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Random matrix approach to scalar fields on fuzzy spaces

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Abstract

We formulate theory of interacting scalar field on the fuzzy sphere as a random matrix model. We then analyze the expectation values of observables of the theory in the large N limit and we demonstrate that the eigenvalue distribution of the matrix M remains the polynomially deformed Wigner semicircle. We also compute distributions involving the matrix Laplacian of M and we show that the correlation between the eigenvalues of these two is different from the free field case.

1 Introduction

Matrix models have established a firm place in the modern physics. Starting with pioneering work of Wigner in description of the spectrum of heavy nuclei, they have emerged in areas of string theory as finite approximations to Riemann surfaces [2], condensed matter system as the Calogero model [3] or chaos in quantum systems [4]. Fuzzy spaces are non-commutative spaces with a finite dimensional underlying Hilbert space [5]. It is possible to describe them by finite dimensional matrices and they become their commutative counterpart in the limit of these matrices being very large. The physical motivation of using fuzzy spaces is in regulating the divergences without breaking the isometries of the space-time, which is especially appealing to quantum gravity considerations [6]. Fuzzy spaces also arise as brane solutions in string theory and in M -theory [7].

In [8], it was shown that matrix models and fuzzy spaces are related. One can generalize the standard Gaussian matrix ensembles by addition of matrix Laplacian. This procedure is motivated by the scalar field theories on fuzzy spaces and introduces new observables involving matrix derivatives. Averages of some of these were considered. This was then used to show, that the eigenvalue distributions of the matrix and its Laplacian are correlated and the joint distribution was computed. From physical point of view, such ensembles provide new computational tools for the fuzzy theories, for example to compute distributions of various observables.

In this paper, we present some further work along these lines. In the first part, we compute expectation values and derive the distributions of more general observables in such defined random matrix ensembles. This shows that this framework is very robust and can be used beyond simple observables of the form $M^m B^b$, where B is the matrix Laplacian of M . We also show, that there is a different scaling of the terms in the action, under which the contribution of the mass term survives the limit of very large matrices and keeps the correlation between M and B finite. Moreover, the correlation between the distributions of M and B then remains finite for any form of the kinetic term in this limit. In the second part, we concentrate on the matrix ensemble motivated by an interacting field theory. We show that, as was the case before, the distribution of eigenvalues of M is altered only by a rescaling of the variable. We then investigate the properties of observables involving B . We show that the distribution of the eigenvalues of B and the joint distribution of M and B are connected and we compute each of these up to the second order in the MB correlation. We also show that the interaction brings some new features into the problem, which were not present in the free case.

Some aspects of this problem were approached by other authors from different points of view. Authors of [9] treated the Laplacian term as a perturbation and integrated out the angular degrees of freedom. The eigenvalue problem was then solved using the standard saddle point methods. After the corresponding approximation is made, our results presented here agree with the results computed using this method in [10]. In [11], the kinetic term and added interaction term are treated exactly, but the eigenvalue distribution is said to have the same destiny as in the free case and different aspects of the results are considered. Here, we present more complete calculation of the distribution and we also give calculation

of expectation values of observables involving the Laplacian.

2 Results for the free theory

We start with the summary of the results of [8], which we are going to need later. We will consider the Euclidean theory of a real scalar field, show how it corresponds to a random matrix ensemble and we will sketch the derivation of the eigenvalue distribution of such matrix. Moreover, we will then compute distributions of more involved observables demonstrating the robustness of the method. We will consider interacting fields and corresponding matrix ensembles in the following sections.

Coordinates on the fuzzy sphere [6] are given by $N \times N$ matrices X_α , such that

$$\begin{aligned} X_\alpha &= \frac{2r}{N} L_\alpha, \\ [L_\alpha, L_\beta] &= i\varepsilon_{\alpha\beta\gamma} L_\gamma, \\ \sum_\alpha X_\alpha^2 &= \left(1 - \frac{1}{N^2}\right) r^2, \end{aligned} \tag{1}$$

where $L_\alpha, \alpha = 1, 2, 3$ are the generators of the N dimensional representation of $SU(2)$. r corresponds to the radius of the sphere and $\theta = 2r^2/N$ is the non-commutativity parameter. Derivatives correspond to L commutators and the integral over sphere corresponds to a matrix trace. Therefore the real scalar field on a fuzzy sphere, which is a $N \times N$ hermitian matrix M , is governed by the following free field action

$$S_0(M) = \frac{1}{2} \text{Tr} (M \mathcal{K} M) + \frac{1}{2} \mu^2 \text{Tr} (M^2). \tag{2}$$

We have absorbed the volume factor into the definition of M . The standard Laplacian kinetic term is given by $\mathcal{K}M = [L_\alpha, [L_\alpha, M]]$ and we denote $\mathcal{K}M = B$. Note, that using different matrices in the definition of the underlying fuzzy space (1) and using appropriate kinetic term $\mathcal{K}M$, we could alter this approach for this different fuzzy space.

The dynamics of the theory is given by the correlation functions of observables $\mathcal{O}(M)$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int dM \mathcal{O}(M) e^{-S_0}, \tag{3}$$

where $Z = \int dM e^{-S_0}$. The integration measure in these integrals is the Dyson measure on the set of hermitian matrices

$$dM = \prod_{i \leq j} d\text{Re} M_{ij} \prod_{i < j} d\text{Im} M_{ij}. \tag{4}$$

These expectation values are then computed using Wick contractions. From (3), it is clear that the scalar field corresponds to a modified Gaussian random matrix ensemble with an extra term in the measure, which corresponds to the kinetic term of the S_0 . This term breaks the $SU(N)$ invariance of the action and one can not use the standard method of

diagonalization of M . We will therefore proceed along a different path, deriving recurrence relations for the moments $\text{Tr}(M^{2m})$ and solving these.

We introduce a basis for fields on the fuzzy sphere in terms of $N \times N$ matrices

$$T_m^l, \quad l = 0, 1, \dots, N-1, \quad m = -l, -l+1, \dots, l-1, l, \quad (5)$$

normalized as $\text{Tr}(T_m^l T_{m'}^{l'}) = \delta^{ll'} \delta_{mm'}$ and with $[L_\alpha, [L_\alpha, T_m^l]] = l(l+1)T_m^l$. We can expand the matrix M in terms of this basis as

$$M = \sum_{l,m} c_m^l T_m^l. \quad (6)$$

We assume that the action of the kinetic term on the basis matrices T_m^l does not depend on m and the correlator of two components of M is $\langle c_m^l c_{m'}^{l'} \rangle = \delta^{ll'} \delta_{mm'} G(l)$ where the propagator $G(l)$ depends on the form of the kinetic term. For example for the standard kinetic term it is $G(l) = 1/(\mu^2 + l(l+1))$.

2.1 Previous results

In this section, we will normalize the distributions of M and B to have unit radius. This eliminates some cumbersome factors of 2 in final formulas. In the sections to follow, we will however change this normalization to follow the standard convention. This should be kept in mind when comparing results from these sections 2 and 4.

We define the following two-point functions

$$\langle (MM)_{ij} \rangle = f \delta_{ij}, \quad \langle (BB)_{ij} \rangle = g \delta_{ij}, \quad \langle (MB)_{ij} \rangle = h \delta_{ij}. \quad (7)$$

Also $f = \langle \text{Tr}(MM) \rangle / N$ and similarly for g and h .

We define the normalized correlators

$$W_{m,b} = \frac{1}{N} \left\langle \text{Tr} \left[\left(\frac{M}{2\sqrt{f}} \right)^m \left(\frac{B}{2\sqrt{g}} \right)^b \right] \right\rangle. \quad (8)$$

We will now describe in a little more detail the procedure of deriving the recurrence relations for the correlators of this form presented in [8]. In the rest of this paper, the very same procedure in different settings is going to be used. We start with the expression

$$\left\langle (M^{m-1})_{ij} M_{jk} (B^b)_{ki} \right\rangle, \quad (9)$$

where we have singled out one of the matrices M . Consider the possible contractions of this matrix. If it is contracted with a different matrix M , this results into contribution

$$\left\langle (M^{m-2-p})_{il} \dot{M}_{lr} (M^p)_{rj} \dot{M}_{jk} (B^p)_{ki} \right\rangle = \frac{f}{N} \langle \text{Tr}(M^p) \rangle \left\langle \text{Tr}(M^{m-2-p} B^b) \right\rangle \quad (10)$$

where the overdot denotes the two contracted matrices. To obtain this formula, we have used the fact, that the non-planar contractions between the two parts do not contribute in

the large N limit and an important result of [8], which is that for any matrix V which is product of M 's and B 's we get $\langle V_{ij} \rangle = \langle \text{Tr} V \rangle \delta_{ij} / N$. This also means, that the correlators (7) are the only quantities we will need when doing the Wick contractions. Similarly, if we consider the contraction of the matrix M with one of the B matrices, we get

$$\left\langle (M^{m-1})_{ij} \dot{M}_{jk} (B^p)_{kl} \dot{B}_{lr} (B^{b-1-p})_{ri} \right\rangle = \frac{h}{N} \left\langle \text{Tr} \left(M^{m-1} B^{b-p-1} \right) \right\rangle \langle \text{Tr} (B^p) \rangle. \quad (11)$$

Summing over all possible contractions yields

$$\begin{aligned} 4W_{m,b} &= \sum_{p=0}^{m-2} W_{p,0} W_{m-2-p,b} + \gamma \sum_{p=0}^{b-1} W_{m-1,b-1-p} W_{0,p} \quad m \geq 1, \\ 4W_{m,b} &= \sum_{p=0}^{b-2} W_{0,p} W_{m,b-p-2} + \gamma \sum_{p=0}^{m-1} W_{m-1-p,b-1} W_{p,0} \quad b \geq 1, \end{aligned} \quad (12)$$

where $\gamma = h/\sqrt{fg}$. The second equation comes from considering the contractions of one of the B matrices. Defining the generating function

$$\phi(t, s) = \sum_{m,b=0}^{\infty} W_{m,b} t^m s^b, \quad (13)$$

these recursion rules become equations for $\phi(t, s)$

$$\begin{aligned} 4\left(\phi(t, s) - \phi(0, s)\right) &= t^2 \phi(t, s) \phi(t, 0) + \gamma t s \phi(t, s) \phi(0, s), \\ 4\left(\phi(t, s) - \phi(t, 0)\right) &= s^2 \phi(t, s) \phi(0, s) + \gamma t s \phi(t, s) \phi(t, 0). \end{aligned} \quad (14)$$

These can be solved as follows. From $\phi(0, 0) = 1$ and from $s = 0$ in the first and $t = 0$ in the second equation we obtain equations for $\phi(t, 0)$ and $\phi(0, s)$. These are easy to solve and we can then solve for $\phi(t, s)$. We have the final formulas

$$\phi(t, 0) \equiv \phi(t) = 2 \frac{1 - \sqrt{1 - t^2}}{t^2} = \frac{2}{1 + \sqrt{1 - t^2}} \quad , \quad \phi(0, s) = \phi(s) \quad , \quad (15)$$

$$\phi(t, s) = \frac{\phi(t) \phi(s)}{1 - \frac{1}{4} \gamma t s \phi(s) \phi(t)}. \quad (16)$$

As a consistency check, we see that either of (14) gives the same expression for $\phi(t, s)$. We now need to find the corresponding distributions. $\phi(t)$ is the moment generating function for the eigenvalue distribution of M , i.e. $W_{m,0} = \int dx x^m \rho(x)$. Similarly $\phi(s)$ is the moment generating function for the eigenvalue distribution of B . $\phi(t, s)$ is the generating function for the joint probability distribution of the eigenvalues of M and B , which we denote $\rho(x, y)$. This means that $W_{m,b} = \int dx dy x^m y^b \rho(x, y)$. To invert this generating function we use the method described latter in this section in equations (38)-(40) and we arrive at the final formula for the distribution function

$$\rho(x, y) = \rho(x) \rho(y) \frac{1 - \gamma^2}{(1 - \gamma^2)^2 - 4\gamma(1 + \gamma^2)xy + 4\gamma^2(x^2 + y^2)}, \quad (17)$$

where $\rho(x) = 2\sqrt{1-x^2}/\pi$ is the Wigner semicircle distribution generated by $\phi(t)$. This distribution is positive for $\gamma^2 < 1$ and in the limit $\gamma \rightarrow \pm 1$ becomes $\rho(x)\delta(x-y)$, so the two matrices are completely correlated/anti-correlated, as expected.

Looking at the definition of the moments (8), the distribution of eigenvalues of the unnormalized M has radius $2\sqrt{f}$, distribution of eigenvalues of B has radius $2\sqrt{g}$.

2.2 Joint distribution of eigenvalues of three matrices

Since the two point functions of the matrices M and B were the only relevant quantities for computation of the joint distribution of the previous section, we can define 'linear' functions of matrix M as

$$A_1, A_2 = \sum_{m,l} c_m^l a_{1,2}(l, m) T_m^l, \quad (18)$$

where $a_{1,2}$ are some functions of l and m . When we compute the distributions for these matrices the whole procedure will go through basically intact and we recover the same results. Namely the joint distribution of the eigenvalues of A_1 and A_2 is going to be given by

$$\rho(x, y) = \rho(x)\rho(y) \frac{1 - \gamma_{12}^2}{(1 - \gamma_{12}^2)^2 - 4\gamma_{12}(1 + \gamma_{12}^2)xy - 4\gamma_{12}^2(x^2 + y^2)} \quad (19)$$

and

$$\gamma_{12} = \frac{\langle \text{Tr}(A_1 A_2) \rangle}{\sqrt{\alpha_1 \alpha_2}}, \quad (20)$$

where $\alpha_i = \langle \text{Tr}(A_i^2) \rangle$. The distributions of eigenvalues of matrices A_1, A_2 have radius $2\sqrt{\alpha_i}$. In terms of the original definition we have

$$\langle \text{Tr}(A_1 A_2) \rangle = \frac{1}{N} \sum_{l=0}^{N-1} G(l) \sum_{m=-l}^l a_1(l, m) a_2(l, m), \quad (21)$$

$$\alpha_i = \frac{1}{N} \sum_{l=0}^{N-1} G(l) \sum_{m=-l}^l a_i^2(l, m). \quad (22)$$

We now turn to computation of the joint distribution of three matrices of the form (18), i.e. we look for $\rho(x, y, z)$ such that

$$W_{a,b,c} = \frac{1}{N} \left\langle \text{Tr} \left[\left(\frac{A_1}{2\sqrt{\alpha_1}} \right)^a \left(\frac{A_2}{2\sqrt{\alpha_2}} \right)^b \left(\frac{A_3}{2\sqrt{\alpha_3}} \right)^c \right] \right\rangle = \int dx dy dz x^a y^b z^c \rho(x, y, z). \quad (23)$$

Our approach will be again to write down recursion rules for the moments $W_{a,b,c}$ coming from the Wick contractions. Focusing on one of the matrices A_1 , there are three different

types of contractions

$$\begin{aligned}
\langle \text{Tr} (A_1^a A_2^b A_3^c) \rangle &\rightarrow \frac{\langle \text{Tr} (A_1^2) \rangle}{N} \langle \text{Tr} (A_1^{p-2}) \rangle \langle \text{Tr} (A_2^b A_3^c) \rangle \\
&\rightarrow \frac{\sqrt{\langle \text{Tr} (A_1 A_2) \rangle}}{N} \langle \text{Tr} (A_1^{a-1} A_2^p) \rangle \langle \text{Tr} (A_2^{b-p-1} A_3^c) \rangle \\
&\rightarrow \frac{\sqrt{\langle \text{Tr} (A_1 A_3) \rangle}}{N} \langle \text{Tr} (A_1^{a-1} A_2^b A_3^p) \rangle \langle \text{Tr} (A_3^p) \rangle.
\end{aligned} \tag{24}$$

Summing over all possible contractions we obtain

$$\begin{aligned}
4W_{a,b,c} &= \sum_{p=0}^{a-2} W_{p,0,0} W_{a-p-2,b,c} + \gamma_{12} \sum_{p=0}^{b-1} W_{a-1,p,0} W_{0,b-p-1,c} + \\
&+ \gamma_{13} \sum_{p=0}^{c-1} W_{0,0,p} W_{a-1,b,c-p-1} \quad a \geq 1.
\end{aligned} \tag{25}$$

This rule holds only for $a \geq 1$. The quantities of the form $W_{0,0,p}$ and $W_{0,b,c}$ are moments of one and two matrix distributions and are given by (12). In this recursion rule they enter as the initial condition.

As before in the case of two matrices, there are more recursion formulas to write. Considering contractions of one of the A_2 matrices and one of the A_3 matrices we obtain two more equations

$$\begin{aligned}
4W_{a,b,c} &= \sum_{p=0}^{b-2} W_{0,p,0} W_{a,b-p-2,c} + \gamma_{12} \sum_{p=0}^{a-1} W_{p,0,0} W_{a-p-1,b-1,c} + \\
&+ \gamma_{23} \sum_{p=0}^{c-1} W_{a,0,p} W_{0,b-1,c-p-1} \quad b \geq 1,
\end{aligned} \tag{26}$$

$$\begin{aligned}
4W_{a,b,c} &= \sum_{p=0}^{b-2} W_{0,0,p} W_{a,b,c-p-2} + \gamma_{13} \sum_{p=0}^{a-1} W_{0,p,0} W_{a-p-1,b,c-1} + \\
&+ \gamma_{23} \sum_{p=0}^{b-1} W_{a,p,0} W_{0,b-1-p,c-1} \quad c \geq 1.
\end{aligned} \tag{27}$$

We solve these by defining the generating function

$$\phi(t, s, u) = \sum_{a,b,c} W_{a,b,c} t^a s^b u^c \tag{28}$$

and by rewriting the recursion relations as equations for $\phi(t, s, u)$

$$4\left(\phi(t, s, u) - \phi(0, s, u)\right) = t^2\phi(t, 0, 0)\phi(t, s, u) + \gamma_{12}ts\phi(t, s, 0)\phi(0, s, u) + \gamma_{13}tu\phi(0, 0, u)\phi(t, s, u), \quad (29)$$

$$4\left(\phi(t, s, u) - \phi(t, 0, u)\right) = s^2\phi(0, s, 0)\phi(t, s, u) + \gamma_{12}ts\phi(t, 0, 0)\phi(t, s, u) + \gamma_{23}su\phi(t, 0, u)\phi(0, t, u), \quad (30)$$

$$4\left(\phi(t, s, u) - \phi(0, s, u)\right) = u^2\phi(0, 0, u)\phi(t, s, u) + \gamma_{13}tu\phi(0, s, 0)\phi(t, s, u) + \gamma_{23}su\phi(t, s, 0)\phi(0, t, u). \quad (31)$$

We require $\phi(0, 0, 0) = W_{0,0,0} = 1$ as a normalization condition. Now, choosing appropriate variables to be zero, we can solve these equations. For example in the first relation, setting $s = u = 0$ we get

$$4\phi(t, 0, 0) - 4 = t^2\phi^2(t, 0, 0), \quad (32)$$

which is the same equation we have arrived at in the two matrix case and has solution

$$\phi(t, 0, 0) \equiv \phi(t) = \frac{2}{1 + \sqrt{1 - t^2}}. \quad (33)$$

Similarly for the case of $\phi(0, s, 0)$ and $\phi(0, 0, u)$.

Setting $u = 0$ in (29) we get equation for $\phi(t, s, 0)$ which solves again for the formula obtained in the previous section

$$\phi(t, s, 0) = \frac{4\phi(s)}{4 - t^2\phi(t) - \gamma ts\phi(s)} = \frac{4\phi(s)}{\frac{4}{\phi(t)} - \gamma ts\phi(s)} = \frac{\phi(t)\phi(s)}{1 - \frac{1}{4}\gamma_{12}ts\phi(t)\phi(s)}. \quad (34)$$

The same quantity could be obtained from (30) by setting $u = 0$. Doing this, we get the same result, which again serves as a consistency check. Next, we set $s = 0$ in (29) and obtain

$$\phi(t, 0, u) = \frac{\phi(t)\phi(u)}{1 - \frac{1}{4}\gamma_{13}tu\phi(t)\phi(u)}. \quad (35)$$

Again, setting $s = 0$ in (31) gives the same formula as a check. Finally $t = 0$ in (30) or (31) yields

$$\phi(0, s, u) = \frac{\phi(s)\phi(u)}{1 - \frac{1}{4}\gamma_{23}su\phi(s)\phi(u)}. \quad (36)$$

Plugging these into one of the original equations we get the final formula for the generating function

$$\phi(t, s, u) = \frac{\phi(t)\phi(s)\phi(u)}{\left(1 - \frac{1}{4}st\gamma_{12}\phi(s)\phi(t)\right)\left(1 - \frac{1}{4}tu\gamma_{13}\phi(t)\phi(u)\right)\left(1 - \frac{1}{4}su\gamma_{23}\phi(s)\phi(u)\right)}. \quad (37)$$

With no surprise, this is the formula we obtain from either of the three equations.

We will now describe a general method that can be used to invert generating functions of this form to obtain the corresponding distribution. We present the proof in the appendix A. If the generating function is expressed as

$$\phi(t_1, \dots, t_n) = f\left(t_1\phi(t_1), \dots, t_n\phi(t_n)\right)\phi(t_1) \dots \phi(t_n), \quad (38)$$

then the corresponding distribution is given by

$$\rho(x_1, \dots, x_n) = \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \dots \varepsilon_n F(e^{i\varepsilon_1\theta_1}, \dots, e^{i\varepsilon_n\theta_n}), \quad (39)$$

where

$$F(z_1, \dots, z_n) = \left(\prod_{j=1}^n \frac{z_j}{i\pi} \right) f(2z_1, \dots, 2z_n) \quad (40)$$

and $x_n = \cos \theta_n$. Looking at (37), we see that $\phi(t, s, u)$ is of the form (38). Applying this procedure yields a very complicated formula of the form

$$\rho(x, y, z) = \rho(x)\rho(y)\rho(z) \times \tilde{\rho}_3(x, y, z). \quad (41)$$

Explicit formula for $\tilde{\rho}_3(x, y, z)$ is given in the appendix B.

However couple of important observations can be made. In the case $\gamma_{12,23,13} = 0$ the factor $\tilde{\rho}_3(x, y, z)$ becomes 1 and the matrices are correctly uncorrelated. In the case of two of the three γ 's vanishing the factor becomes (if x and y are the two variables for which the correlation does not vanish)

$$\tilde{\rho}_3(x, y, z) = \frac{1 - \gamma^2}{(1 - \gamma^2)^2 - 4\gamma(1 + \gamma^2)xy + 4\gamma^2(x^2 + y^2)} \quad (42)$$

and we obtain $\rho(x, y, z) = \rho(x, y) \times \rho(z)$. If $\gamma_{12,23,31} \rightarrow 1$ we get $\rho(x, y, z) \rightarrow \rho(x)\delta(x-y)\delta(x-z)$ and if $\gamma_{12,23} \rightarrow -1, \gamma_{23} \rightarrow 1$ we get $\rho(x, y, z) \rightarrow \rho(x)\delta(x+y)\delta(x+z)$, i.e. fully correlated or anti-correlated distributions, as expected. Finally in the case of $\gamma_{12} = \gamma_{13} = \gamma, \gamma_{23} \rightarrow 1$ we obtain $\rho(x, y, z) \rightarrow \rho(x, y)\delta(y-z)$.

2.3 MBMB joint distribution

To get the four point joint distribution of matrices *MBMB*, we need to compute the following quantities

$$W_{a,b,c,d} = \frac{1}{N} \left\langle \text{Tr} \left[\left(\frac{M}{2\sqrt{f}} \right)^a \left(\frac{B}{2\sqrt{g}} \right)^b \left(\frac{M}{2\sqrt{f}} \right)^c \left(\frac{B}{2\sqrt{g}} \right)^d \right] \right\rangle. \quad (43)$$

Using the same procedure with the explicit Wick contractions of one of the matrices in the M^a group and planarity of the diagrams we find the following large N recursion rule

$$\begin{aligned}
4W_{a,b,c,d} &= \sum_{p=0}^{a-2} W_{p,0,0,0} W_{a-p-2,b,c,d} + \gamma \sum_{p=0}^{b-1} W_{0,p,0,0} W_{a-1,b-p-1,c,d} + \\
&+ \sum_{p=0}^{c-1} W_{0,b,p,0} W_{a-1,0,c-p-1,d} + \gamma \sum_{p=0}^{b-1} W_{a-1,0,0,p} W_{0,b,c,d-p-1}
\end{aligned} \tag{44}$$

for $a \geq 1$. Introducing the generating function

$$\phi(t, s, u, v) = \sum_{a,b,c,d} W_{a,b,c,d} t^a s^b u^c v^d \tag{45}$$

this becomes

$$\begin{aligned}
4\left(\phi(t, s, u, v) - \phi(0, s, u, v)\right) &= t^2 \phi(t, 0, 0, 0) \phi(t, s, u, v) + \gamma t s \phi(0, s, 0, 0) \phi(t, s, u, v) + \\
&+ t u \phi(0, s, u, 0) \phi(t, 0, u, v) + \gamma t v \phi(t, 0, 0, v) \phi(0, s, u, v)
\end{aligned} \tag{46}$$

As before, we can consider contractions of a matrix from other groups of matrices and we get three more equations for the generating function. These are

$$\begin{aligned}
4\left(\phi(t, s, u, v) - \phi(t, 0, u, v)\right) &= s^2 \phi(0, s, 0, 0) \phi(t, s, u, v) + \gamma s u \phi(0, 0, u, 0) \phi(t, s, u, v) + \\
&+ s v \phi(0, 0, u, v) \phi(t, s, 0, v) + \gamma s t \phi(t, s, 0, 0) \phi(t, 0, u, v),
\end{aligned} \tag{47}$$

$$\begin{aligned}
4\left(\phi(t, s, u, v) - \phi(t, s, 0, v)\right) &= u^2 \phi(0, 0, u, 0) \phi(t, s, u, v) + \gamma u v \phi(0, 0, 0, v) \phi(t, s, u, v) + \\
&+ u t \phi(t, 0, 0, v) \phi(t, s, u, 0) + \gamma u s \phi(0, s, u, 0) \phi(t, s, 0, v),
\end{aligned} \tag{48}$$

$$\begin{aligned}
4\left(\phi(t, s, u, v) - \phi(t, s, u, 0)\right) &= v^2 \phi(0, 0, 0, v) \phi(t, s, u, v) + \gamma v t \phi(t, 0, 0, 0) \phi(t, s, u, v) + \\
&+ v s \phi(t, s, 0, 0) \phi(0, s, u, v) + \gamma v u \phi(0, 0, u, v) \phi(t, s, u, 0)
\end{aligned} \tag{49}$$

We again have $\phi(0, 0, 0, 0) = W_{0,0,0,0} = 1$. Then, choosing appropriate variables to be zero, we can solve these for the generating function pretty much the same way we did in the case of three matrices. The equations for $\phi(t, 0, 0, 0)$, $\phi(t, s, 0, 0)$, $\phi(t, s, u, 0)$ are going to be the very same ones as for $\phi(t, 0, 0)$, $\phi(t, s, 0)$, $\phi(t, s, u)$ and will give the same results. We can make the notation more compact by defining

$$\psi(t, s) = 1 - \frac{1}{4} \gamma_{ts} t s \phi(t) \phi(s), \tag{50}$$

where γ_{ts} is the correlation parameter between the matrices corresponding to the variables t and s . This way

$$\phi(\tau, \sigma) = \frac{\phi(\tau) \phi(\sigma)}{\psi(\tau, \sigma)}, \tag{51}$$

$$\phi(\tau, \sigma, \rho) = \frac{\phi(\tau) \phi(\sigma) \phi(\rho)}{\psi(\tau, \sigma) \psi(\tau, \rho) \psi(\sigma, \rho)}, \tag{52}$$

where τ, σ, ρ are any of t, s, u, v . Finally $\phi(t, s, u, v)$ becomes

$$\begin{aligned} \phi(t, s, u, v) &= \frac{\phi(t)\phi(s)\phi(u)\phi(v)}{\psi(t, s)\psi(t, v)\psi(u, v)\psi(s, u)\psi(s, v)\psi(t, u)} \\ &\times \left[1 - \frac{tsuv}{16}\phi(t)\phi(s)\phi(u)\phi(v) \right]. \end{aligned} \quad (53)$$

Using the method described in detail in the previous section and summarized in equation (39) yields the four point distribution of the form

$$\rho(x, y, z, w) = \rho(x)\rho(y)\rho(z)\rho(w) \times \tilde{\rho}_4(x, y, z, w), \quad (54)$$

with explicit formula for $\tilde{\rho}_4(x, y, z, w)$ given in the appendix B. Here we just observe that the factor $\tilde{\rho}_4(x, y, z, w)$ is 1 when $\gamma = 0$, which gives two independent distributions. In the case of $\gamma \rightarrow 1$ we get $\rho(x, y, z, w) \rightarrow \rho(x)\delta(x-y)\delta(z-w)\delta(x-w)$ and in the case of $\gamma \rightarrow -1$ $\rho(x, y, z, w) \rightarrow \rho(x)\delta(x+y)\delta(z+w)\delta(x+w)$, i.e. the variables correctly correlate or anti-correlate.

3 Mass rescaling and the correlation γ

The large N structure of correlation parameter γ was discussed in [8]. Most importantly, it was shown, that in the case of the standard Laplacian kinetic term the correlation vanishes as $1/\sqrt{\log N}$. In this section, we show that there is a different scaling of the kinetic and the mass terms of the action and that upon a rescaling of μ the correlation γ is finite.

The MM correlator is given by

$$f = \frac{1}{N} \langle \text{Tr}(MM) \rangle = \frac{1}{N} \sum_{l=0}^{N-1} (2l+1)G(l) \approx \int_0^1 2NxG(Nx), \quad (55)$$

where we have introduced $x = l/N$ and changed the summation to integration $\frac{1}{N} \sum_l \rightarrow \int dx$ and we will use $\approx$ when we will want to stress that the equality holds only in the leading order in N . In the case of Laplacian kinetic term, the large N limit of this expression is

$$f = \int_0^1 dx \frac{2Nx}{(Nx)^2 + \mu^2}, \quad (56)$$

which has the $\log N/N$ behavior and this leads to logarithmic vanishing of γ . In this scaling, the mass term does not play any role and the theory is effectively massless in the large N limit. However, if we rescale the mass $\mu^2 \rightarrow N^2 \tilde{\mu}^2$, both terms are going to contribute.¹ Namely

$$f = \int_0^1 dx \frac{2Nx}{(Nx)^2 + N^2 \tilde{\mu}^2} = \frac{1}{N} \log \left(1 + \frac{1}{\tilde{\mu}^2} \right). \quad (57)$$

¹Using the Euler–Maclaurin formula, we check that this scaling does not make any previously subdominant terms large enough to contribute.

Where we have kept only the leading order contribution. Similar calculation then yields also

$$h = \frac{1}{N} \sum_{l=0}^{N-1} (2l+1)l(l+1)G(l) \approx N \left[1 - \tilde{\mu}^2 \log \left(1 + \frac{1}{\tilde{\mu}^2} \right) \right], \quad (58)$$

$$g = \frac{1}{N} \sum_{l=0}^{N-1} (2l+1) \left(l(l+1) \right)^2 G(l) \approx N^3 \left[\frac{1}{2} - \tilde{\mu}^2 + \tilde{\mu}^4 \log \left(1 + \frac{1}{\tilde{\mu}^2} \right) \right] \quad (59)$$

and finally finite

$$\gamma = \frac{\left[1 - \tilde{\mu}^2 \log \left(1 + \frac{1}{\tilde{\mu}^2} \right) \right]}{\sqrt{\log \left(1 + \frac{1}{\tilde{\mu}^2} \right) \left[\frac{1}{2} - \tilde{\mu}^2 + \tilde{\mu}^4 \log \left(1 + \frac{1}{\tilde{\mu}^2} \right) \right]}}. \quad (60)$$

Note that this is finite in the limit of very large $\tilde{\mu}$ and tends to $\sqrt{3}/2$, which is the correlation in the case of no kinetic term. The same value was obtained in [8].

4 Interacting theory

We are now ready to introduce the interaction to the free action (2). We will consider a quartic interaction potential

$$S_{int} = \tilde{g} \text{Tr} (M^4) \quad , \quad \tilde{g} = g/N. \quad (61)$$

The factor of N in the definition of the coupling constant will become clear shortly. As mentioned in the introduction, the case without the kinetic term is well known [12],[13] and the result is a polynomial correction to the Wigner semicircle distribution.

Expanding the interaction part in power series in $\tilde{g}$ yields for an average of an observable $\mathcal{O}(M)$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{a=0}^{\infty} \frac{(-\tilde{g})^a}{a!} \int dM e^{-S_0(M)} \mathcal{O}(M) [\text{Tr} (M^4)]^a = \frac{1}{Z} \sum_{a=0}^{\infty} \frac{(-\tilde{g})^a}{a!} \langle \mathcal{O}(M) [\text{Tr} (M^4)]^a \rangle, \quad (62)$$

where

$$Z = \sum_{a=0}^{\infty} \frac{(-\tilde{g})^a}{a!} \int dM e^{-S_0(M)} [\text{Tr} (M^4)]^a = \sum_{a=0}^{\infty} \frac{(-\tilde{g})^a}{a!} \langle [\text{Tr} (M^4)]^a \rangle. \quad (63)$$

We see that evaluating the average in the interacting theory can be done using averages of the free theory. We just need to pick correct diagrams that contribute to the expectation value on the RHS of (62). The diagrams containing vacuum bubbles, i.e. parts, where some of the vertexes from $\text{Tr} (M^4)^a$ contract only among themselves, will be canceled by the $1/Z$ factor. Therefore we can write

$$\langle \mathcal{O} \rangle = \sum_{a=0}^{\infty} \frac{(-\tilde{g})^a}{a!} \langle \mathcal{O}(M) [\text{Tr} (M^4)]^a \rangle_{0,con}, \quad (64)$$

where the subscript will indicate that we consider only diagrams that do not contain disconnected vacuum bubbles and that the contractions are to be taken using the free theory measure.

Now we also see the motivation for the $1/N$ factor in the definition of the coupling constant $\tilde{g}$. Since each trace in $\langle \mathcal{O}(M) [\text{Tr}(M^4)]^a \rangle$ raises large N dependence of this expression by one power of N , the different expectation values in the sum (64) are going to be all of the same order, given by $\mathcal{O}(M)$. This means that all the terms of the sum will contribute in the large N limit.

Before we proceed with computation of the eigenvalue distribution of M , let us stress one point. The contractions in (64) are done using the free measure. But we have already seen that in the free theory, the kinetic term only rescaled the radius of the original distribution. And therefore we expect the same in the interacting case, namely that all the distributions of the M^4 -theory with no kinetic term will survive also in the full theory, only with a rescaled variable and possibly rescaled coupling g .

4.1 Eigenvalue distribution of the matrix M

As in the previous section, to obtain the eigenvalue distribution of M , we will first compute the moment generating function

$$\phi_1(t) = \sum_{m=0}^{\infty} t^m F_m = \sum_{m=0}^{\infty} t^{2m} F_{2m}. \quad (65)$$

where F_m are properly rescaled moments $\langle \text{Tr}(M^{2m}) \rangle$. The potential is even and therefore odd moments vanish.

From (64) we see that we need to investigate quantities of the form

$$F_a^{2m} = \frac{1}{N^{1+a}} \left\langle \text{Tr} \left[\left(\frac{M}{\sqrt{f}} \right)^{2m} \right] \text{Tr} \left[\left(\frac{M}{\sqrt{f}} \right)^4 \right] \dots \text{Tr} \left[\left(\frac{M}{\sqrt{f}} \right)^4 \right] \right\rangle_{0, \text{con}}. \quad (66)$$

Expressions

$$F_{2m} = \sum_{a=0}^{\infty} \frac{(-g)^a}{a!} F_a^{2m} \quad (67)$$

are then going to be finite and will give $2m$ -point correlators of M 's in the interacting theory. Let us stress again that in this section, recursion rules (12) for the free correlators change a little, due to a different normalization of the distribution of M .

We will now write down the recursion rules for F_a^{2m} in a very similar fashion we did in previous cases. The contributing planar diagrams are going to be of the form of $2m$ points on a circle with four-point vertexes inside this circle. The number of vertexes is a and the points connect with the vertexes with no connecting lines intersecting. If we look at one point on the circle, there are two different types of contractions we can make. First is with

Figure 1: Two types of terms contributing to (68).

a vertex, then three legs of the vertex become effectively three new points on the circle, which now has $2m+2$ points, but there are only $a-1$ vertexes left and there are $4a$ different legs that we can connect the first point to. This contraction gives the first line of (68). The second type of contraction is with a different point on the circle, then the circle splits into two circles, each new circle can have different number of vertexes in it, one circle α and the other $a-\alpha$, with a proper combinatorial factor. These contractions give the second line of (68). The two types of contractions are illustrated in the figure 4.1. We need to be a little careful. If there are no points on any circle to connect with, we can not have any vertexes in this circle, as this would produce a non-connected diagram. Therefore we need to set $F_a^0 = 0$ for any $a \neq 0$ and $F_0^0 = 1$.

Summing over all possible contractions, we obtain

$$\begin{aligned} \langle \text{Tr}(M^{2m}) [\text{Tr}(M^4)]^a \rangle &= 4afN \langle \text{Tr}(M^{2m+2}) [\text{Tr}(M^4)]^{a-1} \rangle + \\ &+ \frac{f}{N} \sum_{p=1}^m \sum_{\alpha=0}^a \binom{a}{\alpha} \langle \text{Tr}(M^{2(p-1)}) [\text{Tr}(M^4)]^\alpha \rangle \langle \text{Tr}(M^{2(m-p)}) [\text{Tr}(M^4)]^{a-\alpha} \rangle \end{aligned} \quad (68)$$

and using the definition (66) this becomes

$$F_a^{2m} = 4aF_{a-1}^{2m+2} + \sum_{p=1}^m \sum_{\alpha=0}^a \frac{a!}{\alpha!(a-\alpha)!} F_\alpha^{2(p-1)} F_{a-\alpha}^{2(m-p)}. \quad (69)$$

Immediately we note, that if we define $\tilde{F}_a^{2m} = F_a^{2m}/a!$ this simplifies into

$$\tilde{F}_a^{2m} = 4\tilde{F}_{a-1}^{2(m+1)} + \sum_{p=0}^{m-1} \sum_{\alpha=0}^a \tilde{F}_\alpha^{2p} \tilde{F}_{a-\alpha}^{2(m-1-p)}. \quad (70)$$

Multiply the recurrence relation by $(-g)^a$ and sum over $a = 1$ to ∞ to obtain

$$F_{2m} - F_0^{2m} = -4gF_{2(m+1)} + \sum_{p=0}^{m-1} \left[F_{2p}F_{2(m-1-p)} - F_0^{2p}F_0^{2(m-1-p)} \right]. \quad (71)$$

We know from the free theory that the moments of the Wigner distribution are the Catalan numbers, $F_0^{2m} = c_n$, and using the identity for the Catalan numbers $c_n = \sum_{p=0}^{n-1} c_p c_{n-1-p}$ the extra terms cancel. Note that this is consequence of the free case recursion rules (12), and similar terms will cancel for this reason also later. We therefore get

$$F_{2(m+1)} = \frac{1}{4g} \left[\sum_{p=0}^{m-1} F_{2p} F_{2(m-1-p)} - F_{2m} \right]. \quad (72)$$

This expression holds however only for $m \geq 1$, so we need to specify F_0 and F_2 , which enter here as the as an initial condition and need to be extracted directly from (70). From the definition of F_a^0 it is clear that $F_0 = 1$, consistent with the normalization of the distribution. F_2 is essentially the dressed propagator of the interacting theory and for a while lets go further without specifying it.

We multiply the previous formula by $t^{2(m+1)}$ and sum over $m = 1$ to ∞ to obtain

$$\phi_1(t) - 1 - t^2 F_2 = \frac{1}{4g} [t^4 \phi_1^2(t) - t^2 (\phi_1(t) - 1)] \quad (73)$$

or

$$t^4 \phi_1^2 - (4g + t^2) \phi_1 + (4g + t^2 + 4gt^2 F_2) = 0 \quad (74)$$

which gives

$$\phi_1(t) = \frac{4g + t^2 - \sqrt{(4g + t^2)^2 - 4t^4(4g + t^2) - 16F_2gt^6}}{2t^4}. \quad (75)$$

So we are left to specify the two point function F_2 .

At this point, we are going to take F_2 to be the expression obtained by the standard methods [12],[13]. This formula is explicitly

$$F_2 = \frac{(1 + 48g)^{3/2} - 1 - 72g}{864g} = \frac{2}{3} a^2 (4 - a^2), \quad (76)$$

where $a^2 = (\sqrt{1 + 48g} - 1)/24g$. It might seem that we are assuming something we are trying to prove, but this is not the case. We assume that the initial condition we are about to use in (72) is the same as for the case of no kinetic term. We then compute the generating function $\phi_1(t)$ that is given by the moments (72) and if it turns out to be the generating function of [12], we conclude that this assumption leads to all the correlators being of the form as in the theory with no kinetic term.

Before we show that this is indeed the case, let us go back to (70) for a while. We can use this recursion rule to generate number of $\tilde{F}_a^{2m}$ and check that the first couple of $\tilde{F}_a^{2m}$ do match the coefficients of the expansion of (76), which can be taken as a motivation for using this expression. After some trial and error using the integer factorization of the terms we can even guess the formula for $\tilde{F}_a^{2m}$ in the following form

$$\tilde{F}_a^{2m} = 12^a \left(\frac{(2m)!}{m!(m-1)!} \right) \left(\frac{(m+2a-1)!}{(m+a+1)!a!} \right). \quad (77)$$

Not surprisingly this is indeed the formula of g expansion of the $2m$ -point correlator given in the appendix of [12]. It would be interesting to see, whether one can extract F_2 in a closed form directly from the recursion rule (70) without solving for $\tilde{F}_a^{2m}$. Or one could try to prove that (77) solves the recursion rule (70) by explicit computation or some inductive method. We have attempted this, but the problem is more complicated and we will proceed with assumption (76).

We plug the formula (76) for F_2 into the the generating function (75), which after some algebra can be brought into the form

$$\phi_1(t) = \frac{t^2 + 4g}{2t^4} - \frac{1}{t^2} \left(\frac{1}{2} + 4ga^2 + \frac{2g}{t^2} \right) \sqrt{1 - 4a^2t^2}. \quad (78)$$

This is the standard generating function for the distribution of M with no kinetic term and $\text{Tr}(M^4)$ interaction. Now, the discussion goes along the usual lines, the resolvent is

$$\omega(\lambda) = \frac{1}{\lambda} \phi_1(1/\lambda) = \int_{-2a}^{2a} dx \frac{\rho_1(x)}{\lambda - x} \quad (79)$$

and the distribution is given by the discontinuity equation²

$$\rho(x) = -\frac{1}{2\pi i} [\omega(x + i\varepsilon) - \omega(x - i\varepsilon)]. \quad (80)$$

This yields the distribution

$$\rho_1(x) = \frac{1}{\pi} \left(\frac{1}{2} + 4ga^2 + 2gx^2 \right) \sqrt{4a^2 - x^2}. \quad (81)$$

This is the polynomial deformation to the Wigner semicircle distribution, with the radius $2a$. Looking back at (66), the variable in the case of the unscaled matrix is $x \rightarrow x/\sqrt{f}$ and we need to replace $g \rightarrow f^2g$,³ i.e. the final formula is

$$\rho_1(x) = \frac{1}{\pi} \left(\frac{1}{2f} + 4gfa^2 + 2gx^2 \right) \sqrt{4a^2f - x^2}. \quad (82)$$

Therefore starting from the recurrence relation (69) we have been able to recover the distribution of eigenvalues of the random matrix ensemble with weight (61). In the next section, we will generalize this approach to different observables of the interacting theory.

As mentioned in the introduction, the authors of [10] have computed the same distribution treating the kinetic part of the action as a perturbation. After using the explicit formula (56) for f , the expression (82) reduces to their result. To see this, one has to introduce a parameter ε in front of the kinetic term, do the expansion of f in powers of ε and take $\varepsilon = 1$ at the end of the calculation.

²For more details, see e.g. [13]

³To see this better, we should carry the explicit factors of f in the calculation. Such calculation would yield an extra factor of f^{m+2a} in (77). From the definition of $\phi_1(t)$ we can see that f^m rescales t by $\sqrt{f}$ and from (67) we can see that f^{2a} rescales g by f^2 .

4.2 Eigenvalue distribution of the matrix B and the joint MB distribution

As in the free case, the theory now includes new observables involving the matrix $B = \mathcal{K}M$. In the free case, this matrix followed the Wigner semicircle distribution, as did the underlying matrix M . Now, the situation is going to be different, since contractions of B with matrix M in the interaction vertex are going to turn even pure B correlators into mixed MB ones.

We will discuss the distribution of eigenvalues of B and the joint distribution for M and B and we will derive the recursion rules for the two generating functions

$$\phi_2(s) = \sum s^{2n} G_{2n}, \quad (83)$$

$$\phi(t, s) = \sum_{m,b} t^m s^b W_{m,b} \quad (84)$$

and

$$G_{2b} = \sum_{a=0}^{\infty} \frac{(-g)^a}{a!} G_a^{2b} = \sum_{a=0}^{\infty} (-g)^a \tilde{G}_a^{2b} \quad , \quad W_{m,b} = \sum_{a=0}^{\infty} \frac{(-g)^a}{a!} W_a^{m,b} = \sum_{a=0}^{\infty} (-g)^a \tilde{W}_a^{m,b} \quad (85)$$

are the moments of the distributions, with

$$G_a^{2b} = \frac{1}{N^{1+a}} \left\langle \text{Tr} \left[\left(\frac{B}{\sqrt{g}} \right)^{2b} \right] \text{Tr} \left[\left(\frac{M}{\sqrt{f}} \right)^4 \right] \dots \text{Tr} \left[\left(\frac{M}{\sqrt{f}} \right)^4 \right] \right\rangle_{0,con}, \quad (86)$$

$$W_a^{m,b} = \frac{1}{N^{1+a}} \left\langle \text{Tr} \left[\left(\frac{M}{\sqrt{f}} \right)^m \left(\frac{B}{\sqrt{g}} \right)^b \right] \text{Tr} \left[\left(\frac{M}{\sqrt{f}} \right)^4 \right] \dots \text{Tr} \left[\left(\frac{M}{\sqrt{f}} \right)^4 \right] \right\rangle_{0,con} \quad (87)$$

Using the same approach as before, it is now quite easy to write down the recurrence rule for $\tilde{G}_a^{2m}$

$$\tilde{G}_a^{2b} = 4\gamma \tilde{W}_{a-1}^{3,2b-1} + \sum_{p=0}^{b-1} \sum_{\alpha=0}^a \tilde{G}_\alpha^{2p} \tilde{G}_{a-\alpha}^{2(b-1-p)}, \quad (88)$$

where the first term comes from the contraction of B with a vertex and the second term from the contraction of B with a different matrix B . This holds only for $b \geq 1$. It is also not too difficult to write down the recursion rules for $\tilde{W}$'s. Here, we can obtain two different recursions, considering contraction of the first M matrix or the first B matrix. These two are

$$\tilde{W}_a^{m,b} = \sum_{p=0}^{m-2} \sum_{\alpha=0}^a \tilde{F}_\alpha^p \tilde{W}_{a-\alpha}^{m-p-2,b} + \gamma \sum_{p=0}^{b-1} \sum_{\alpha=0}^a \tilde{G}_\alpha^p \tilde{W}_{a-\alpha}^{m-1,b-1-p} + 4\gamma \tilde{W}_{a-1}^{m+2,b} \quad (89)$$

holding for $m \geq 1, a \geq 1$ and

$$\tilde{W}_a^{m,b} = \sum_{p=0}^{b-2} \sum_{\alpha=0}^a \tilde{G}_\alpha^p \tilde{W}_{a-\alpha}^{m,b-p-2} + \gamma \sum_{p=0}^{m-1} \sum_{\alpha=0}^a \tilde{F}_\alpha^p \tilde{W}_{a-\alpha}^{m-1-p,b-1} + 4\gamma \tilde{W}_{a-1}^{m+3,b-1} \quad (90)$$

for $b \geq 1, a \geq 1$. Note that for $b = 0$ in the first and $m = 0$ in the second we recover the recursion rules for $\tilde{F}$'s and $\tilde{G}$'s respectively. Also in these expressions $\tilde{W}_0^{m,b}$ are treated as initial values, given by the recursion rules of the free case (12).

Setting $m = 1$ in the first of the recursion we get

$$4\gamma\tilde{W}_{a-1}^{3,b} = \gamma\tilde{W}_a^{1,2b-1} - \gamma^2 \sum_{k=0}^{b-1} \sum_{\alpha=0}^a \tilde{G}_\alpha^{2k} \tilde{G}_{a-\alpha}^{2(b-1-k)}, \quad (91)$$

where we have used the fact, that $W_a^{0,b} \equiv G_a^b$ is nonzero only for even b . Using this in the the recursion rule for $\tilde{G}$ (88) we find

$$\tilde{G}_a^{2b} = \gamma\tilde{W}_a^{1,2b-1} + (1 - \gamma^2) \sum_{p=0}^{b-1} \sum_{\alpha=0}^a \tilde{G}_\alpha^{2p} \tilde{G}_{a-\alpha}^{2(b-1-p)}. \quad (92)$$

Multiplying this equation by $(-g)^a$ and summing over all a yields

$$G_{2b} = \gamma W_{1,2b-1} + (1 - \gamma^2) \sum_{p=0}^{b-1} G_{2p} G_{2(b-1-p)}. \quad (93)$$

Note that for the cancellation of the "0"-terms, we had to use $W_0^{1,2b-1} = \gamma G_0^{2b}$, which is going to be discussed shortly.

$$\phi_2(s) - 1 = (1 - \gamma^2)s^2\phi_2^2(s) + \gamma s \left[\sum_{b=1}^{\infty} W_{1,2b-1} s^{2b-1} \right] = (1 - \gamma^2)s^2\phi_2^2(s) + \gamma s \underbrace{\left[\sum_{b=0}^{\infty} W_{1,b} s^b \right]}_{W_1(s)}, \quad (94)$$

where we have used the fact that $W_{1,b}$ vanishes for even b . We see that this is very different from the equation for $\phi_1(t)$, as expected due to the different role of M and B in the interaction. The solution is given by

$$\phi_2(s) = \frac{1 - \sqrt{1 - 4s^2(1 - \gamma^2)(1 + \gamma s W_1)}}{2(1 - \gamma^2)s^2}. \quad (95)$$

From the condition $\phi_2 \rightarrow \phi_0$ in the limit of $g \rightarrow 0$ we see that in this limit one has to have $W_1 \rightarrow \gamma s \phi_0^2$. We will prove that this is indeed the case in the next section. Note, that $W_0^{1,2b-1} = \gamma G_0^{2b}$ is a consequence of this.

In the following, we will drop the argument of ϕ_1 and ϕ_2 . It will be understood that the former is always functions of t and the latter function of s .

From the recursion rules for $\tilde{W}$'s, we can derive an equation for the generating function of the two point distribution $\phi(t, s)$. To do this, we express (89) as

$$\begin{aligned} W_{m,b} - W_0^{m,b} &= \sum_{p=0}^{m-2} \left[F_p W_{m-p-2,b} - F_0^p W_0^{m-p-2,b} \right] + \gamma \sum_{p=0}^{b-1} \left[G_p W_{m-1,b-1-p} - G_0^p W_0^{m-1,b-p-1} \right] \\ &- 4g W_{m+2,b}. \end{aligned} \quad (96)$$

The terms with a subscript 0 will cancel, since they follow the relations for the free quantities (12). Continuing the procedure we arrive at

$$\phi(t, s) - \phi_2 = t^2 \phi_1 \phi(t, s) + \gamma t s \phi_2 \phi(t, s) - \frac{4g}{t^2} [\phi(t, s) - \phi_2 - t W_1(s) - t^2 W_2(s)], \quad (97)$$

where we have denoted $W_2(s) = \sum_{b=0}^{\infty} W_{2,b} s^b$. A similar equation can be derived from the second recurrence rule

$$\phi(t, s) - \phi_1 = s^2 \phi_2 \phi(t, s) + \gamma t s \phi_1 \phi(t, s) - \frac{4g\gamma s}{t^3} [\phi(t, s) - \phi_2 - t W_1(s) - t^2 W_2(s)]. \quad (98)$$

Before, we have used multiple equations for the generating functions as a consistency check. At this point, we trust the procedure enough to use these as independent equations and to eliminate one of the unknown functions. Doing this and using the equation (94) we get the final formula for $\phi(t, s)$

$$\phi(t, s) = \frac{\phi_2 (t \phi_1 - \gamma s \phi_2)}{t - \gamma s \phi_2 + \gamma s t W_1}. \quad (99)$$

So to compute $\phi_2(s)$ and $\phi(t, s)$ we need to specify the function $W_1(s)$.

Let us stress here, that the coefficient γ in these expressions is the very same correlation as in expressions of section 2. This shows, that even though we consider different matrix ensemble, it still knows about the underlying fuzzy sphere it was built on, which is therefore encoded solely in the kinetic term of (2).

4.3 Leading order $W_1(s)$ in γ

In the case of $\tilde{F}_a^{2m}$, we have been able to guess the solution of the recursion rule. However for the case of $\tilde{W}_a^{1,2b+1}$ the situation is more involved and no easy guess is possible. Moreover, simple analysis of the integer factorization of the first couple of terms shows, that such guess might be very difficult and the explicit formula more complicated than a simple product of factorials.

Therefore we will compute $\tilde{W}_a^{1,2b+1}$ only in the leading order in γ to get the first non-trivial contribution due to the MB contraction. In such case, the contributing diagrams have only one MB contraction, leading to one factor of γ . Let us stress that the results we will obtain are still exact in g and we make no assumption about the magnitude of the coupling.

As figure 4.3 illustrates, we get the following expression for the diagrams with one MB contraction

$$\tilde{W}_a^{1,2b+1} = 4\gamma \tilde{W}_{a-1}^{4,0} \sum_{p=0}^b c_p c_{b-p} = \gamma \tilde{F}_a^2 c_{b+1}, \quad (100)$$

where we have used the recursion rule (70). This holds for $a \geq 1$. For $a = 0$ we simply have $W_0^{1,2b+1} = \gamma \sum c_p c_{b-p} = \gamma c_{b+1} = \gamma F_0^2 c_{b+1}$. This then yields

$$W_{1,2b+1} = \sum_{a=0}^{\infty} (-g)^a W_a^{1,2b+1} = \gamma c_{b+1} F_2 \quad (101)$$

Figure 2: Triangles represent matrices B . With only one MB contraction, the diagram splits into two parts of matrices B and contractions among M 's. There are no contractions of B 's with the lone matrix M on the circle, since then we would have a vacuum bubble or another factor of γ

and

$$W_1 = \sum_{b=0}^{\infty} s^b W_{1,b} = \sum_{b=0}^{\infty} s^{2b+1} W_{1,2b+1} = \gamma F_2 \frac{\phi_0(s) - 1}{s} = \gamma s \phi_0^2(s) F_2. \quad (102)$$

In the last step, we have used the equation for the generating function of the Wigner distribution $\phi_0(s) - s^2 \phi_0^2(s) = 1$. Note, that in the free theory case, there are no vertexes to contract with and the first order in γ is the full formula. In this limit $F_2 \rightarrow 1$ and we get the advertised behavior of W_1 in the free field case. We use this result in the equation (95) to obtain the leading contribution to the distribution ϕ_2

$$\phi_2(s) = \phi_0(s) + \gamma^2 \phi_0(s) \frac{1 - \phi_0(s) + s^2 \phi_0^2(s) F_2}{2 - \phi_0(s)} = \phi_0(s) + \gamma^2 (F_2 - 1) \phi_0(s) \frac{s^2 \phi_0^2(s)}{1 - s^2 \phi_0^2(s)}. \quad (103)$$

Expanding (99) up to second order in γ yields

$$\begin{aligned} \phi(t, s) &= \phi_0(s) \phi_1(t) + \gamma s \phi_0^2(s) \frac{\phi_1(t) - 1}{t} + \\ &+ \gamma^2 \left[\phi_1(t) (F_2 - 1) \phi_0(s) \frac{s^2 \phi_0^2(s)}{1 - s^2 \phi_0^2(s)} + \frac{(1 - F_2 t^2) \phi_1(t) - 1}{t^2} s^2 \phi_0^3(s) \right]. \end{aligned} \quad (104)$$

We now need to invert these to obtain the distributions. We can use the standard approach of defining the resolvent (79) and then using the discontinuity (80). But it is easier to use a fact mentioned in the appendix A that the non-singular part of $\phi(t)/t^n$ generates the distribution $x^n \rho(x)$, where $\rho(x)$ is generated by $\phi(t)$. This way we find out that

$$s^2 \phi_0^3(s) = \frac{(1 - s^2) \phi_0(s) - 1}{s^2}, \quad (105)$$

generates $y^2(1-y^2)\rho_0(y)$ and similarly $((1-F_2t^2)\phi_1(t)-1)/t^2$ generates $x^2(1-F_2x^2)\rho_1(x)$. To invert the generating function $\phi_2(s)$, we use the formula (39) again. After some algebra, we find final result for the distribution of the eigenvalues of matrix B and the joint distribution of M and B , valid up to second order in the correlation γ to be

$$\begin{aligned}\rho_2(y) &= \rho_0(y) \left(1 + \gamma^2(1-F_2)\frac{y^2-2}{y^2-4} \right), \\ \rho(x,y) &= \rho_1(x)\rho_2(y) \left(1 + \gamma xy + \gamma^2 x^2(1-F_2x^2)y^2(1-y^2) \right).\end{aligned}\tag{106}$$

As in the case of the distribution of M , we have to change $g \rightarrow f^2g, x \rightarrow x/\sqrt{f}, y \rightarrow y\sqrt{g}$ to get the distributions of the unscaled matrices. Note that both of these become the appropriate free expressions but there is something new in the second formula. One could guess that the interacting result for $\rho(x,y)$ would be just the free case with the one matrix marginals replaced by the interacting expressions. The extra factor of F_2 shows that this is not the case.

5 Conclusions and Outlook

The main results we present are twofold. First, we have shown that it is possible to compute distributions of more general observables in the matrix ensemble discussed in [8]. Also, we show how rescaling of the mass puts all the terms on the same footing in the large N limit of the theory with the Laplacian kinetic term. Second, we have shown that the matrix ensemble corresponding to the interacting scalar field on the fuzzy sphere does reproduce the distribution of eigenvalues of the model without the kinetic term, which is the polynomially deformed Wigner semicircle. The radius of the distribution gets renormalized by a factor of $\sqrt{f}$ as was the case for the free field ensemble and this therefore seems to be a generic feature. We have obtained the equations for the distribution of matrix B and the joint distribution of M and B and we have solved these up to an unknown function W_1 . We have then computed the first non-trivial contribution, which enabled us to compute these distributions to the second order in γ . The results we have obtained show, that the interaction introduces a novel features into the joint distribution of M and B beyond changing the one matrix marginals.

There are two points that deserve further treatment. First of all, the proof that the formula (77) does indeed solve the recursion rule (70), or equivalently recovering the expression (76) for F_2 directly from the recursion. Also more systematic treatment of the γ expansion of $\phi_2(s)$ and $\phi(t,s)$ is needed and could lead to better understanding and possibly complete solution for $W_1(s)$. One possible strategy could be the following. Express the generating functions and W_1 and W_2 a power series in γ and then solve (97) and (98) order by order. Since W_1 includes only odd and W_2 only even powers of γ , these two equations, together with (95), are enough to provide all the information. Using this method, it is straightforward to check that we recover the same $\gamma^0, \gamma^1, \gamma^2$ order results as we did in the previous section.

With the results presented and possibly some future progress there are several issues

to explore. The presented results treat the kinetic term non-perturbatively, therefore it is going to be interesting to analyze the phase diagram of the theory and to compare the findings with previous numerical work [14]. It has been also shown that a modification of the kinetic term can remove the tadpole diagrams responsible for the UV/IR mixing [15]. Our method is well suited for such modification and one will be able to study signatures of this removal in phase diagram of the theory.

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A Proof of the formula (39)

We will prove the one dimensional case. The generalization to more dimensions is then straightforward. The presented proof is a modification of approach used in [8].

Let us have a generating function of the form

$$\Phi(t) = f(t\phi(t))\phi(t), \quad (107)$$

where $\phi(t) = 2/(1 + \sqrt{1 - t^2})$ is the generating function for the Wigner semicircle distribution. We expand this as

$$\Phi(t) = \sum_{n=0}^{\infty} a_n t^n \phi^{n+1}(t). \quad (108)$$

Using the explicit formula for $\phi(t)$ we obtain

$$t^n \phi^{n+1}(t) = \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} \left(\frac{2}{t}\right)^n (1-t^2)^k \phi(t) - 2 \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \left(\frac{2}{t}\right)^n (1-t^2)^k. \quad (109)$$

Now follows a crucial observation. If a generating function $g(t)$ corresponds to the distribution $\rho(x)$, then the non-singular part of $g(t)/t^n$ generates $x^n \rho(x)$ for any $n \geq 0$. In previous expression, the second term contains only singular terms. However the left hand side is clearly non-singular and therefore the second term cancels the singular part of the first term but does not contribute otherwise. After resuming the first term of (109), we see, that $t^n \phi^{n+1}(t)$ generates the following distribution

$$\begin{aligned} \rho_n(x) &= \rho_0(x) 2^n x^n \frac{1}{2\sqrt{1 - \frac{1}{x^2}}} \left[\left(1 + \sqrt{1 - \frac{1}{x^2}}\right)^{n+1} - \left(1 - \sqrt{1 - \frac{1}{x^2}}\right)^{n+1} \right] = \\ &= \frac{2^n}{i\pi} \left[\left(x + i\sqrt{1 - x^2}\right)^{n+1} - \left(x - i\sqrt{1 - x^2}\right)^{n+1} \right]. \end{aligned} \quad (110)$$

Since $|x| \leq 1$, we can write $x = \cos \theta$ for some $\theta \in [0, \pi]$ and $x \pm i\sqrt{1-x^2} = e^{\pm i\theta}$. This yields

$$\rho_n(x) = \frac{2^n e^{i(n+1)\theta} - 2^n e^{-i(n+1)\theta}}{i\pi}. \quad (111)$$

And the final distribution is then

$$\rho(x) = \sum_{n=0}^{\infty} a_n \rho_n(x) = \frac{1}{i\pi} \left(e^{i\theta} f(2e^{i\theta}) - e^{-i\theta} f(2e^{-i\theta}) \right), \quad (112)$$

which is the desired formula (40) for the one dimensional case.

B Explicit formulas for three and four matrix distributions

The factor $\tilde{\rho}_3(x, y, z)$ multiplying $\rho(x)\rho(y)\rho(z)$ in formula (41) is given by fraction with the following numerator

$$\begin{aligned} & 1 - g_{12}^2 - g_{13}^2 + g_{12}^2 g_{13}^2 - g_{12} g_{13} g_{23} + g_{12}^3 g_{13} g_{23} + g_{12} g_{13}^3 g_{23} - g_{12}^3 g_{13}^3 g_{23} - g_{23}^2 + g_{12}^2 g_{23}^2 + \\ & + g_{13}^2 g_{23}^2 - g_{12}^2 g_{13}^2 g_{23}^2 + g_{12} g_{13} g_{23}^3 - g_{12}^3 g_{13} g_{23}^3 - g_{12} g_{13}^3 g_{23}^3 + g_{12}^3 g_{13}^3 g_{23}^3 - 4g_{12}^2 g_{13}^2 x^2 + \\ & + 4g_{12} g_{13} g_{23} x^2 + 4g_{12}^2 g_{13}^2 g_{23}^2 x^2 - 4g_{12} g_{13} g_{23}^3 x^2 + 4g_{12} g_{13}^2 x y - 4g_{13} g_{23} x y - 4g_{12}^2 g_{13} g_{23} x y + \\ & + 4g_{12}^2 g_{13}^3 g_{23} x y + 4g_{12} g_{23}^2 x y - 4g_{12} g_{13}^2 g_{23}^2 x y - 4g_{12}^3 g_{13}^2 g_{23}^2 x y + 4g_{12}^2 g_{13} g_{23}^3 x y + 4g_{12} g_{13} g_{23} y^2 - \\ & 4g_{12} g_{13}^3 g_{23} y^2 - 4g_{12}^2 g_{23}^2 y^2 + 4g_{12}^2 g_{13}^2 g_{23}^2 y^2 + 4g_{12}^2 g_{13} x z - 4g_{12} g_{23} x z - 4g_{12} g_{13}^2 g_{23} x z \\ & + 4g_{12}^3 g_{13}^2 g_{23} x z + 4g_{13} g_{23}^2 x z - 4g_{12}^2 g_{13} g_{23}^2 x z - 4g_{12}^2 g_{13}^3 g_{23}^2 x z + 4g_{12} g_{13}^2 g_{23}^3 x z - 4g_{12} g_{13} y z \\ & + 4g_{12}^2 g_{23} y z + 4g_{13}^2 g_{23} y z - 4g_{12}^2 g_{13}^2 g_{23} y z - 4g_{12} g_{13} g_{23}^2 y z + 4g_{12}^3 g_{13} g_{23}^2 y z \\ & + 4g_{12} g_{13}^3 g_{23}^2 y z - 4g_{12}^2 g_{13}^3 g_{23}^2 y z + 4g_{12} g_{13} g_{23} z^2 - 4g_{12}^3 g_{13} g_{23} z^2 - 4g_{13}^3 g_{23}^2 z^2 + 4g_{12}^2 g_{13}^2 g_{23}^2 z^2 \end{aligned}$$

and denominator

$$\begin{aligned} & 1 - 2\gamma_{12}^2 + \gamma_{12}^4 - 2\gamma_{13}^2 + 4\gamma_{12}^2 \gamma_{13}^2 - 2\gamma_{12}^4 \gamma_{13}^2 + \gamma_{13}^4 - 2\gamma_{12}^2 \gamma_{13}^4 + \gamma_{12}^4 \gamma_{13}^4 - 2\gamma_{23}^2 + 4\gamma_{12}^2 \gamma_{23}^2 - 2\gamma_{12}^4 \gamma_{23}^2 + 4\gamma_{13}^2 \gamma_{23}^2 \\ & - 8\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^2 + 4\gamma_{12}^4 \gamma_{13}^2 \gamma_{23}^2 - 2\gamma_{13}^4 \gamma_{23}^2 + 4\gamma_{12}^2 \gamma_{13}^4 \gamma_{23}^2 - 2\gamma_{12}^4 \gamma_{13}^4 \gamma_{23}^2 + \gamma_{23}^4 - 2\gamma_{12}^2 \gamma_{23}^4 + \gamma_{12}^4 \gamma_{23}^4 - 2\gamma_{13}^2 \gamma_{23}^4 \\ & + 4\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^4 - 2\gamma_{12}^4 \gamma_{13}^2 \gamma_{23}^4 + \gamma_{13}^4 \gamma_{23}^4 - 2\gamma_{12}^2 \gamma_{13}^4 \gamma_{23}^4 + \gamma_{12}^4 \gamma_{13}^4 \gamma_{23}^4 + 4\gamma_{12}^2 x^2 + 4\gamma_{13}^2 x^2 - 16\gamma_{12}^2 \gamma_{13}^2 x^2 + 4\gamma_{12}^4 \gamma_{13}^2 x^2 \\ & + 4\gamma_{12}^2 \gamma_{13}^4 x^2 - 8\gamma_{12}^2 \gamma_{23}^2 x^2 - 8\gamma_{13}^2 \gamma_{23}^2 x^2 + 32\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^2 x^2 - 8\gamma_{12}^4 \gamma_{13}^2 \gamma_{23}^2 x^2 - 8\gamma_{12}^2 \gamma_{13}^4 \gamma_{23}^2 x^2 + 4\gamma_{12}^2 \gamma_{23}^4 x^2 \\ & + 4\gamma_{13}^2 \gamma_{23}^4 x^2 - 16\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^4 x^2 + 4\gamma_{12}^4 \gamma_{13}^2 \gamma_{23}^4 x^2 + 4\gamma_{12}^2 \gamma_{13}^4 \gamma_{23}^4 x^2 + 16\gamma_{12}^2 \gamma_{13}^2 x^4 - 32\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^2 x^4 + 16\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^4 x^4 \\ & - 4\gamma_{12} x y - 4\gamma_{12}^3 x y + 8\gamma_{12} \gamma_{13}^2 x y + 8\gamma_{12}^3 \gamma_{13}^2 x y - 4\gamma_{12} \gamma_{13}^4 x y - 4\gamma_{12}^3 \gamma_{13}^4 x y + 8\gamma_{12} \gamma_{23}^2 x y + 8\gamma_{12}^3 \gamma_{23}^2 x y \\ & - 16\gamma_{12} \gamma_{13}^2 \gamma_{23}^2 x y - 16\gamma_{12}^3 \gamma_{13}^2 \gamma_{23}^2 x y + 8\gamma_{12} \gamma_{13}^4 \gamma_{23}^2 x y + 8\gamma_{12}^3 \gamma_{13}^4 \gamma_{23}^2 x y - 4\gamma_{12} \gamma_{23}^4 x y - 4\gamma_{12}^3 \gamma_{23}^4 x y + 8\gamma_{12} \gamma_{13}^2 \gamma_{23}^4 x y \\ & + 8\gamma_{12}^3 \gamma_{13}^2 \gamma_{23}^4 x y - 4\gamma_{12} \gamma_{13}^4 \gamma_{23}^4 x y - 4\gamma_{12}^3 \gamma_{13}^4 \gamma_{23}^4 x y - 16\gamma_{12} \gamma_{13}^2 x^3 y - 16\gamma_{12}^3 \gamma_{13}^2 x^3 y + 32\gamma_{12} \gamma_{13}^2 \gamma_{23}^2 x^3 y \\ & + 32\gamma_{12}^3 \gamma_{13}^2 \gamma_{23}^2 x^3 y - 16\gamma_{12} \gamma_{13}^2 \gamma_{23}^4 x^3 y - 16\gamma_{12}^3 \gamma_{13}^2 \gamma_{23}^4 x^3 y + 4\gamma_{12}^2 y^2 - 8\gamma_{12}^2 \gamma_{13}^2 y^2 + 4\gamma_{12}^2 \gamma_{13}^4 y^2 + 4\gamma_{12}^2 y^2 - 16\gamma_{12}^2 \gamma_{23}^2 y^2 \\ & + 4\gamma_{12}^4 \gamma_{23}^2 y^2 - 8\gamma_{13}^2 \gamma_{23}^2 y^2 + 32\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^2 y^2 - 8\gamma_{12}^4 \gamma_{13}^2 \gamma_{23}^2 y^2 + 4\gamma_{13}^4 \gamma_{23}^2 y^2 - 16\gamma_{12}^2 \gamma_{13}^4 \gamma_{23}^2 y^2 + 4\gamma_{12}^4 \gamma_{13}^4 \gamma_{23}^2 y^2 \\ & + 4\gamma_{12}^2 \gamma_{23}^4 y^2 - 8\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^4 y^2 + 4\gamma_{12}^2 \gamma_{13}^4 \gamma_{23}^4 y^2 + 16\gamma_{12}^2 \gamma_{13}^2 x^2 y^2 + 16\gamma_{12}^2 \gamma_{23}^2 x^2 y^2 + 16\gamma_{13}^2 \gamma_{23}^2 x^2 y^2 - 96\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^2 x^2 y^2 + \end{aligned}$$

$$\begin{aligned}
& 16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2x^2y^2+16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2x^2y^2+16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4x^2y^2+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^4y^2-16\gamma_{12}\gamma_{23}^2xy^3-16\gamma_{12}^3\gamma_{23}^2xy^3 \\
& +32\gamma_{12}\gamma_{13}^2\gamma_{23}^2xy^3+32\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2xy^3-16\gamma_{12}\gamma_{13}^4\gamma_{23}^2xy^3-16\gamma_{12}^3\gamma_{13}^4\gamma_{23}^2xy^3-64\gamma_{12}\gamma_{13}^2\gamma_{23}^2x^3y^3-64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2x^3y^3 \\
& +16\gamma_{12}^2\gamma_{23}^2y^4-32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2y^4+16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2y^4+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2y^4-4\gamma_{13}xz+8\gamma_{12}^2\gamma_{13}xz-4\gamma_{12}^4\gamma_{13}xz-4\gamma_{13}^3xz \\
& +8\gamma_{12}^2\gamma_{13}^3xz-4\gamma_{12}^4\gamma_{13}^3xz+8\gamma_{13}\gamma_{23}^2xz-16\gamma_{12}^2\gamma_{13}\gamma_{23}^2xz+8\gamma_{12}^4\gamma_{13}\gamma_{23}^2xz+8\gamma_{13}^3\gamma_{23}^2xz-16\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2xz \\
& +8\gamma_{12}^4\gamma_{13}^3\gamma_{23}^2xz-4\gamma_{13}\gamma_{23}^4xz+8\gamma_{12}^2\gamma_{13}\gamma_{23}^4xz-4\gamma_{12}^4\gamma_{13}\gamma_{23}^4xz-4\gamma_{13}^3\gamma_{23}^4xz+8\gamma_{12}^2\gamma_{13}^3\gamma_{23}^4xz \\
& -4\gamma_{12}^4\gamma_{13}^3\gamma_{23}^4xz-16\gamma_{12}^2\gamma_{13}x^3z-16\gamma_{12}^2\gamma_{13}^3x^3z+32\gamma_{12}^2\gamma_{13}\gamma_{23}^3x^3z+32\gamma_{12}^2\gamma_{13}^3\gamma_{23}^3x^3z-16\gamma_{12}^2\gamma_{13}\gamma_{23}^4x^3z \\
& -16\gamma_{12}^2\gamma_{13}^3\gamma_{23}^4x^3z-4\gamma_{23}yz+8\gamma_{12}^2\gamma_{23}yz-4\gamma_{12}^4\gamma_{23}yz+8\gamma_{13}^2\gamma_{23}yz-16\gamma_{12}^2\gamma_{13}^2\gamma_{23}yz+8\gamma_{12}^4\gamma_{13}^2\gamma_{23}yz-4\gamma_{13}^4\gamma_{23}yz \\
& +8\gamma_{12}^2\gamma_{13}^4\gamma_{23}yz-4\gamma_{12}^4\gamma_{13}^4\gamma_{23}yz-4\gamma_{23}^3yz+8\gamma_{12}^2\gamma_{23}^3yz-4\gamma_{12}^4\gamma_{23}^3yz+8\gamma_{13}^2\gamma_{23}^3yz-16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3yz \\
& +8\gamma_{12}^4\gamma_{13}^2\gamma_{23}^3yz-4\gamma_{12}^4\gamma_{13}^3\gamma_{23}^3yz+8\gamma_{12}^2\gamma_{13}^4\gamma_{23}^3yz-4\gamma_{12}^4\gamma_{13}^4\gamma_{23}^3yz+16\gamma_{12}\gamma_{13}x^2yz+16\gamma_{12}^3\gamma_{13}x^2yz \\
& +16\gamma_{12}\gamma_{13}^3x^2yz+16\gamma_{12}^3\gamma_{13}^3x^2yz-16\gamma_{12}^2\gamma_{23}x^2yz-16\gamma_{13}^2\gamma_{23}x^2yz+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}x^2yz-16\gamma_{12}^4\gamma_{13}^2\gamma_{23}x^2yz \\
& -16\gamma_{12}^2\gamma_{13}^4\gamma_{23}x^2yz-32\gamma_{12}\gamma_{13}\gamma_{23}^2x^2yz-32\gamma_{12}^3\gamma_{13}\gamma_{23}^2x^2yz-32\gamma_{12}\gamma_{13}^3\gamma_{23}^2x^2yz-32\gamma_{12}^3\gamma_{13}^3\gamma_{23}^2x^2yz \\
& -16\gamma_{12}^2\gamma_{23}^3x^2yz-16\gamma_{13}^2\gamma_{23}^3x^2yz+64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3x^2yz-16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^3x^2yz-16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^3x^2yz+16\gamma_{12}\gamma_{13}\gamma_{23}^4x^2yz \\
& +16\gamma_{12}^3\gamma_{13}\gamma_{23}^4x^2yz+16\gamma_{12}\gamma_{13}^3\gamma_{23}^4x^2yz+16\gamma_{12}^3\gamma_{13}^3\gamma_{23}^4x^2yz-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4yz-64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4yz \\
& -16\gamma_{12}^2\gamma_{13}xy^2z-16\gamma_{12}^2\gamma_{13}^3xy^2z+16\gamma_{12}\gamma_{23}xy^2z+16\gamma_{12}^3\gamma_{23}xy^2z-32\gamma_{12}\gamma_{13}^2\gamma_{23}xy^2z-32\gamma_{12}^3\gamma_{13}^2\gamma_{23}xy^2z \\
& +16\gamma_{12}\gamma_{13}^4\gamma_{23}xy^2z+16\gamma_{12}^3\gamma_{13}^4\gamma_{23}xy^2z-16\gamma_{13}\gamma_{23}^2xy^2z+64\gamma_{12}^2\gamma_{13}\gamma_{23}^2xy^2z-16\gamma_{12}^4\gamma_{13}\gamma_{23}^2xy^2z-16\gamma_{13}^3\gamma_{23}^2xy^2z \\
& +64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2xy^2z-16\gamma_{12}^4\gamma_{13}^3\gamma_{23}^2xy^2z+16\gamma_{12}\gamma_{23}^3xy^2z+16\gamma_{12}^3\gamma_{23}^3xy^2z-32\gamma_{12}\gamma_{13}^2\gamma_{23}^3xy^2z-32\gamma_{12}^3\gamma_{13}^2\gamma_{23}^3xy^2z \\
& +16\gamma_{12}\gamma_{13}^4\gamma_{23}^3xy^2z+16\gamma_{12}^3\gamma_{13}^4\gamma_{23}^3xy^2z-16\gamma_{12}^2\gamma_{13}\gamma_{23}^4xy^2z-16\gamma_{12}^2\gamma_{13}^3\gamma_{23}^4xy^2z+64\gamma_{12}\gamma_{13}^2\gamma_{23}^3x^3y^2z \\
& +64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^3x^3y^2z-64\gamma_{12}^2\gamma_{13}\gamma_{23}^3x^3y^2z-64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^3x^3y^2z+64\gamma_{12}\gamma_{13}^2\gamma_{23}^3x^3y^2z+64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^3x^3y^2z \\
& -16\gamma_{12}^2\gamma_{23}y^3z+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}y^3z-16\gamma_{12}^2\gamma_{13}^4\gamma_{23}y^3z-16\gamma_{12}^2\gamma_{23}^3y^3z+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3y^3z-16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^3y^3z \\
& -64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2y^3z+64\gamma_{12}\gamma_{13}\gamma_{23}^2x^2y^3z+64\gamma_{12}^3\gamma_{13}\gamma_{23}^2x^2y^3z+64\gamma_{12}\gamma_{13}^3\gamma_{23}^2x^2y^3z+64\gamma_{12}^3\gamma_{13}^3\gamma_{23}^2x^2y^3z \\
& -64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3x^2y^3z-64\gamma_{12}^2\gamma_{13}\gamma_{23}^2xy^4z-64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2xy^4z+4\gamma_{13}^2z^2-8\gamma_{12}^2\gamma_{13}^2z^2+4\gamma_{12}^4\gamma_{13}^2z^2 \\
& +4\gamma_{23}^2z^2-8\gamma_{12}^2\gamma_{23}^2z^2+4\gamma_{12}^4\gamma_{23}^2z^2-16\gamma_{13}^2\gamma_{23}^2z^2+32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2z^2-16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2z^2+4\gamma_{13}^4\gamma_{23}^2z^2 \\
& -8\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2z^2+4\gamma_{12}^4\gamma_{13}^4\gamma_{23}^2z^2+4\gamma_{13}^2\gamma_{23}^4z^2-8\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4z^2+4\gamma_{12}^4\gamma_{13}^2\gamma_{23}^4z^2+16\gamma_{12}^2\gamma_{13}^2x^2z^2 \\
& +16\gamma_{12}^2\gamma_{23}^2x^2z^2+16\gamma_{13}^2\gamma_{23}^2x^2z^2-96\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2z^2+16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2x^2z^2+16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2x^2z^2+16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4x^2z^2 \\
& +64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4x^4z^2-16\gamma_{12}\gamma_{13}^2xyz^2-16\gamma_{12}^3\gamma_{13}^2xyz^2+16\gamma_{13}\gamma_{23}xyz^2-32\gamma_{12}^2\gamma_{13}\gamma_{23}xyz^2+16\gamma_{12}^4\gamma_{13}\gamma_{23}xyz^2 \\
& +16\gamma_{13}^3\gamma_{23}xyz^2-32\gamma_{12}^2\gamma_{13}^3\gamma_{23}xyz^2+16\gamma_{12}^4\gamma_{13}^3\gamma_{23}xyz^2-16\gamma_{12}\gamma_{23}^2xyz^2-16\gamma_{12}^3\gamma_{23}^2xyz^2+64\gamma_{12}\gamma_{13}^2\gamma_{23}xyz^2 \\
& +64\gamma_{12}^3\gamma_{13}^2\gamma_{23}xyz^2-16\gamma_{12}\gamma_{13}^4\gamma_{23}xyz^2-16\gamma_{12}^3\gamma_{13}^4\gamma_{23}xyz^2+16\gamma_{13}\gamma_{23}^3xyz^2-32\gamma_{12}^2\gamma_{13}\gamma_{23}^3xyz^2+16\gamma_{12}^4\gamma_{13}\gamma_{23}^3xyz^2 \\
& +16\gamma_{13}^3\gamma_{23}^3xyz^2-32\gamma_{12}^2\gamma_{13}^3\gamma_{23}^3xyz^2+16\gamma_{12}^4\gamma_{13}^3\gamma_{23}^3xyz^2-16\gamma_{12}\gamma_{13}^2\gamma_{23}^4xyz^2-16\gamma_{12}^3\gamma_{13}^2\gamma_{23}^4xyz^2 \\
& +64\gamma_{12}^2\gamma_{13}\gamma_{23}x^3yz^2+64\gamma_{12}^2\gamma_{13}^3\gamma_{23}x^3yz^2-64\gamma_{12}\gamma_{13}^2\gamma_{23}^2x^3yz^2-64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2x^3yz^2+64\gamma_{12}^2\gamma_{13}\gamma_{23}^3x^3yz^2+ \\
& 64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^3x^3yz^2+16\gamma_{12}^2\gamma_{13}^2y^2z^2+16\gamma_{12}^2\gamma_{23}^2y^2z^2+16\gamma_{13}^2\gamma_{23}^2y^2z^2-96\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2y^2z^2+ \\
& 16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2y^2z^2+16\gamma_{12}^2\gamma_{13}^4\gamma_{23}^2y^2z^2+16\gamma_{12}^2\gamma_{13}^2\gamma_{23}^4y^2z^2-64\gamma_{12}\gamma_{13}\gamma_{23}x^2y^2z^2-64\gamma_{12}^3\gamma_{13}\gamma_{23}x^2y^2z^2-
\end{aligned}$$

$$\begin{aligned}
& 64\gamma_{12}\gamma_{13}^3\gamma_{23}x^2y^2z^2 - 64\gamma_{12}^3\gamma_{13}^3\gamma_{23}x^2y^2z^2 + 128\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2y^2z^2 - 64\gamma_{12}\gamma_{13}\gamma_{23}^3x^2y^2z^2 - 64\gamma_{12}^3\gamma_{13}\gamma_{23}^3x^2y^2z^2 - \\
& 64\gamma_{12}\gamma_{13}^3\gamma_{23}^3x^2y^2z^2 - 64\gamma_{12}^3\gamma_{13}^3\gamma_{23}^3x^2y^2z^2 + 64\gamma_{12}^2\gamma_{13}\gamma_{23}xy^3z^2 + 64\gamma_{12}^2\gamma_{13}^3\gamma_{23}xy^3z^2 - 64\gamma_{12}\gamma_{13}^2\gamma_{23}^2xy^3z^2 - \\
& 64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2xy^3z^2 + 64\gamma_{12}^2\gamma_{13}\gamma_{23}^3xy^3z^2 + 64\gamma_{12}^3\gamma_{13}^3\gamma_{23}xy^3z^2 + 64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2y^4z^2 - 16\gamma_{13}\gamma_{23}^2xz^3 \\
& + 32\gamma_{12}^2\gamma_{13}\gamma_{23}^2xz^3 - 16\gamma_{12}^4\gamma_{13}\gamma_{23}^2xz^3 - 16\gamma_{13}^3\gamma_{23}^2xz^3 + 32\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2xz^3 - 16\gamma_{12}^4\gamma_{13}^3\gamma_{23}^2xz^3 - 64\gamma_{12}^2\gamma_{13}\gamma_{23}^2x^3z^3 - \\
& 64\gamma_{12}^2\gamma_{13}^3\gamma_{23}^2x^3z^3 - 16\gamma_{13}^2\gamma_{23}yz^3 + 32\gamma_{12}^2\gamma_{13}^2\gamma_{23}yz^3 - 16\gamma_{12}^4\gamma_{13}^2\gamma_{23}yz^3 - 16\gamma_{13}^2\gamma_{23}^3yz^3 + 32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3yz^3 - \\
& 16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^3yz^3 - 64\gamma_{12}^2\gamma_{13}^2\gamma_{23}x^2yz^3 + 64\gamma_{12}\gamma_{13}\gamma_{23}^2x^2yz^3 + 64\gamma_{12}^3\gamma_{13}\gamma_{23}^2x^2yz^3 + 64\gamma_{12}\gamma_{13}^3\gamma_{23}^2x^2yz^3 + \\
& 64\gamma_{12}^3\gamma_{13}^3\gamma_{23}^2x^2yz^3 - 64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3x^2yz^3 + 64\gamma_{12}\gamma_{13}^2\gamma_{23}xy^2z^3 + 64\gamma_{12}^3\gamma_{13}^2\gamma_{23}xy^2z^3 - 64\gamma_{12}^2\gamma_{13}\gamma_{23}^2xy^2z^3 - \\
& 64\gamma_{12}^3\gamma_{13}^3\gamma_{23}^2xy^2z^3 + 64\gamma_{12}\gamma_{13}^2\gamma_{23}^3xy^2z^3 + 64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^3xy^2z^3 - 64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3y^3z^3 - 64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^3y^3z^3 + 16\gamma_{13}^2\gamma_{23}^2z^4 \\
& - 32\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2z^4 + 16\gamma_{12}^4\gamma_{13}^2\gamma_{23}^2z^4 + 64\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2x^2z^4 - 64\gamma_{12}\gamma_{13}^2\gamma_{23}^2xyz^4 - 64\gamma_{12}^3\gamma_{13}^2\gamma_{23}^2xyz^4 + 64\gamma_{12}^2\gamma_{13}\gamma_{23}^2y^2z^4
\end{aligned}$$

The factor $\tilde{\rho}_4(x, y, z, w)$ multiplying $\rho(x)\rho(y)\rho(z)\rho(w)$ in formula (54) is given by

$$\frac{(1 - \gamma^2)^3 [1 + \gamma^6 - \gamma^2(1 + \gamma^2)(1 + 4xz + 4yw) + 4\gamma^3(x + z)(w + y)]}{denominator}$$

with the denominator

$$\begin{aligned}
& (1 + \gamma^2)^8 - 4\gamma(1 + \gamma^{14})(x + z)(w + y)8\gamma^2(1 + \gamma^{12}) \left[x^2 + y^2 + z^2 + w^2 + 2(xy + zw)(xw + yz) + 4xyzw \right] \\
& - 4\gamma^3(1 + \gamma^{10})(x + z)(w + y) \left[-5 + 4(x^2 + y^2 + z^2 + w^2) + 4(xz + wy) + 16xyzw \right] \\
& + 16\gamma^4(1 + \gamma^8) \left[-3(x^2 + y^2 + z^2 + w^2) + (x^4 + y^4 + z^4 + w^4) + 3(x^2 + z^2)(y^2 + w^2) + 4(x^2z^2 + y^2w^2) \right. \\
& - 4xyzw - 2(xy + zw)(xw + yz) + 8xyzw(x^2 + y^2 + z^2 + w^2) + 4(x^2 + z^2)(y^2 + w^2)(xz + wy) \\
& \left. + 16xyzw(xz + wy) + 16x^2y^2z^2w^2 \right] \\
& - 4\gamma^5(1 + \gamma^6)(x + z)(w + y) \left[9 - 12(x^2 + y^2 + z^2 + w^2 + xz + wy) + 16(x^2 + z^2)(y^2 + w^2) \right. \\
& \left. + 16(x^2z^2 + y^2w^2) + 16xz(x^2 + z^2) + 16yw(y^2 + w^2) + 48xyzw + 64xyzw(xz + wy) \right] \\
& 8\gamma^6(1 + \gamma^4) \left[15(x^2 + y^2 + z^2 + w^2) - 8(x^4 + y^4 + z^4 + w^4) - 2(xy + zw)(xw + yz) \right. \\
& - 24(x^2 + z^2)(y^2 + w^2) - 32(x^2z^2 + y^2w^2) - 4xyzw + 32(x^2y^2z^2 + \text{perm}) + 16x^2z^2(x^2 + z^2) \\
& + 16y^2w^2(y^2 + w^2) + 8(x^2 + z^2)(y^4 + w^4) + 8(x^4 + z^4)(y^2 + w^2) \\
& + 32xyzw(xz(x^2 + z^2) + yw(y^2 + w^2)) + 32xyzw(x^2 + z^2)(y^2 + w^2) \\
& + 64xyzw(x^2z^2 + y^2w^2) + 64xyzw(xy + zw)(xw + yz) + 32(x^3y^2z^3 + \text{cycl}) + 128x^2y^2z^2w^2 \left. \right] \\
& - 4\gamma^7(1 + \gamma^2)(x + z)(w + y) \left[-5 + 8(x^2 + y^2 + z^2 + w^2) + 8(xz + yw) \right. \\
& - 16(x^2 + z^2)(y^2 + w^2) - 16xz(x^2 + z^2 + xz) - 16yw(y^2 + w^2 + yw) - 64xyzw \\
& \left. + 64xyzw(x^2 + y^2 + z^2 + w^2) + 128xyzw(xz + yw) + 64(x^3z^3 + y^3w^3) + 64(x^2y^2z^2 + \text{perm}) \right]
\end{aligned}$$

$$\begin{aligned}
& 32\gamma^8 \Big[-5(x^2 + y^2 + z^2 + w^2) + 3(x^4 + y^4 + z^4 + w^4) + 9(x^2 + z^2)(y^2 + w^2) \\
& + 2(xy + zw)(xw + yz) + 12(x^2z^2 + y^2w^2) + 4xyzw - 8x^2z^2(x^2 + z^2) - 8y^2w^2(y^2 + w^2) \\
& - 4(x^2 + z^2)(y^4 + w^4) - 4(x^4 + z^4)(y^2 + w^2) - 8xyzw(x^2 + y^2 + z^2 + w^2) - 16xyzw(xz + yw) \\
& - 16(x^2y^2z^2 + \text{perm}) - 4(x^2 + z^2)(y^2 + w^2)(xz + wy) + 8(x^4z^4 + y^4w^4) + 64x^2y^2z^2w^2 \\
& + 32xyzw(x^2z^2 + y^2w^2) + 16xyzw(x^2 + z^2)(y^2 + w^2) + 16xyzw(xz(x^2 + z^2) + yw(y^2 + w^2)) \\
& + 32xyzw(xy + zw)(xw + yz) + 16(x^3y^2z^3 + \text{cycl}) + 8(x^2y^2z^4 + x^2y^4z^2 + x^4y^2z^2 + \text{perm}) \Big]
\end{aligned}$$

Note that all the polynomials have the desired symmetry $x \leftrightarrow z, y \leftrightarrow w, (x, y) \leftrightarrow (z, w)$.

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