Hybrid r-vacua in N=2 supersymmetric QCD: Universal condensate formula

M. Shifman and A. Yung

Phys. Rev. D **87**, 085044 — Published 30 April 2013
DOI: 10.1103/PhysRevD.87.085044

FTPI-MINN-13/07, UMN-TH-3139/13

Hybrid r-Vacua in $\mathcal{N} = 2$ Supersymmetric QCD: Universal Condensate Formula

M. Shifman[a] **and A. Yung**[a,b]

[a] *William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA*
[b] *Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia*

Abstract

We derive an exact unified formula for all condensates (quark and monopole) in the hybrid r vacua in $\mathcal{N} = 2$ supersymmetric QCD slightly deformed by a $\mu\mathcal{A}^2$ term. The gauge group is assumed to be U(N) and the number of the quark flavors N_f subject to the condition $N < N_f < 2N$. In the r vacua r quarks and $N - r - 1$ monopoles from non-overlapping subgroups of U(N) develop vacuum expectation values (VEVs) ($r < N$). We then briefly review possible dynamical regimes (confinement, screening, and "instead of confinement") in the hybrid r vacua in μ-deformed $\mathcal{N} = 2$ SQCD (the small-μ limit).

1 Introduction

The main goal of this paper is to derive a unified formula for the quark and monopole vacuum condensates in an arbitrary r vacuum in $\mathcal{N}=2$ supersymmetric QCD (SQCD) in terms of the roots of the Seiberg-Witten curve [1]. Following Seiberg and Witten we deform $\mathcal{N}=2$ SQCD by a small mass term μ for the adjoint field. We will show that all the condensates reduce to effective parameters ξ_P,

$$\xi_P = -2\sqrt{2}\,\mu\,\sqrt{(e_P - e_N^+)(e_P - e_N^-)} \tag{1.1}$$

where the subscript $P = 1, ..., N-1$ marks the appropriate condensates (quark or monopole), e_1, e_2, ..., e_{N-1} are the double roots of the Seiberg-Witten curve corresponding to the quark and monopole condensation, while $e_N^{\pm}$ are two unpaired roots present in any $r < N$ vacuum in the case of the $\mu\,\mathrm{Tr}\,\mathcal{A}^2$ perturbation. If P lies in the interval $[1, r]$, Eq. (1.1) describes the quark vacuum expectation values (VEVs) [2], while for $r+1 \le P \le N-1$ it gives the monopole VEVs.

For generic values of the quark masses the theories we discuss support BPS-saturated non-Abelian magnetic strings [3, 4, 5, 6]. These strings confine monopoles. The tensions of these strings are [7, 8]

$$T_P = 2\pi|\xi_P|\,, \qquad P = 1, ..., r\,. \tag{1.2}$$

For $r+1 \le P \le N-1$ the same expression gives the tensions of the Abelian electric strings, which confine quarks. The value of the P-th condensate is $\xi_P/2$ (see below for a more precise definition).

Let us briefly outline our basic model (a more detailed description and all relevant notation can be found in our previous original publications [5, 8] and the review papers [7]).

The gauge group of $\mathcal{N}=2$ SQCD under consideration is U(N). We introduce N_f quark flavors ($N < N_f < 2N$) endowed with mass terms and then perturb $\mathcal{N}=2$ SQCD by a small mass term $\mu\mathcal{A}^2$ for the adjoint matter (part of the $\mathcal{N}=2$ gauge supermultiplet).

At generic quark masses this theory has a number of isolated vacua where r flavors of (s)quarks condense, $r \le N$ (the so called r vacua). The $r = N$ vacuum, with the maximal possible number of condensed quarks, was studied more than others (for a review see [7]). Non-Abelian flux tubes (strings) confining monopoles were shown to exist [3, 4, 5, 6] in this vacuum, see [9, 10, 7] for extensive reviews. Massless r-vacua with $r < N$ were studied in [11, 12] in the SU(N) version of the theory.[1]

Extensions to U(N) were discussed recently for the $r > N_f/2$ and, in particular, $r = N-1$ and $r = N$ cases [13, 14, 2]. Confinement of monopoles at weak coupling was demonstrated to survive in the strong coupling regime at small values of the quark VEVs given by $\xi/2 \sim \mu m$ where m is a typical quark mass. The latter was described in terms of the so-called r-duality

[1]If quark mass terms vanish certain r vacua coalesce, and the Higgs branches develop from the common roots. The $r < N$ vacua correspond to roots of the nonbaryonic Higgs branches, while the $r = N$ vacuum to a root of the baryonic Higgs branch in the SU(N) theory [11]. We consider nonvanishing, nondegenerative quark masses.

and was found to be an "instead-of-confinement" phase: the screened quarks decay into monopole-antimonopole pairs with the monopoles confined by non-Abelian strings. One of the results of [2] was the expression for the quark condensates in the low-energy theory in terms of the roots of the Seiberg-Witten curve, see Eq. (1.1). In this paper we continue this line of research and consider the monopole $r = 0$ as well as hybrid r vacua with r quarks and $(N - r - 1)$ monopoles (from the orthogonal subgroups of $\mathrm{U}(N)$) condensing.[2] Equation (1.1) proves to be valid for all condensates in all vacua. Although our derivation will be carried out in particular examples the assertion is universal.

The paper is organized as follows. In Sec. 2 we discuss the r-vacuum structure and review Eq. (1.1) for $r > N_f/2$. In Sec. 3 we present a detailed analysis of the monopole ($r = 0$) vacuum and derive Eq. (1.1) in this case. As a byproduct we observe that Eq. (1.1) reproduces the famous sine formula for the string tensions [15] in the limit of large quark masses, when the theory under consideration reduces to pure gauge theory.[3] Section 4 is devoted to the hybrid r-vacua with $r < N_f/2$. Equation (1.1) for the quark and monopole condensates is derived in certain examples. Finally, Section 5.1 presents an overall picture of confinement and screening in the hybrid r vacua. In Sec. 5 we also summarize various phases exhibiting themselves in different r vacua. Appendix contains details pertinent to the VEVs calculation in a hybrid vacuum.

2 μ-Deformed SQCD: vacuum structure

2.1 The model

In the absence of deformation the model under consideration is $\mathcal{N} = 2$ SQCD with N_f massive quark hypermultiplets. We assume that $N_f > N$ but $N_f < 2N$ where N refers to the gauge group, $\mathrm{U}(N)$. The latter inequality ensures our theory to be asymptotically free. In addition, we will introduce a small mass term $\mu\mathcal{A}^2$ for the adjoint matter breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$.

The field content is as follows. In addition to the $\mathrm{SU}(N)$ and $\mathrm{U}(1)$ $\mathcal{N} = 2$ gauge supermultiplets we have N_f quark multiplets consisting of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and their fermion superpartners — all in the fundamental representation of the $\mathrm{SU}(N)$ gauge group. Here $k = 1, ..., N$ is the color index while A is the flavor index, $A = 1, ..., N_f$. We will treat q^{kA} and $\tilde{q}_{Ak}$ as rectangular matrices with N rows and N_f columns.

The superpotential of the undeformed theory is

$$\mathcal{W}_{\mathcal{N}=2} = \sqrt{2}\sum_{A=1}^{N_f}\left(\frac{1}{2}\,\tilde{q}_A\mathcal{A}q^A + \tilde{q}_A\mathcal{A}^a\,T^a q^A + m_A\,\tilde{q}_A q^A\right)\,, \tag{2.1}$$

[2]A certain aspect of the large-μ limit was not quite adequately treated in [2]. This will be corrected in a separate publication. In the present paper we limit ourselves to the small-μ limit.

[3]For a related discussion see [16].

where $\mathcal{A}$ and $\mathcal{A}^a$ are chiral $\mathcal{N}=1$ superfields, the $\mathcal{N}=2$ superpartners of the gauge bosons, while m_A are the quark mass terms. Then we add a single trace deformation

$$\mathcal{W}_{\rm br} = \mu \,{\rm Tr}\, \Phi^2, \tag{2.2}$$

where

$$\Phi = \frac{1}{2}\,\mathcal{A} + T^a\,\mathcal{A}^a\,, \tag{2.3}$$

and T^a stand for the SU(N) generators. Generally speaking, (2.2) breaks[4] $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$. We assume the deformation (2.2) to be weak,

$$|\mu| \ll \Lambda\,, \tag{2.4}$$

where Λ is the scale of the $\mathcal{N}=2$ theory. Thus, we consider the theory close to its $\mathcal{N}=2$ limit.

2.2 Vacua

The number of isolated $r=N$ vacua is

$$\mathcal{N}_{r=N} = C^N_{N_f} = \frac{N_f!}{N!(N_f-N)!}\,. \tag{2.5}$$

This is the maximal number of quark fields that can develop VEVs, see [7]. All gauge bosons are completely Higgsed and the theory is in the color-flavor locking phase (assuming quark masses to be close to each other). The quark VEVs are determined by ξ_P's ($P=1,...,N$) of the order of μm_P. For large values of ξ the theory is at weak coupling and can be studied semiclassically. In particular, non-Abelian strings are known to exist which confine monopoles [3, 4, 5, 6].

If we reduce ξ the theory undergoes a crossover transition from weak to strong coupling regime, described in terms of a weakly coupled infrared-free dual theory [13] with the U($\tilde N$) gauge group and N_f light quark-like dyon flavors, $\tilde N = N_f - N$. The dyon condensation leads to confinement of monopoles too. The quarks and gauge bosons of the original theory are in the “instead-of-confinement” phase [13, 2].

The number of the r vacua[5] with $r<N$ is [12]

$$\mathcal{N}_{r<N} = \sum_{r=0}^{N-1} (N-r)\, C^r_{N_f} = \sum_{r=0}^{N-1} (N-r)\,\frac{N_f!}{r!(N_f-r)!}\,, \tag{2.6}$$

representing the number of choices one can pick up r condensing quarks out of N_f quarks times the Witten index in the classically unbroken SU($N-r$) pure gauge theory.

[4]For small μ and equal quark masses (2.2) reduces to the Fayet-Iliopoulos F-term [17] which does not break $\mathcal{N}=2$ supersymmetry, see [18, 19, 8].

[5]Our definition of r refers to the large quark mass domain. In fact, effectively r depends on the quark masses, see [20].

Consider a particular vacuum in which the first r quarks develop VEVs. We denote it as $(1, ..., r)$. Quasiclassically, at large masses, the adjoint scalar VEVs are

$$\langle \Phi \rangle \approx -\frac{1}{\sqrt{2}} \,\mathrm{diag}\, [m_1, ..., m_r, 0, ..., 0] \,, \tag{2.7}$$

where the last $(N - r)$ entries classically vanish. In quantum theory the vanishing entries become of the order of Λ, generally speaking. The classically unbroken U$(N - r)$ gauge sector gets Higgsed through the Seiberg–Witten mechanism [1], first down to U$(1)^{N-r}$ and then almost completely by condensation of $(N - r - 1)$ monopoles. A single U(1) factor remains unbroken, as the monopoles are charged with respect to the Cartan generators of the SU$(N - r)$ group.

The presence of the unbroken U$(1)^{\mathrm{unbr}}$ symmetry makes the $r < N$ vacua qualitatively different from the $r = N$ vacuum: the latter has no massless gauge bosons. According to [21], these sets of vacua belong to two different "branches."

The low-energy theory in the r vacuum has the gauge group

$$\mathrm{U}(r) \times \mathrm{U}(1)^{N-r} \,, \tag{2.8}$$

with N_f quark flavors charged under the U(r) factor and $(N - r - 1)$ monopoles charged under the U(1) factors.

2.3 $r > N_f/2$

For $r > N_f/2$ and large ξ the SU(r) non-Abelian quark sector is at weak coupling since it is asymptotically free.[6] The action of this theory is presented in [2] for a particular example, the $r = N - 1$ vacuum. The quark condensates can be read-off from the superpotentials (2.1) and (2.2) using (2.7). They are

$$\begin{aligned} \langle q^{kA} \rangle &= \langle \bar{\tilde{q}}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \sqrt{\xi_r} & 0 & \ldots & 0 \end{pmatrix}, \\ k &= 1, ..., r \,, \qquad A = 1, ..., N_f \,. \end{aligned} \tag{2.9}$$

The first r parameters ξ in the quasiclassical approximation are

$$\xi_P \approx 2\, \mu m_P, \qquad P = 1, ..., r. \tag{2.10}$$

In quantum theory the parameters ξ_P determining the quark condensates are connected with the roots of the Seiberg-Witten curve [8, 14, 2] which in the theory at hand takes the form [11]

$$y^2 = \prod_{P=1}^{N} (x - \phi_P)^2 - 4 \left(\frac{\Lambda}{\sqrt{2}} \right)^{2N - N_f} \prod_{A=1}^{N_f} \left(x + \frac{m_A}{\sqrt{2}} \right). \tag{2.11}$$

[6]The opposite case $r < N_f/2$ is discussed in Sec. 4.

Here ϕ_P are gauge invariant parameters on the Coulomb branch. Semiclassically,

$$\Phi \approx \mathrm{diag}\,[\phi_1, ..., \phi_N]\,. \tag{2.12}$$

In the $r < N$ vacuum (more exactly, in the $(1, ..., r)$ vacuum) we have

$$\phi_P \approx -\frac{m_P}{\sqrt{2}}, \qquad P = 1, ..., r\,, \qquad \phi_P \sim \Lambda_{\mathcal{N}=2}, \qquad P = r+1, ..., N \tag{2.13}$$

in the large m_A limit, see (2.7).

To identify the $r < N$ vacuum in terms of the curve (2.11) it is necessary to find such values of ϕ_P which ensure the Seiberg-Witten curve to have $N-1$ double roots, with r parameters ϕ_P determined by the quark masses in the semiclassical limit, see (2.13). The above $N-1$ double roots will be associated with the r condensed quarks and $(N-r-1)$ condensed monopoles – altogether $N-1$ condensed states.

In contrast, in the $r = N$ vacuum the maximal possible number of condensed states (quarks) in the U(N) theory is N. As was mentioned, this difference is related to the the unbroken $\mathrm{U}(1)^{\mathrm{unbr}}$ gauge group in the $r < N$ vacua [21]. In the classically unbroken (after the quark condensation) U($N-r$) gauge group, $N-r-1$ monopoles condense at a quantum level, breaking the non-Abelian SU($N-r$) subgroup. One U(1) factor remains unbroken because the monopoles are not coupled to this U(1).

Thus in the $r < N$ vacua with the quadratic deformation superpotential (2.2) the Seiberg–Witten curve factorizes [22],

$$y^2 = \prod_{P=1}^{r}(x - e_P)^2 \prod_{K=r+1}^{N-1}(x - e_K)^2\,(x - e_N^+)(x - e_N^-)\,. \tag{2.14}$$

The first r double roots in the large mass limit are given by the mass parameters, $\sqrt{2}e_P \approx -m_P$, $P = 1, ..., r$. Other $(N-r-1)$ double roots associated with light monopoles are much smaller and determined by Λ. The last two roots are also much smaller.

For the single-trace deformation superpotential (2.2) the sum of the unpaired roots vanishes [22],

$$e_N^+ + e_N^- = 0\,. \tag{2.15}$$

The root e_N^+ determines the value of the gaugino condensate [21].

Now, Eq. (1.1) was derived in one of our previous papers [2] for the case of the quark condensate namely, for $P = 1, ..., r$.

In the remainder of this paper we demonstrate that the monopole condensates in the monopole vacuum ($r = 0$) or hybrid r vacua are also determined by the same formula with the replacement of the quark double roots by the monopole double roots, so that the subscript P in (1.1) can run over monopole double roots $P = (r+1), ..., (N-1)$ too. Thus Eq. (1.1) is very general and determines VEVs of any condensed field independently of its nature.

3 $r = 0$: the monopole vacuum

In this section we consider the monopole vacuum with $r = 0$ and show that the monopole condensates are still given by Eq. (1.1). Then, we demonstrate that for the above monopole vacuum (in the limit of large quark masses, i.e. when the theory at hand reduces to pure gauge theory) Eq. (1.1) gives the famous sine formula for the monopole VEVs and, hence, the electric string tensions [15].

3.1 Monopole VEVs

Consider the simplest example: the $r = 0$ vacuum in U(2) SQCD with N_f quark flavors. It is a straightforward generalization of the SU(2) theory studied in [1, 23]. The low-energy gauge group is U(1)× U(1) where the first U(1) factor is associated with, say, the τ_3 generator of SU(2). In this case the light matter sector consists of one monopole singlet M and $\tilde{M}$ charged with respect to the first U(1) factor [1]. The relevant F-terms in the scalar potential are

$$\begin{aligned} V(M, \tilde{M}, a_3^D, a) &= 2g_D^2 \left| \tilde{M}M + \frac{\mu}{\sqrt{2}} \frac{\partial u_2}{\partial a_3^D} \right|^2 + g_1^2 \left| \mu \frac{\partial u_2}{\partial a} \right|^2 \\ &+ 2\left| a_3^D M \right|^2 + 2\left| a_3^D \bar{\tilde{M}} \right|^2 + \cdots , \end{aligned} \tag{3.1}$$

where we denote the light adjoint scalar of the dual gauge multiplet associated with τ_3 by a_3^D, while a stands for the neutral scalar in the U(1) gauge multiplet of U(2). The corresponding coupling constants are g_D and g_1, respectively. We also define

$$u_k = \left\langle \mathrm{Tr} \left(\frac{1}{2}\, a + T^a\, a^a \right)^k \right\rangle , \qquad k = 1, ..., N . \tag{3.2}$$

Thus, the deformation superpotential (2.2) is proportional to u_2. From the potential (3.1) it is easy to derive for the monopole vacuum

$$\langle \tilde{M}M \rangle = -\frac{\mu}{\sqrt{2}} \frac{\partial u_2}{\partial a_3^D} ; \qquad \frac{\partial u_2}{\partial a} = 0, \quad a_3^D = 0 . \tag{3.3}$$

The Seiberg-Witten curve in this case factorizes as follows:

$$y^2 = (x - e_1)^2\, (x - e_2^+)(x - e_2^-), \tag{3.4}$$

see (2.14). Here the double root at $x = e_1$ corresponds to a single condensed monopole in the $r = 0$ vacuum, while two other roots (subject to the condition (2.15)) determine the gaugino condensate.

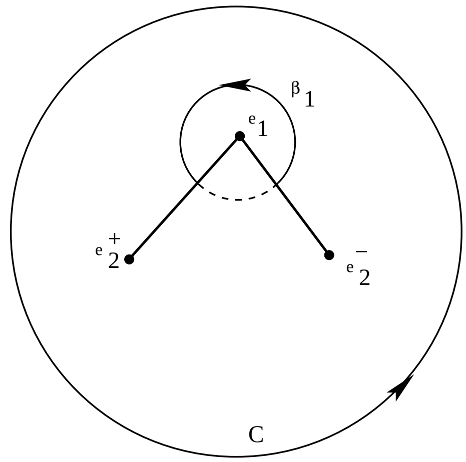

Figure 1: β_1 and C-contours in the x plane in the U(2) theory. Solid straight lines denote cuts.

The exact solution of the theory on the Coulomb branch relates the fields a_3^D and a to contour integrals running along the contours β_1 in the x plane encircling the double root e_1 and the contour C at infinity, see Fig 1.

Using explicit the expressions from [24, 25, 26, 27] and their generalizations to the U(N) case [8] we arrive at

$$\frac{\partial a_3^D}{\partial u_2} = \frac{1}{2}\frac{1}{2\pi i}\oint_{\beta_1}\frac{dx}{y}\,, \qquad \frac{\partial a_3^D}{\partial u_1} = \frac{1}{2\pi i}\oint_{\beta_i}\frac{dx}{y}\left[x-(e_1+e_2)\right]\,,$$
$$\frac{\partial a}{\partial u_2} = \frac{1}{2}\frac{1}{2\pi i}\oint_{C}\frac{dx}{y}\,, \qquad \frac{\partial a}{\partial u_1} = \frac{1}{2\pi i}\oint_{C}\frac{dx}{y}\left[x-(e_1+e_2)\right]\,, \tag{3.5}$$

where the variables u_1 and u_2 are given in Eq. (3.2), while

$$e_2 = \frac{1}{2}\left(e_2^+ + e_2^-\right). \tag{3.6}$$

In fact, e_2 should vanish due to the condition (2.15). We will see shortly that this is indeed the case.

For the factorized curve (3.4) the integrals (3.5) can be easily evaluated. In particular, the integral along the β_1 contour is given by its pole contributions. This gives

$$\frac{\partial a_3^D}{\partial u_2} = \frac{1}{2}\frac{1}{\sqrt{(e_1-e_2^+)(e_1-e_2^-)}}, \qquad \frac{\partial a}{\partial u_2} = 0,$$
$$\frac{\partial a_3^D}{\partial u_1} = -\frac{e_2}{\sqrt{(e_1-e_2^+)(e_1-e_2^-)}}, \qquad \frac{\partial a}{\partial u_1} = 1\,. \tag{3.7}$$

Inverting this matrix we get

$$\frac{\partial u_2}{\partial a_3^D} = 2\sqrt{(e_1-e_2^+)(e_1-e_2^-)}, \qquad \frac{\partial u_2}{\partial a} = 2e_2. \tag{3.8}$$

Now from (3.3) we see that indeed

$$e_2 = 0\,, \tag{3.9}$$

i.e. the condition (2.15) is automatically met. The monopole VEV is[7]

$$\langle M\rangle = \langle \bar{\tilde{M}}\rangle = \sqrt{\frac{\xi_1}{2}} \tag{3.10}$$

with

$$\xi_1 = -2\sqrt{2}\,\mu\,\sqrt{(e_1-e_2^+)(e_1-e_2^-)}. \tag{3.11}$$

We see that the monopole condensate in the $r=0$ vacuum is determined by the same Eq. (1.1) in much the same way as the quark condensates, see (2.9). Straightforward generalization of this result to arbitrary N gives for for elementary monopole condensates

$$\langle M_{P(P+1)}\rangle = \langle \bar{\tilde{M}}_{P(P+1)}\rangle = \sqrt{\frac{\xi_P}{2}}, \tag{3.12}$$

where the parameters ξ_P are again determined by the general formula (1.1) ($P = 1, ..., (N-1)$). Here $M_{PP'}$ denotes the monopole with the charge given by the root $\alpha_{PP'} = w_P - w_{P'}$ of the SU(N) algebra with weights w_P, $P < P'$.

3.2 The sine formula

The famous sine formula for the k-string tensions (and, hence, condensates) was derived in [15] in the $\mathcal{N}=2$ limit of pure gluodynamics. The latter can be obtained from our model by tending the quark masses to infinity, where they decouple.

Consider the $r=0$ monopole vacuum in the U(N) gauge theory with heavy quarks, $m_A \to \infty$. The Seiberg-Witten curve in this case takes the form

$$y^2 = \prod_{P=1}^{N}(x-\phi_P)^2 - 4\left(\frac{\Lambda_0}{\sqrt{2}}\right)^{2N}\,, \tag{3.13}$$

where the scale Λ_0 is

$$\Lambda_0^{2N} = \Lambda^{2N-N_f}\prod_{A=1}^{N_f} m_A\,. \tag{3.14}$$

The corresponding expressions for ϕ_P's, double monopole roots e_P and two unpaired roots

[7]Here we also use the D-term condition requiring $|M| = |\tilde{M}|$.

$e_N^{\pm}$ are [15]

$$
\begin{aligned}
\phi_P &= 2\cos\frac{\pi(P-\frac{1}{2})}{N}\,\frac{\Lambda_0}{\sqrt{2}}, \qquad P=1,...,N, \\
e_P &= 2\cos\frac{\pi P}{N}\,\frac{\Lambda_0}{\sqrt{2}}, \qquad P=1,...,(N-1), \\
e_N^{\pm} &= \pm\, 2\,\frac{\Lambda_0}{\sqrt{2}}.
\end{aligned} \tag{3.15}
$$

Substituting these roots in the formula (1.1) we arrive at the following monopole VEVs:

$$
\langle \tilde{M}_{P(P+1)} M_{P(P+1)} \rangle = \frac{\xi_P}{2} = -2i\mu\Lambda_0\ \sin\frac{\pi P}{N}\,, \tag{3.16}
$$

The same monopole VEVs determine the tensions of the Abelian electric strings,

$$
T_P = 2\pi|\xi_P|\,, \qquad P=1,...,N-1\,. \tag{3.17}
$$

Our general expression (1.1) reproduces the sign formula! The string described by (3.16) can be viewed [18] as the so-called "k strings," see [16] and references therein.

In much the same way as the magnetic non-Abelian strings appearing upon the quark condensation in the r vacua, these strings are BPS to the leading order in μ [18, 19]. These Abelian electric strings confine quarks.

4 Hybrid r vacua

As was already mentioned, the low-energy gauge group in the hybrid r vacuum is (2.8), while the light matter sector consist of N_f quark flavors charged under the U(r) gauge subgroup, plus $(N-r-1)$ singlet Abelian monopoles. The quarks and monopoles are charged with respect to orthogonal subgroups of U(N). Hence, they are mutually local (i.e. can be described by a local Lagrangian). If in Sec. 2.3 we discussed the case $r > N_f/2$, now we turn to the opposite case $r < N_f/2$.

In these vacua the low-energy theory is infrared free and it is at weak coupling once the quark and monopole VEVs are small. To ensure this condition we assume all parameters ξ_P given by (1.1) to be small enough.

For example, for large and (almost) equal quark masses the effective scale of the non-Abelian SU(r) subgroup of (2.8) is

$$
\Lambda_{SU(r)}^{N_f-2r} = \frac{m^{2(N-r)}}{\Lambda^{2N-N_f}} \tag{4.1}
$$

where m is the common mass, and $|\xi_P| \ll \Lambda^2_{\mathrm{SU}(r)}$, $P=1,...,r$. For simplicity here and in Sec. 5.1 we assume m to be large and hence quarks have only electric color charges. For a discussion of the small mass limit see Sec. 5.3.

As an example we choose for our analysis the $r = 1$ vacuum in the U(3) gauge theory with N_f quark flavors. The light matter sector consists of a single color component of N_f quark flavors and a monopole singlet. We can choose color charges of quarks and monopole as follows (see (2.7)):

$$\vec{n}_{q^{1A}} = \left(\frac{1}{2}, 0;\ \frac{1}{2}, 0;\ \frac{1}{2\sqrt{3}}, 0\right), \quad \vec{n}_{M_{23}} = \left(0, 0;\ 0, -\frac{1}{2};\ 0, \frac{\sqrt{3}}{2}\right), \tag{4.2}$$

respectively, where we use the notation

$$\vec{n} = \left(n_e, n_m;\ n_e^3, n_m^3;\ n_e^8, n_m^8\right), \tag{4.3}$$

and n_e and n_m denote the electric and magnetic charges of a given state with respect to the U(1) gauge group. Moreover, n_e^3, n_m^3 and n_e^8, n_e^8 stand for the electric and magnetic charges with respect to the Cartan generators of the SU(3) gauge group. The charges chosen in (4.2) correspond to taking the quark charge equal to the weight w_1 and monopole charge equal to the orthogonal root $\alpha_{23} = w_2 - w_3$ of SU(3) subgroup of U(3), see Fig. 2.

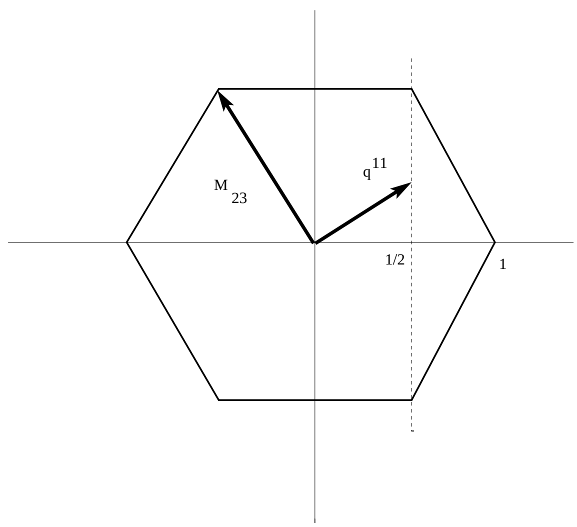

Figure 2: Projection of charges (4.2) of the condensed quark and monopole states onto SU(3) subalgebra of U(3).

From Eq. (4.2) we see that the quarks interact with U(1) gauge field

$$A_\mu^q = \sqrt{\frac{3}{7}}\left(A_\mu + A_\mu^3 + \frac{1}{\sqrt{3}}A_\mu^8\right) \tag{4.4}$$

with the charge

$$n_q \equiv \frac{1}{2}\sqrt{\frac{7}{3}}. \tag{4.5}$$

At the same time, the monopoles interact with the U(1) gauge field

$$A_\mu^D = \frac{1}{2}\left(A_\mu^{D3} + \sqrt{3}\, A_\mu^{D8}\right) \tag{4.6}$$

with the charge $n_M = 1$, while the orthogonal combination

$$A_\mu^{\rm unbr} = \frac{3}{2\sqrt{7}}\left(-\frac{4}{3}A_\mu + A_\mu^3 + \frac{1}{\sqrt{3}}\, A_\mu^8\right) \tag{4.7}$$

is the gauge field of the unbroken $\mathrm{U}(1)^{\rm unbr}$ always present in all $r < N$ vacua. Here A_μ^{Da} denote dual gauge potentials associated with the Cartan generators of SU(3).

Relevant F-terms in the scalar potential of the low-energy theory are

$$\begin{aligned} V &= 2g_q^2\left|n_q\,\tilde{q}_{A1}q^{1A} + \frac{\mu}{\sqrt{2}}\,\frac{\partial u_2}{\partial a_q}\right|^2 \\ &+ 2g_M^2\left|\tilde{M}_{23}M_{23} + \frac{\mu}{\sqrt{2}}\,\frac{\partial u_2}{\partial a^D}\right|^2 + g_{\rm unbr}^2\left|\mu\,\frac{\partial u_2}{\partial a_{\rm unbr}}\right|^2 \\ &+ 2\left|\left(n_q\,a_q + \frac{m_A}{\sqrt{2}}\right)q^{1A}\right|^2 + 2\left|\left(n_q\,a_q + \frac{m_A}{\sqrt{2}}\right)\bar{\tilde{q}}^{1A}\right|^2 \\ &+ 2\left|a^D M\right|^2 + 2\left|a^D \bar{\tilde{M}}\right|^2 + \cdots, \end{aligned} \tag{4.8}$$

where a_q, a^D and $a_{\rm unbr}$ are scalar superpartners of the gauge potentials in (4.4), (4.6) and (4.7), while g_q, g_M and $g_{\rm unbr}$ are the corresponding U(1) gauge couplings. The dots represent the D terms. From Eq. (4.8) we learn that

$$\begin{aligned} n_q\,\langle\,\tilde{q}_{A1}q^{1A}\rangle &= -\frac{\mu}{\sqrt{2}}\,\frac{\partial u_2}{\partial a_q}, \\ \langle\tilde{M}_{23}M_{23}\rangle &= -\frac{\mu}{\sqrt{2}}\,\frac{\partial u_2}{\partial a^D}, \\ \frac{\partial u_2}{\partial a_{\rm unbr}} &= 0, \end{aligned} \tag{4.9}$$

while $a^D = 0$ and $\sqrt{2}\,n_q\,a_q = -m_1$. All derivatives in Eqs. (4.9) can be calculated from the Seiberg-Witten curve which factorizes in the $r = 1$ vacuum at hand as follows:

$$y^2 = (x - e_1)^2\,(x - e_2)^2\,(x - e_3^+)(x - e_3^-). \tag{4.10}$$

Double roots at $x = e_1$ and $x = e_2$ are associated with the light quark q^{11} and light monopole M_{23}, respectively. Details of this calculation can be found in Appendix. The result is

$$\langle\tilde{q}_{11}q^{11}\rangle = \frac{\xi_1}{2}, \qquad \langle\tilde{M}_{23}M_{23}\rangle = \frac{\xi_2}{2}, \tag{4.11}$$

while the last equation in (4.9) ensures that $e_3^+ + e_3^- = 0$, see (2.15). Here ξ_1 and ξ_2 are given by (1.1).

Again we see that all condensates, independently on their nature, are determined by the same universal formula (1.1). Above we analyzed only a few particular examples. Extension to the general case is straightforward, however.

5 Dynamical regimes and dualities in the r vacua

5.1 Confinement and screening

In the hybrid r vacua both quarks and monopoles charged with respect to orthogonal subgroups of U(N) condense. As a result, both the non-Abelian magnetic strings [3, 4, 5, 6] and the Abelian Abrikosov-Nielsen-Olesen electric strings develop supported by the quark and monopole condensates, respectively. Clearly, the magnetic strings confine monopoles while the electric strings confine quarks. Now we focus on large quark masses, with the quarks possessing pure color-electric charges.[8]

Let us turn again to the simplest example of the $r = 1$ vacuum in the U(3) gauge theory and show how confinement and screening of different states work in this case. A similar discussion for the $r = 1$ vacuum in the SU(3) gauge theory can be found in [28].

All charges of condensed quark q^{11} and monopole M_{23} are given in Eq. (4.2). Now we calculate the fluxes of the strings formed due to condensation of these states. Consider first the magnetic strings.

Since we have only one condensed quark q^{11} in $r = 1$ vacuum we deal with a single Abelian magnetic string, to be referred to as S_m. Suppose the q^{11} quark has a winding

$$q^{11} \sim \sqrt{\frac{\xi_1}{2}}\, e^{i\alpha}, \qquad M_{23} \sim \sqrt{\frac{\xi_2}{2}} \tag{5.1}$$

at $r \to \infty$ (see (4.11)), where r and α are the polar coordinates in the plane $i = 1, 2$ orthogonal to the string axis. Equations (5.1) imply the following behavior of the gauge potentials at $r \to \infty$:

$$\begin{aligned} &\frac{1}{2}A_i + \frac{1}{2}A_i^3 + \frac{1}{2\sqrt{3}}A_i^8 \sim \partial_i\alpha\,, \\ &-\frac{1}{2}A_i^3 + \frac{\sqrt{3}}{2}A_i^8 \sim 0\,, \end{aligned} \tag{5.2}$$

as follows from the quark and monopole charges in (4.2). In the $r = 1$ vacuum we have to supplement these conditions with one extra condition ensuring that the combination (4.7)

[8] The string formation and confinement in the r dual theories at small quark masses due to the quark-like dyon condensation was studied in [13, 2], see Sec. 5.3.

of the gauge potentials, which interacts neither with the quark nor monopole, is not excited, namely,

$$-\frac{4}{3}A_i + A_i^3 + \frac{1}{\sqrt{3}}A_i^8 \sim 0\,. \tag{5.3}$$

The solution to these equations is

$$A_i \sim \frac{6}{7}\,\partial_i\alpha\,, \qquad A_i^3 \sim \frac{6}{7}\,\partial_i\alpha\,, \qquad A_i^8 \sim \frac{6}{7\sqrt{3}}\,\partial_i\alpha\,. \tag{5.4}$$

It determines the gauge fluxes $\int dx_i A_i$, $\int dx_i A_i^3$ and $\int dx_i A_i^8$ of the string S_m, respectively. The integration above is performed over a large circle in the $(1,2)$ plane.

Next, we define the string charges [13] as

$$\int dx_i(A_i^D,\, A_i;\, A_i^{3D},\, A_i^3;\, A_i^{8D},\, A_i^8) \equiv 4\pi\,(-n_e,\, n_m;\, -n_e^3,\, n_m^3;\, -n_e^8,\, n_m^8)\,. \tag{5.5}$$

This definition guarantees that the string has the same charge as a probe monopole which can be attached to the string endpoint. In other words, the flux of the given string is the flux of the probe monopole sitting on string's end with the charge defined by (5.5). Note, that this probe monopole does not necessarily exist in the theory under consideration. For example, the monopoles from the SU(r) sector are rather string junctions, so they are attached to two strings, [5, 13]. We will see below that the charges of the physical monopoles confined in the hybrid vacuum differ from the charge of the probe monopoles.

In particular, according to this definition, the charge of the string with the fluxes (5.4) is

$$\vec{n}_{S_m} = \left(0,\, \frac{3}{7};\, 0,\, \frac{3}{7};\, 0,\, \frac{3}{7\sqrt{3}}\right). \tag{5.6}$$

Since this string is associated with the quark winding, it is magnetic.

Now let us consider the electric string existing due to the winding of the monopole M_{23}. In the vacuum at hand we have

$$q^{11} \sim \sqrt{\frac{\xi_1}{2}}, \qquad M_{23} \sim \sqrt{\frac{\xi_2}{2}}\,e^{i\alpha} \tag{5.7}$$

at $r \to \infty$. Therefore,

$$-\frac{1}{2}A_i^{3D} + \frac{\sqrt{3}}{2}A_i^{8D} \sim \partial_i\alpha\,, \qquad \frac{1}{2}A_i^{3D} + \frac{1}{2\sqrt{3}}A_i^{8D} \sim 0\,. \tag{5.8}$$

Solution to these equation is

$$A_i^{3D} \sim -\frac{1}{2}\,\partial_i\alpha\,, \qquad A_i^{8D} \sim \frac{\sqrt{3}}{2}\,\partial_i\alpha\,. \tag{5.9}$$

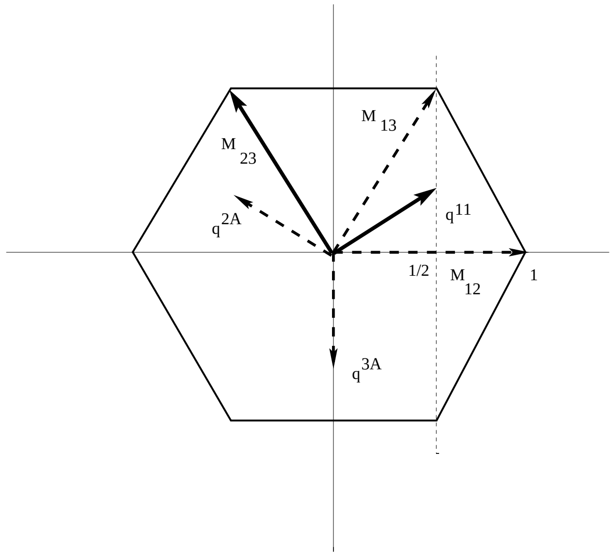

Figure 3: Projection of charges of different quark and monopole states to SU(3) subalgebra of U(3). Charges of condensed states are shown by solid arrows, while charges of confined states are shown by dashed arrows.

The gauge potential A_i^D is not excited. This gives the charge of the S_e string,

$$\vec{n}_{S_e} = \left(0,\, 0;\, \frac{1}{4},\, 0;\, -\frac{\sqrt{3}}{4},\, 0\right). \tag{5.10}$$

Since this string is associated with the monopole winding, it is electric.

It is instructive to check that all quarks and elementary monopoles are either screened or confined in the hybrid vacuum under consideration. Clearly the quarks q^{1A} and monopoles M_{23} are screened. Let us analyze other quarks q^{2A}, q^{3A} as well as monopoles M_{12}, M_{13}. The SU(3) projections of the charges of these states are shown in Fig. 3. Note, that these states are heavy and are not included in the low-energy theory.

Start with the quark q^{2A}. It should be confined by the electric sting S_e. It is not difficult to verify this. Indeed, the charge of this quark can be represented as

$$\vec{n}_{q^{2A}} = \left(\frac{1}{2},\, 0;\, -\frac{1}{2},\, 0;\, \frac{1}{2\sqrt{3}}, 0\right) = -\vec{n}_{S_e} + \frac{1}{7}\,\vec{n}_{q^{11}} + \frac{9}{7}\,\vec{n}^e_{\rm unbr}, \tag{5.11}$$

where

$$\vec{n}^e_{\rm unbr} = \left(\frac{1}{3},\, 0;\, -\frac{1}{4},\, 0;\, -\frac{1}{4\sqrt{3}}, 0\right) \tag{5.12}$$

is the source for the electric $\mathrm{U}(1)^{\rm unbr}$ gauge field (4.7). This U(1) is unbroken.

We see that the q^{2A} quark is confined. Part of its electric flux is confined by the electric string (5.10). Another part is screened by the q^{11} condensate. What is left is precisely the flux of the unbroken gauge field $\mathrm{U}(1)^{\rm unbr}$.

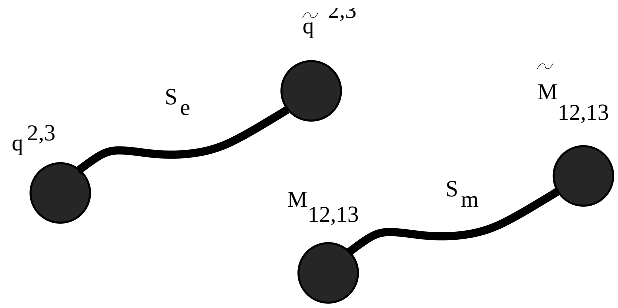

Figure 4: Stringy mesons made of quarks and monopoles.

Of course, any three-dimensional vector of the quark q^{2A} charges can always be written as a linear combination of three orthogonal vectors. What is nontrivial in Eq. (5.11), however, is the coefficient in front of the string charge: it should be integer to ensure confinement.

As a result of confinement and screening stringy mesons made of quarks and antiquarks q^{2A} connected by strings S_e are formed, see Fig. 4. The string endpoints emit electric fluxes of the unbroken $\mathrm{U}(1)^{\mathrm{unbr}}$. This makes this meson a dipole-like configuration, cf. [2]. All other color fluxes are either confined or screened inside the meson.

Analogously we can convince ourselves that the quark q^{3A} is confined too. To check this we represent the charge of this quark as

$$\vec{n}_{q^{3A}} = \left(\frac{1}{2},\, 0;\, 0,\, 0;\, -\frac{1}{\sqrt{3}},\, 0\right) = \vec{n}_{S_e} + \frac{1}{7}\,\vec{n}_{q^{11}} + \frac{9}{7}\,\vec{n}^{\,e}_{\mathrm{unbr}}\,. \tag{5.13}$$

Thus, the q^{3A} quark is obviously confined by the electric string S_e. The unconfined part of its flux is screened by the q^{11} condensate while the remainder coincides with the flux of unbroken $\mathrm{U}(1)^{\mathrm{unbr}}$.

Now we will pass to confinement of the monopoles. Decomposing

$$\vec{n}_{M_{12}} = (0,\, 0;\, 0,\, 1;\, 0,\, 0\,) = \vec{n}_{S_m} - \frac{1}{2}\,\vec{n}_{M_{23}} - \frac{9}{7}\,\vec{n}^{\,m}_{\mathrm{unbr}} \tag{5.14}$$

we see that the part of the monopole M_{12} flux is confined by the magnetic string S_m (see (5.6)), while the the second term is screened by the M_{23} condensate. The remainder of the flux is proportional to

$$\vec{n}^{\,m}_{\mathrm{unbr}} = \left(0,\, \frac{1}{3};\, 0,\, -\frac{1}{4};\, 0,\, -\frac{1}{4\sqrt{3}}\right), \tag{5.15}$$

which is the source for the unbroken magnetic gauge field $\mathrm{U}(1)^{\mathrm{unbr}}$.

As a result, a meson formed by the magnetic string S_m with the M_{12} monopole and its antimonopole attached to the endpoints appears in the physical spectrum. This meson is a dipole-like configuration emitting magnetic fluxes of the unbroken gauge field $\mathrm{U}(1)^{\mathrm{unbr}}$, see Fig. 4.

For the M_{13} monopole we have

$$\vec{n}_{M_{13}} = \left(0,\,0;\,0,\,\frac{1}{2};\,0,\,\frac{\sqrt{3}}{2}\right) = \vec{n}_{S_m} + \frac{1}{2}\,\vec{n}_{M_{23}} - \frac{9}{7}\,\vec{n}^{\,m}_{\rm unbr}\,, \tag{5.16}$$

This monopole is apparently confined by the same S_m magnetic string.

Note, that in the simple case at hand ($r = 1$) we have a single condensed quark and a single condensed monopole ($N - r - 1 = 1$). Therefore other (confined) quarks and monopoles play a role of the endpoints of electric and magnetic strings, respectively. In the case of generic r, with r condensed quarks, we have r elementary magnetic non-Abelian strings. Hence, the confined elementary monopoles of the SU(r) subgroup become junctions of two "neighboring" strings [5, 2]. Similarly, for a generic value of $(N-r-1)$ (i.e. $N-r-1$ condensed monopoles) we have $(N - r - 1)$ Abelian electric strings, thus certain confined quarks become junctions of two different elementary electric strings [18].

5.2 r Duality in $\mathcal{N} = 2$

In Sec. 5.3 we will briefly analyze various phases attainable in $\mathcal{N} = 2$ SQCD in the limit of small quark masses. It is instructive to discuss now the transition to this limit.

From Sec. 2.3 we know that the low-energy theory in the r vacuum with $r > N_f/2$ is at weak coupling because the quark masses are large and hence $\sqrt{\xi} \gg \Lambda$. However, if we reduce the quark masses making the parameters ξ small the quark sector runs to strong coupling, and the theory undergoes a crossover transition.

At small values of ξ low-energy physics can be described by a dual weakly coupled infrared free r-dual theory [2]. The gauge group of the r-dual theory is

$$U(\nu) \times U(1)^{N-\nu}, \qquad \nu = \left\{ \begin{array}{cc} r, & r \le \frac{N_f}{2} \\ N_f - r, & r > \frac{N_f}{2} \end{array} \right. . \tag{5.17}$$

The light matter sector of the r-dual theory is represented by N_f flavors of non-Abelian quark-like dyons charged with respect to the gauge group SU(ν) (as well as a combination of Abelian factors in (5.17)), plus $(r-\nu)$ singlet quarks and $(N-r-1)$ monopoles charged with respect to different Abelian factors in (5.17). The color charges of the non-Abelian quark-like dyons are identical to those of quarks.[9] However, they belong to a different representation of the global color-flavor locked group. VEVs of both non-Abelian quark-like dyons and quark singlets are still given by Eq. (1.1) with $P = 1, ..., r$ [2].

Upon condensation of the quark-like dyons in the U(ν) sector of the r-dual theory non-Abelian string are formed. These strings still confine monopoles, rather than quarks [13, 2]. Thus, r duality is not electromagnetic.

[9]Because of monodromies, the quarks (preserving their weight-like electric charges) pick up certain root-like magnetic charges at strong coupling.

At strong coupling where the dual description is applicable, the quarks and gauge bosons of the original theory from the U(ν) sector are in the "instead-of-confinement" phase. Namely, the Higgs-screened quarks and gauge bosons decay into monopole-antimonopole pairs on the curves of marginal stability (CMS) [13, 29]. The (anti)monopoles pair is confined. In other words, the original quarks and gauge bosons evolve at small ξ into monopole-antimonopole stringy mesons (presumably forming the Regge trajectories).

Note, that the presence of the SU(ν)$\times$U(1)$^{N_f-\nu}$ gauge groups at the roots of the Higgs branches in massless ($\xi = 0$) $\mathcal{N} = 2$ SU(N) SQCD was first recognized long ago in [11], see also [12].

5.3 Phases of $\mathcal{N} = 2$ SQCD at small masses

In this section we summarize for completeness the phases of μ-deformed $\mathcal{N} = 2$ QCD with small quark masses (and small μ). First, we will discuss the small -r vacua, namely, $r < N_f/2$.

As we reduce the quark masses, the quantum numbers of the light states change due to monodromies [1, 23, 30]. In particular, the quarks pick up root-like color-magnetic charges in addition to their weight-like color-electric charges. Still (in the $r < N_f/2$ vacua) there is no crossover, the low-energy theory remains the same: infrared free U(r)$\times$U(1)$^{N-r}$ gauge theory with N_f quarks (or, more exactly, what becomes of quarks) and $N - r - 1$ singlet monopoles [31]. It is at weak coupling provided the parameters ξ_P are small enough.

The quarks from the U(r) sector and monopoles form the orthogonal U(1)$^{N-r}$ still develop VEVs determined by Eq. (1.1). Physics of screening and confinement also remains intact at small m_A. Say, if a given monopole state (charged with respect to the Cartan generators of SU(r)) is confined by the quark condensation at large masses, this confinement property does not change when we follow this given state to the small mass domain, although the quark color charges change [31]. If quarks are screened in the r vacuum at large masses they (or what becomes of quarks) are still screened in the same vacuum in the limit of small masses. Monodromies are nothing other than the relabeling of states, they do not change physics.

In the r vacua with $r > N_f/2$ physics is quite different, see [13, 2] and Sec. 2.3 above. With decreasing ξ the theory undergoes a crossover transition. At small ξ physics can be described by weakly coupled infrared free r-dual theory with the gauge group U(ν)$\times$U(1)$^{N-\nu}$ and $\nu = N_f - r$. The quarks from U(ν) sector are in the "instead-of-confinement" phase: the Higgs-screened quarks decay into the monopole-antimonopole pairs confined by the non-Abelian strings. The singlet quarks from the U(1)$^{r-\nu}$ sector and the monopoles from U(1)$^{N-r}$ sector are Higgs-screened. Other monopoles charged with respect to Cartan generators of SU(r) and heavy quarks charged with respect to the orthogonal U(1)$^{N-r}$ are confined.

6 Conclusions

Our main result is the demonstration of the fact that VEVs of all condensates – quark and monopole – in the hybrid r vacua of $\mathcal{N}=2$ SQCD are given by the unified exact formula (1.1). In the limit of infinitely heavy quarks, when the theory under consideration becomes pure glue, this formula implies the will-known sine formula for the string tensions. (The P strings appearing in (3.16) are usually referred to as k strings.)

In Sec. 5 we briefly discuss dynamical regimes and dualities in the hybrid r vacua. Due to the condensation of r quarks and $(N-r-1)$ monopoles we have r non-Abelian magnetic and $(N-r-1)$ Abelian Abrikosov-Nielsen-Olesen electric strings in such vacua. Magnetic strings confine monopoles, while electric strings confine quarks. We calculate the fluxes of the confining strings. A similar discussion in the SU(N) theory was presented in [28].

Dynamical regimes and their change crucially depend on the value of r. In the $r < N_f/2$ vacua the small quark mass domain does not qualitatively differ from the large quark mass domain: confinement and screening are essentially the same. In $r > N_f/2$ vacua the physics is rather different. With decreasing m_A (and hence decreasing ξ) the theory undergoes a crossover transition and at small ξ can be described using r duality.

Acknowledgments

This work is supported in part by DOE grant DE-FG02-94ER40823. The work of A.Y. was supported by FTPI, University of Minnesota, by RFBR Grant No. 13-02-00042a and by Russian State Grant for Scientific Schools RSGSS-657512010.2.

Appendix: The $r = 1$ vacuum in U(3)

In this Appendix we calculate the derivatives $\partial u_2/\partial a_q$ and $\partial u_2/\partial a_D$ which appear in the right-hand sides of Eqs. (4.9) for the quark and monopole condensates in the $r=1$ vacuum of the U(3) theory. This calculation is quite similar to the calculation in the $r=0$ vacuum in the U(2) theory in Sect. 3 and in the $r=3$ vacuum in the U(3) theory in [8]. Therefore, we will be brief.

Explicit expressions from [24, 25, 26, 27] generalized to the U(N) case [8] imply

$$\frac{\partial \Phi_1}{\partial u_k} = \frac{1}{2\pi i}\oint_{\alpha_1} \frac{dx}{y}\, P_k(x) + \delta_{k1},$$

$$\frac{\partial a^D}{\partial u_k} = \frac{1}{2\pi i}\oint_{\beta_2} \frac{dx}{y}\, P_k(x) + \delta_{k1},$$

$$\frac{\partial\left(\Phi_1+\Phi_2+\Phi_3\right)}{\partial u_k} = \frac{1}{2\pi i}\oint_{C} \frac{dx}{y}\, P_k(x) + 3\delta_{k1}, \tag{A.1}$$

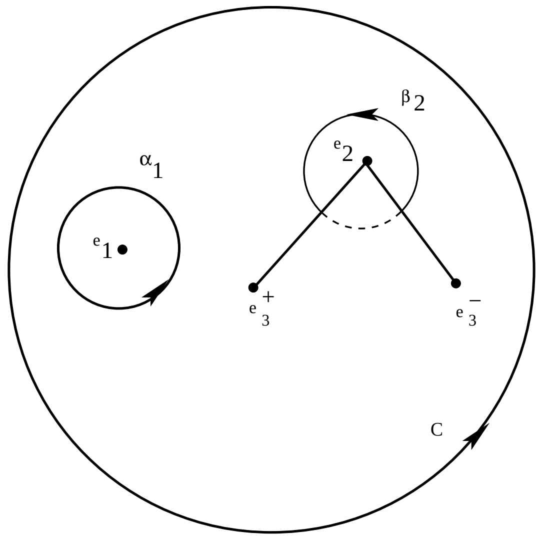

Figure 5: α_1, β_2 and C-contours in x-plane for the U(3) theory. Solid straight lines denote cuts.

where Φ_1, Φ_2 and Φ_3 are diagonal elements of the matrix Φ, see (2.3), while the polynomials $P_k(x)$, $k = 1, 2, 3$ are given by

$$\begin{aligned} P_3(x) &= \frac{1}{3} \\ P_2(x) &= \frac{1}{2}\left[x - \frac{1}{3}(e_1 + e_2 + e_3)\right], \\ P_1(x) &= -2\left[x^2 - \frac{1}{2}x\,(e_1 + e_2 + e_3) + \frac{1}{9}(e_1 + e_2 + e_3)^2\right] \end{aligned} \tag{A.2}$$

and $e_3 = (e_3^+ + e_3^-)/2$. Here the contours α_1 and β_2 encircle the double roots e_1 and e_2 of the Seiberg-Witten curve (4.10) associated with the light quark q^{11} and the light monopole M_{23}, respectively, while C is the contour at infinity, see Fig. 5.

The contour integrals in (A.1) can be readily calculated, in particular the integrals along the contours α_1 and β_2 are given by their pole contributions. These integrals determine the derivatives of a_q and a_{unbr} with respect to u_k since $\Phi_1 = n_q\, a_q$, while a_{unbr} is a linear combination of of a_q and $(\Phi_1 + \Phi_2 + \Phi_3) = 3a/2$, see Eq. (4.7). Inverting the matrix $\partial(a_q, a^D, a_{\text{unbr}})/\partial u_k$ we get the desired expressions for $\partial u_2/\partial a_q$, $\partial u_2/\partial a^D$ and $\partial u_2/\partial a_{\text{unbr}}$ in terms of the roots of the Seiberg-Witten curve.

Ommiting details presented in Sect. 3 and [8] for similar cases we arrive at the results for the quark and monopole VEVs quoted in Eq. (4.11). Also, the last equation in (4.9) gives $e_3 = 0$, in accordance with (2.15).

References

[1] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994), (E) **B430**, 485 (1994) [hep-th/9407087].

[2] M. Shifman and A. Yung, Phys. Rev. D **86**, 025001 (2012) arXiv:1204.4165 [hep-th].

[3] A. Hanany and D. Tong, JHEP **0307**, 037 (2003) [hep-th/0306150].

[4] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B **673**, 187 (2003) [hep-th/0307287].

[5] M. Shifman and A. Yung, Phys. Rev. D **70**, 045004 (2004) [hep-th/0403149].

[6] A. Hanany and D. Tong, JHEP **0404**, 066 (2004) [hep-th/0403158].

[7] M. Shifman and A. Yung, *Supersymmetric Solitons,* Rev. Mod. Phys. **79** 1139 (2007) [arXiv:hep-th/0703267]; an expanded version in Cambridge University Press, 2009.

[8] M. Shifman and A. Yung, Phys. Rev. D **82**, 066006 (2010) [arXiv:1005.5264 [hep-th]].

[9] D. Tong, *TASI Lectures on Solitons,* arXiv:hep-th/0509216.

[10] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A **39**, R315 (2006) [arXiv:hep-th/0602170].

[11] P. Argyres, M. Plesser and N. Seiberg, Nucl. Phys. **B471**, 159 (1996) [hep-th/9603042].

[12] G. Carlino, K. Konishi and H. Murayama, Nucl. Phys. B **590**, 37 (2000) [hep-th/0005076].

[13] M. Shifman and A. Yung, Phys. Rev. D **79**, 125012 (2009) [arXiv:0904.1035 [hep-th]].

[14] M. Shifman and A. Yung, Phys. Rev. D **83**, 105021 (2011) [arXiv:1103.3471 [hep-th]].

[15] M. R. Douglas and S. H. Shenker, Nucl. Phys. B **447**, 271 (1995) [hep-th/9503163].

[16] A. Armoni and M. Shifman, Nucl. Phys. B **664**, 233 (2003) [hep-th/0304127]; Nucl. Phys. B **671**, 67 (2003) [hep-th/0307020].

[17] P. Fayet and J. Iliopoulos, Phys. Lett. B **51**, 461 (1974).

[18] A. Hanany, M. J. Strassler and A. Zaffaroni, Nucl. Phys. B **513**, 87 (1998) [hep-th/9707244].

[19] A. I. Vainshtein and A. Yung, Nucl. Phys. B **614**, 3 (2001) [hep-th/0012250].

[20] M. Shifman and A. Yung, Phys. Rev. D **86**, 065003 (2012) [arXiv:1204.4164 [hep-th]].

[21] F. Cachazo, N. Seiberg and E. Witten, JHEP **0304**, 018 (2003) [hep-th/0303207].

[22] F. Cachazo, K. A. Intriligator and C. Vafa, Nucl. Phys. B **603**, 3 (2001) [hep-th/0103067];

[23] N. Seiberg and E. Witten, Nucl. Phys. **B431**, 484 (1994) [hep-th/9408099].

[24] P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. **74**, 3931 (1995) [hep-th/9411057].

[25] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. B **344**, 169 (1995) [hep-th/9411048].

[26] P. C. Argyres, M. R. Plesser, and A. Shapere, Phys. Rev. Lett. **75**, 1699 (1995) [hep-th/9505100].

[27] A. Hanany and Y. Oz, Nucl. Phys. B **452**, 283 (1995) [hep-th/9505075].

[28] A. Marshakov and A. Yung, Nucl. Phys. B **647**, 3 (2002) [arXiv:hep-th/0202172].

[29] M. Shifman and A. Yung, Phys. Rev. D **81**, 085009 (2010) [arXiv:1002.0322 [hep-th]].

[30] A. Bilal and F. Ferrari, Nucl. Phys. B **516**, 175 (1998) [arXiv:hep-th/9706145].

[31] A. Marshakov and A. Yung, Nucl. Phys. B **831**, 72 (2010) [arXiv:0912.1366[hep-th]].