Proposed proper Engle-Pereira-Rovelli-Livine vertex amplitude

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A proposed proper EPRL vertex amplitude

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Abstract

As established in a prior work of the author, the linear simplicity constraints used in the construction of the so-called 'new' spin-foam models mix three of the five sectors of Plebanski theory as well as two dynamical orientations, and this is the reason for multiple terms in the asymptotics of the EPRL vertex amplitude as calculated by Barrett et al. Specifically, the term equal to the usual exponential of i times the Regge action corresponds to configurations either in sector (II+) with positive orientation or sector (II-) with negative orientation. The presence of the other terms beyond this cause problems in the semiclassical limit of the spin-foam model when considering multiple 4-simplices due to the fact that the different terms for different 4-simplices mix in the semi-classical limit, leading in general to a non-Regge action and hence non-Regge and non-gravitational configurations persisting in the semiclassical limit.

To correct this problem, we propose to modify the vertex so its asymptotics include only the one term of the form $e^{iS_{\text{Regge}}}$. To do this, an explicit classical discrete condition is derived that isolates the desired gravitational sector corresponding to this one term. This condition is quantized and used to modify the vertex amplitude, yielding what we call the 'proper EPRL vertex amplitude.' This vertex still depends only on standard $SU(2)$ spin-network data on the boundary, is $SU(2)$ gauge-invariant, and is linear in the boundary state, as required. In addition, the asymptotics now consist in the single desired term of the form $e^{iS_{\text{Regge}}}$, and all degenerate configurations are exponentially suppressed. A natural generalization to the Lorentzian signature is also presented.

1 Introduction

At the heart of the path integral formulation of quantum mechanics [1, 2] is the prescription that the contribution to the transition amplitude by each classical trajectory should be the exponential of i times the classical action. The use of such an expression has roots tracing back to Paul Dirac's *Principles of Quantum Mechanics* [3], and is central to the successful derivation of the classical limit of the path integral, using the fact that the classical equations of motion are the stationary points of the classical action.

The modern spin-foam program [4–6] aims to provide a definition, via path integral, of the dynamics of loop quantum gravity (LQG) [4, 6–8], a background independent canonical quantization of general relativity. The only spin-foam model to so far match the kinematics of loop quantum gravity and therefore achieve this goal is the so-called EPRL model [9–12], which, for Barbero-Immirzi parameter less than 1 is equal to the FK model [13].

In loop quantum gravity, geometric operators have *discrete* spectra. The basis of states diagonalizing the area and other geometric operators are the *spin-network states*. The spin-foam path integral

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consists in a sum over amplitudes associated to histories of such states, called *spin-foams*. Each spin-foam in turn can be interpreted in terms of a Regge geometry on a simplicial lattice. The simplest amplitude provided by a spin-foam model is the so-called *vertex amplitude* which gives the probability amplitude for a set of quantum data on the boundary of single 4-simplex.

The semiclassical (i.e. large quantum number, equivalent to $\hbar \to 0$) limit [14] of the EPRL vertex amplitude, however, is *not* equal to the exponential of i times the Regge action as one would desire, but includes other terms as well.[1] As a consequence, when considering multiple 4-simplices, the semi-classical limit of the amplitude has cross-terms, each of which consists in the exponential of a sum of terms, one for each 4-simplex, equal to the Regge action for that 4-simplex times differing coefficients, yielding what can be called a 'generalized Regge action' [16, 17]. The stationary point equations of this 'generalized Regge action' are *not* the Regge equations of motion and hence *not* those of general relativity, whence general relativity will fail to be recovered in the classical limit. As presented in the recent work [18, 19], the extra terms causing this problem correspond to different sectors of Plebanski theory, as well as different orientations of the space-time. These various sectors and orientations are present in the spin-foam sum because the so-called *linear simplicity constraint* — the constraint which is also used in the Freidel-Krasnov model [13] — allows them.

In this paper, we propose a modification to the EPRL vertex amplitude which solves this problem. We begin by deriving, at the classical discrete level, a condition which isolates the sector corresponding only to the first term in the asymptotics, the exponential of i times the Regge action. We call this sector the 'Einstein-Hilbert' sector, because it is the sector of Plebanski theory in which the BF action reduces to the Einstein-Hilbert action. More specifically, this sector consists in configurations which are either in (what is called) Plebanski sector (II+) with positive space-time orientation, or (what is called) Plebanski sector (II-) with negative orientation.[2] This condition is then appropriately quantized and inserted into the expression for the vertex, leading to a modification of the EPRL vertex amplitude. The resulting vertex continues to be a function of a loop quantum gravity boundary state and hence may still be used to define dynamics *for loop quantum gravity*. It furthermore remains linear in the boundary state and fully $SU(2)$ invariant — two conditions forming a non-trivial requirement restricting the possible expressions for the vertex. It is also in a precise sense $Spin(4)$ invariant. Lastly, as is shown in the final section of this paper, for a complete set of boundary states, the asymptotics of the vertex include only a single term, equal to the exponential of i times the Regge action, enabling the correct equations of motion to dominate in the classical limit. We call the resulting vertex amplitude the *proper EPRL vertex amplitude*. A natural generalization to the Lorentzian case is presented in section 4.4. A summary of these results can be found in [20].

We begin the paper with a review of the classical discrete framework underlying the spin-foam model and derive the condition isolating the Einstein-Hilbert sector. Then, after briefly reviewing the existing EPRL vertex amplitude, the definition of the new proper vertex is introduced. The last half of the paper is then spent proving the properties summarized above. We then close with a discussion.

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[1] From [15], this is true also for the Freidel-Krasnov model, as must be the case as it is equal to EPRL for $\gamma < 1$. In [15], one finds two terms, not one, in the asymptotics. Furthermore, the presence of only two terms is likely due to their reformulating the model as a discrete first order path integral and then imposing non-degeneracy, a procedure whose equivalent in the spin-foam language, needed for contact with canonical states, is not known [15].

[2] In a prior version of this article, the sector corresponding to the first term in the asymptotics was mischaracterized as the (II+) sector, whereas in fact it is the combination of sectors stated here. This mistake was due to an error in the prior work [18] which was corrected in [19]. The correction of this error did not at all change the proper vertex or its motivation rooted in the semiclassical limit, but only changed the interpretation in terms of Plebanski sectors and orientations.

2 Classical analysis

2.1 Background

2.1.1 Generalities

We use the same definitions as in [18]. Let $\tau^i := \frac{-i}{2}\sigma^i$ $(i = 1, 2, 3)$, where σ^i are the Pauli matrices. For each element $\lambda \in \mathfrak{su}(2)$, $\lambda^i \in \mathbb{R}^3$ shall denote its components with respect to the basis τ^i. Let I denote the 2×2 identity matrix. We also freely use the isomorphism between $\mathfrak{spin}(4) := \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{so}(4)$, $(J_-, J_+) \equiv (J_-^i \tau_i, J_+^i \tau_i) \leftrightarrow J^{IJ}$ $(I, J = 0, 1, 2, 3)$, explicitly given by

$$\begin{aligned} J^{ij} &= \epsilon^{ij}{}_k (J_+^k + J_-^k) \\ J^{0i} &= J_+^i - J_-^i. \end{aligned} \tag{2.1}$$

J_+^i and J_-^i are called the self-dual and anti-self-dual parts of J^{IJ}. Furthermore, we remind the reader [14] of the explicit expression for the action of $Spin(4) = SU(2) \times SU(2)$ group elements on $\mathbb{R}^4$. For each $V^I \in \mathbb{R}^4$ define

$$\zeta(V) := V^0 I + i\sigma_i V^i. \tag{2.2}$$

Then the action of $G = (X^-, X^+)$ is given by

$$\zeta(G \cdot V) = X^- \zeta(V)(X^+)^{-1}. \tag{2.3}$$

2.1.2 Discrete classical framework

Spin-foam models of quantum gravity are based on a formulation of gravity as a *constrained BF theory*, using the ideas of Plebanski [21]. In the continuum, the basic variables are an $\mathfrak{so}(4)$ connection ω_μ^{IJ} and an $\mathfrak{so}(4)$-valued two-form $B_{\mu\nu}^{IJ}$, which we call the *Plebanski two-form*, where lower case greek letters are used for space-time manifold indices. The action is

$$S = \frac{1}{2\kappa} \int (B + \frac{1}{\gamma} {}^\star B)_{IJ} \wedge F^{IJ}, \tag{2.4}$$

with $F := \mathrm{d}\omega + \omega \wedge \omega$ the curvature of ω, * the Hodge dual on internal indices $I, J, K \ldots$, $\kappa := 8\pi G$, and $\gamma \in \mathbb{R}^+$ the Barbero-Immirzi parameter. If $B_{\mu\nu}^{IJ}$ satisfies what we call the Plebanski constraint [22,23], it must be one of the five forms

(I±) $B^{IJ} = \pm e^I \wedge e^J$ for some e_μ^I

(II±) $B^{IJ} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL} e^K \wedge e^L$ for some e_μ^I

(deg) $\epsilon_{IJKL}\eta^{\mu\nu\rho\sigma} B_{\mu\nu}^{IJ} B_{\rho\sigma}^{KL} = 0$ (degenerate case)

which we call *Plebanski sectors*. Here ϵ_{IJKL} denotes the internal Levi-Civita array, and $\eta^{\mu\nu\rho\sigma}$ denotes the Levi-Civita tensor of density weight 1. In sectors (II±), the BF action reduces to a sign times the *Holst action* for gravity [24],

$$S_{Holst} = \frac{1}{4\kappa} \int \left(\epsilon_{IJKL} e^K \wedge e^L + \frac{2}{\gamma} e_I \wedge e_J \right) \wedge F^{IJ}, \tag{2.5}$$

the Legendre transform of which forms the starting point for loop quantum gravity [7, 24].

In spin-foam quantization, one usually introduces a simplicial discretization of space-time. However, in this paper we concern ourselves with the so-called 'vertex amplitude', which may be thought of as the transition amplitude for a single 4-simplex. For clarity, we thus focus on a single oriented 4-simplex S. The EPRL model has also been generalized to general cell-complexes [12]; however because we use the work [14], and because we introduce formulae that, so far, apply only to 4-simplices, we restrict the discussion to the case of a 4-simplex. In S, number the tetrahedra $a = 0, \ldots, 4$,[3] and let Δ_{ab} denote the triangle between tetrahedra a and b, oriented as part of the boundary of a. One thinks of each tetrahedron, as well as the 4-simplex itself, as having its own 'frame' [10]. One has a parallel transport map from each tetrahedron to the 4-simplex frame, yielding in our case 5 parallel transport maps $G_a = (X_a^-, X_a^+) \in Spin(4)$, $a = 0, \ldots, 4$. The continuum two-form B is then represented by the algebra elements $B_{ab} = \int_{\Delta_{ab}} B$, where each element is treated as being 'in the frame at a.' For each ab, in terms of self-dual and anti-self-dual parts, these elements are related to the momenta conjugate to the parallel transports (see section 3.2) by [9, 18]

$$(J_{ab}^{\pm})^i = \left(\frac{\gamma \pm 1}{\kappa\gamma}\right)(B_{ab}^{\pm})^i. \tag{2.6}$$

We call B_{ab} and J_{ab} the *canonical bivectors* due to their role in the canonical theory in section 3.2.

From the discrete data $\{B_{ab}^{IJ}, G_a\}$ one can reconstruct the continuum two-form $B_{\mu\nu}^{IJ}$ as follows. Fix a flat connection ∂_μ on the 4-simplex S, such that S is the convex hull of its vertices as determined by the affine structure defined by ∂_μ; we say such a flat connection is *adapted* to S. The choice of such a connection is unique up to diffeomorphism and hence is a pure gauge choice (see appendix A). If the data $\{B_{ab}^{IJ}, G_a\}$ satisfy (1.) closure, $\sum_{b\neq a} B_{ab}^{IJ} = 0$, and (2.) orientation, $G_a \triangleright B_{ab} = -G_b \triangleright B_{ba}$, then it has been proven [18, 25] that there exists a unique two-form field $B_{\mu\nu}^{IJ}$ on the manifold S, constant with respect to ∂_μ, such that

$$\mathcal{B}_{ab} := G_a \triangleright B_{ab} = \int_{\Delta_{ab}} B \tag{2.7}$$

for all $a \neq b$. Here the left hand side is the parallel transport of the bivectors B_{ab}^{IJ} to the '4-simplex frame', henceforth denoted $\mathcal{B}_{ab}$, and $\triangleright$ here and throughout the rest of the paper denotes the adjoint action. Both closure (1.) and orientation (2.) are imposed in the EPRL vertex in the sense that violations are suppressed exponentially [14]. In addition, the EPRL model imposes (3.) *linear simplicity*,

$$C_{ab}^I := \frac{1}{2}\mathcal{N}_J \epsilon^{JI}{}_{KL} B_{ab}^{KL} \approx 0, \tag{2.8}$$

where $\mathcal{N}^I := (1, 0, 0, 0)$, as a restriction on the allowed boundary states for each 4-simplex, as shall be reviewed in the quantum theory below. From (2.8), it follows that the continuum two-form $B_{\mu\nu}^{IJ}$ defined by (2.7) is in Plebanski sector (II+), (II-) or (deg) [18]. We represent this sector by a function $\nu(B_{\mu\nu})$, defined to be $+1$ if $B_{\mu\nu}$ is in (II+), -1 if $B_{\mu\nu}$ is in (II-), and 0 if $B_{\mu\nu}$ is degenerate. If $\nu(B_{\mu\nu}) \neq 0$, $B_{\mu\nu}$ furthermore defines an *orientation* of S, which can either agree or disagree with the fixed orientation of S used to define form integrals. We represent this dynamically defined orientation by its sign relative to the fixed orientation $\mathring{\epsilon}^{\mu\nu\rho\sigma}$ of S:

$$\omega(B_{\mu\nu}) := \mathrm{sgn}(\mathring{\epsilon}^{\mu\nu\rho\sigma}\epsilon_{IJKL}B_{\mu\nu}^{IJ}B_{\rho\sigma}^{KL}), \tag{2.9}$$

where, for convenience, sgn$(\cdot)$ is defined to be zero when its argument is zero. Because the only arbitrary choice in the construction of $B_{\mu\nu}^{IJ}$, that of the flat connection ∂_μ, is unique upto diffeomorphism, a diffeomorphism which, when chosen to preserve each face of S, must be orientation preserving, and

[3] In the prior work [18], the order of the numbering was used to code the orientation of S. This was done by imposing an *ordering condition* correlating the orientation of S to the numbering. However, in this paper, we present things in such a way that one does not need to code the orientation in the numbering, and so the numbering is left arbitrary.

because each Plebanski sector as well as the dynamically determined orientation is invariant under such diffeomorphisms, the functions $\nu(B_{\mu\nu}(\{\mathcal{B}_{ab}\},\partial))$ and $\omega(B_{\mu\nu}(\{\mathcal{B}_{ab}\},\partial))$ are independent of the choice of connection ∂_μ adapted to S, so that one can write simply $\nu(\{\mathcal{B}_{ab}\})$ and $\omega(\{\mathcal{B}_{ab}\})$. (For a more detailed derivation of this fact, see appendix A.) This reviews the sense, established in [18], in which the *classical* constraints imposed *quantum* mechanically in the EPRL model admit the three distinct, well-defined Plebanski sectors (II+), (II-), and (deg), as well as two possible dynamical orientations.[4]

2.1.3 Reduced boundary data

The set of canonical bivectors B^{IJ}_{ab} satisfying linear simplicity is parameterized by what we call *reduced boundary data* — one unit 3-vector n^i_{ab} per ordered pair ab, and one area A_{ab} per triangle (ab) — via

$$B_{ab} = \frac{1}{2}A_{ab}(-n_{ab}, n_{ab}). \tag{2.10}$$

From (2.6) and (2.10), the generators of internal spatial rotations in terms of the reduced boundary data are

$$L^i_{ab} = (J^-)^i_{ab} + (J^+)^i_{ab} = \frac{1}{\kappa\gamma}A_{ab}n^i_{ab}. \tag{2.11}$$

The corresponding bivectors in the 4-simplex frame then take the form

$$\mathcal{B}_{ab} = B^{\text{phys}}_{ab}(A_{ab}, n_{ab}, G_a) := \frac{1}{2}A_{ab}(-X^-_a \triangleright n_{ab}, X^+_a \triangleright n_{ab}). \tag{2.12}$$

We call (2.12) the 'physical' bivectors reconstructed from A_{ab}, n_{ab}, G_a. In terms of the reduced boundary data, closure and orientation become the conditions $\sum_{b\neq a} A_{ab}n_{ab} = 0$ and $X^\pm_a \triangleright n_{ab} = -X^\pm_b \triangleright n_{ba}$.

2.1.4 Reconstruction theorem

In addition to reconstructing the 2-form field $B^{IJ}_{\mu\nu}$ from the bivectors $\mathcal{B}_{ab} = B^{\text{phys}}_{ab}(A_{ab}, n_{ab}, G_a)$, one can also reconstruct a geometrical 4-simplex in $\mathbb{R}^4$. This will be needed in the present paper. Let M denote $\mathbb{R}^4$ as an oriented manifold, equipped with the canonical $\mathbb{R}^4$ metric. A *geometrical 4-simplex* σ in M is the convex hull of 5 points, called vertices, in M, not all of which lie in the same 3-plane. We define a *numbered 4-simplex* σ to be a geometrical 4-simplex with tetrahedra numbered $0, \ldots 4$. Given a numbered 4-simplex in M, the associated *geometrical bivectors* $(B^{\text{geom}}_{ab})^{IJ}$ are defined as $(B^{\text{geom}}_{ab})^{IJ} := A(\Delta_{ab})\frac{(N_a\wedge N_b)^{IJ}}{|N_a\wedge N_b|}$, where $A(\Delta_{ab})$ is the area of the triangle Δ_{ab} shared by tetrahedra a and b, and N^I_a is the outward unit normal to tetrahedron a, $(N_a \wedge N_b)^{IJ} := 2N^{[I}_a N^{J]}_b$, and $|X^{IJ}|^2 := \frac{1}{2}X^{IJ}X_{IJ}$.

A set of reduced boundary data $\{A_{ab}, n_{ab}\}$ is *non-degenerate* if, for each a, the span of the vectors n_{ab} with $b \neq a$ is three dimensional. We call two sets of $SU(2)$ group elements $\{U^1_a\}, \{U^2_a\}$ *equivalent*, $\{U^1_a\} \sim \{U^2_a\}$, if $\exists Y \in SU(2)$ and five signs ϵ_a such that

$$U^2_a = \epsilon_a Y U^1_a. \tag{2.13}$$

For the proof of the following partial version of theorem 3 in [14], see [14, 18].

[4] In a different sense based on discrete analogies, an awareness of the presence of the three Plebanski sectors was implicit already in [15].

Theorem 1 (Partial version of the reconstruction theorem)**.** *Let a set of non-degenerate reduced boundary data* $\{A_{ab}, n_{ab}\}$ *satisfying closure be given, as well as a set* $\{G_a\} \subset Spin(4)$, $a = 0, \ldots, 4$, *solving the orientation constraint, such that* $\{X_a^-\} \not\sim \{X_a^+\}$*. Then there exists a numbered 4-simplex* σ *in* $\mathbb{R}^4$*, unique up to inversion and translation, such that*

$$B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, G_a) = \mu B_{ab}^{\text{geom}}(\sigma) \tag{2.14}$$

for some $\mu = \pm 1$*, with* μ *independent of the ambiguity in* σ*.*

The sign μ in the above theorem is uniquely determined by the data $\{A_{ab}, n_{ab}, G_a\}$. In fact, as shown in [19], it is equal to the *product* of the sign corresponding to the Plebanski sector $\nu(B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, G_a))$ and the sign of the orientation $\omega(B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, G_a))$. Recall we have defined the Einstein-Hilbert sector to consist in two-forms $B_{\mu\nu}$ which are either in Plebanski sector (II+) with positive orienation or in Plebanski sector (II-) with negative orientation. The continuum two form $B_{\mu\nu}$ reconstructed from the bivectors $\{B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, G_a)\}$ will thus be in the Einstein-Hilbert sector (in which case we also say the bivectors are in the Einstein-Hilbert sector) *if and only if* $\mu = \nu\omega = +1$.

2.2 Explicit classical expression for the geometrical bivectors, and the restriction to the Einstein-Hilbert sector

We now come to the new part of the classical analysis.

Lemma 1. *Let* $\{A_{ab}, n_{ab}, G_a\}$ *be given satisfying the hypotheses of theorem 1 and let* σ *be the numbered 4-simplex gauranteed to exist by this theorem. Let* $\{N_a^I\}$ *denote the outward pointing normals to the tetrahedra of* σ*. Then*

$$N_a^I = \alpha_a (G_a \cdot \mathcal{N})^I \tag{2.15}$$

for some set of signs α_a*.*

Proof. We first note that

$$\begin{aligned}
(N_a \wedge N_b)^{IJ} (G_a \cdot \mathcal{N})_J &\propto B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_{ab}^{\pm})^{IJ} (G_a \cdot \mathcal{N})_J \\
&\propto \left[G_a \triangleright (-n_{ab}, n_{ab})\right]^{IJ} (G_a \cdot \mathcal{N})_J \\
&= (G_a)^I{}_K (G_a)^J{}_L (-n_{ab}, n_{ab})^{KL} (G_a)_{IM} \mathcal{N}^M \\
&= (G_a)^J{}_L (-n_{ab}, n_{ab})^{KL} \mathcal{N}_K = (G_a)^J{}_L (-n_{ab}, n_{ab})^{0L} = 0
\end{aligned}$$

where (2.14) was used in the first line, and (2.1) was used in the last line. Since this holds for all b, it follows that $G_a \cdot \mathcal{N}$ is proportional to N_a; as both of these vectors are unit, the the coefficient of proportionality must be ± 1 for each a. ■

For the following theorem and throughout the rest of the paper, let $\widehat{=}$ denote equality modulo multiplication by a *positive* real number.

Theorem 2. *Let* $\{A_{ab}, n_{ab}, G_a\}$ *be given satisfying the hypotheses of theorem 1 and let* σ *be the numbered 4-simplex gauranteed to exist by this theorem. Then*

$$B_{ab}^{\text{geom}}(\sigma) \widehat{=} \beta_{ab}(\{G_{a'b'}\}) (G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N}) \tag{2.16}$$

where

$$\beta_{ab}(\{G_{a'b'}\}) := -\text{sgn}\left[\epsilon_{ijk} (G_{ac} \cdot \mathcal{N})^i (G_{ad} \cdot \mathcal{N})^j (G_{ae} \cdot \mathcal{N})^k \epsilon_{lmn} (G_{bc} \cdot \mathcal{N})^l (G_{bd} \cdot \mathcal{N})^m (G_{be} \cdot \mathcal{N})^n\right] \tag{2.17}$$

with $\{c, d, e\} = \{0, \ldots, 4\} \setminus \{a, b\}$ *in any order, and* sgn *is defined to be zero when its argument is zero.*

Proof. Let $\{N_a^I\}$ be the outward pointing normals to the tetrahedra of σ. Then they satisfy the four-dimensional closure relation (see appendix B)

$$\sum_a V_a N_a^I = 0 \tag{2.18}$$

where $V_a > 0$ is the volume of the ath tetrahedron, implying

$$N_a^I = \frac{1}{V_a} \sum_{a' \neq a} V_{a'} N_{a'}^I. \tag{2.19}$$

Thus

$$\begin{aligned}
0 \quad &< \quad \epsilon(N_a, N_c, N_d, N_e)^2 = -\frac{V_b}{V_a}\epsilon(N_b, N_c, N_d, N_e)\epsilon(N_a, N_c, N_d, N_e) \\
&\widehat{=} \quad -\alpha_a\alpha_b\epsilon(G_b \cdot \mathcal{N}, G_c \cdot \mathcal{N}, G_d \cdot \mathcal{N}, G_e \cdot \mathcal{N})\epsilon(G_a \cdot \mathcal{N}, G_c \cdot \mathcal{N}, G_d \cdot \mathcal{N}, G_e \cdot \mathcal{N}) \\
&= \quad -\alpha_a\alpha_b\epsilon(\mathcal{N}, G_{bc} \cdot \mathcal{N}, G_{bd} \cdot \mathcal{N}, G_{be} \cdot \mathcal{N})\epsilon(\mathcal{N}, G_{ac} \cdot \mathcal{N}, G_{ad} \cdot \mathcal{N}, G_{ae} \cdot \mathcal{N}) \\
&= \quad -\alpha_a\alpha_b\epsilon_{ijk}(G_{bc} \cdot \mathcal{N})^i(G_{bd} \cdot \mathcal{N})^j(G_{be} \cdot \mathcal{N})^k\epsilon_{lmn}(G_{ac} \cdot \mathcal{N})^l(G_{ad} \cdot \mathcal{N})^m(G_{ae} \cdot \mathcal{N})^n
\end{aligned}$$

where $\{\alpha_a\}$ are the signs in lemma 1. Therefore

$$\beta_{ab}(\{G_{a'b'}\}) = \alpha_a\alpha_b \tag{2.20}$$

where $\beta_{ab}(\{G_{a'b'}\})$ is as in (2.17). We thus have

$$B_{ab}^{\text{geom}}(\sigma)\widehat{=}N_a \wedge N_b = \alpha_a\alpha_b(G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N}) = \beta_{ab}(\{G_{a'b'}\})(G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N}).$$

■

Throughout this paper, let $\beta_{ab}(\{G_{a'b'}\})$ be defined by (2.17), and for convenience we define $\tilde{B}_{ab}^{\text{geom}}(G_{a'}) := \beta_{ab}(\{G_{a'b'}\})(G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N})$, the right hand side of (2.16).

Because the expression $(G \cdot \mathcal{N})^i$ used above will appear often, it is useful to stop for a moment to prove some facts about it. From (2.2) and (2.3),

$$(G_{ab}\mathcal{N})^0 I + i\sigma_i(G_{ab} \cdot \mathcal{N})^i = \zeta(G_{ab} \cdot \mathcal{N}) = X_{ab}^- X_{ba}^+,$$

from which one obtains the alternate expression

$$(G_{ab} \cdot \mathcal{N})^i = \text{tr}(\tau^i X_{ab}^- X_{ba}^+). \tag{2.21}$$

The meaning of this latter expression in turn is made clear in the following definition.

Definition 1. *Given $g \in SU(2)$ not equal to $\pm I$, there exists a unique unit vector $n[g]^i \in \mathbb{R}^3$ and $\alpha[g] \in (0, 2\pi)$ satisfying*

$$g = \exp(\alpha[g] \cdot n[g] \cdot \tau) = \cos\left(\frac{\alpha[g]}{2}\right) + in[g] \cdot \sigma \sin\left(\frac{\alpha[g]}{2}\right). \tag{2.22}$$

We call $n[g]^i$ the proper axis of g.

In terms of the above definition, one has

$$(G_{ab} \cdot \mathcal{N})^i = \text{tr}(\tau_i X_{ab}^- X_{ba}^+) = \sin\left(\frac{\alpha[X_{ab}^- X_{ba}^+]}{2}\right) n[X_{ab}^- X_{ba}^+]^i. \tag{2.23}$$

Lemma 2. *Let $\{A_{ab}, n_{ab}, G_a\}$ be given satisfying the hypotheses of theorem 1 and let σ be the numbered 4-simplex thereby gauranteed to exist. Then*

$$\mu = B^{\mathrm{geom}}_{ab}(\sigma)_{IJ} B^{\mathrm{phys}}_{ab}(A_{ab}, n_{ab}, G_a)^{IJ} \widehat{=} \beta_{ab}(\{G_{a'b'}\}) \mathrm{tr}(\tau_i X^-_{ab} X^+_{ba}) L^i_{ab}. \tag{2.24}$$

Proof. Starting from (2.14) and theorem 2,

$$\begin{aligned}
\mu &= B^{\mathrm{geom}}_{ab}(\sigma)_{IJ} B^{\mathrm{phys}}_{ab}(A_{ab}, n_{ab}, G_a)^{IJ} \\
&\widehat{=} \beta_{ab}(\{G_{a'b'}\}) \left[(G_a \cdot \mathcal{N}) \wedge (G_b \cdot \mathcal{N})\right]_{IJ} \frac{1}{2} A_{ab} \left[G_a \triangleright (-n_{ab}, n_{ab})\right]^{IJ} \\
&= \frac{1}{2} A_{ab} \beta_{ab}(\{G_{a'b'}\}) \left[\mathcal{N} \wedge (G_{ab} \cdot \mathcal{N})\right]_{IJ} \left[-n_{ab}, n_{ab}\right]^{IJ} \\
&= A_{ab} \beta_{ab}(\{G_{a'b'}\}) \left[\mathcal{N} \wedge (G_{ab} \cdot \mathcal{N})\right]_{0i} \left[-n_{ab}, n_{ab}\right]^{0i} \\
&= 2 A_{ab} \beta_{ab}(\{G_{a'b'}\}) (G_{ab} \cdot \mathcal{N})_i n^i_{ab} \widehat{=} \beta_{ab}(\{G_{a'b'}\}) (G_{ab} \cdot \mathcal{N})_i L^i_{ab} \\
&= \beta_{ab}(\{G_{a'b'}\}) \mathrm{tr}(\tau_i X^-_{ab} X^+_{ba}) L^i_{ab}.
\end{aligned}$$

■

We now come to the classical condition isolating the Einstein-Hilbert sector.

Theorem 3. *Let a set of non-degenerate reduced boundary data $\{A_{ab}, n_{ab}\}$ satisfying closure be given, as well as a set $\{G_a\} \subset Spin(4), a = 0, \ldots 4$ solving the orientation constraint. Then $B^{\mathrm{phys}}_{ab}(A_{ab}, n_{ab}, G_a)$ is in the Einstein-Hilbert sector iff*

$$\beta_{ab}(\{G_{a'b'}\}) \mathrm{tr}(\tau_i X^-_{ab} X^+_{ba}) L^i_{ab} > 0 \tag{2.25}$$

for any one pair a, b.

Proof.

($\Rightarrow$) Suppose $B^{\mathrm{phys}}_{ab}(A_{ab}, n_{ab}, G_a)^{IJ}$ is in the Einstein-Hilbert sector. Then by theorem 3 in [18,19], $\{X^-_a\} \not\sim \{X^+_a\}$, so that μ exists, and $\mu = 1$. Lemma 2 then implies (2.25).

($\Leftarrow$) Suppose (2.25) holds. Suppose by way of contradiction $\{X^-_a\} \sim \{X^+_a\}$. Then $\mathrm{tr}(\tau^i X^-_{ab} X^+_{ba}) = 0$ contradicting (2.25). Therefore $\{X^-_a\} \not\sim \{X^+_a\}$. Lemma 2 together with (2.25) then implies $\mu = +1$, so that theorem 3 in [18, 19] implies $B^{\mathrm{phys}}_{ab}(A_{ab}, n_{ab}, G_a)^{IJ}$ is in the Einstein-Hilbert sector. ■

3 Review of quantum framework and the EPRL vertex

3.1 Notation for $SU(2)$ and $Spin(4)$ structures.

Let V_j denote the carrying space for the spin j representation of $SU(2)$, and $\rho_j(g), \rho_j(x)$ the representation of $g \in SU(2)$ and $x \in \mathfrak{su}(2)$ thereon, with the j subscript dropped when it is clear from the context. Let $\hat{L}^i := i\rho(\tau^i)$ denote the generators in each of these representation according to the context. Let $\epsilon : V_j \times V_j \to \mathbb{C}$ denote the invariant bilinear epsilon inner product, and $\langle \cdot, \cdot \rangle$ the hermitian inner product, on V_j [4, 14]. These inner products determine an antilinear structure map $J : V_j \to V_j$ by $\epsilon(\psi, \phi) = \langle J\psi, \phi \rangle$. J commutes with all group representation matrices, so that it anti-commutes with all generators.

Let $V_{j^-,j^+} = V_{j^-} \otimes V_{j^+}$ denote the carrying space for the spin (j^-, j^+) representation of $Spin(4) \equiv SU(2) \times SU(2)$, and $\rho_{j^-,j^+}(X^-, X^+) := \rho_{j^-}(X^-) \otimes \rho_{j^+}(X^+)$ the representation of $(X^-, X^+) \in$

$Spin(4)$ thereon, again with the subscript dropped when it is clear from the context. $\hat{J}^i_- := i\rho(\tau^i)\otimes I_{j^+}$ and $\hat{J}^i_+ := iI_{j^-}\otimes\rho(\tau^i)$ are then the anti-self-dual and self-dual generators respectively, so that $\hat{L}^i := \hat{J}^i_- + \hat{J}^i_+$ are the generators of spatial rotations on V_{j^-,j^+}. Define the bilinear form $\epsilon : V_{j^+,j^-}\times V_{j^+,j^-}\to\mathbb{C}$ by $\epsilon(\psi^+\otimes\psi^-,\phi^+\otimes\phi^-) := \epsilon(\psi^+,\phi^+)\epsilon(\psi^-,\phi^-)$, and the antilinear map $J : V_{j^-,j^+}\to V_{j^-,j^+}$ by $J : \psi^+\otimes\psi^-\mapsto(J\psi^+)\otimes(J\psi^-)$, so that

$$\epsilon(\Psi,\Phi) = \langle J\Psi,\Phi\rangle. \tag{3.1}$$

As in the case of the $SU(2)$ representations, all $Spin(4)$ representation operators commute with J, and all generators anticommute with J. Lastly, let $\iota_k^{j^-,j^+}$ denote the intertwining map from V_k to $V_{j^-}\otimes V_{j^+}$, scaled such that it is isometric in the Hilbert space inner products.

3.2 Canonical phase space, kinematical quantization, and the EPRL vertex

In the general boundary formulation of quantum mechanics [4], one associates to the boundary of any 4-dimensional region a *phase space*, whose quantization yields the *boundary Hilbert space* of the theory for that region. In the present case, the region is the 4-simplex S. The boundary data consists in the algebra elements B_{ab} and J_{ab} in the frame of each tetrahedron a, and for each pair of tetrahedra a,b one has a parallel transport map G_{ab} from b to a, related to the G_a introduced in section 2.1.2 by $G_{ab} = (G_a)^{-1}G_b$. These boundary data form a classical phase space isomorphic to the cotangent bundle over any choice of ten independent parallel transport maps $G_{ab} = (X^+_{ab}, X^-_{ab})$, $\Gamma = T^*(Spin(4)^5) = T^*((SU(2)\times SU(2))^5)$, which for simplicity we choose to be the ten with $a<b$. For $a<b$, $J_{ab} = (J^-_{ab}, J^+_{ab})$ and $J_{ba} = (J^-_{ba}, J^+_{ba})$ respectively generate right and left translations on G_{ab}.

The boundary Hilbert space of states $\mathcal{H}^{Spin(4)}_{\partial S}$ is the L^2 space over the ten $G_{ab} = (X^-_{ab}, X^+_{ab})\in Spin(4)$ with $a<b$. The momenta operators $\hat{J}^\pm_{ab}$ and $\hat{J}^\pm_{ba}$ then act by i times right and left invariant vector fields, respectively, on the elements $X^\pm_{ab}$, and, in terms of these, $\hat{L}^i_{ab} := (\hat{J}^-_{ab})^i + (\hat{J}^+_{ab})^i$. One can define an overcomplete basis of $\mathcal{H}^{Spin(4)}_{\partial S}$, the *projected spin-network states* (see [26, 27]), each element of which is labeled by four spins $j^\pm_{ab}, k_{ab}, k_{ba}$ and two states $\psi_{ab}\in V_{k_{ab}}, \psi_{ba}\in V_{k_{ba}}$ per triangle:

$$\Psi_{\{j^\pm_{ab},k_{ab},\psi_{ab}\}}(G_{ab}) := \prod_{a<b}\epsilon(\iota^{j^-_{ab},j^+_{ab}}_{k_{ab}}\psi_{ab}, \rho(G_{ab})\iota^{j^-_{ab},j^+_{ab}}_{k_{ba}}\psi_{ba}). \tag{3.2}$$

When acting on such a state, the operators $\hat{L}^i_{ab}, \hat{L}^i_{ba}$ act specifically on the irreducible representation (irrep) vectors ψ_{ab},ψ_{ba}:

$$\hat{L}^i_{ab}\Psi_{\{j^\pm_{cd},k_{cd},\psi_{cd}\}} = \epsilon(\iota^{j^-_{ab},j^+_{ab}}_{k_{ab}}\hat{L}^i\psi_{ab}, \rho(G_{ab})\iota^{j^-_{ab},j^+_{ab}}_{k_{ba}}\psi_{ba})\prod_{c<d,(cd)\neq(ab)}\epsilon(\iota^{j^-_{cd},j^+_{cd}}_{k_{cd}}\psi_{cd}, \rho(G_{cd})\iota^{j^-_{cd},j^+_{cd}}_{k_{dc}}\psi_{dc}), \tag{3.3}$$

$$\hat{L}^i_{ba}\Psi_{\{j^\pm_{cd},k_{cd},\psi_{cd}\}} = \epsilon(\iota^{j^-_{ab},j^+_{ab}}_{k_{ab}}\psi_{ab}, \rho(G_{ab})\iota^{j^-_{ab},j^+_{ab}}_{k_{ba}}\hat{L}^i\psi_{ba})\prod_{c<d,(cd)\neq(ab)}\epsilon(\iota^{j^-_{cd},j^+_{cd}}_{k_{cd}}\psi_{cd}, \rho(G_{cd})\iota^{j^-_{cd},j^+_{cd}}_{k_{dc}}\psi_{dc}). \tag{3.4}$$

In terms of the projected spin-network over-complete basis, the linear simplicity constraint, when quantized as in [9], is equivalent to

$$k_{ab} = \frac{2j^-_{ab}}{|1-\gamma|} = \frac{2j^+_{ab}}{|1+\gamma|} = k_{ba} \tag{3.5}$$

for all $a\neq b$. The projected spin networks satisfying linear simplicity are thus parameterized by one spin k_{ab} and two states $\psi_{ab},\psi_{ba}\in V_{k_{ab}}$ per triangle (ab), the same parameters specifying a generalized

$SU(2)$ *spin-network state* of LQG:

$$\Psi_{\{k_{ab},\psi_{ab}\}}(X_{ab}) := \prod_{a<b} \epsilon(\psi_{ab}, \rho(X_{ab})\psi_{ba}) \in \mathcal{H}^{LQG}_{\partial S} \equiv L^2(SU(2)^{10}). \tag{3.6}$$

Because $j^{\pm}_{ab} = \frac{1}{2}|1 \pm \gamma|k_{ab}$ are always half-integers, one deduces that only certain values of the spins k_{ab} are allowed; let $\mathcal{K}_\gamma$ be this set of allowable values, and let $\mathcal{H}^{\gamma}_{\partial S}$ be the span of the $SU(2)$ spin-networks (3.6) with $\{k_{ab}\} \subset \mathcal{K}_\gamma$. One has an embedding

$$\begin{array}{rcl} \iota : \mathcal{H}^{\gamma}_{\partial S} & \to & \mathcal{H}^{Spin(4)}_{\partial S} \\ \Psi_{\{k_{ab},\psi_{ab}\}} & \mapsto & \Psi_{\{s^{\pm}_{ab},k_{ab},\psi_{ab}\}} \end{array} \tag{3.7}$$

where here, and throughout the rest of the paper, we set

$$s^{\pm} := \frac{1}{2}|1 \pm \gamma|k. \tag{3.8}$$

Due to (3.3) and (3.4) (and because the $SU(2)$ spin-networks satisfy a similar property), this embedding in fact *intertwines* the spatial rotation generators $\hat{L}^i_{ab}$ in the $Spin(4)$ and $SU(2)$ theories. Through the embedding ι, the operators $\hat{L}^i_{ab}$ in the $SU(2)$ theory thus have the same physical meaning as the corresponding operators in the $Spin(4)$ boundary theory.

Having reviewed the above, the EPRL vertex for a given LQG boundary state $\Psi^{LQG}_{\{k_{ab},\psi_{ab}\}} \in \mathcal{H}^{\gamma}_{\partial S} \subset \mathcal{H}^{LQG}_{\partial S}$ is then

$$\begin{aligned} A_v(\{k_{ab},\psi_{ab}\}) &:= A_v(\Psi_{\{k_{ab},\psi_{ab}\}}) = \int_{\mathrm{Spin}(4)^5} \prod_a \mathrm{d}G_a (\iota\Psi_{\{k_{ab},\psi_{ab}\}})(G_{ab}) \\ &= \int_{\mathrm{Spin}(4)^5} \prod_a \mathrm{d}G_a \prod_{a<b} \epsilon(\iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} \psi_{ab}, \rho(G_{ab}) \iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} \psi_{ba}). \end{aligned} \tag{3.9}$$

4 Proposed proper EPRL vertex

4.1 Definition

Let us consider the structure of the original EPRL vertex amplitude (3.9): The integration over the group elements G_a can, in a precise sense, be interpreted as a "sum over histories" of parallel transports from the tetrahedra frames to the 4-simplex frames. This integration over the G_a's inside the vertex amplitude can be thought of as a remnant of the process of integrating out the discrete connection used to obtain the initial BF spin-foam model (see [28]). Furthermore, in the semiclassical analysis [14], one sees that the G_a's over which one integrates in (3.9) play precisely the role of such parallel transports. Given this interpretation of the G_a's, in order to impose the desired restriction to the Einstein-Hilbert sector, one must restrict the discrete history data G_a so that they satisfy the inequality (2.25):

$$\beta_{ab}(\{G_{a'b'}\}) \mathrm{tr}(\tau_i X^-_{ab} X^+_{ba}) L^i_{ab} > 0. \tag{4.1}$$

Normally one would do this by inserting into the path integral

$$\Theta(\beta_{ab}(\{G_{a'b'}\}) \mathrm{tr}(\tau_i X^-_{ab} X^+_{ba}) L^i_{ab}) \tag{4.2}$$

where Θ is the Heaviside function, defined to be zero when its argument is zero. However, in the integral (3.9), it is not the classical quantity L^i_{ab} that appears, but rather *states* ψ_{ab} in irreducible representations of the corresponding operators $\hat{L}^i_{ab}$.[5] As noted in equations (3.3) and (3.4), $\hat{L}^i_{ab}$ acts on ψ_{ab} via the $SU(2)$ generators $\hat{L}^i$. Therefore, we partially 'quantize' the expression (4.2) by replacing L^i_{ab} with the generators $\hat{L}^i$, yielding the following G_a-dependent operator acting in the spin k_{ab} representation of $SU(2)$:

$$P_{ba}(\{G_{a'b'}\}) := P_{(0,\infty)}\left(\beta_{ab}(\{G_{a'b'}\})\mathrm{tr}(\tau_i X^-_{ba} X^+_{ab})\hat{L}^i\right), \tag{4.3}$$

where $P_{\mathcal{S}}(\hat{O})$ denotes the spectral projector onto the portion $\mathcal{S} \subset \mathbb{R}$ of the spectrum of the operator $\hat{O}$. Inserting (4.3) into the face factors of (3.9) we obtain what we call the *proper EPRL vertex amplitude*:

$$A_v^{(+)}(\{k_{ab},\psi_{ab}\}) := \int_{\mathrm{Spin}(4)^5} \prod_a \mathrm{d}G_a \prod_{a<b} \epsilon(\iota^{s^-_{ab}s^+_{ab}}_{k_{ab}}\psi_{ab}, \rho(G_{ab})\iota^{s^-_{ab}s^+_{ab}}_{k_{ab}} P_{ba}(\{G_{a'b'}\})\psi_{ba}). \tag{4.4}$$

Let us stop for a moment and remark on the properties of this vertex amplitude. First, as the EPRL vertex, it depends on an $SU(2)$ spin network boundary state and hence may be used to construct a spin-foam model *for loop quantum gravity.* It is linear in the SU(2) boundary state, as required for the final spin-foam amplitude to be linear in the initial state and anti-linear in the final state. Furthermore, as we will show in the next subsection, it is invariant under SU(2) gauge transformations. Finally, and most importantly, as we will show in the next section, its asymptotics only include the single term $e^{iS_{\mathrm{Regge}}}$, as desired.

Throughout the rest of this paper, the notation $P_{ba}(\{G_{a'b'}\})$ introduced in (4.3) will also refer to the projector acting in the spin (s^-_{ab}, s^+_{ab}) representation of $Spin(4)$, defined by the same expression (4.3). In each statement using the notation $P_{ba}(\{G_{a'b'}\})$, either the context will determine which projector is intended, or the statement will hold for both projectors.

Finally, let us briefly note two ways to rewrite the proper vertex: (1.) It may at first appear arbitrary that the projector was inserted on the right side of each face factor in equation (4.4). However, in fact, one can put the projector (appropriately transformed) anywhere in each face-factor, and the vertex amplitude doesn't change. See appendix D. (2.) We note that, using equation (3.1), one has the following equivalent expression for the proper vertex:

$$A_v^{(+)}(\{k_{ab},\psi_{ab}\}) := \int_{\mathrm{Spin}(4)^5} \prod_a \mathrm{d}G_a \prod_{a<b} \langle J\iota^{s^-_{ab}s^+_{ab}}_{k_{ab}}\psi_{ab}, \rho(G_{ab})\iota^{s^-_{ab}s^+_{ab}}_{k_{ab}} P_{ba}(\{G_{a'b'}\})\psi_{ba}\rangle. \tag{4.5}$$

4.2 Proof of invariance under $SU(2)$ gauge transformations

Theorem 4. *The proper EPRL vertex is invariant under arbitrary $SU(2)$ gauge transformations at the tetrahedra.*

Proof. Let $\{k_{ab},\psi_{ab}\}$ be the data for a given spin network on the boundary, and let five $SU(2)$ elements h_a, one at each tetrahedron, be given. We wish to show
$A_v^{(+)}(\Psi_{\{k_{ab},\rho(h_a)\psi_{ab}\}}) = A_v^{(+)}(\Psi_{\{k_{ab},\psi_{ab}\}})$.

[5] If one uses coherent boundary data as will be done in the next section, then one does have a classical label L^i_{ab} present, but one would still not be able to simply insert the factor (4.2), as, due to the overcompleteness of the set of coherent states, this would lead to a vertex amplitude that is not linear in the boundary state, something necessary to ensure the final transition amplitude defined by the spin-foam sum is linear in the boundary state.

First, define $\tilde{G}_{ab} := (h_a, h_a)^{-1} \circ G_{ab} \circ (h_b, h_b)$. Then

$$\begin{aligned}
\left(\tilde{G}_{ab} \cdot \mathcal{N}\right)^i &= \mathrm{tr}(\tau^i \tilde{X}^-_{ab} \tilde{X}^+_{ba}) = \mathrm{tr}(\tau^i h_a^{-1} X^-_{ab} X^+_{ba} h_a) \\
&= \mathrm{tr}((h_a \tau^i h_a^{-1}) X^-_{ab} X^+_{ba}) = h_a \triangleright \mathrm{tr}(\tau^i X^-_{ab} X^+_{ba}) \\
&= h_a \triangleright (G_{ab} \cdot \mathcal{N})^i .
\end{aligned} \tag{4.6}$$

From this and the $SO(3)$ invariance of ϵ_{ijk}, it follows that

$$\beta_{ab}(\{\tilde{G}_{a'b'}\}) = \beta_{ab}(\{G_{a'b'}\}). \tag{4.7}$$

We thus have

$$\begin{aligned}
\rho(h_b)^{-1} P_{ba}(\{G_{a'b'}\}) \rho(h_b) &= \rho(h_b)^{-1} P_{(0,\infty)} \left(\beta_{ab}(\{G_{a'b'}\})(G_{ba} \cdot \mathcal{N})_i L^i\right) \rho(h_b) \\
&= P_{(0,\infty)} \left(\beta_{ab}(\{G_{a'b'}\})[(h_b)^{-1} \triangleright (G_{ba} \cdot \mathcal{N})_i] L^i\right) \\
&= P_{(0,\infty)} \left(\beta_{ab}(\{\tilde{G}_{a'b'}\})(\tilde{G}_{ba} \cdot \mathcal{N})_i L^i\right) \\
&= P_{ba}(\{\tilde{G}_{a'b'}\})
\end{aligned} \tag{4.8}$$

where lemma 10 has been used in the second line, and (4.6) and (4.7) have been used in the third. Using (4.8), we finally have

$$\begin{aligned}
A_v^{(+)}(\{k_{ab}, \rho(h_a)\psi_{ab}\}) &:= \int \left(\prod_{a<b} \mathrm{d}G_{ab}\right) \prod_{a<b} \epsilon \left(\iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} \rho(h_a)\psi_{ab}, \rho(G_{ab}) \iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} P_{ba}(\{G_{a'b'}\}) \rho(h_b) \psi_{ba}\right) \\
&= \int \left(\prod_{a<b} \mathrm{d}G_{ab}\right) \prod_{a<b} \epsilon \left(\iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} \rho(h_a)\psi_{ab}, \rho(G_{ab}) \iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} \rho(h_b) P_{ba}(\{\tilde{G}_{a'b'}\}) \psi_{ba}\right) \\
&= \int \left(\prod_{a<b} \mathrm{d}G_{ab}\right) \prod_{a<b} \epsilon \left(\iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} \psi_{ab}, \rho(h_a, h_a)^{-1} \rho(G_{ab}) \rho(h_b, h_b) \iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} P_{ba}(\{\tilde{G}_{a'b'}\}) \psi_{ba}\right) \\
&= \int \left(\prod_{a<b} \mathrm{d}G_{ab}\right) \prod_{a<b} \epsilon \left(\iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} \psi_{ab}, \rho(\tilde{G}_{ab}) \iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} P_{ba}(\{\tilde{G}_{a'b'}\}) \psi_{ba}\right) \\
&= \int \left(\prod_{a<b} \mathrm{d}\tilde{G}_{ab}\right) \prod_{a<b} \epsilon \left(\iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} \psi_{ab}, \rho(\tilde{G}_{ab}) \iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} P_{ba}(\{\tilde{G}_{a'b'}\}) \psi_{ba}\right) \\
&= A_v^{(+)}(\{k_{ab}, \psi_{ab}\})
\end{aligned}$$

where we have used in the third line the intertwining property of $\iota^{s^-_{ab} s^+_{ab}}_{k_{ab}}$ and in the second to last line the right and left invariance of the Haar measure. ■

4.3 $Spin(4)$ invariance

As mentioned in section 2, in defining the classical discrete variables $\{G_a, B_{ab}\}$, one thinks of each tetrahedron as having its own 'frame'. Concretely, this is manifested in the fact that there exists a local $Spin(4)$ gauge transformation acting at each tetrahedron. Given a choice of $Spin(4)$ group element H_a at each tetrahedron a, one has the following gauge transformation:

$$(\{H_{a'}\}) \cdot G_a = G_a H_a, \qquad (\{H_{a'}\}) \cdot B_{ab} = H_a \triangleright B_{ab}. \tag{4.9}$$

The definition of the proper vertex (4.4) makes key use of a fixed internal direction $\mathcal{N}^I = (1,0,0,0)$. This vector is used to impose the simplicity constraints (2.8) at each tetrahedron, and superficially breaks the above $Spin(4)$ gauge symmetry. Furthermore, in order to embed LQG states into BF states solving simplicity, the proper vertex uses the map $\iota_k^{s^- s^+}$, which is defined using a specific embedding $h : g \mapsto (g,g)$ of $SU(2)$ into $Spin(4)$ via the symmetry condition

$$\iota_k^{s^- s^+} \circ \rho(g) = \rho(h(g,g)) \circ \iota_k^{s^- s^+}. \tag{4.10}$$

This use of h also also seems to break the above $Spin(4)$ symmetry. The fixed vector $\mathcal{N}^I$ and embedding h are related by the fact that the $SO(4)$ action of every element in the image of h preserves $\mathcal{N}^I$. (The original EPRL vertex amplitude uses these two exact same extra structures [9,11].)

$Spin(4)$ acts on the unit vector $\mathcal{N}^I$ by its $SO(4)$ action, while it acts on the map $\iota_k^{s^- s^+}$ via

$$(\Lambda \cdot \iota)_k^{s^-,s^+} := \rho(\Lambda) \circ \iota_k^{j^-,j^+} \tag{4.11}$$

for $\Lambda \in Spin(4)$. The transformed map $(\Lambda \cdot \iota)_k^{j^-,j^+} : V_k \to V_{j^-,j^+}$ still satisfies a symmetry condition similar to (4.10), but with a different embedding $(\Lambda \cdot h) : SU(2) \to Spin(4)$:

$$(\Lambda \cdot \iota)_k^{s^-,s^+} \circ \rho(g) = \rho((\Lambda \cdot h)(g)) \circ (H \cdot \iota)_k^{s^-,s^+} \tag{4.12}$$

where $(\Lambda \cdot h)(g) := \Lambda h(g) \Lambda^{-1}$.

In this section we consider what happens when, in the definition of the proper vertex, the unit vector $\mathcal{N}^I$ and the map $\iota_k^{s^- s^+}$ are replaced, at each tetrahedron a, by their transformation under an arbitrary $Spin(4)$ element Λ_a. The resulting, a priori possibly modified proper vertex amplitude we denote by ${}^{\{\Lambda_a\}}A_v^{(+)}$. An arbitrary $Spin(4)$ gauge transformation $\{H_a\}$ then acts on ${}^{\{\Lambda_a\}}A_v^{(+)}$ via

$${}^{\{\Lambda_a\}}A_v^{(+)} \mapsto {}^{\{H_a \Lambda_a\}}A_v^{(+)}. \tag{4.13}$$

We shall prove that the generalized proper vertex ${}^{\{\Lambda_a\}}A_v^{(+)}$ is in fact independent of $\{\Lambda_a\}$, and so is trivially invariant under the above action and in this sense is $Spin(4)$ invariant at each tetrahedron. This result is similar to that in [29].

We begin by noting how to write $A_v^{(+)}$ in a way that makes its dependence on $\mathcal{N}^I$ explicit, which then allows us to write down explicitly the generalized proper vertex ${}^{\{\Lambda_a\}}A_v^{(+)}$, after which we prove its independence of $\{\Lambda_a\}$. From the first line of equation (D.5),

$$A_v^{(+)}(\{k_{ab}, \psi_{ab}\}) = \int_{\mathrm{Spin}(4)^5} \prod_a \mathrm{d}G_a \prod_{a<b} \epsilon(\iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ab}, \rho(G_{ab}) P_{ba}(\{G_{a'b'}\}) \iota_{k_{ba}}^{s_{ba}^- s_{ba}^+} \psi_{ba}). \tag{4.14}$$

The above projector $P_{ba}(\{G_{a'b'}\})$ on $V_{s_{ab}^-, s_{ab}^+}$ can be written

$$P_{ba}(\{G_{a'b'}\}) := P_{(0,\infty)}(\beta_{ba}(\{G_{a'b'}\}) \epsilon_{IJKL} \mathcal{N}^I (G_{ba} \cdot \mathcal{N})^J \hat{J}^{KL}) \tag{4.15}$$

with

$$\begin{aligned} \beta_{ba}(\{G_{a'b'}\}) \quad := \quad & -\mathrm{sgn}\left[\epsilon_{IJKL} \mathcal{N}^I (G_{ac} \cdot \mathcal{N}^J (G_{ad} \cdot \mathcal{N})^K (G_{ae} \cdot \mathcal{N})^L \cdot \right. \\ & \left. \cdot \epsilon_{MNPQ} \mathcal{N}^M (G_{bc} \cdot \mathcal{N})^N (G_{bd} \cdot \mathcal{N})^P (G_{be} \cdot \mathcal{N})^Q \right] \end{aligned}$$

with $\{c,d,e\} = \{0,\ldots,4\} \setminus \{a,b\}$ in any order. This immediately yields the following expression for the generalized proper vertex:

$${}^{\{\Lambda_{a'}\}}A_v^{(+)}(\{k_{ab}, \psi_{ab}\}) = \int_{\mathrm{Spin}(4)^5} \prod_a \mathrm{d}G_a \prod_{a<b} \epsilon(\rho(\Lambda_a) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ab}, \rho(G_{ab}) \, {}^{\{\Lambda_{a'}\}}P_{ba}(\{G_{a'b'}\}) \rho(\Lambda_b) \iota_{k_{ba}}^{s_{ba}^- s_{ba}^+} \psi_{ba}). \tag{4.16}$$

where

$$
{}^{\{\Lambda_{a'}\}}P_{ba}(\{G_{a'b'}\}) := P_{(0,\infty)}({}^{\{\Lambda_{a'}\}}\beta_{ba}(\{G_{a'b'}\})\epsilon_{IJKL}(\Lambda_b \cdot \mathcal{N})^I (G_{ba}\Lambda_a \cdot \mathcal{N})^J \hat{J}^{KL}) \tag{4.17}
$$

with

$$
\begin{aligned}
{}^{\{\Lambda_{a'}\}}\beta_{ba}(\{G_{a'b'}\}) \;\; := \;\; -\mathrm{sgn}\big[&\epsilon_{IJKL}(\Lambda_a \cdot \mathcal{N})^I (G_{ac}\Lambda_c \cdot \mathcal{N})^J (G_{ad}\Lambda_d \cdot \mathcal{N})^K (G_{ae}\Lambda_e \cdot \mathcal{N})^L \,\cdot \\
&\cdot\epsilon_{MNPQ}(\Lambda_b \cdot \mathcal{N})^M (G_{bc}\Lambda_c \cdot \mathcal{N})^N (G_{bd}\Lambda_d \cdot \mathcal{N})^P (G_{be}\Lambda_e \cdot \mathcal{N})^Q\big]
\end{aligned}
$$

Theorem 5. ${}^{\{\Lambda_{a'}\}}A_v^{(+)}(\{k_{ab}, \psi_{ab}\}) = A_v^{(+)}(\{k_{ab}, \psi_{ab}\})$ *for all* $\{\Lambda_{a'}\} \subset Spin(4)$.

Proof.

Let $\tilde{G}_a := \Lambda_a G_a$. We begin by proving (i.) ${}^{\{\Lambda_{a'}\}}\beta_{ba}(\{G_{ab}\}) = \beta_{ba}(\{\tilde{G}_{ab}\})$, and (ii.) ${}^{\{\Lambda_{a'}\}}P_{ba}(\{G_{a'b'}\}) = \rho(\Lambda_b) \circ P_{ba}(\{\tilde{G}_{a'b'}\}) \circ \rho(\Lambda_b)^{-1}$. Using these facts in (4.16), together with the $Spin(4)$ invariance of the ϵ-inner product on V_{j^-,j^+} and right invariance of the Haar measure, then yields the result.

(i.)

$$
\begin{aligned}
{}^{\{\Lambda_{a'}\}}\beta_{ba}(\{G_{a'b'}\}) \;\; &:= \;\; -\mathrm{sgn}\big[\epsilon_{IJKL}(\Lambda_a \cdot \mathcal{N}_a)^I (G_{ac}\Lambda_c \cdot \mathcal{N}_c)^J (G_{ad}\Lambda_d \cdot \mathcal{N}_d)^K (G_{ae}\Lambda_e \cdot \mathcal{N}_e)^L \cdot \\
&\qquad \cdot\epsilon_{MNPQ}(\Lambda_b \cdot \mathcal{N}_b)^M (G_{bc}\Lambda_c \cdot \mathcal{N}_c)^N (G_{bd}\Lambda_d \cdot \mathcal{N}_d)^P (G_{be}\Lambda_e \cdot \mathcal{N}_e)^Q\big] \\
&= \;\; -\mathrm{sgn}\Big[\epsilon_{IJKL}(\Lambda_a \cdot \mathcal{N}_a)^I (\Lambda_a\tilde{G}_{ac} \cdot \mathcal{N}_c)^J (\Lambda_a\tilde{G}_{ad} \cdot \mathcal{N}_d)^K (\Lambda_a\tilde{G}_{ae} \cdot \mathcal{N}_e)^L \cdot \\
&\qquad \cdot\epsilon_{MNPQ}(\Lambda_b \cdot \mathcal{N}_b)^M (\Lambda_b\tilde{G}_{bc} \cdot \mathcal{N}_c)^N (\Lambda_b\tilde{G}_{bd} \cdot \mathcal{N}_d)^P (\Lambda_b\tilde{G}_{be} \cdot \mathcal{N}_e)^Q\Big] \\
&= \;\; -\mathrm{sgn}\Big[\epsilon_{IJKL}\mathcal{N}_a^I (\tilde{G}_{ac} \cdot \mathcal{N}_c)^J (\tilde{G}_{ad} \cdot \mathcal{N}_d)^K (\tilde{G}_{ae} \cdot \mathcal{N}_e)^L \,\cdot \\
&\qquad \cdot\epsilon_{MNPQ}\mathcal{N}_b^M (\tilde{G}_{bc} \cdot \mathcal{N}_c)^N (\tilde{G}_{bd} \cdot \mathcal{N}_d)^P (\tilde{G}_{be} \cdot \mathcal{N}_e)^Q\Big] \\
&= \;\; \beta_{ba}(\{\tilde{G}_{a'b'}\})
\end{aligned}
$$

where the $SO(4)$ invariance of ϵ_{IJKL} was used.

(ii.)

$$
\begin{aligned}
{}^{\{\Lambda_{a'}\}}P_{ba}(\{G_{a'b'}\}) \;\; &:= \;\; P_{(0,\infty)}(\beta_{ba}(\{\tilde{G}_{a'b'}\})\epsilon_{IJKL}(\Lambda_b \cdot \mathcal{N})^I (G_{ba}\Lambda_a \cdot \mathcal{N})^J \hat{J}^{KL}) \\
&:= \;\; P_{(0,\infty)}(\beta_{ba}(\{\tilde{G}_{a'b'}\})\epsilon_{IJKL}(\Lambda_b \cdot \mathcal{N})^I (\Lambda_b\tilde{G}_{ba} \cdot \mathcal{N})^J \hat{J}^{KL}) \\
&:= \;\; P_{(0,\infty)}(\beta_{ba}(\{\tilde{G}_{a'b'}\})\epsilon_{IJKL}\mathcal{N}^I (\tilde{G}_{ba} \cdot \mathcal{N})^J (\Lambda_b^{-1})^K{}_M (\Lambda_b^{-1})^L{}_N \hat{J}^{MN}) \\
&:= \;\; P_{(0,\infty)}(\rho(\Lambda_b)\beta_{ba}(\{\tilde{G}_{a'b'}\})\epsilon_{IJKL}\mathcal{N}^I (\tilde{G}_{ba} \cdot \mathcal{N})^J \hat{J}^{MN} \rho(\Lambda_b)^{-1}) \\
&:= \;\; \rho(\Lambda_b) \circ P_{(0,\infty)}(\beta_{ba}(\{\tilde{G}_{a'b'}\})\epsilon_{IJKL}\mathcal{N}^I (\tilde{G}_{ba} \cdot \mathcal{N})^J \hat{J}^{MN}) \circ \rho(\Lambda_b)^{-1} \\
&:= \;\; \rho(\Lambda_b) \circ P_{ba}(\{\tilde{G}_{a'b'}\}) \circ \rho(\Lambda_b)^{-1}
\end{aligned}
$$

where result (i.) was used in the first line, the $SO(4)$ invariance of ϵ_{IJKL} in the third line, and the $Spin(4)$ covariance of the generators $\hat{J}^{KL}$ in the fourth line. ■

4.4 Lorentzian generalization

We close this section by noting that there is an obvious generalization of the expression (4.4) of the proper vertex to the Lorentzian signature. In the Lorentzian EPRL model [9,30], one uses the unitary representations of $SL(2,\mathbb{C})$, which are labeled by a real number ρ together with an integer n. Denote the carrying space for such representations by $V^{\rm Lor}_{\rho,n}$, and let $\rho(G)$ denote the representation thereon of $G \in SL(2,\mathbb{C})$. $V^{\rm Lor}_{\rho,n}$ decomposes into an infinite direct sum of irreducible representations of $SU(2)$:

$$V^{\rm Lor}_{\rho,n} = \oplus_{k=n/2}^{\infty} V_k \tag{4.18}$$

where in the sum j is incremented in steps of 1. The analogue of the embedding $\iota_k^{s^- s^+} : V_k \to V_{s^-,s^+}$ in the Lorentzian case is the embedding $\mathcal{I}_k : V_k \to V^{\rm Lor}_{2\gamma k, 2k}$ mapping V_k into the lowest k component of $V^{\rm Lor}_{2\gamma k,2k}$ in the sum (4.18). The elements in the image of this embedding satisfy a quantization of the simplicity constraints just as those of $\iota_k^{s^- s^+}$ do in the Euclidean case [9]. Furthermore, just as one has the invariant bilinear form ϵ on V_{s^-,s^+}, related to the hermitian inner product via the anti-linear map J, so too one has an invariant bilinear form β on $V^{\rm Lor}_{\rho,n}$, related to the hermitian inner product on $V^{\rm Lor}_{\rho,n}$ via an anti-linear map $\mathcal{J}$ in the same way [31]. For simple representations, $(j^+, j^-) = (s^-, s^+)$, $(\rho, n) = (2\gamma k, 2k)$, ϵ and β furthermore have the same (anti-)symmetry properties: $\epsilon(\psi,\phi) = (-1)^{2k}\epsilon(\phi,\psi)$, $\beta(\psi,\phi) = (-1)^{2k}\beta(\phi,\psi)$. In terms of these structures, the expression for the Lorentzian EPRL vertex amplitude is exactly analogous to the Euclidean expression (3.9) [9,31]:

$$A_v^{\rm Lor}(\{k_{ab},\psi_{ab}\}) = \int_{{\rm SL}(2,\mathbb{C})^4} \prod_{a\neq 4} {\rm d}G_a \prod_{a<b} \beta(\mathcal{I}_{k_{ab}}\psi_{ab}, \rho(G_{ab})\mathcal{I}_{k_{ab}}\psi_{ba}), \tag{4.19}$$

the only notable difference being that one of the group integrations is dropped in order to ensure finiteness of the amplitude [32, 33]. One can then modify this vertex amplitude in exactly the same way as was done in the Euclidean case, to yield a Lorentzian version of the proper EPRL vertex:

$$A_v^{(+),{\rm Lor}}(\{k_{ab},\psi_{ab}\}) := \int_{{\rm SL}(2,\mathbb{C})^4} \prod_{a\neq 4} {\rm d}G_a \prod_{a<b} \beta(\mathcal{I}_{k_{ab}}\psi_{ab}, \rho(G_{ab})\mathcal{I}_{k_{ab}} P_{ba}(\{G_{a'b'}\})\psi_{ba}). \tag{4.20}$$

where

$$P_{ba}(\{G_{a'b'}\}) := P_{(0,\infty)}\left(\beta_{ab}(\{G_{a'b'}\})(G\cdot\mathcal{N})_i \hat{L}^i\right), \tag{4.21}$$

with

$$\beta_{ab}(\{G_{a'b'}\}) := -{\rm sgn}\left[\epsilon_{ijk}(G_{ac}\cdot\mathcal{N})^i(G_{ad}\cdot\mathcal{N})^j(G_{ae}\cdot\mathcal{N})^k \epsilon_{lmn}(G_{bc}\cdot\mathcal{N})^l(G_{bd}\cdot\mathcal{N})^m(G_{be}\cdot\mathcal{N})^n\right] \tag{4.22}$$

with $\{c,d,e\} = \{0,\ldots,4\}\setminus\{a,b\}$ in any order, and where $G\cdot\mathcal{N}$ denotes the $SO(1,3)$ action of G on $\mathcal{N}^I = (1,0,0,0)$. Though this generalization of the proper vertex to the Lorentzian signature is natural, and it is difficult to imagine how otherwise to generalize to this case, nevertheless one should justify this generalization more systematically, by quantizing an appropriate classical condition isolating the Lorentzian Einstein-Hilbert sector. One should also check whether the above generalization has the required semi-classical limit, as we will prove below is the case for the Euclidean proper vertex. (Henceforth in this paper, unless otherwise indicated, "proper vertex" shall again always refer to "Euclidean proper vertex.")

5 Asymptotics

In the following we state and prove the asymptotics of the proper vertex, using key results from [14].

5.1 Statement of the formula

It will be useful for later purposes to define the following before defining coherent states.

Definition 2. *Given any unit $n^i \in \mathbb{R}^3$, let $|n; k, m\rangle$ denote the eigenstate of $n \cdot \hat{L}$ in V_k with eigenvalue m, and $|n; j^-, j^+, k, m\rangle$ the eigenstate of $\hat{L}^2$ and $n \cdot \hat{L}$ in V_{j^-, j^+} with eigenvalues $k(k+1)$ and m, with phase fixed arbitrarily for each set of labels.*

Definition 3. *Given a unit 3-vector n, a spin j, and a phase θ, we define the corresponding coherent state as*

$$|n, \theta\rangle_j := e^{i\theta}|n; j, j\rangle. \tag{5.1}$$

The θ argument represents a phase freedom, and will usually be suppressed. Additionally, when the spin is clear from the context, it will be omitted. Such coherent states were first used in quantum gravity by Livine and Speziale [34].

We call an assignment of one spin $k_{ab} \in \mathcal{K}_\gamma$ and two unit 3-vectors n^i_{ab}, n^i_{ba} to each triangle (ab) in S a set of *quantum boundary data*. Given such data, the corresponding boundary *state* in the $SU(2)$ boundary Hilbert space of S is

$$\Psi_{\{k_{ab}, n_{ab}\}, \theta} := \Psi_{\{k_{ab}, \psi_{ab}\}} \qquad \text{with} \qquad |\psi_{ab}\rangle := |n_{ab}, \theta_{ab}\rangle_{k_{ab}} \tag{5.2}$$

where the θ_{ab} are any phases summing to θ modulo 2π. The phase θ will usually be suppressed. The state $\Psi_{\{k_{ab}, n_{ab}\}}$ so defined is a coherent boundary state corresponding to the classical reduced boundary data $A_{ab} = A(k_{ab}) := \kappa\gamma k_{ab}$ and n_{ab}.

When $\{A(k_{ab}), n_{ab}\}$ is non-degenerate and satisfies closure, we also say that $\{k_{ab}, n_{ab}\}$ is non-degenerate and satisfies closure. In this case, for each tetrahedron a, there exists a geometrical tetrahedron in $\mathbb{R}^3$, unique up to translations, such that $\{A(k_{ab})\}_{b \neq a}$ and $\{n^i_{ab}\}_{b \neq a}$ are the areas and outward unit normals, respectively, of the four triangular faces, which we denote by $\{\Delta^t_{ab}\}_{b \neq a}$. If these five geometrical tetrahedra can be glued together consistently to form a 4-simplex, we say that the boundary data $\{k_{ab}, n_{ab}\}$ is *Regge-like*. For such data, there exists a set of $SU(2)$ elements $\{g_{ab} = g_{ba}^{-1}\}$, unique up to a $\mathbb{Z}_2$ lift ambiguity [14], such that the adjoint action of each g_{ab} on $\mathbb{R}^3$ maps (1.) Δ^t_{ab} into Δ^t_{ba}, and (2.) n_{ba} into $-n_{ab}$. These group elements can be used to completely remove the phase ambiguity in the boundary state (5.2), by requiring the phase of the coherent states to be chosen such that $g_{ab}|n_{ba}\rangle_{k_{ab}} = J|n_{ab}\rangle_{k_{ab}}$, where J is as defined in section 3.1. The resulting boundary state $\Psi_{\{k_{ab}, n_{ab}\}}$ is called the *Regge state* determined by $\{k_{ab}, n_{ab}\}$, and is denoted by $\Psi^{\text{Regge}}_{\{k_{ab}, n_{ab}\}}$.

The following theorem, as theorem 1 in [14], uses the fact that, because the boundary data $\{k_{ab}, n_{ab}\}$ determine the geometry of all boundary tetrahedra, it also determines the geometry of the 4-simplex itself [14, 35], and hence, in particular, the dihedral angles $\Theta_{ab} \in [0, \pi]$ via the equation $N_a \cdot N_b = \cos \Theta_{ab}$ where N_a and N_b are the outward pointing normals to the ath and bth tetrahedra, respectively.

Theorem 6 (Proper EPRL asymptotics)**.** *If $\{k_{ab}, n_{ab}\}$ is boundary data representing a non-degenerate Regge geometry, then*

$$A^{(+)}_v(\Psi^{\text{Regge}}_{\lambda k_{ab}, n_{ab}}) \sim \left(\frac{2\pi}{\lambda}\right)^{12} N^\gamma_{+-} \exp\left(i \sum_{a<b} A(\lambda k_{ab})\Theta_{ab}\right) \tag{5.3}$$

where N^γ_{+-} is the Hessian factor calculated in [14]. If $\{k_{ab}, n_{ab}\}$ does not represent a non-degenerate Regge geometry, then $A^{(+)}_v(\Psi_{\lambda k_{ab}, n_{ab}, \theta})$ decays exponentially with large λ for any choice of phase θ.

To prove this theorem, in manner similar to [14], we cast the proper vertex in appropriate integral form $A^{(+)}_v = \int \mathrm{d}\mu(x) e^{S_{\gamma<1}(x)}$ and $A^{(+)}_v = \int \mathrm{d}\mu(x) e^{S_{\gamma>1}(x)}$, separately for the cases $\gamma < 1$ and $\gamma > 1$, where $S_{\gamma<1}$ and $S_{\gamma>1}$ are "actions". We then determine the critical points for each action. In proving this theorem, we are interested in critical points whose contributions are not exponentially suppressed. For this reason, we define the term "critical point" to mean points where the action is stationary and its real part is *non-negative*. If a point in the domain of integration is such that the real part of the action is an absolute maximum and is non-negative, we shall say it is a *maximal point*.

5.2 Integral expressions and critical points

In the following, whenever we say the words "critical points" with no other qualification, we refer to critical points of the proper EPRL vertex (4.4).

5.2.1 The case $\gamma < 1$

The relevant integral form of the proper vertex in this case is

$$A_v^{(+)}(\Psi_{\{k_{ab},n_{ab}\},\theta}) = \int \prod_a \mathrm{d}G_a \exp(S_{\gamma<1}) \tag{5.4}$$

where

$$\exp(S_{\gamma<1}) = \prod_{a<b} \langle J \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ab}, \quad \rho(G_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ba}(\{G_{a'b'}\}) n_{ba} \rangle. \tag{5.5}$$

The action $S_{\gamma<1}$ is, as in [14], generally complex. The two conditions that determine critical points are maximality and stationarity. In both proving the equations for maximality and checking stationarity, it will be simplest to reuse the results in [14]. This will highlight the simplicity of the additional steps necessary for the present modification. Recall from [14] that the action for $\gamma < 1$ for the original EPRL model is

$$\exp(S_{\gamma<1}^{\mathrm{EPRL}}) = \prod_{a<b} \langle J \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ab}, \quad \rho(G_{ab})^{-1} \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ba} \rangle. \tag{5.6}$$

For the purpose of the following lemmas and the rest of this section, a set of group elements together with boundary data $\{G_a, k_{ab}, n_{ab}\}$ is said to satisfy *proper orientation* if, for all $a \neq b$, $\beta_{ab}(\{G_{a'b'}\}) \mathrm{tr}(\tau_i X_{ab}^- X_{ba}^+) n_{ab}^i > 0$.

Lemma 3. *Given boundary data $\{k_{ab}, n_{ab}\}$, $\{G_a\}$ is a maximal point of $S_{\gamma<1}$ iff orientation and proper orientation are satisfied.*

Proof. Because (5.5) and (5.6) only differ by insertion of the projectors $P_{ba}(\{G_{a'b'}\})$, and recalling from [14] that $|\exp(S_{\gamma<1}^{\mathrm{EPRL}})| \leq 1$, one immediately has

$$\exp(2\mathrm{Re}\, S_{\gamma<1}) = |\exp(S_{\gamma<1})| \leq |\exp(S_{\gamma<1}^{\mathrm{EPRL}})| \leq 1 \tag{5.7}$$

From [14], the second $\leq$ is an equality iff orientation is satisfied. The first $\leq$ is an equality iff the inserted projectors act as unity, i.e., iff $P_{ba}(\{G_{a'b'}\})|n_{ba}\rangle = |n_{ba}\rangle$, which, if orientation holds, is equivalent to proper orientation, $\beta_{ab}(\{G_{a'b'}\}) \mathrm{tr}(\tau_i X_{ab}^- X_{ba}^+) n_{ab}^i > 0$. Thus, $\{G_{a'b'}\}$ is a maximal point, so that both inequalities are saturated, iff orientation and proper orientation hold. ■

Lemma 4. *Let boundary data $\{k_{ab}, n_{ab}\}$ be given, and suppose $\{G_a\}$ is a maximal point of $S_{\gamma<1}$. Then it is also a stationary point of $S_{\gamma<1}$ iff closure is additionally satisfied.*

Proof. If δ is any variation of the group elements G_a, from (5.5), (5.6) and the fact that $\{G_a\}$ is maximal, one immediately has

$$\delta \exp(S_{\gamma<1}) = \delta \exp(S_{\gamma<1}^{\mathrm{EPRL}}) + \prod_{a<b} \langle J \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} n_{ab}, \quad \rho(G_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \left(\delta P_{ba}(\{G_{a'b'}\})\right) n_{ba} \rangle. \tag{5.8}$$

From lemma 9.c,

$$P_{ba}(\{G_{a'b'}\}) \circ \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} = \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \circ P_{ba}(\{G_{a'b'}\}). \tag{5.9}$$

Taking the variation of both sides, and using the result in (5.8),

$$\delta \exp(S_{\gamma<1}) = \delta \exp(S^{\mathrm{EPRL}}_{\gamma<1}) + \prod_{a<b} \langle J \iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} n_{ab}, \quad \rho(G_{ab}) \left(\delta P_{ba}(\{G_{a'b'}\})\right) \iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} n_{ba} \rangle. \tag{5.10}$$

From lemma 3, as $\{G_a\}$ is a maximal point, orientation is satisfied. Using this,

$$\begin{aligned} \rho(G_{ba}) J \iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} |n_{ab}\rangle \quad &\propto \quad \rho(G_{ba}) J |n_{ab}; s^-_{ab}, s^+_{ab}, k_{ab}, k_{ab}\rangle \ \propto \ \rho(G_{ba}) | -n_{ab}; s^-_{ab}, s^+_{ab}, k_{ab}, k_{ab}\rangle \\ &\propto \quad \rho(G_{ba}) \left[| -n_{ab}; s^-_{ab}, s^-_{ab}\rangle \otimes | -n_{ab}; s^+_{ab}, s^+_{ab}\rangle \right] \\ &\propto \quad | -X^-_{ba} \triangleright n_{ab}; s^-_{ab}, s^-_{ab}\rangle \otimes | -X^+_{ba} \triangleright n_{ab}; s^+_{ab}, s^+_{ab}\rangle \\ &\propto \quad |n_{ba}; s^-_{ab}, s^-_{ab}\rangle \otimes |n_{ba}; s^+_{ab}, s^+_{ab}\rangle \propto |n_{ba}; s^-_{ab}, s^+_{ab}, k_{ab}, k_{ab}\rangle \end{aligned} \tag{5.11}$$

Where lemma 9.b was used in line 1 and $k_{ab} = s^-_{ab} + s^+_{ab}$, was used in lines 2 and 4.

We next claim that $|n_{ba}; s^-_{ab}, s^+_{ab}, k_{ab}, k_{ab}\rangle$ is an eigenstate of $\mathrm{tr}(\tau_i X^-_{ab} X^+_{ba}) \hat{L}^i$. If $X^-_{ab} X^+_{ba} = \pm I$, then $\mathrm{tr}(\tau_i X^-_{ab} X^+_{ba}) = 0$ and the claim is trivially true. If, on the other hand, $X^-_{ab} X^+_{ba} \neq \pm I$, then from equation (52) in [14], $X^-_{ab} X^+_{ba} = \exp(\lambda_{ab} n_{ab} \cdot \tau)$ for some λ_{ab}, so that $\pm n^i_{ab} = n[X^-_{ab} X^+_{ba}]^i \widehat{=} \mathrm{tr}(\tau_i X^-_{ab} X^+_{ba})$, and the claim follows from the definition of $|n_{ba}; s^-_{ab}, s^+_{ab}, k_{ab}, k_{ab}\rangle$. In either case, the claim is proven. This, along with (5.11), $\iota^{s^-_{ab} s^+_{ab}}_{k_{ab}} |n_{ab}\rangle = |n_{ba}; s^-_{ab}, s^+_{ab}, k_{ab}, k_{ab}\rangle$, $P_{ba} = P_{(0,\infty)}(\beta_{ab} n[X^-_{ba} X^+_{ab}] \cdot L)$ and corollary 11 in appendix C, implies that the second term in (5.10) is zero. As proven in [14], using that orientation is satisfied, the remaining term in (5.10) is zero iff closure is satisfied. ■

Theorem 7. *Given boundary data $\{k_{ab}, n_{ab}\}$, $\{G_a\}$ is a critical point of $S_{\gamma<1}$ iff closure, orientation, and proper orientation are satisfied.*

Proof.

($\Rightarrow$) Suppose $\{G_a\}$ is a critical point of $S_{\gamma<1}$. Then lemma 3 implies that orientation and proper orientation are satisfied, and lemma 4 implies that closure is satisfied.

($\Leftarrow$) Suppose closure, orientation, and proper orientation are satisfied. Then by lemma 3, $\{G_a\}$ is a maximal point of $S_{\gamma<1}$, and by lemma 4 it is a stationary point of $S_{\gamma<1}$. ■

5.2.2 The case $\gamma > 1$

For this case, we derive from scratch an expression for the proper vertex analogous to (18) and (19) in [14]. In doing this, we use the spinorial form of the irreps of $SU(2)$. Let $A, B, C, \dots = 0, 1$ denote spinor indices. The carrying space V_j can then be realized as the space of symmetric spinors of rank $2j$ (see, for example, [4]). Let n^A denote the spinor corresponding to the coherent state $|n\rangle_{\frac{1}{2}}$. As in [14, 36], the key property of coherent states we use is that, in their spinorial form, the higher spin coherent states are given by

$$(|n\rangle_j)^{A_1 \cdots A_{2j}} = n^{A_1} \cdots n^{A_{2j}}. \tag{5.12}$$

From the relation (3.8) between k and s^+, s^- for a given triangle, one deduces for $\gamma > 1$ that $s^+ = s^- + k$. For this case, the explicit expression for $\iota^{s^- s^+}_k$ in terms of symmetric spinors is given in equations (A.12) and (A.13) of [4][6]. Let $v^{A_1 \cdots A_{2k}} \in V_k$ be given. For $\gamma > 1$, one has

$$\iota^{s^- s^+}_k (v)^{A_1 \cdots A_{2s^+} B_1 \cdots B_{2s^-}} = v^{(A_1 \cdots A_{2k}} \epsilon^{A_{2k+1}|B_1|} \cdots \epsilon^{A_{2s^+})B_{2s^-}} \tag{5.13}$$

[6] In (A.13) of [4], symmetrization over the A group, B group, and C group of indices was forgotten but was clear from the context.

where the symmetrization is over the A indices only. In order to impose the symmetrization over the A indices, similar to [14], on the left of each $\iota_k^{s^- s^+}$, acting in the self-dual part of the co-domain, we insert a resolution of the identity on V_{s^+} into coherent states:

$$d_{s^+} \int \mathrm{d}m |m\rangle_{s^+ \, s^+}\langle m| = I_{s^+} \tag{5.14}$$

where $\mathrm{d}m$ is the measure on the metric 2-sphere normalized to unit area, and $d_s := 2s+1$. In spinorial notation

$$d_{s^+} \int \mathrm{d}m \; m^{A_1} \cdots m^{A_{2s^+}} m^\dagger_{B_1} \cdots m^\dagger_{B_{2s^+}} = \delta^{(A_1}_{B_1} \cdots \delta^{A_{2s^+})}_{B_{2s^+}}. \tag{5.15}$$

where $m^\dagger_A := ({}_{\frac{1}{2}}\langle m|)_A$. Starting from equation (4.4) with $\psi_{ab} = |n_{ab}\rangle_{k_{ab}} = n_{ab}^{A_1} \cdots n_{ab}^{A_{2k_{ab}}}$, writing out all spinor indices explicitly, we insert two resolutions of the identity (5.15) into each face factor in (4.4), one after each $\iota_k^{s^- s^+}$. Denote the integration variables m_{ab} and m_{ba} respectively for the left and right insertions. Writing out the ϵ-inner product in terms of alternating tensors ϵ_{AB}, using $m^\dagger_A = -\epsilon_{AB}(Jm)^B$, simplifying, and then writing the final expression again in terms of hermitian inner products, one obtains

$$A_v^{(+)} = \int \prod_a \mathrm{d}G_a \left(\prod_{a<b} (-1)^{2s^-_{ab}} d^2_{s^+_{ab}} \mathrm{d}m_{ab} \mathrm{d}m_{ba} \right) \exp(S_{\gamma>1}) \tag{5.16}$$

where

$$\begin{aligned} \exp(S_{\gamma>1}) \quad &= \quad \prod_{a<b} {}_{k_{ab}}\langle m_{ab}|n_{ab}\rangle_{k_{ab}} \, {}_{s^+_{ab}}\langle Jm_{ab}|\rho(X^+_{ab})|m_{ba}\rangle_{s^+_{ab}} \\ &\qquad {}_{k_{ab}}\langle m_{ba}|P_{ba}(\{G_{a'b'}\})|n_{ba}\rangle_{k_{ab}} \overline{{}_{s^-_{ab}}\langle Jm_{ab}|\rho(X^-_{ab})|m_{ba}\rangle_{s^-_{ab}}}. \end{aligned} \tag{5.17}$$

Recall from [14] that the action for $\gamma > 1$ for the original EPRL model is[7]

$$\begin{aligned} \exp(S^{\mathrm{EPRL}}_{\gamma>1}) \quad &= \quad \prod_{a<b} {}_{k_{ab}}\langle m_{ab}|n_{ab}\rangle_{k_{ab}} \, {}_{s^+_{ab}}\langle Jm_{ab}|\rho(X^+_{ab})|m_{ba}\rangle_{s^+_{ab}} \\ &\qquad {}_{k_{ab}}\langle m_{ba}|n_{ba}\rangle_{k_{ab}} \overline{{}_{s^-_{ab}}\langle Jm_{ab}|\rho(X^-_{ab})|m_{ba}\rangle_{s^-_{ab}}}. \end{aligned} \tag{5.18}$$

Lemma 5. *Given boundary data $\{k_{ab}, n_{ab}\}$, $\{G_a, m_{ab}\}$ is a maximal point of $S_{\gamma>1}$ iff orientation and proper orientation are satisfied and $m_{ab} = n_{ab}$ for all $a \neq b$.*

Proof. Because (5.17) and (5.18) only differ by insertion of the projectors P_{ba}, and recalling from [14] that $|\exp(S^{\mathrm{EPRL}}_{\gamma>1})| \leq 1$, one has

$$\exp(2\mathrm{Re}\, S_{\gamma>1}) = |\exp(S_{\gamma>1})| \leq |\exp(S^{\mathrm{EPRL}}_{\gamma>1})| \leq 1. \tag{5.19}$$

From [14], the second $\leq$ is an equality iff orientation is satisfied and $m_{ab} = n_{ab}$ for all $a \neq b$. As in the $\gamma < 1$ case, the first $\leq$ is an equality iff the inserted projectors act as unity, which, if orientation is satisfied, is equivalent to proper orientation. It follows that $\{G_{a'b'}\}$ is a maximal point, so that both inequalities are saturated, iff orientation, proper orientation, and $m_{ab} = n_{ab}$ for all $a \neq b$, are satisfied. ■

[7]The coherent state $|m_{ab}\rangle$ used here is related to the corresponding coherent state used in [14] by the action of J.

Lemma 6. *Let boundary data $\{k_{ab}, n_{ab}\}$ be given, and suppose $\{G_a, m_{ab}\}$ is a maximal point of $S_{\gamma>1}$. Then it is also a stationary point of $S_{\gamma>1}$ iff closure is additionally satisfied.*

Proof. If δ is any variation of G_a and m_{ab}, from (5.17) and (5.18) one has

$$\begin{aligned} \delta \exp(S_{\gamma>1}) &= \delta \exp(S^{\mathrm{EPRL}}_{\gamma>1}) + \prod_{a<b} {}_{k_{ab}}\langle m_{ab}|n_{ab}\rangle_{k_{ab}} {}_{s^+_{ab}}\langle Jm_{ab}|\rho(X^+_{ab})|m_{ba}\rangle_{s^+_{ab}} \\ &\qquad {}_{k_{ab}}\langle m_{ba}|(\delta P_{ba}(\{G_{a'b'}\}))|n_{ba}\rangle_{k_{ab}} \overline{{}_{s^-_{ab}}\langle Jm_{ab}|\rho(X^-_{ab})|m_{ba}\rangle_{s^-_{ab}}} \end{aligned} \tag{5.20}$$

Because $\{G_a, m_{ab}\}$ is a maximal point, from lemma 5, orientation and proper orientation are satisfied, and $m_{ab} = n_{ab}$ for all $a \neq b$. It follows that $|n_{ba}\rangle_{k_{ab}} = |m_{ba}\rangle_{k_{ab}}$ and both are eigenstates of $P_{ba}(\{G_{a'b'}\})$ with eigenvalue 1, so that by corollary 11 in appendix D, the second term above is zero. As proven in [14], because orientation is satisfied and $m_{ab} = n_{ab}$ for all $a \neq b$, it follows that the remaining term in (5.20) is zero iff closure is satisfied. ■

Theorem 8. *Given boundary data $\{k_{ab}, n_{ab}\}$, $\{G_a, m_{ab}\}$ is a critical point of $S_{\gamma>1}$ iff closure, orientation, and proper orientation are satisfied, and $m_{ab} = n_{ab}$ for all $a \neq b$.*

Proof.

($\Rightarrow$) Suppose $\{G_a, m_{ab}\}$ is a critical point of $S_{\gamma>1}$. Then lemma 5 implies that orientation and proper orientation are satisfied and $m_{ab} = n_{ab}$ for all $a \neq b$, and lemma 6 implies that closure is satisfied.

($\Leftarrow$) Suppose closure, orientation, and proper orientation are satisfied and $m_{ab} = n_{ab}$ for all $a \neq b$. Then by lemma 5, $\{G_a, m_{ab}\}$ is a maximal point of $S_{\gamma>1}$, and by lemma 6 it is a stationary point of $S_{\gamma>1}$. ■

Thus, though in the $\gamma > 1$ case one has an extra set of variables $\{m_{ab}\}$, these are restricted to be equal to $\{n_{ab}\}$ by the critical point equations, allowing one to treat the $\gamma < 1$ and $\gamma > 1$ cases in a unified way. The remaining critical point conditions on $\{G_a, n_{ab}\}$ (given in theorem 7) have a symmetry: if $\{G_a\}$ form a solution, then so does the set of group elements $\{\tilde{G}_a = (\tilde{X}^-_a, \tilde{X}^+_a)\}$ with

$$\tilde{X}^\pm_a = \epsilon^\pm_a Y^\pm X^\pm_a \tag{5.21}$$

for any $(Y^-, Y^+) \in Spin(4)$ and any set of ten signs $\epsilon^\pm_a$. This transformation is also a symmetry of the actions (5.5) and (5.17). If two solutions $\{G_a\}$, $\{\tilde{G}_a\}$ are related by such a symmetry transformation, we call them *equivalent* and write $\{G_a\} \sim \{\tilde{G}_a\}$.

5.3 Proof of the asymptotic formula

Using the above results, we proceed to prove theorem 6.

Before getting into the details of the proof, we summarize its general structure. As already mentioned, the critical point equations for the proper vertex integrals (5.4) and (5.16) have a set of symmetries (5.21), of which the global $Spin(4)$ symmetry is the only continuous one. In order to apply the stationary phase method to calculate the asymptotics, the critical points must be isolated, and hence this continuous symmetry must be removed. As in [14], we do this by performing the change of variables $\tilde{G}_a := (G_0)^{-1} G_a$ for $a = 1, \ldots 4$. Then G_0 no longer appears in the integrand, so that the G_0 integral drops out. Upon removing the tilde labels, the remaining integrand is the same as the original integrand except with G_0 replaced by the identity. In what follows $G_a = (X^-_a, X^+_a)$ shall denote these "gauge-fixed" group elements, with $G_0 \equiv \mathrm{id}$, in terms of which the continuous symmetry has been removed.

The proof then has two steps, the first of which has already been done in theorems 7 and 8 above: (1.) prove that the critical points of proper EPRL are precisely the subset of critical points of original EPRL at which proper orientation is satisfied. (2.) prove that, given a set of $SU(2)$ boundary data $\{k_{ab}, n_{ab}\}$, the critical points of original EPRL at which proper orientation is satisfied are all equivalent and are precisely the critical points which give rise to the asymptotic term (5.3) in the original EPRL asymptotics [14]. Because proper orientation is satisfied, the projector P_{ba} will act as the identity, and the value of the proper EPRL action at these critical points will be the same as the value of the original EPRL action at these points, yielding precisely the asymptotic behavior (5.3) claimed.

Let us begin by reviewing the results from theorems 7 and 8. The critical point equations for $\gamma < 1$ and $\gamma > 1$ are equivalent: the only difference is that for $\gamma > 1$ one integrates over extra variables, m_{ab}, which, however, come with the critical point equations $m_{ab} = n_{ab}$, eliminating them. This allows us to effectively consider both the $\gamma < 1$ case and $\gamma > 1$ case simultaneously in the following. As given in theorems 7 and 8, the remaining critical point equations are

$$X_a^{\pm} \triangleright n_{ab} = -X_b^{\pm} \triangleright n_{ba} \tag{5.22}$$

and

$$\beta_{ab} \mathrm{tr}(\tau_i X_{ab}^- X_{ba}^+) L_{ab}^i > 0 \tag{5.23}$$

for all $a < b$. The first of these, (5.22), is of the same form for both $\{X_a^+\}$ and $\{X_a^-\}$:

$$U_a \triangleright n_{ab} = -U_b \triangleright n_{ba}. \tag{5.24}$$

One therefore proceeds by finding the solutions $\{U_a\}$ to (5.24) for a given set of $SU(2)$ boundary data $\{k_{ab}, n_{ab}\}$, and then from these one constructs the solutions $\{G_a\}$ to (5.22), and then one checks which among these, if any, solves (5.23) in order to determine the critical points of the vertex integral.

The solutions to (5.24) have already been analyzed by [14]. To use the results of this analysis, one needs the notion of a *vector geometry*: A set of boundary data $\{k_{ab}, n_{ab}\}$ is called a *vector geometry* if it satisfies closure and there exists $\{h_a\} \subset SO(3)$ such that $(h_a \triangleright n_{ab})^i = -(h_b \triangleright n_{ba})^i$ for all $a \neq b$. The authors of [14] then proceed by considering separately the three cases in which the boundary data (i.) does not define a vector geometry (ii.) defines a vector geometry which is, however, not a non-degenerate 4-simplex geometry, and (iii.) defines a non-degenerate 4-simplex geometry. We use this same division and consider each of these three cases in turn.

Case (i.): Not a vector geometry.

In this case, as proven in [14], there are no solutions to (5.24) and hence no solutions to (5.22), and hence no critical points. The vertex integral therefore decays exponentially with λ.

Case (ii.): A vector geometry, but no non-degenerate 4-simplex geometry.

In this case, as proven in [14], there is exactly one solution to (5.24), upto the equivalence (2.13). The only solution to (5.22) is therefore $(X_a^-, X_a^+) = (U_a, \epsilon_a Y U_a)$. But then $X_{ba}^- X_{ab}^+ = \pm I$, so that this solution fails to satisfy condition (5.23), so that there are no critical points. The vertex integral therefore decays exponentially with λ.

Case (iii.): A non-degenerate 4-simplex geometry.

In this case, as proven in [14], (5.24) has two inequivalent solutions $\{U_a^1\}$ and $\{U_a^2\}$, so that there are four inequivalent solutions to (5.22): $(X_a^-, X_a^+) = (U_a^1, U_a^1), (U_a^2, U_a^2), (U_a^1, U_a^2), (U_a^2, U_a^1)$. Neither (U_a^1, U_a^1) nor (U_a^2, U_a^2), nor any solution equivalent to these, satisfies (5.23), again because $X_{ba}^- X_{ab}^+ = \pm I$. It remains only to consider the solutions

$$\begin{aligned} (\overset{1}{X}{}_a^-, \overset{1}{X}{}_a^+) &= (U_a^1, U_a^2) \\ (\overset{2}{X}{}_a^-, \overset{2}{X}{}_a^+) &= (U_a^2, U_a^1). \end{aligned} \tag{5.25}$$

Because $\overset{1}{X}{}^{-}_{ab}\overset{1}{X}{}^{+}_{ba} = \left(\overset{2}{X}{}^{-}_{ab}\overset{2}{X}{}^{+}_{ba}\right)^{-1}$, the proper axes $n[\overset{1}{X}{}^{-}_{ab}\overset{1}{X}{}^{+}_{ba}]^i$, $n[\overset{2}{X}{}^{-}_{ab}\overset{2}{X}{}^{+}_{ba}]^i$ defined in (2.22) are equal and opposite, so that

$$\mathrm{tr}(\tau_i \overset{1}{X}{}^{-}_{ab}\overset{1}{X}{}^{+}_{ba}) = -\mathrm{tr}(\tau_i \overset{2}{X}{}^{-}_{ab}\overset{2}{X}{}^{+}_{ba}). \tag{5.26}$$

From this one deduces

$$\beta_{ab}(\{\overset{1}{G}_{a'b'}\}) = \beta_{ab}(\{\overset{2}{G}_{a'b'}\}) \tag{5.27}$$

which gives us

$$\beta_{ab}(\{\overset{1}{G}_{a'b'}\})\mathrm{tr}(\tau_i \overset{1}{X}{}^{-}_{ab}\overset{1}{X}{}^{+}_{ba})L^i_{ba} = -\beta_{ab}(\{\overset{2}{G}_{a'b'}\})\mathrm{tr}(\tau_i \overset{2}{X}{}^{-}_{ab}\overset{2}{X}{}^{+}_{ba})L^i_{ba}. \tag{5.28}$$

Because $\{U^1_a\} \not\sim \{U^2_a\}$, we have $\{\overset{1}{X}{}^{+}_a\} \not\sim \{\overset{1}{X}{}^{-}_a\}$ and $\{\overset{2}{X}{}^{+}_a\} \not\sim \{\overset{2}{X}{}^{-}_a\}$, so that both $\{\overset{1}{G}_a\}$ and $\{\overset{2}{G}_a\}$ satisfy the hypotheses of lemma 2, implying that neither side of (5.28) is zero. It follows that exactly one of $\beta_{ab}(\{\overset{\alpha}{G}_{a'b'}\})\mathrm{tr}(\tau_i \overset{\alpha}{X}{}^{-}_{ab}\overset{\alpha}{X}{}^{+}_{ba})L^i_{ba}$, $\alpha = 1, 2$, is positive, so that exactly one of $\{\overset{1}{G}_a\}$, $\{\overset{2}{G}_a\}$ satisfies proper orientation and so is a critical point. Furthermore, at this one critical point, $\mu = 1$, so that, because the value of the action (5.5) (respectively (5.17)) for the proper vertex is equal to the value of the action (5.6)(respectively (5.18)) for the original vertex, from the analysis of [14], this one critical point gives rise to precisely the desired asymptotics stated in theorem 6.

6 Conclusions

The original EPRL model, as shown and emphasized in [18, 19], due to the fact that it is based on the linear simplicity constraints, necessarily mixes three of what we call Plebanski sectors as well as two dynamically determined orientations. This mixing of sectors was identified as the precise reason for the multiplicity of terms in the asymptotics of the EPRL vertex calculated in [14]. Furthermore, when multiple 4-simplices are considered, asymptotic analysis thus far [16, 17] indicates that critical configurations contribute in which these sectors can *vary locally* from 4-simplex to 4-simplex. The asymptotic amplitude for such configurations is the exponential of i times an action which is not Regge, but rather a sort of 'generalized Regge action'. The stationary points of this 'generalized' action are not in general solutions to the Regge equations of motion, and thus one has sectors in the semi-classical limit which do not represent general relativity.

In this paper, a solution to this problem is found. We began by deriving a classical discrete condition that isolates the sector corresponding to the first term in the asymptotics — what we have called the Einstein-Hilbert sector, in which the BF action is equal to the Einstein-Hilbert action including sign. Equivalently, this is the sector in which the sign of the Plebanski sector (II±) matches the sign of the dynamical orientation. By appropriately quantizing this condition and using it to modify the EPRL vertex amplitude, we have constructed what we call the *proper* EPRL vertex amplitude. This vertex amplitude continues to be a function of $SU(2)$ spin-network data, so that it may continue to be used to define dynamics for LQG. We have shown that the proper vertex is $SU(2)$ gauge invariant and is linear in the boundary state, as required to ensure that the final transition amplitude is linear in the initial state and antilinear in the final state. It is furthermore $Spin(4)$-invariant in the sense that, similar to the original EPRL model [29], it is independent of the choice of extra structures used in its definition which seem to break $Spin(4)$ symmetry. Finally, it has the correct asymptotics with the *single* term consisting in the exponential of i times the Regge action.

Two interesting further research directions would be (1.) to justify the Lorentzian signature generalization given in equation (4.20), via a quantization of the Lorentzian Einstein-Hilbert sector, and to verify that (4.20) also has the desired single-termed semi-classical limit and (2.) to generalize the present work to the amplitude for an arbitrary 4-cell, which might be used in a spin-foam model involving arbitrary cell-complexes, similar to the generalization [12] of Kamiński, Kisielowski, and Lewandowski. The first of these tasks should be straightforward. The second, however, seems to require a new way of thinking about the discrete constraint (2.25) used to isolate the Einstein-Hilbert sector. For, the β_{ab} sign factor involved in this condition uses in a central way the fact that there are 5 tetrahedra in each 4-simplex.

Lastly, it is important to understand if and how the graviton propagator calculations [37–41], and spin-foam cosmology calculations [42–44] will change if the presently proposed proper vertex is used in place of the original EPRL vertex. In the case of the graviton propagator, only the leading order term in the vertex expansion has thus far been calculated [40]. To this order, only one 4-simplex is involved, and one does not expect the use of the proper vertex to change the results, because the coherent boundary state used in this work already suppresses all but the one desired term (5.3) in the asymptotics. However, higher order terms in the propagator may very well be affected by the use of the proper vertex. We leave this and similar such questions for future investigations.

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A Well-definedness of orientation and Plebanski sectors

As in section 2 in the main text, we let $\mathcal{B}_{ab}$ denote the bivectors $G_a \triangleright B_{ab}$ in the 4-simplex frame. Throughout this appendix we assume that G_a and B_{ab} satisfy closure, orientation, and linear simplicity, implying corresponding restrictions on $\mathcal{B}_{ab}$.[8] As mentioned in section 2, for each choice of flat connection ∂_μ adapted to S, the discrete variables $\{\mathcal{B}_{ab}\}$ determine a unique continuum two form $B_{\mu\nu}(\{\mathcal{B}_{ab}\}, \partial)$ via the conditions $\partial_\sigma B_{\mu\nu}(\{\mathcal{B}_{ab}\}, \partial) = 0$ and $\int_{\Delta_{ab}(S)} B(\{\mathcal{B}_{cd}\}, \partial) = \mathcal{B}_{ab}$. This continuum two form in turn determines a dynamical orientation of S, as well as determining one of three Plebanski sectors, represented respectively by the functions $\omega(B_{\mu\nu})$ and $\nu(B_{\mu\nu})$, defined in section 2.1.2, taking values in $\{0, 1, -1\}$. We here prove that the orientation and Plebanski sector of $B_{\mu\nu}(\{\mathcal{B}_{ab}\}, \partial)$ are independent of the choice of ∂_μ adapted to S. (In the paper [18,19], a slightly different but equivalent way of reconstructing $B_{\mu\nu}$ was used. The well-definedness of orientation and Plebanski sector of $B_{\mu\nu}$ as reconstructed there was proven in that paper. We here prove it anew for the new present reconstruction of $B_{\mu\nu}$, for completeness.)

In the following, we denote the vertex of S opposite each tetrahedron $a \in \{0, \ldots, 4\}$ by p_a. We also use the term 'face' in the general sense of any lower dimensional simplex which forms part of the boundary of a higher dimensional simplex. Bold lower case Greek letters, $\boldsymbol{\mu}, \boldsymbol{\nu} = 0, 1, 2, 3$, shall be used to label different coordinates of a given coordinate system.

[8] The consequences of linear simplicity will only be used in the final lemma.

Lemma 7. *Given a flat connection ∂_μ adapted to S, there exists a unique coordinate system $x^{\boldsymbol{\mu}}$ such that (1.) ∂_μ is the associated coordinate derivative operator and (2.) the values of the coordinates at the five vertices of S, p_0, p_1, p_2, p_3, p_4, are respectively given by*

$$(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), \text{ and } (0,0,0,0). \tag{A.1}$$

Furthermore, the range of possible values of the 4-tuple of coordinates $(x^{\boldsymbol{\mu}})$ coincides precisely with the linear 4-simplex in $\mathbb{R}^4$ with these five vertices, which we refer to as the 'canonical 4-simplex' in $\mathbb{R}^4$. We call $\{x^{\boldsymbol{\mu}}\}$ 'the coordinate system determined by ∂_μ'.

Proof. For each $\boldsymbol{\mu} = 0,1,2,3$, let $V^\mu_{\boldsymbol{\mu}}$ denote a vector in $T_{p_4}S$ tangent to the edge $\overline{p_4 p_{\boldsymbol{\mu}}}$, pointing away from p_4. This vector is unique upto scaling by a positive number. Fix this scaling freedom by first parallel transporting $V^\mu_{\boldsymbol{\mu}}$ along the edge $\overline{p_4 p_{\boldsymbol{\mu}}}$, and then requiring that the affine length of $\overline{p_4 p_{\boldsymbol{\mu}}}$ with respect to $V^\mu_{\boldsymbol{\mu}}$ be 1.

Because the connection ∂_μ is flat, one can use ∂_μ to unambiguously parallel transport $V^\mu_{\boldsymbol{\mu}}$ to all of S, yielding a vector field $V^\mu_{\boldsymbol{\mu}}$ on S for each $\boldsymbol{\mu} = 0,1,2,3$. Because the vectors $\{V^\mu_{\boldsymbol{\mu}}(p_4)\}$ at p_4 were chosen linearly independent, the vectors $\{V^\mu_{\boldsymbol{\mu}}(p)\}$ at each point $p \in S$ form a basis of T_pS. Let $\{\lambda^{\boldsymbol{\mu}}_\mu(p)\}$ denote the basis dual to $\{V^\mu_{\boldsymbol{\mu}}(p)\}$ at each p. For each $\boldsymbol{\mu}$, the resulting one form $\lambda^{\boldsymbol{\mu}}_\mu$ then satisfies $\partial_\mu \lambda^{\boldsymbol{\mu}}_\nu = 0$, implying $\partial_{[\mu}\lambda^{\boldsymbol{\mu}}_{\nu]} = 0$; because S is simply connected, this implies that, for each $\boldsymbol{\mu}$, there exists a function $x^{\boldsymbol{\mu}}$, unique upto addition of a constant, such that $\lambda^{\boldsymbol{\mu}}_\mu = \partial_\mu x^{\boldsymbol{\mu}}$. Fix this freedom in each $x^{\boldsymbol{\mu}}$ by requiring $x^{\boldsymbol{\mu}}(p_4) = 0$. Because $\lambda^{\boldsymbol{\mu}}_\mu = \partial_\mu x^{\boldsymbol{\mu}}$ are everywhere linearly independent, $\{x^{\boldsymbol{\mu}}\}$ forms a good coordinate system on S. Furthermore, from $V^\mu_{\boldsymbol{\mu}}\partial_\mu x^{\boldsymbol{\nu}} = V^\mu_{\boldsymbol{\mu}}\lambda^{\boldsymbol{\nu}}_\mu = \delta^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}$, one has $V^\mu_{\boldsymbol{\mu}} = \left(\frac{\partial}{\partial x^{\boldsymbol{\mu}}}\right)^\mu$. Because ∂_μ annihilates $\lambda^{\boldsymbol{\mu}}_\mu = \partial_\mu x^{\boldsymbol{\mu}}$ (and $V^\mu_{\boldsymbol{\mu}} = \left(\frac{\partial}{\partial x^{\boldsymbol{\mu}}}\right)^\mu$), ∂_μ is the coordinate derivative operator for $\{x^{\boldsymbol{\mu}}\}$.

Consider the differential equation $V^\mu_{\boldsymbol{\mu}}\partial_\mu x^{\boldsymbol{\nu}} = \delta^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}$ for a given fixed $\boldsymbol{\mu}$. Because $V^\mu_{\boldsymbol{\mu}}$ is tangent to the edge $\overline{p_4 p_{\boldsymbol{\mu}}}$, this equation dictates how to evolve each of the four coordinates $x^{\boldsymbol{\nu}}$ along $\overline{p_4 p_{\boldsymbol{\mu}}}$ from its starting value $x^{\boldsymbol{\nu}} = 0$ at p_4, thereby determining its value at $p_{\boldsymbol{\mu}}$. For $\boldsymbol{\nu} \neq \boldsymbol{\mu}$ this implies $x^{\boldsymbol{\nu}} = 0$ at $\overline{p_4 p_{\boldsymbol{\mu}}}$. For $\boldsymbol{\nu} = \boldsymbol{\mu}$, the differential equation simply expresses that $x^{\boldsymbol{\mu}}$ is an affine coordinate for $V^\mu_{\boldsymbol{\mu}}$ along $\overline{p_4 p_{\boldsymbol{\mu}}}$, so that, by construction, $x^{\boldsymbol{\mu}} = 1$ at $p_{\boldsymbol{\mu}}$.

Now, the coordinates $x^{\boldsymbol{\mu}}$ provide an embedding Φ of S into $\mathbb{R}^4$, $\Phi : p \mapsto x^{\boldsymbol{\mu}}(p)$. By construction the point p_4 maps to $(0,0,0,0)$, whereas, as just shown, the points p_0, p_1, p_2, p_3 map to the points $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$. Because ∂_μ is adapted to S, S is the convex hull of its vertices as determined by the affine structure defined by ∂_μ. But this affine structure is the same as that defined by the coordinates $x^{\boldsymbol{\mu}}$, so that $\Phi[S]$ is the convex hull, in $\mathbb{R}^4$, of the points (A.1). That is, $\Phi[S]$ is the linear 4-simplex in $\mathbb{R}^4$ with vertices (A.1). ■

For the purposes of the following, the action of a diffeomorphism φ on a derivative operator ∂_μ is defined by $(\varphi \cdot \partial)_\mu \lambda_\nu := (\varphi^{-1})^* \partial_\mu (\varphi^* \lambda_\nu)$. The resulting action of $(\varphi \cdot \partial)$ on a general tensor $t^{\alpha\cdots\gamma}{}_{\rho\ldots\sigma}$ is then given by $(\varphi \cdot \partial)_\mu t^{\alpha\cdots\gamma}\rho\ldots\sigma := \varphi \cdot (\partial_\mu(\varphi^{-1} \cdot t^{\alpha\cdots\gamma}\rho\ldots\sigma))$ where $\varphi \cdot$ denotes the left action of φ on the tensor in question (thus, push-forward for contravariant indices and pull-back via φ^{-1} for covariant indices [45]).

Lemma 8. *Given any two flat connections ∂_μ, $\tilde{\partial}_\mu$ adapted to S, there exists an orientation preserving diffeomorphism $\varphi : S \to S$ mapping each face of S to itself, and mapping ∂_μ to $\tilde{\partial}_\mu$.*

Proof.

Let $x^{\boldsymbol{\mu}}$ and $\tilde{x}^{\boldsymbol{\mu}}$ denote the coordinate systems on S determined by ∂_μ and $\tilde{\partial}_\mu$ respectively, in the manner described in the foregoing lemma. From this lemma, the range of the coordinates in these two systems are exactly the same, so that one can define a diffeomorphism $\varphi : S \to S$ by the condition $x^{\boldsymbol{\mu}}(p) = \tilde{x}^{\boldsymbol{\mu}}(\varphi(p))$. For each face f in S, because the range of values of the coordinates $x^{\boldsymbol{\mu}}$ and $\tilde{x}^{\boldsymbol{\mu}}$ over f are the same — namely the points in the corresponding face of the canonical 4-simplex in $\mathbb{R}^4$ — φ maps f back to itself.

Furthermore, the action of $(\varphi\cdot\partial)$ on the coordinate gradients $(\partial_\nu \tilde{x}^{\boldsymbol{\mu}})$ is given by

$$(\varphi\cdot\partial)_\mu(\partial_\nu\tilde{x}^{\boldsymbol{\mu}}) := (\varphi^{-1})^*(\partial_\mu(\varphi^*(\partial_\nu\tilde{x}^{\boldsymbol{\mu}}))) = (\varphi^{-1})^*(\partial_\mu\partial_\nu(\varphi^*\tilde{x}^{\boldsymbol{\mu}})) = (\varphi^{-1})^*(\partial_\mu\partial_\nu x^{\boldsymbol{\mu}}) = 0 \qquad \text{(A.2)}$$

where, in the last step, the fact that ∂_μ is the coordinate derivative for $x^{\boldsymbol{\mu}}$ was used. Equation (A.2) implies that $(\varphi\cdot\partial)_\mu$ is the coordinate derivative for $\tilde{x}^{\boldsymbol{\mu}}$, whence $(\varphi\cdot\partial)_\mu = \tilde{\partial}_\mu$.

It remains only to show that φ is orientation preserving. This can be seen from the fact that φ maps $\frac{\partial}{\partial x^{\mathbf{0}}}^{[\alpha}\cdots\frac{\partial}{\partial x^{\mathbf{3}}}^{\delta]}$ to $\frac{\partial}{\partial \tilde{x}^{\mathbf{0}}}^{[\alpha}\cdots\frac{\partial}{\partial \tilde{x}^{\mathbf{3}}}^{\delta]}$. Specifically, because these two inverse 4-forms are nowhere vanishing, there exists a nowhere vanishing function λ, which therefore doesn't change sign, such that

$$\frac{\partial}{\partial \tilde{x}^{\mathbf{0}}}^{[\alpha}\cdots\frac{\partial}{\partial \tilde{x}^{\mathbf{3}}}^{\delta]} = \lambda \frac{\partial}{\partial x^{\mathbf{0}}}^{[\alpha}\cdots\frac{\partial}{\partial x^{\mathbf{3}}}^{\delta]}. \qquad \text{(A.3)}$$

To find the sign of λ, it is sufficient to find its sign at a single point. At p_4, for each $\boldsymbol{\mu}$, by construction, $\frac{\partial}{\partial x^{\boldsymbol{\mu}}}$ and $\frac{\partial}{\partial \tilde{x}^{\boldsymbol{\mu}}}$ are both tangent to $\overline{p_4 p_{\boldsymbol{\mu}}}$ and oriented in the direction of $p_{\boldsymbol{\mu}}$. It follows that the coefficient λ in equation (A.3) is positive at p_4, and thus positive throughout S. Thus the push-forward action of φ maps $\frac{\partial}{\partial x^{\mathbf{0}}}^{[\alpha}\cdots\frac{\partial}{\partial x^{\mathbf{3}}}^{\delta]}$ to itself times an everywhere positive function, so that φ is orientation preserving. ■

Theorem 9. *$\omega(B^{IJ}_{\mu\nu}(\{\mathcal{B}_{ab}\},\partial))$ and $\nu(B^{IJ}_{\mu\nu}(\{\mathcal{B}_{ab}\},\partial))$ are independent of the choice of ∂_μ adapted to S.*

Proof. Let ∂_μ and $\tilde{\partial}_\mu$ be two flat connections adapted to S. Then by the previous lemma, there exists an orientation preserving diffeomorphism $\varphi : S \to S$ mapping ∂_μ to $\tilde{\partial}_\mu$ and such that φ preserves each face of S, where 'face' includes in its meaning tetrahedra, triangles, edges, and vertices on the boundary. In particular, for each $a,b\in\{0,1,2,3,4\}$, φ preserves $\Delta_{ab}(S)$. Because it also preserves tetrahedron a and b and the fixed orientation of S, it in fact also preserves the orientation of $\Delta_{ab}(S)$ [18]. Using this fact and the diffeomorphism covariance of the form integral, one has

$$\int_{\Delta_{ab}(S)} \varphi^* B(\{\mathcal{B}_{cd}\},\tilde{\partial}) = \int_{\Delta_{ab}(S)} B(\{\mathcal{B}_{cd}\},\tilde{\partial}) = \mathcal{B}_{ab} \qquad \text{(A.4)}$$

where the definition of $B_{\mu\nu}(\{\mathcal{B}_{cd}\},\tilde{\partial})$ was used in the last step. Furthermore,

$$\partial_\sigma(\varphi^* B_{\mu\nu}(\{\mathcal{B}_{ab}\},\tilde{\partial}) = \varphi^*\circ(\varphi^{-1})^*\partial_\sigma(\varphi^* B_{\mu\nu}(\{\mathcal{B}_{ab}\},\tilde{\partial}) = \varphi^*\tilde{\partial}_\sigma B_{\mu\nu}(\{\mathcal{B}_{ab}\},\tilde{\partial}) = 0 \qquad \text{(A.5)}$$

where again the definition of $B_{\mu\nu}(\{\mathcal{B}_{ab}\},\tilde{\partial})$ was used in the last step. Equations (A.4) and (A.5) then imply

$$B_{\mu\nu}(\{\mathcal{B}_{ab}\},\partial) = \varphi^* B_{\mu\nu}(\{\mathcal{B}_{ab}\},\tilde{\partial}). \qquad \text{(A.6)}$$

Because φ is orientation preserving, and both the orientation and Plebanski sector of $B_{\mu\nu}$ are invariant under orientation preserving diffeomorphisms, one has

$$\omega(B_{\mu\nu}(\{\mathcal{B}_{ab}\},\partial)) = \omega(B_{\mu\nu}(\{\mathcal{B}_{ab}\},\tilde{\partial})) \qquad \text{and} \qquad \nu(B_{\mu\nu}(\{\mathcal{B}_{ab}\},\partial)) = \nu(B_{\mu\nu}(\{\mathcal{B}_{ab}\},\tilde{\partial})),$$

proving the theorem. ■

B Four dimensional closure

The following property is mentioned, for example, in [15, 46, 47].

Theorem 10. *For any geometrical 4-simplex σ in $\mathbb{R}^4$,*

$$\sum_t V_t N_t^I = 0 \tag{B.1}$$

where the sum is over tetrahedra, and V_t and N_t^I are the volume and outward normal to tetrahedron t.

Proof. Define ${}^3\epsilon^I$ to be the three-form on $\mathbb{R}^4$ with components $({}^3\epsilon^I)_{JKL} = \epsilon^I{}_{JKL}$. Then

$$\mathrm{d}{}^3\epsilon^I = 0.$$

Thus,

$$0 = \int_\sigma \mathrm{d}{}^3\epsilon^I = \sum_t \int_t {}^3\epsilon^I. \tag{B.2}$$

Let ${}^t\epsilon$ denote the volume form for t, so that for each t,

$$\epsilon_{IJKL} = 4(N_t)_{[I}({}^t\epsilon)_{JKL]}.$$

Pulling back JKL to tetrahedron t, it follows that

$$\underset{t\leftarrow}{{}^3\epsilon^I} = N_t^I({}^t\epsilon)$$

which combined with B.2 yields the result. ■

C Properties of embeddings and projectors

Recall $|n;k,m\rangle$ denotes the eigenstate of $n \cdot \hat{L}$ in V_k with eigenvalue m, and $|n;j^-,j^+,k,m\rangle$ the eigenstate of $\hat{L}^2$ and $n \cdot \hat{L}$ in $V_{j^-.j^+}$ with eigenvalues $k(k+1)$ and m,

Lemma 9.

(a.)

$$\hat{L}^i \circ \iota_k^{j^-,j^+} = \iota_k^{j^-,j^+} \circ \hat{L}^i \tag{C.1}$$

(b.) For each unit $n^i \in \mathbb{R}^3$ and each k, m, there exists $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$ such that

$$\iota_k^{j^-,j^+}|n;k,m\rangle = e^{i\theta}|n;j^-,j^+,k,m\rangle. \tag{C.2}$$

(c.) For any $\mathcal{S} \subseteq \mathbb{R}$,

$$P_\mathcal{S}(n\cdot\hat{L}) \circ \iota_k^{j^-,j^+} = \iota_k^{j^-,j^+} \circ P_\mathcal{S}(n\cdot\hat{L}). \tag{C.3}$$

Proof.

Proof of (a.):

From the intertwining property of $\iota_k^{j^-,j^+}$,

$$\rho(e^{t\tau^i}, e^{t\tau^i})\iota_k^{j^-,j^+} = \iota_k^{j^-,j^+}\rho(e^{t\tau^i}). \tag{C.4}$$

Taking $i\frac{\mathrm{d}}{\mathrm{d}t}$ of both sides and setting $t=0$ yields the result.

Proof of (b.):
Using part (a.),

$$\begin{aligned}\left(n \cdot \hat{L}\right) \iota_k^{j^-,j^+} |n; k, m\rangle &= \iota_k^{j^-,j^+} \left(n \cdot \hat{L}\right) |n; k, m\rangle = m\iota_k^{j^-,j^+}|n; k, m\rangle, \qquad \text{and} \\ \hat{L}^2 \iota_k^{j^-,j^+} |n; k, m\rangle &= \iota_k^{j^-,j^+} \hat{L}^2 |n; k, m\rangle = k(k+1)\iota_k^{j^-,j^+}|n; k, m\rangle.\end{aligned}$$

The result follows.

Proof of (c.):
We have for each m,

$$\begin{aligned}P_{\mathcal{S}}(n \cdot \hat{L})\iota_k^{j^-,j^+}|n; k, m\rangle &= e^{i\theta} P_{\mathcal{S}}(n \cdot \hat{L})|n; j^-, j^+, k, m\rangle = e^{i\theta}\chi_{\mathcal{S}}(m)|n; j^-, j^+, k, m\rangle \\ &= \chi_{\mathcal{S}}(m)\iota_k^{j^-,j^+}|n; k, m\rangle = \iota_k^{j^-,j^+} P_{\mathcal{S}}(n \cdot \hat{L})|n; k, m\rangle\end{aligned}$$

where $\chi_{\mathcal{S}}(m)$ denotes the characteristic function for $\mathcal{S}$. ■

Lemma 10. *In any irreducible representation (irrep) of Spin(4), for any two* $(v_-)^i, (v_+)^i \in \mathbb{R}^3$

$$\rho(X^-, X^+) \circ P_{\mathcal{S}}(v_- \cdot \hat{J}^- + v_+ \cdot \hat{J}^+) = P_{\mathcal{S}} \left((X^- \triangleright v_-) \cdot \hat{J}^- + (X^+ \triangleright v_+) \cdot \hat{J}^+\right) \rho(X^-, X^+) \tag{C.5}$$

Proof. Let $j^\pm, m^\pm$ be given. Write $v_\pm^i = \lambda_\pm n_\pm^i$ with $\lambda_\pm \geq 0$ and $n_\pm^i$ unit. Using that $\rho(X^\pm)|n_\pm; j^\pm, m^\pm\rangle = e^{i\theta^\pm}|X^\pm \triangleright n_\pm; j^\pm, m^\pm\rangle$ for some $\theta^\pm$, we have

$$\begin{aligned}&\rho(X^-, X^+)P_{\mathcal{S}}(v_- \cdot \hat{J}^- + v_+ \cdot \hat{J}^+)|n_-; j^-, m^-\rangle \otimes |n_+; j^+, m^+\rangle \\ &\quad= \chi_{\mathcal{S}}(\lambda_- m^- + \lambda_+ m^+)\rho(X^-, X^+)|n_-; j^-, m^-\rangle \otimes |n_+; j^+, m^+\rangle \\ &\quad= e^{i(\theta^- + \theta^+)}\chi_{\mathcal{S}}(\lambda_- m^- + \lambda_+ m^+)|X^- \triangleright n_-; j^-, m^-\rangle \otimes |X^+ \triangleright n_+; j^+, m^+\rangle \\ &\quad= e^{i(\theta^- + \theta^+)} P_{\mathcal{S}} \left((X^- \triangleright v_-) \cdot \hat{J}^- + (X^+ \triangleright v_+) \cdot \hat{J}^+\right) |X^- \triangleright n_-; j^-, m^-\rangle \otimes |X^+ \triangleright n_+; j^+, m^+\rangle \\ &\quad= P_{\mathcal{S}} \left((X^- \triangleright v_-) \cdot \hat{J}^- + (X^+ \triangleright v_+) \cdot \hat{J}^+\right) \rho(X^-, X^+)|n_-; j^-, m^-\rangle \otimes |n_+; j^+, m^+\rangle\end{aligned}$$

■

Lemma 11. *Let $\hat{O}_t$ be any one-parameter family of self-adjoint operators on a Hilbert space $\mathcal{H}$. For each t, let ψ_t be a normalized eigenstate of $\hat{O}_t$ such that all ψ_t have the same eigenvalue $\lambda \in \mathbb{R}$. Then*

$$\langle\psi_t| \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_t\right) |\psi_t\rangle = 0. \tag{C.6}$$

Proof.

$$\langle\psi_t|\hat{O}_t|\psi_t\rangle = \lambda \tag{C.7}$$

for all t. Taking $\frac{\mathrm{d}}{\mathrm{d}t}$ of both sides,

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_t|\right) \hat{O}_t|\psi_t\rangle + \langle\psi_t| \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_t\right) |\psi_t\rangle + \langle\psi_t|\hat{O}_t \frac{\mathrm{d}}{\mathrm{d}t}|\psi_t\rangle &= 0 \\ \lambda \frac{\mathrm{d}}{\mathrm{d}t} \left(\langle\psi_t, \psi_t\rangle\right) + \langle\psi_t| \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_t\right) |\psi_t\rangle &= 0 \\ \langle\psi_t| \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_t\right) |\psi_t\rangle &= 0\end{aligned}$$

■

Applying this to the family of operators $\hat{O}_t = n_t \cdot \hat{L}$ on V_{j^-,j^+} and the states $|n_t; j^-, j^+, k, m\rangle$, and to the family of operators $\hat{O}_t = n_t \cdot \hat{L}$ on V_k and the states $|n_t; k, m\rangle$, yields the following.

Corollary 11. *For any variation δ of n, any j^-, j^+, k, any $m \in \{-k, -k+1, \ldots, k\}$, and any set $\mathcal{S} \subset \mathbb{R}$, one has*

$$\langle n; j^-, j^+, k, m|\delta P_{\mathcal{S}}(n \cdot \hat{L})|n; j^-, j^+, k, m\rangle = 0. \tag{C.8}$$

and

$$\langle n; k, m|\delta P_{\mathcal{S}}(n \cdot \hat{L})|n; k, m\rangle = 0. \tag{C.9}$$

D Expression for vertex with projectors on the left

Lemma 12. *For each unit $n^i \in \mathbb{R}^3$, $g \in SU(2)$, k, and m, there exists $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$ such that*

$$\rho(h)|n; k, m\rangle = e^{i\theta}|h \triangleright n; k, m\rangle \tag{D.1}$$

Proof.

$$\left((h \triangleright n) \cdot \hat{L}\right) \rho(h)|n; k, m\rangle = \rho(h)(n \cdot \hat{L})|n; k, m\rangle = m\rho(h)|n; k, m\rangle.$$

■

Lemma 13. *For any $\mathcal{S} \subseteq \mathbb{R}$, and in any irrep of Spin(4), and any $v^i \in \mathbb{R}^3$,*

$$P_{\mathcal{S}}(v \cdot \hat{L}) \circ J = J \circ P_{\mathcal{S}}(-v \cdot \hat{L}) \tag{D.2}$$

Proof. Let $v^i =: \lambda n^i$ with $\lambda \geq 0$ and n^i unit. Using that J anti-commutes with $\hat{L}^i$, for any n and k,

$$\left(n \cdot \hat{L}\right) J|n; j^-, j^+, k, m\rangle = -J\left(n \cdot \hat{L}\right)|n; j^-, j^+, k, m\rangle = -mJ|n; j^-, j^+, k, m\rangle \tag{D.3}$$

whence

$$J|n; j^-, j^+, k, m\rangle = e^{i\theta_m}|n; j^-, j^+, k, -m\rangle \tag{D.4}$$

for some $\{\theta_m\} \subset \frac{\mathbb{R}}{2\pi\mathbb{Z}}$, so that, for all m,

$$\begin{aligned} P_{\mathcal{S}}(v \cdot \hat{L})J|n; j^-, j^+, k, m\rangle &= e^{i\theta_m} P_{\mathcal{S}}(v \cdot \hat{L})|n; j^-, j^+, k, -m\rangle \\ &= \chi_{\mathcal{S}}(-\lambda m)J|n; j^-, j^+, k, m\rangle = JP_{\mathcal{S}}(-v \cdot \hat{L})|n; j^-, j^+, k, m\rangle. \end{aligned}$$

■

Theorem 12. *The vertex amplitude (4.4) can also be written with the projector, appropriately transformed, moved to anywhere in each face factor:*

$$\begin{aligned} A_v^{(+)}(\{k_{ab}, \psi_{ab}\}) &= \int_{\mathrm{Spin}(4)^5} \prod_a \mathrm{d}G_a \prod_{a<b} \epsilon(\iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ab}, \rho(G_{ab}) P_{ba}(\{G_{a'b'}\}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ba}) \\ &= \int_{\mathrm{Spin}(4)^5} \prod_a \mathrm{d}G_a \prod_{a<b} \epsilon(P_{ab}(\{G_{a'b'}\}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ab}, \rho(G_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ba}) \\ &= \int_{\mathrm{Spin}(4)^5} \prod_a \mathrm{d}G_a \prod_{a<b} \epsilon(\iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} P_{ab}(\{G_{a'b'}\}) \psi_{ab}, \rho(G_{ab}) \iota_{k_{ab}}^{s_{ab}^- s_{ab}^+} \psi_{ba}). \end{aligned} \tag{D.5}$$

Proof. One starts from (4.5), and uses lemma 9.c, lemma 10, the hermicity of orthogonal projectors, and lemma 13 in succession, as well as using the fact that if $X^i{}_j$ denotes the adjoint action of $X = SU(2)$ then $X^i{}_j \tau^j = X^{-1} \tau^i X$. ■

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