-x)^{\Delta+1}(y_2-x)^{\Delta-1}} \\ &+ \; \frac{1}{(y_1-y_2)^{2\Delta_1-\Delta+1}(y_1-x)^{\Delta-1}(y_2-x)^{\Delta+1}} \end{aligned}$$

The two terms on the right are the three-point functions of scalar operators of dimensions $(\Delta,\Delta_1+1,\Delta_1)$ and $(\Delta,\Delta_1,\Delta_1+1)$ respectively.[11]

In [18] the smearing function for uplifting a non-conserved primary current of dimension Δ to a massive vector field in the bulk was found to be

$$zA_\mu(z,x)=\int K_\Delta(z,x,x')j_\mu(x')+\frac{z}{2(\Delta-d/2+1)}\int K_{\Delta+1}(z,x,x')\partial_\mu A_z^0(x') \tag{73}$$

for the μ components and

$$A_z(z,x)=\int K_\Delta(z,x,x')A_z^0(x') \tag{74}$$

[11] This is a simplification that only occurs when the two scalar operators are of the same dimension, but since this is the interesting case for a conserved current we only treat this case.

for the z component. Here $A^0_z(x) = \frac{1}{d-\Delta-1}\partial^\mu j_\mu(x)$ and $K_\Delta(z,x,x')$ is the scalar smearing function appropriate for a scalar primary of dimension Δ. Since A_z is smeared into the bulk with a scalar smearing function, we can borrow our scalar result (13), (20) to get

$$
\begin{aligned}
&< A_z(x,z)\mathcal{O}(y_1)\bar{\mathcal{O}}(y_2) > \\
&= \frac{-1}{(y_1-y_2)^{2\Delta_1+1-\Delta}}\left(\frac{\chi}{\chi-1}\right)^{\frac{\Delta+1}{2}} F(\frac{\Delta+1}{2},\frac{\Delta-d+3}{2},\Delta-\frac{d}{2}+1,\frac{1}{1-\chi}) \\
&\left[\left(\frac{z}{z^2+(x-y_1)^2}\right)^{\frac{\Delta+1}{2}}\left(\frac{z}{z^2+(x-y_2)^2}\right)^{\frac{\Delta-1}{2}} - \left(\frac{z}{z^2+(x-y_1)^2}\right)^{\frac{\Delta-1}{2}}\left(\frac{z}{z^2+(x-y_2)^2}\right)^{\frac{\Delta+1}{2}}\right]
\end{aligned}
$$

where

$$\chi = \frac{[(x-y_1)^2+z^2][(x-y_2)^2+z^2]}{z^2(y_1-y_2)^2} \tag{75}$$

is invariant under conformal transformations. After some algebra and using

$$F(a,b,c,x) = (1-x)^{c-a-b}F(c-a,c-b,c,x) \tag{76}$$

this can be written as the z component of the quantity

$$
\begin{aligned}
&< A_M(x,z)\mathcal{O}(y_1)\bar{\mathcal{O}}(y_2) > \qquad (77)\\
&\equiv \frac{-1}{2(y_1-y_2)^{2\Delta_1}(\chi-1)^{\frac{\Delta-1}{2}}}F(\frac{\Delta-1}{2},\frac{\Delta-d+1}{2},\Delta-\frac{d}{2}+1,\frac{1}{1-\chi})\partial^x_M \ln\frac{(x-y_1)^2+z^2}{(x-y_2)^2+z^2}
\end{aligned}
$$

where M is a vector index in the bulk.

Although we've only calculated the z component, this must be the complete result, since (77) has the correct behavior under conformal transformations to represent the three-point function of a bulk massive vector with two boundary scalars. But as a check of this result, and to develop some formulas that will be useful in the sequel, we now show that (77) gives the correct $y_2 \to \infty$ asymptotic behavior for the μ components of the bulk vector.

The leading behavior of (72) as $y_2 \to \infty$ is

$$< j_\mu(x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) > \sim \frac{1}{y_2^{2\Delta_1}}\frac{1}{1-\Delta}\partial_\mu\frac{1}{x^{\Delta-1}} \tag{78}$$

We will also need the leading behavior

$$< A^0_z(x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) > \sim \frac{1}{y_2^{2\Delta_1}}\frac{1}{x^{\Delta+1}} \tag{79}$$

Using the massive vector smearing function (73) one finds in the large y_2 limit

$$< zA_\mu(z,x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) > \sim \frac{1}{y_2^{2\Delta_1}}\frac{\Gamma(\Delta-d/2+1)}{\pi^{d/2}\Gamma(\Delta-d+1)}\partial_\mu\left(\frac{I_1}{1-\Delta}+\frac{zI_2}{2(\Delta-d+1)}\right) \tag{80}$$

where

$$\begin{aligned} I_1 &= \int_{t'^2+y^2\le z^2} d^{d-1}y dt' \left(\frac{z^2-t'^2-y^2}{z}\right)^{\Delta-d} \frac{1}{((\vec{x}+i\vec{y})^2-(t+t')^2)^{\frac{\Delta-1}{2}}} \\ I_2 &= \int_{t'^2+y^2\le z^2} d^{d-1}y dt' \left(\frac{z^2-t'^2-y^2}{z}\right)^{\Delta-d+1} \frac{1}{((\vec{x}+i\vec{y})^2-(t+t')^2)^{\frac{\Delta+1}{2}}} \end{aligned}$$

These integrals give

$$\begin{aligned} I_1 &= \frac{\mathrm{vol}(S^{d-2})\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2})\Gamma(\Delta-d+1)}{2\Gamma(\Delta-\frac{d}{2}+1)} \frac{z^\Delta}{x^{\Delta-1}} F(\frac{\Delta-1}{2},\frac{\Delta-d+1}{2},\Delta-\frac{d}{2}+1,-\frac{z^2}{x^2}) \\ I_2 &= \frac{\mathrm{vol}(S^{d-2})\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2})\Gamma(\Delta-d+2)}{2\Gamma(\Delta-\frac{d}{2}+2)} \frac{z^{\Delta+1}}{x^{\Delta+1}} F(\frac{\Delta+1}{2},\frac{\Delta-d+3}{2},\Delta-\frac{d}{2}+2,-\frac{z^2}{x^2}) \end{aligned} \tag{81}$$

Putting this all together we find

$$< zA_\mu(z,x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) >\sim \frac{1}{y_2^{2\Delta_1}}\frac{1}{1-\Delta}\partial_\mu\left[\frac{z^\Delta}{x^{\Delta-1}}F(\frac{\Delta-1}{2},\frac{\Delta-d+3}{2},\Delta-\frac{d}{2}+1,-\frac{z^2}{x^2})\right] \tag{82}$$

Using the hypergeometric identity

$$x\frac{d}{dx}F(a,b,c,x) = a(F(a+1,b,c,x)-F(a,b,c,x)) \tag{83}$$

this can be written as

$$< zA_\mu(z,x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) >\sim \frac{1}{y_2^{2\Delta_1}}\frac{x_\mu z^\Delta}{x^{\Delta+1}}F(\frac{\Delta+1}{2},\frac{\Delta-d+3}{2},\Delta-\frac{d}{2}+1,-\frac{z^2}{x^2}) \tag{84}$$

Finally using the identity

$$F(a,b,c,x) = (1-x)^{c-a-b}F(c-a,c-b,c,x) \tag{85}$$

we find agreement with the $y_2 \to \infty$ limit of (77). This shows that (77) has the correct asymptotic behavior for the μ components of the bulk vector.

For later use we record the 3-point function of a massive vector field strength with two boundary scalars. It follows from (77) that

$$\begin{aligned} &< F_{MN}(x,z)\mathcal{O}(y_1)\bar{\mathcal{O}}(y_2) > \\ &= \frac{\frac{\Delta-1}{4}}{(y_1-y_2)^{2\Delta_1}}\frac{1}{(\chi-1)^{\frac{\Delta+1}{2}}}F(\frac{\Delta+1}{2},\frac{\Delta-d+1}{2},\Delta-\frac{d}{2}+1,\frac{1}{1-\chi}) \\ &\times\left[\partial^x_M\chi\partial^x_N\ln\frac{(x-y_1)^2+z^2}{(x-y_2)^2+z^2} - M\leftrightarrow N\right] \end{aligned} \tag{86}$$

Now let's look at the singularity structure of these correlators, and see if there are higher-dimension vector operators we can add to cancel any unwanted singularities. As in the scalar case [13] a non-zero commutator is generated between the bulk field and the boundary operators in the region $0 < \chi < 1$, corresponding to bulk spacelike separation. Near $\chi = 1$ the 3-point function (77) has an expansion for even d of the form (we present only the non-analytic terms)

$$\sum_{k=0}^{d/2-1} b_{k\Delta}(1-\chi)^{-\frac{d}{2}+k} + \ln(\chi-1)\sum_{k=0}^{\infty} a_{k\Delta}(1-\chi)^k \tag{87}$$

while for odd d the expansion has the form

$$\frac{1}{(\chi-1)^{\frac{d}{2}}}\sum_{k=0}^{\infty} a_{k\Delta}(1-\chi)^k \tag{88}$$

For any non-conserved primary current the expansion has the same form, just with different coefficients $a_{k\Delta}$ and $b_{k\Delta}$. For $0 < \chi < 1$ this gives a non-zero commutator which is a power series in $(1-\chi)$. Thus if we re-define our bulk massive vector field to include a sum of smeared non-conserved primary currents with arbitrarily high dimension, we can cancel the commutator to whatever order in $(1-\chi)$ we choose. In this way we can make the bulk massive vector obey micro-causality to an arbitrarily good approximation.

At leading order in $1/N$ the higher dimension non-conserved currents we add are double-trace operators built from the two scalars appearing in the 3-point function and their derivatives. For instance the lowest-dimension primary current built in this way is

$$\bar{\mathcal{O}}\partial_\mu\mathcal{O} - \mathcal{O}\partial_\mu\bar{\mathcal{O}} \tag{89}$$

In the large N limit this operator has dimension $2\Delta_1 + 1$. The next operator one can write is

$$\alpha(\nabla^2\bar{\mathcal{O}})\partial_\mu\mathcal{O} + \beta\bar{\mathcal{O}}\partial_\mu\nabla^2\mathcal{O} + \gamma(\partial^\nu\bar{\mathcal{O}})\partial_\mu\partial_\nu\mathcal{O} - (\mathcal{O}\leftrightarrow\bar{\mathcal{O}}) \tag{90}$$

This will be a primary current of dimension $2\Delta_1 + 3$ if

$$\begin{aligned}
\alpha &= \frac{1}{4\Delta_1^2(\Delta_1+1)(d-2\Delta_1-2)} \\
\beta &= \frac{1}{4\Delta_1(\Delta_1+1)(d-2\Delta_1-2)} \\
\gamma &= \frac{1}{4\Delta_1(\Delta_1+1)}
\end{aligned} \tag{91}$$

A similar construction can be carried out at leading order in the $1/N$ expansion to build primary non-conserved currents of dimension $2\Delta_1 + 1 + 2n$ for any n.

6.2 Bulk gauge fields

Finally we turn to massless gauge fields in the bulk, where the smearing function in holographic gauge is [18]

$$zA_\mu(t,\vec{x},z) = \frac{1}{\mathrm{vol}(S^{d-1})} \int\limits_{t'^2+|\vec{y}'|^2=z^2} dt' d^{d-1}y'\, j_\mu(t+t',\vec{x}+i\vec{y}') \tag{92}$$

We wish to determine the higher-dimension operators which are necessary to achieve bulk locality. We first discuss correlators involving the field strength, then consider the gauge field itself.

The three-point function of a bulk field strength F_{MN} with two boundary scalars can be obtained from the 3-point function of a massive field strength with two scalars by analytically continuing $\Delta \to d-1$. From (86) this gives

$$< F_{MN}(x,z)\mathcal{O}(y_1)\bar{\mathcal{O}}(y_2) >= \frac{\frac{d-2}{4}}{(y_1-y_2)^{2\Delta_1}} \frac{1}{(\chi-1)^{\frac{d}{2}}} \left[\partial^x_M \chi \partial^x_N \ln\frac{(x-y_1)^2+z^2}{(x-y_2)^2+z^2} - M \leftrightarrow N\right] \tag{93}$$

This has the same singularity structure as the 3-point function for a massive field strength, so it can be made local in the bulk in exactly the same way, by adding appropriate smeared higher-dimension field strengths to our definition of the bulk F_{MN}. [The justification for this analytic continuation is somewhat subtle, since the massive vector smearing function (73), (74) does not smoothly go over to the massless result (92). The first term in (73), after integrating against a CFT correlator, can be analytically continued to $\Delta = d-1$ to get the same result one would obtain from (92).[12] The second term in (73), in the limit $\Delta = d-1$, can be eliminated by a gauge transformation with gauge parameter

$$\lambda = \frac{\Gamma(d/2)}{2\pi^{d/2}} \int\limits_{t'^2+|\vec{y}'|^2<z^2} dt' d^{d-1}y'\, A^0_z(t',\vec{y}') \tag{94}$$

This gauge transformation also has the effect of setting A_z in (74) to zero, i.e. it's exactly what's needed to impose holographic gauge.]

Now consider correlators involving the bulk gauge field itself. For simplicity we work in the limit $y_2 \to \infty$. In this limit the CFT 3-point function can be obtained from (78) by setting $\Delta = d-1$. Up to an overall constant this gives

$$< j_\mu(x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) > \sim \frac{1}{y_2^{2\Delta_1}} \partial_\mu \frac{1}{x^{d-2}}. \tag{95}$$

[12] As an example of this sort of continuation, up to an overall normalization the result I_1 for a massive vector (81) can be analytically continued to $\Delta = d-1$ to reproduce the massless vector result (98) obtained below.

Smearing the current into the bulk using (92) gives

$$< zA_\mu(z,x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) > \sim \frac{1}{y_2^{2\Delta_1}}\partial_\mu h(z,x) \tag{96}$$

where

$$h(z,x) = \frac{1}{\mathrm{vol}(S^{d-1})}\int_{t'^2+|\vec{y}|^2=z^2} d^{d-1}y dt' \frac{1}{((\vec{x}+i\vec{y})^2-(t+t')^2)^{\frac{d-2}{2}}} \tag{97}$$

Doing the integral in the same way as before we find

$$h(z,x) = \frac{z^{d-1}}{x^{d-2}} \tag{98}$$

Thus as $y_2 \to \infty$ we have the asymptotic behavior

$$\begin{aligned} &< F_{\mu\nu}(x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) > \sim 0 \\ &< F_{\mu z}(x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) > \sim \frac{x_\mu z^{d-3}}{y_2^{2\Delta_1}x^d} \end{aligned} \tag{99}$$

This agrees with the $\mathcal{O}(1/y^{2\Delta_1})$ asymptotic behavior of (93) for $y_1 = 0$. Since (93) is AdS covariant and has the correct asymptotic behavior, it must be the exact result. This is another way of seeing that the analytic continuation we made to obtain (93) is legitimate.

Now let's see what we can achieve by adding higher-dimension operators to our definition of a bulk gauge field. We already saw that for correlators involving the field strength we could achieve bulk locality by adding suitable higher-dimension currents to our definition of a bulk vector; the massive and massless cases proceeded in an identical manner. However the singularity in a gauge field correlator is not the same as for a massive vector. This can be seen in the results above. In the limit $y_2 \to \infty$, the gauge field correlator (96) has singular behavior near $x = 0$, namely

$$< zA_\mu(z,x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) > \sim \frac{1}{y_2^{2\Delta_1}} z^{d-1}\frac{x_\mu}{x^d} \tag{100}$$

In contrast, for $y_2 \to \infty$ and $x \sim 0$ the massive vector correlator (84) behaves as

$$< zA_\mu(z,x)\mathcal{O}(0)\bar{\mathcal{O}}(y_2) > \sim \frac{1}{y_2^{2\Delta_1}} z^{d-3}\frac{x_\mu}{x^{d-2}} \tag{101}$$

(The difference can be traced back to a cancellation between I_1 and I_2 in (80).) This means that for a gauge field one cannot hope to cancel the boundary light-cone singularity which is present in (100) by adding massive vector fields.

This is surprising because, from the bulk point of view, we'd expect a gauge field in the bulk to commute at spacelike separation with a charged scalar on the boundary.[13] Fortunately the requirement of having F_{MN} be local with scalars on the boundary, together with

[13]Note the step functions in (5), reflecting the fact that the Wilson lines extend towards $z = 0$.

the gauge condition $A_z = 0$, is enough to uniquely define the bulk gauge field in terms of smeared CFT operators. Suppose we add a higher-dimension (hence non-conserved) primary current j^i_μ to our definition of a bulk gauge field, with a coefficient chosen to make F_{MN} local. Acting with the smearing function (73), (74) it would seem we generate a non-zero A_z in the bulk. We can restore holographic gauge by making a gauge transformation with parameter

$$\lambda = \int_0^z dz' \int dx' K_{\Delta_i}(z', x|x') A_z^{i,0}(x') \tag{102}$$

Here Δ_i is the dimension of the current and $A_z^{i,0} = \frac{1}{d-\Delta_i-1}\partial^\mu j^i_\mu$. So in order to build a bulk massless vector field from boundary operators, where the three-point function of F_{MN} is local, one has to add to the free-field definition of A_μ an infinite tower of contributions coming from primary currents of increasing dimensions built from $\mathcal{O}, \bar{\mathcal{O}}$ and their derivatives, of the form

$$\sum_i a_i \left[\frac{1}{z}\int K_{\Delta_i} j^i_\mu + \partial_\mu \left(\frac{1}{2(\Delta_i - d/2+1)}\int K_{\Delta_i+1} A_z^{i,0} - \int_0^z dz' \int K_{\Delta_i} A_z^{i,0}\right)\right] \tag{103}$$

The same structure appeared in (12). Overall the correction (103) should make A_μ commute with boundary scalars at spacelike separation.

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