Self-accelerating massive gravity: Time for field fluctuations

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Self-accelerating Massive Gravity: Time for Field Fluctuations

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The ghost-free theory of massive gravity has exact solutions where the effective stress energy generated by the graviton mass term is a cosmological constant for any isotropic metric. Since they are exact, these solutions mimic a cosmological constant in the presence of any matter-induced isotropic metric perturbation. In the Stückelberg formulation, this stress energy is carried entirely by the spatial Stückelberg field. We show that any stress energy carried by fluctuations in the spatial field away from the exact solution always decays away in an expanding universe. However, the dynamics of the spatial Stückelberg field perturbation depend on the background temporal Stückelberg field, which is equivalent to the unitary gauge time coordinate. This dependence resolves an apparent conflict in the existing literature by showing that there is a special unitary time choice for which the field dynamics and energy density perturbations vanish identically. In general, the isotropic system has a single dynamical degree of freedom requiring two sets of initial data; however, only one of these initial data choices will affect the observable metric. Finally, we construct cosmological solutions with a well-defined perturbative initial value formulation and comment on alternate solutions that evolve to singularities.

I. INTRODUCTION

Constructed to remove the Boulware-Deser ghost [1], the theory of massive gravity [2–5] also possesses solutions that accelerate the cosmological expansion in the absence of a true cosmological constant [6–16]. In a previous paper [13], we demonstrated that all of these apparently distinct solutions are part of a single class of isotropic solutions in the Stückelberg formulation of the theory. This class possesses several key features. The spatial Stückelberg fields follow the radial coordinate of the metric itself and produce the same effective cosmological constant in the presences of any isotropic metric. The temporal Stückelberg field is inhomogeneous in isotropic coordinates but does not play a direct role in establishing the effective cosmological constant. Solutions for this field are not uniquely specified, leading to seemingly distinct versions of self-acceleration.

Once the Stückelberg fields are set on a self-accelerating solution, they remain on it for any evolution of the matter fields that remains isotropic, including cosmological expansion and radial collapse of matter perturbations. Given the inherent interest of self-accelerating cosmological solutions, it is important to understand whether Stückelberg field perturbations around them are themselves stable and healthy. This question has now been widely studied, with different authors using different approaches. Confusingly, these different approaches seem to give different conclusions. Study of perturbations in a decoupling limit showed a potentially healthy scalar degree of freedom but potentially problematic (strongly coupled or ghost-like) vector degrees of freedom [6, 17, 18]. A local patch expansion approach, which one might expect to be qualitatively similar to a decoupling limit, found no propagating Stückelberg degrees of freedom at all [19]. Finally, study of perturbations in the full theory around a particular solution for the temporal Stückelberg field similarly found no propagating degrees of freedom [20][21].

To resolve this issue, we study spherically symmetric Stückelberg perturbations around the full theory for the whole class of solutions. We find that in general the spatial Stückelberg field fluctuation does possess dynamics and carries stress-energy fluctuations in addition to the background constant. There are special choices of the temporal Stückelberg field background that eliminate the dynamics but also eliminate the possibility of energy density fluctuations. In all cases, isotropic energy density fluctuations are stable and from general initial conditions decay back to the cosmological constant in an expanding universe. Our approach allows us to show that the decoupling limit findings [6, 17] and specific temporal solutions in the full theory [20] are not, after all, in conflict. We have not been able to harmonize our findings with Ref. [19], who finds no dynamics for either temporal Stückelberg field choice. Because we restrict ourselves to isotropic perturbations, we are not able to address the generality of recent calculations [22, 23] that demonstrate that at least some of these solutions are unstable to anisotropic perturbations.

The structure of this paper is as follows. In §II, we briefly review the theory of massive gravity [5] and the class of isotropic self-accelerating solutions [13]. We then consider the equations of motion and action governing spherically symmetric field perturbations around this class of solutions in §III. Finally in §IV we show how the choice of the temporal Stückelberg field in the background affects the dynamics of the perturbations and their contribution to stress energy. We close with a discussion and summary of these results in §V.

II. SELF-ACCELERATION

In the section, we review the action, equations of motion, and stress energy for the massive gravity model [4, 5]. We specialize these quantities for the class of exact isotropic self-accelerating solutions [13] which will form the basis of the perturbation studies that follow.

A. Massive Stress Energy

The Lagrangian density [4, 5]

$$\mathcal{L}_G = \frac{M_{\rm pl}^2}{2}\sqrt{-g}\left[R - \frac{m^2}{4}\mathcal{U}(g_{\mu\nu},\Sigma_{\mu\nu})\right], \qquad (1)$$

represents a covariant theory of massive gravity constructed so as to eliminate the Boulware-Deser ghost [1, 24, 25]. Here m is the graviton mass and $M_{\rm pl}$ is the reduced Planck mass. The potential $\mathcal{U}$ is constructed in a covariant manner by making it a function of the so-called fiducial metric $\Sigma_{\mu\nu}$

$$\Sigma_{\mu\nu} = \partial_\mu\phi^a\partial_\nu\phi^b\eta_{ab}, \qquad (2)$$

where ϕ^a are the 4 Stückelberg fields. These fields transform as spacetime scalars and hence maintain general covariance.

The potential $\mathcal{U}$ can now be written in terms of the matrix

$$\Sigma^\mu{}_\nu \equiv g^{\mu\alpha}\Sigma_{\alpha\nu}, \qquad (3)$$

as

$$\begin{aligned}\frac{\mathcal{U}}{4} &= -12 + 6[\sqrt{\boldsymbol{\Sigma}}] + [\boldsymbol{\Sigma}] - [\sqrt{\boldsymbol{\Sigma}}]^2 \\ &+ \alpha_3\Big(-24 + 18[\sqrt{\boldsymbol{\Sigma}}] - 6[\sqrt{\boldsymbol{\Sigma}}]^2 + [\sqrt{\boldsymbol{\Sigma}}]^3 \\ &\quad - 3[\boldsymbol{\Sigma}]([\sqrt{\boldsymbol{\Sigma}}]-2) + 2[\boldsymbol{\Sigma}^{3/2}]\Big) \\ &+ \alpha_4\Big(-24 + 24[\sqrt{\boldsymbol{\Sigma}}] - 12[\sqrt{\boldsymbol{\Sigma}}]^2 - 12[\sqrt{\boldsymbol{\Sigma}}][\boldsymbol{\Sigma}] \\ &\quad + 6[\sqrt{\boldsymbol{\Sigma}}]^2[\boldsymbol{\Sigma}] + 4[\sqrt{\boldsymbol{\Sigma}}]^3 + 12[\boldsymbol{\Sigma}] - 3[\boldsymbol{\Sigma}]^2 \\ &\quad - 8[\boldsymbol{\Sigma}^{3/2}]([\sqrt{\boldsymbol{\Sigma}}]-1) + 6[\boldsymbol{\Sigma}^2] - [\sqrt{\boldsymbol{\Sigma}}]^4\Big), \end{aligned} \qquad (4)$$

where brackets denote traces, $[\mathbf{A}] \equiv A^\mu{}_\mu$, and α_3, α_4 are free parameters.

Variation of the action with respect to the metric yields the modified Einstein equations

$$G_{\mu\nu} = m^2 T_{\mu\nu} + \frac{1}{M_{\rm pl}^2}T^{(m)}_{\mu\nu}, \qquad (5)$$

where $G_{\mu\nu}$ is the usual Einstein tensor and $T^{(m)}_{\mu\nu}$ is the matter stress energy tensor. Here

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\,\frac{\mathcal{U}}{4} \qquad (6)$$

is the dimensionless effective stress energy tensor provided by the mass term. An explicit expression in terms of $\boldsymbol{\Sigma}$ is given in Ref. [13].

B. Effective Cosmological Constant

A constant stress energy is an exact solution of massive gravity for any spatially isotropic metric [13],

$$ds^2 = -b^2(r,t)dt^2 + a^2(r,t)(dr^2 + r^2 d\Omega^2). \qquad (7)$$

To see this fact, consider that an isotropic parameterization of the Stückelberg fields

$$\begin{aligned}\phi^0 &= f(t,r), \\ \phi^i &= g(t,r)\frac{x^i}{r}, \end{aligned} \qquad (8)$$

enables the potential to be written more compactly as

$$\frac{\mathcal{U}}{4} = P_0\left(\frac{g}{ar}\right) + \sqrt{X}P_1\left(\frac{g}{ar}\right) + WP_2\left(\frac{g}{ar}\right), \qquad (9)$$

where

$$\begin{aligned}X &\equiv \left(\frac{\dot f}{b} + \mu\frac{g'}{a}\right)^2 - \left(\frac{\dot g}{b} + \mu\frac{f'}{a}\right)^2, \\ W &\equiv \frac{\mu}{ab}\left(\dot f g' - \dot g f'\right), \end{aligned} \qquad (10)$$

and $\mu \equiv {\rm sgn}(\dot f g' - \dot g f')$. Here and below overdots denote derivatives with respect to t and primes denote derivatives with respect to r when acting on Stückelberg or metric fields. Note that W is related to the determinant of $\boldsymbol{\Sigma}^{1/2}$ which we assume is never zero. Correspondingly μ must be either $+1$ or -1 throughout the spacetime. This will be an important consideration in §IV.

The P_n polynomials are

$$\begin{aligned}P_0(x) &= -12 - 2x(x-6) - 12(x-1)(x-2)\alpha_3 \\ &\quad - 24(x-1)^2\alpha_4, \\ P_1(x) &= 2(3-2x) + 6(x-1)(x-3)\alpha_3 + 24(x-1)^2\alpha_4, \\ P_2(x) &= -2 + 12(x-1)\alpha_3 - 24(x-1)^2\alpha_4, \end{aligned} \qquad (11)$$

and satisfy the recursion

$$P_n' = 2P_{n+1} - xP_{n+1}', \qquad (12)$$

where here and throughout $P_n'(x) \equiv dP_n/dx$ and should not be confused with radial derivatives.

Varying the action with respect to f and g yields the Stückelberg field equations

$$\begin{aligned}&\partial_t\left[\frac{a^3r^2}{\sqrt{X}}\left(\frac{\dot f}{b} + \mu\frac{g'}{a}\right)P_1 + \mu a^2r^2 g' P_2\right] \\ &- \partial_r\left[\frac{a^2br^2}{\sqrt{X}}\left(\mu\frac{\dot g}{b} + \frac{f'}{a}\right)P_1 + \mu a^2r^2\dot g P_2\right] = 0, \end{aligned} \qquad (13)$$

and

$$\begin{aligned}&-\partial_t\left[\frac{a^3r^2}{\sqrt{X}}\left(\frac{\dot g}{b} + \mu\frac{f'}{a}\right)P_1 + \mu a^2r^2 f' P_2\right] \\ &+ \partial_r\left[\frac{a^2br^2}{\sqrt{X}}\left(\mu\frac{\dot f}{b} + \frac{g'}{a}\right)P_1 + \mu a^2r^2\dot f P_2\right] \\ &= a^2br\left[P_0' + \sqrt{X}P_1' + WP_2'\right]. \end{aligned} \qquad (14)$$

Eq. (13) has an exact solution given by $P_1(x_0) = 0$. Here

$$x_0 = \frac{1 + 6\alpha_3 + 12\alpha_4 \pm \sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4}}{3(\alpha_3 + 4\alpha_4)} \tag{15}$$

unless $\alpha_3 = \alpha_4 = 0$ in which case $x_0 = 3/2$. Given $g = x_0 ar$, Eq. (14) reduces to

$$\sqrt{X} = \frac{W}{x_0} + x_0, \tag{16}$$

and can be used to define solutions for f (see §IV).

For any such solution, the stress energy tensor takes the cosmological constant form

$$T^\mu{}_\nu = -\rho \delta^\mu{}_\nu, \tag{17}$$

where

$$\rho = -p = \frac{1}{2} P_0(x_0). \tag{18}$$

To restore physical units to the dimensionless stress tensor multiply by $m^2 M_{\rm pl}^2$. The Einstein equations (5) then determine the metric functions a and b jointly in the presence of matter in a manner independent of the solution for f. The self-accelerating stress energy depends only on the universal form for the spatial Stückelberg field, $g = x_0 ar$, common to the whole class of isotropic solutions.

III. FIELD FLUCTUATIONS

In this section we derive a complete set of equations of motion for the observable properties of Stückelberg field fluctuations or equivalently the field equation for spatial Stückelberg fluctuations. We then relate these results to the second order action and its dependence on the temporal Stückelberg background solution and massive gravity parameters.

A. Equations of Motion

Now let us consider spherically symmetric Stückelberg perturbations. Hereafter f and g represent the background solution for the Stückelberg fields, a and b the background metric, while δf and

$$\delta\Gamma = \delta g - x_0 r \delta a, \tag{19}$$

δa and δb quantify the perturbations. Because our solutions are exact for any $a(r,t)$, the metric can have perturbations away from a given background, e.g. the Friedmann-Robertson-Walker background, due to perturbations in the matter sector and still be on the exact self-accelerating solution in the Stückelberg sector. $\delta\Gamma$ therefore is the fluctuation in g away from the self-accelerating solution.

Linearizing Eq. (13) in the fluctuations, we obtain a closed-form equation for $\delta\Gamma$

$$\begin{aligned} 0 &= \delta\Gamma_{\rm eom} \\ &= \partial_t \left[\frac{a^2 r}{\sqrt{X}} \left(\frac{\dot f}{b} + \mu \frac{g'}{a} \right) \delta\Gamma \right] - \partial_r \left[\frac{abr}{\sqrt{X}} \left(\mu \frac{\dot g}{b} + \frac{f'}{a} \right) \delta\Gamma \right] \\ &\quad - \mu a^2 r^2 \left[\frac{(ar)'}{ar} \delta\dot\Gamma - \frac{\dot a}{a} \delta\Gamma' \right]. \end{aligned} \tag{20}$$

This equation for $\delta\Gamma$ contains no reference to matter or metric perturbations. It is also decoupled from the Stückelberg fluctuation δf. The coefficients in this equation do however depend explicitly on the background solution for the metric and both Stückelberg fields.

This equation is first order in time and requires one set of initial data $\delta\Gamma(r, t = 0)$ to solve. However, note that if we choose initial values of $\delta\Gamma(r, 0) = 0$, $\delta\Gamma$ will stay zero regardless of matter-driven perturbations in the metric. In this sense, the Stückelberg fluctuations are decoupled from the matter.

As was the case for the background, the perturbed Eq. (14) defines the evolution of δf and depends explicitly on $\delta\Gamma$ as well as on the metric fluctuations (see Appendix A). Thus to characterize the complete Stückelberg sector requires a second set of initial data $\delta f(r, 0)$. Considered together, the amount of initial data and the two coupled first order systems are indicative of a single propagating degree of freedom. However, these fields are not interdependent in the usual way of a field and its conjugate momentum and in particular do not combine to form a single wave equation: $\delta\Gamma$ does not depend on δf. We will return to the interpretation of these facts in §III B.

Even though the Stückelberg fluctuations are decoupled from the matter, a finite $\delta\Gamma$ generates a metric perturbation, and hence a matter perturbation, due to the effective stress energy it carries. By explicitly evaluating Eq. (6), we can write down the energy density perturbation

$$\delta\rho = -\delta T^0{}_0 = \left(\frac{g'^2 - f'^2 + a^2 W}{a^2 \sqrt{X}} - x_0 \right) \frac{P_1' \delta\Gamma}{2ar}, \tag{21}$$

and the radial momentum density perturbation

$$\delta q = T^r{}_0 = -\frac{\dot f f' - \dot g g'}{a^2 \sqrt{X}} \frac{P_1' \delta\Gamma}{2ar}. \tag{22}$$

Here and below the polynomials $P_i = P_i(x_0)$ are assumed to be evaluated on the background. Note that both quantities are directly proportional to P_1'. Furthermore $\delta\rho$ and δq depend only on $\delta\Gamma$ and not its derivatives, in contrast to the energy and momentum of usual fields, which typically also get contributions from the their kinetic energy.

The energy and momentum density suffice to define the impact of the Stückelberg fields on matter through the two associated Einstein equations. To define the whole stress tensor or use all of the Einstein equations, we can also derive the radial pressure

$$\delta T^r{}_r = -\left(\frac{\dot f^2 - \dot g^2 + b^2 W}{b^2\sqrt{X}} - x_0\right)\frac{P_1'\delta\Gamma}{2ar}. \tag{23}$$

The other components of the stress energy tensor can then be found through conservation of energy

$$\dot{\delta\rho} = -3\frac{\dot a}{a}(\delta\rho + \delta p) + \frac{(ba^3)'}{ba^3}\delta q + \frac{(r^2\delta q)'}{r^2} = 0. \tag{24}$$

Given $\delta\rho$ and δq, this equation defines δp. Since by definition

$$\delta p = \frac{1}{3}(\delta T^r{}_r + \delta T^\theta{}_\theta + \delta T^\phi{}_\phi) \tag{25}$$

and $\delta T^\theta{}_\theta = \delta T^\phi{}_\phi$, the energy equation defines the remaining components. We have verified that direct evaluation of the stress components in Eq. (6) gives the same result once combined with the equations of motion. With the stress tensor fully defined, any two of the Einstein equations completes the dynamics as usual and define δa and δb jointly with the matter stress energy.

Momentum conservation provides another useful auxiliary equation

$$\begin{aligned}\dot{\delta q} = &-\Big(5\frac{\dot a}{a} - \frac{\dot b}{b}\Big)\delta q + \frac{b^2}{a^2}\left[\frac{b'}{b}\Big(\delta\rho + \delta T^r{}_r\Big)\right. \\ &\left.+(\delta T^r{}_r)' + \frac{(ar)'}{ar}\Big(2\delta T^r{}_r - \delta T^\theta{}_\theta - \delta T^\phi{}_\phi\Big)\right].\end{aligned} \tag{26}$$

With an anisotropic stress tensor, balancing radial gradients with the different $T^i{}_j$ components can result in static conditions if

$$(\delta T^r{}_r)' = -\frac{(ar)'}{ar}\Big(2\delta T^r{}_r - \delta T^\theta{}_\theta - \delta T^\phi{}_\phi\Big). \tag{27}$$

Whereas for an isotropic stress tensor, the right hand side vanishes and $(\delta T^r{}_r)' = \delta p'$, whose finite value would generate a momentum density unless balanced by the b'/b gravitational term. The impact of anisotropic equations of state on hydrostatic equilibrium and stability has been studied in a related neutron star context (e.g. [26, 27]).

In summary, similar to what we found for the background solution, the stress energy tensor for the fluctuations depends on the perturbation to the spatial Stückelberg field ($\delta\Gamma$) and not the perturbation to the temporal Stückelberg field (δf). Hence even though the Stückelberg fields require two sets of initial data obeying coupled first order equations, reminiscent of a field and its conjugate momentum, only one of these has any observable impact on the matter. Moreover, the coupling is unidirectional, as $\delta\Gamma$ forms its own autonomous first order system. It can be consistently set to zero by a choice of initial condition.

B. Action

In order to make contact with the literature on Stückelberg field dynamics, it is useful to see how these results arise from the second order action. In particular, apparent differences concerning the sign and value of the time derivative terms in this action have led to seemingly conflicting results about the presence of strongly coupled or ghost-like modes [6, 17, 20].

We expand the action of Eq. (9) to second order in both Stückelberg and metric perturbations around our background solution in the Appendix. It is sufficient here to examine the Stückelberg sector:

$$\begin{aligned}S_{SS} =& \frac{P_1'}{2}M_{\rm pl}^2 m^2 \int dr d\Omega \Big[\delta f \delta\Gamma_{\rm eom} - \frac{ab}{2}\delta\Gamma^2 \\ &- a^2 br\delta\Gamma(\delta\Gamma'\partial_{g'} + \dot{\delta\Gamma}\partial_{\dot g})\sqrt{X}\Big],\end{aligned} \tag{28}$$

where $\delta\Gamma_{\rm eom}$ was defined in Eq. (20) and note that this is the only term that depends on δf in the whole second order action (see Eq. A5). Furthermore since $\delta\Gamma_{\rm eom}$ depends only on $\delta\Gamma$, there are no terms quadratic in δf fluctuations. Varying the action with respect to δf reproduces the perturbed equation of motion $\delta\Gamma_{\rm eom} = 0$ with no dependence on other perturbations.

With the help of Eq. (16), we can see that the coefficient of the $\delta f \dot{\delta\Gamma}$ term in the action is proportional to

$$P_1'\left[\frac{x_0 \dot f}{bW} - \mu\frac{(ar)'}{a}\right], \tag{29}$$

and determines the dynamics of $\delta\Gamma$ through its equation of motion. The quantity in brackets is determined by the background solution for f and there is a special case where it vanishes identically. Note that if we assume that the time dependence of $f \propto a^p$, then $\dot f \propto f\dot a/a$ and $bW \propto f\dot a/a$ so this condition becomes independent of time. We shall see that this very special f-g symmetry is exploited in open universe solutions (see §IV C) and explains their lack of Stückelberg dynamics [20].

While P_1' in Eq. (29) drops out of the equation of motion, it does determine the sign of all Stückelberg coefficients in the action. For the self-accelerating solutions

$$P_1'(x_0) = \pm 4\sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4}, \tag{30}$$

for the two x_0 solutions. If $\alpha_3 = \alpha_4 = 0$ there is only one solution, $P_1' = -4$ and the sign of the coefficient is fixed entirely by the background solution. There is a special choice

$$\alpha_4 = \frac{1}{12}(1 + 3\alpha_3 + 9\alpha_3^2), \tag{31}$$

where $P_1' = 0$. In this case the whole Stückelberg quadratic action vanishes as do all of the stress-energy components. This is the same special choice that was made in Refs. [9, 10], who similarly found that the quadratic action for fluctuations around solutions with this parameter choice vanished.

For general α_3 and α_4, P_1' and hence the action may have either sign. This is consistent with the decoupling limit analysis of the kinetic term where $\alpha_3 \neq 0$ or $\alpha_4 \neq 0$ was necessary to ensure that the helicity 0 scalar fluctuation is not a ghost [6, 17]. In this limit, the scalar, $\delta\pi$, determines the two Stückelberg field fluctuations $\delta\Gamma \to \delta\pi'$, $\delta f \to -\dot{\delta\pi}$ and hence the $\delta f \delta\dot{\Gamma}$ term acts as a kinetic term for $\delta\pi$ after integration by parts (see also Eq. 41).

The decoupling limit form suggests a partial way to interpret the first order nature of the coupled δf-$\delta\Gamma$ system. There δf appear in the action in a way similar to a canonical momentum for the field $\delta\pi$, which is itself related to $\delta\Gamma$ [28]. This viewpoint suggests an interpretation of δf and $\delta\Gamma$ as a pair of "half" degrees of freedom. This counting is related to the absence of the Boulware-Deser ghost: of our original 4 Stückelberg fields, there is one constraint by construction and there are two angular modes that are absent in our analysis because of spherical symmetry.

However, interpreting δf and $\delta\Gamma$ as a simple single propagating scalar does not carry over to the full theory, where the fields are not directly related. This is of course not in itself a contradiction, as the structure of this theory makes a global helicity decomposition generically impossible [29]. In particular δf has no necessary relationship to $\delta\Gamma$ as any choice that solves its equation of motion yields the same solution for $\delta\Gamma$. Furthermore, the dynamics of $\delta\Gamma$ depends on the choice of background solution for f, as we shall see in the next section.

IV. TIME DYNAMICS

The detailed dynamics of Stückelberg field fluctuations depend on the choice of solution for f, the Stückelberg field related to the time coordinate. There are even special cases where the fluctuations have no dynamics. All such choices have the same background metric and self-acceleration. In this section we systematically construct explicit solutions for f and study their implications for the dynamics of field fluctuations and the associated initial value problem. We also generalize many solutions in the literature [6–9, 12, 17] as well as introduce new classes.

A. Unitary Gauges

Since the Stückelberg fields are spacetime scalars, their values, once mapped to the same spacetime point, are coordinate independent. Unitary gauges are defined so their time t_u and radial r_u coordinates are

$$t_u = f(r,t), \qquad r_u = g(r,t). \tag{32}$$

The Stückelberg fields are therefore just the unitary gauge coordinates and their functional form in isotropic coordinates (r,t) is simply the map itself. In unitary gauges, the fiducial metric $\Sigma_{\mu\nu} = \eta_{\mu\nu}$ and this simplification has been crucial for finding most solutions existing in the literature. While we do not utilize a unitary gauge construction in our solution, we will categorize solutions by their unitary gauge correspondence.

All self-accelerating solutions have the common property that $g = x_0 a r$, implying that the radial coordinate of any unitary gauge is simply the same conformally rescaled version of the isotropic radial coordinate, similar to the distinction between physical and comoving coordinates in cosmology.

The different solutions are thus characterized by the choice of time coordinate for the unitary gauge. Each of these choices solves Eq. (16) for the same g; written explicitly,

$$\begin{aligned} b^2 f'^2 + 2ar(a'\dot{f}^2 - \dot{a}\dot{f}f') + r^2(a'\dot{f} - \dot{a}f')^2 &= \\ x_0^2(a'^2 b^2 r^2 + 2a'ab^2 r - \dot{a}^2 a^2 r^2). \end{aligned} \tag{33}$$

Recall that the metric functions a and b do not depend on the solution for f and so may be considered external functions in solving this equation.

There are several properties of this equations that are useful to note. It is a nonlinear partial differential equation in space and time. Its solutions can be characterized by the boundary conditions $f(0,t)$ and $f(r,0)$. The former is equivalent to the local $r = 0$ relationship between t and unitary time f and is our primary classification criterion. The latter allows a family of such solutions associated with a time integration constant. The equation does not guarantee that the map in Eq. (32) is free from singularities everywhere in the spacetime. A singularity corresponds to a change in the sign of the Jacobian determinant $\dot{f}g' - \dot{g}f'$ and hence a change in the sign of μ (see Eq. 10). Solutions should be checked against singularities developed in the course of radial or temporal integration as they signal a breakdown in our treatment of the square roots in the action.

Finally given a solution for the $\alpha_3 = \alpha_4 = 0$ case where $x_0 = 3/2$,

$$f(r,t;\alpha_3,\alpha_4) = \frac{2x_0}{3} f(r,t;0,0) \tag{34}$$

is the solution for the full parameter space. One may similarly scale any other specific α_3 and α_4 case to the general case by use of the ratios of the respective values of x_0. This relationship makes it trivial to generalize special solutions in the literature (cf. [8]).

In our examples, we seek cosmological solutions where

$$b(r,t) = 1, \qquad a(r,t) = \frac{a_F(t)}{1 + Kr^2/4}, \tag{35}$$

where K is the spatial curvature and $a_F(t)$ is the scale factor of a Friedmann-Robertson-Walker metric. Note that in a flat $K = 0$ geometry $a(r,t) = a(t) = a_F(t)$.

B. Unitary Time t

Perhaps the most natural choice of solution is to take unitary time f to be linearly related to t

$$f(0,t) = \frac{x_0}{C} t, \tag{36}$$

where C is a constant. For a flat universe $K = 0$ and we can solve Eq. (33) with this ansatz to $\mathcal{O}(r^4)$

$$f(r,t) = \frac{x_0}{CH}\left(Ht + \frac{1-\sqrt{1-C^2}}{2}(aHr)^2\right), \tag{37}$$

where $H = \dot a/a$. Note that on cosmological scales where $aHr \gtrsim 1$, this temporal field becomes spatially inhomogeneous.

1. de Sitter

The special case that $H =$ const. is of course the late time limit of the self-accelerating solution where the matter becomes subdominant to the effective cosmological constant

$$H = \sqrt{\frac{1}{6}P_0 m}, \tag{38}$$

and we can normalize $a = e^{Ht}$. Solving Eq. (33) for this case with the ansatz

$$f(r,t) = \frac{x_0}{CH}\left[Ht + F(aHr)\right] \tag{39}$$

gives

$$\begin{aligned} f(r,t) &= \frac{x_0}{CH}\left[Ht - y + \ln\frac{1+y}{1-(aHr)^2}\right], \\ y &= \sqrt{1 + C^2(a^2H^2r^2 - 1)}. \end{aligned} \tag{40}$$

These relations are equivalent to the de Sitter solutions of Refs. [7, 8] up to an overall constant and generalizes them to arbitrary α_3, α_4.

Note that if $C = 1$, $f \propto t$ and $g \propto ar \equiv r_p$ near $r = 0$ and can be thought of deriving from a Lorentz scalar in physical coordinates

$$\pi = \frac{1}{2}x_0(r_p^2 - t^2) \tag{41}$$

with $f \approx -\dot\pi$ and $g \approx \partial_{r_p}\pi$. Hence $C \neq 1$ may be said to have a "vector" in the background [8] even though the Stückelberg fields are spacetime scalars.

Given the solution in Eq. (40), the evolution equation for $\delta\Gamma$ becomes

$$\begin{aligned} (y^2 + y)\frac{\dot{\delta\Gamma}}{H} &= \left[y + \frac{y^2}{(aHr)^2}\right] r\delta\Gamma' \\ &+ \left[\frac{2y^2}{(aHr)^2} - 3y^2 + 1\right]\delta\Gamma, \end{aligned} \tag{42}$$

and the energy density, momentum density and pressure are given by

$$\begin{aligned} \delta\rho &= -\frac{(aHr)^2 C^3}{(1+C)(1-y)^2}\frac{x_0 P_1'\delta\Gamma}{2ar}, \\ \delta q &= \frac{y + C^2 - 1}{C^2a^2Hr}\delta\rho, \quad \delta p = \frac{1}{3}\delta\rho\,. \end{aligned} \tag{43}$$

The pressure follows a radiation-like equation of state but itself is composed of anisotropic contributions

$$\delta T^r{}_r = \left(\frac{y + C^2 - 1}{C^2 aHr}\right)^2 \delta\rho \neq \delta p. \tag{44}$$

It is instructive to consider the simplest $C = 1$ case further. Here $y = aHr$, and given an initial value for $\delta\Gamma(r,0)$, the general solution is

$$\delta\Gamma(r,t) = \frac{1}{a^3}\left(1 + \frac{a-1}{aHr}\right)^3 \delta\Gamma\left(r + \frac{a-1}{aH}, 0\right), \tag{45}$$

which then defines the stress components

$$\begin{aligned} \delta\rho &= -\frac{(aHr)^2}{2(1-aHr)^2}\frac{x_0P_1'\delta\Gamma}{2ar}, \\ \delta q &= \frac{1}{a}\delta\rho, \quad \delta T^r{}_r = 3\delta p = \delta\rho. \end{aligned} \tag{46}$$

The angular stresses vanish and the total pressure is given by the radial component. These components have poles in their expressions at the horizon $r = 1/aH$ corresponding to the coordinate singularity in Eq. (40) for unitary time. However, since the solution for $\delta\Gamma$ always maps the initial horizon onto the horizon at a later epoch, if the energy density is finite in the initial conditions there, it remains so. Furthermore for large scales $r \gg 1/H$, which were outside the horizon at the initial epoch, $\delta\rho \propto a^{-4}$; this is consistent with expansion effects and the radiation-like equation of state.

2. Matter and Radiation Domination

While we have solved the initial value problem for fluctuations in the de Sitter limit for $f(0,t) \propto t$, in a cosmological solution these would themselves originate from the dynamics in the preceding radiation and matter dominated epochs.

To construct solutions where $f(0,t) \propto t$ during the matter and radiation epochs, we first try

$$f(r,t) = \frac{x_0 t}{C}F(ar/t), \tag{47}$$

with the assumption that $H \propto a^{-3(1+w)/2}$ for a constant w and C. For any such choice, we can solve differential equation for F implied by Eq. (33). While this construction would seem to admit many solutions, for finite C, μ changes sign between $+1$ for $r \to 0$ and -1 for $r \to \infty$

indicating a singularity in the map between isotropic and unitary coordinates.

On the other hand, the $C \to \infty$ limit does not suffer this problem and yields

$$f(r,t) = x_0\sqrt{\tau^2(t) + r^2 a(t)^2}, \tag{48}$$

where

$$\tau = \sqrt{\frac{9(1+w)^2}{4(2+3w)}}\, t \tag{49}$$

for $w > -2/3$. Eq. (49) is now easy to generalize to an arbitrary expansion history by matching

$$\tau^2(t) = a(t)\int^t \frac{d\tilde{t}}{\dot{a}(\tilde{t})} \tag{50}$$

for $\dot{a} > 0$. This solution includes a universe that evolves from radiation domination through matter domination to self-acceleration. Note that since $w = -1$ during self-acceleration, τ is no longer directly related to t and so this solution is distinct from the solutions of the previous section.

The fluctuation $\delta\Gamma$ evolves under

$$2r^2 a^3 \dot{a}\delta\dot{\Gamma} + [\dot{a}^2\tau^2 - a^2] r\delta\Gamma' = -2\frac{A}{B}\delta\Gamma, \tag{51}$$

where

$$\begin{aligned} A &= \dot{a}^4\tau_r^4 + 2a\dot{a}^3\tau_r^3 + r^2a^2\dot{a}^4\tau_r^2 - 2a^3\dot{a}\tau_r(1 - r^2\dot{a}^2) \\ &\quad + 2r^2a^5\ddot{a} - a^4(1 - \dot{a}^2r^2 + 2\dot{a}^4r^4 - 2\dot{a}\ddot{a}r^2\tau_r), \\ B &= (a + \dot{a}\tau_r)^2 - r^2a^2\dot{a}^2, \end{aligned} \tag{52}$$

with $\tau_r^2 = \tau^2 + r^2a^2$ and the stress energy components are

$$\begin{aligned} \delta\rho &= -\frac{a^2\dot{a}r}{B}\frac{x_0 P_1'\delta\Gamma}{\tau_r}, \quad \delta q = \frac{a^2 - \dot{a}^2\tau^2}{2ra^3\dot{a}}\delta\rho, \\ \delta p &= \frac{a\ddot{a}}{3\dot{a}^2}\delta\rho, \quad \delta T^r{}_r = \left(\frac{a^2 - \dot{a}^2\tau^2}{2ra^2\dot{a}}\right)^2\delta\rho. \end{aligned} \tag{53}$$

Note that for a constant background equation of state w, $\delta p/\delta\rho = -(1+3w)/6$. For $w > -1/9$ expansion thus makes the energy density redshift more slowly than the dominant matter but always faster than the constant background Stückelberg contributions. Thus in linear theory starting from some arbitrary field configuration, we expect that $\delta\rho/\rho$ will be driven rapidly to zero for any matter content leaving just the cosmological constant background.

C. Unitary Time a

Unitary time can also be made proportional to $a(0,t) = a_F(t)$

$$f(0,t) = \frac{x_0}{C}a_F(t), \tag{54}$$

where, again, C is a constant. In this case both f and g share the same temporal dependence at the origin and hence provide the ingredients necessary for static solutions. To order $\mathcal{O}(r^4)$

$$f(r,t) \approx \frac{x_0}{C}a_F\left[1 + \frac{\dot{a}_F^2 \pm \sqrt{(\dot{a}_F^2 - C^2)(\dot{a}_F^2 + K)}}{2}r^2\right]. \tag{55}$$

Here we have kept the possibility that $K \neq 0$ since it allows a special class of solutions.

1. Open Solution

If $K < 0$ then Eq. (55) has a simple solution for $C^2 = -K$. Taking this ansatz, the full solution is [12, 20]

$$f(r,t) = x_0 a_F(t)\sqrt{\frac{1}{-K} + \frac{r^2}{(1 + Kr^2/4)^2}}. \tag{56}$$

Note that the open solution for f, like other solutions, is inhomogeneous in isotropic coordinates. There is nothing special about the open solution with regards to homogeneous and isotropic Stückelberg fields (cf. [12]).

On the other hand, the common separable $a_F(t)$ factor in f, g and a allows this solution to satisfy the static condition for the $\delta\Gamma$ field, Eq. (29). Correspondingly, Eq. (20) becomes

$$(4 + Kr^2)r\delta\Gamma' + 2(4 - Kr^2)\delta\Gamma = 0. \tag{57}$$

Note that $\mu = \mathrm{sgn}(\dot{a}_F)$ and so is $+1$ for an expanding universe. Interestingly the determinant goes to zero if the expansion turns around, e.g. because of a negative true cosmological constant, signaling a breakdown of the solution. We therefore consider only expanding solutions here.

The general solution to this equation is

$$\delta\Gamma(r) \propto \left(\frac{4 + Kr^2}{r}\right)^2. \tag{58}$$

This static Stückelberg field produces no energy density, momentum density or pressure; i.e.,

$$\delta p = \delta q = \delta p = 0. \tag{59}$$

These static conditions are maintained by a delicate balance of the radial and anisotropic stress gradients

$$\delta T^r{}_r = \left(1 - \frac{\dot{a}_F}{\sqrt{-K}}\right)\frac{x_0 P_1'\delta\Gamma}{2ar}, \tag{60}$$

which satisfy the static condition for Eq. (27).

Even though $\delta p = \delta q = 0$, anisotropic stress still has an impact on the metric through the Einstein equations. Moreover, these types of solutions are potentially unstable to anisotropic perturbations [22, 23].

2. Flat Solution

If $K = 0$, then there is still one simple solution to Eq. (55) where $C^2 \ll \dot{a}_F^2$. In that case

$$f(r,t) \approx \frac{x_0}{C} a(t). \tag{61}$$

To promote this to an exact solution we follow Ref. [11] and generalize their approach to arbitrary α_3, α_4

$$f(r,t) = \frac{x_0}{C} a(t) \left[1 + \frac{C^2}{4} r^2 + \frac{C^2}{4} \frac{\tau^2(t)}{a^2(t)} \right], \tag{62}$$

which recovers the approximate scaling for $r \ll 1/C$ and $t \ll a/C$. Eq. (20) becomes

$$-2C^2 r^2 a\dot{a}\delta\dot{\Gamma} + [C^2 + (C^2 r^2 - 4)\dot{a}^2] r\delta\Gamma' = 2\frac{A}{B}\delta\Gamma \tag{63}$$

with

$$\begin{aligned} A &= \mu C^4 + 4C^3\dot{a} - 16C\dot{a}^3 - \mu(16 - C^4 r^4)\dot{a}^4 \\ &\quad -2C^3 r^2 a(C\mu + 2\dot{a})\ddot{a}, \\ B &= -\mu C^2 - 4C\dot{a} - \mu(4 - C^2 r^2)\dot{a}^2. \end{aligned} \tag{64}$$

Despite solving Eq. (33), there is a problem with this form as a global solution since

$$\mu = \begin{cases} 1 & aHr < \sqrt{1 + 4H^2/C^2} \\ -1 & aHr > \sqrt{1 + 4H^2/C^2} \end{cases}. \tag{65}$$

The sign change in μ indicates the mapping to unitary gauge is singular unless $C \to 0$.

In this limit we are again left with a static equation

$$-r\delta\Gamma' = 2\delta\Gamma, \tag{66}$$

whose solution is $\delta\Gamma \propto 1/r^2$. Thus the $K \to 0$ flat limit of the open solution is the same as the $C \to 0$ solution of the flat solution. More generally, the stress-energy components become

$$\begin{aligned} \delta\rho &= \frac{C^3 r^2 \dot{a}}{B} \frac{x_0 P_1' \delta\Gamma}{2ar}, \quad \delta q = \frac{C^2 - (4 - C^2 r^2)\dot{a}^2}{2C^2 r a\dot{a}} \delta\rho, \\ \delta p &= \frac{a\ddot{a}}{3\dot{a}^2}\delta\rho, \quad \delta T^r{}_r = \left[\frac{C^2 - (4 - C^2 r^2)\dot{a}^2}{2C^2 r\dot{a}} \right]^2 \delta\rho. \end{aligned} \tag{67}$$

Again for the $C \to 0$ case, $\delta\rho$, δq, δp are all suppressed whereas $\delta T^r{}_r$, $\delta T^\theta{}_\theta$, $\delta T^\phi{}_\phi$ are unsuppressed and contain radial gradients that delicately balance each other.

D. Unitary Time $\sqrt{a}$

While $f(0,t) \propto t$ or a are obvious choices for solutions, they do not exhaust the possibilities. We have seen that the former generally require dynamical Stückelberg fluctuations whereas the latter admit static ones. However those static cases are only formally well-defined as global solutions for $K < 0$.

To see whether there exists a well-defined static solution for $K = 0$ we construct solutions where $f(0,t) \propto \sqrt{a}$ is the limiting form. Similarly to the flat $\propto a$ case, we find an exact flat solution

$$f(t,r) = x_0 \sqrt{\frac{a(t)}{C^2} + \tau^2(t) + a^2(t) r^2} \tag{68}$$

for any constant C. Note that if $C^2 \gg \tau^2/a$, this solution recovers the general $f(0,t) \propto t$ solution of §IV B 2. Since we have already considered the general case of this limit, it is interesting to consider the opposite one. Note that in a cosmological solution, once self acceleration sets in $\tau^2/a \to$ const. of order H^{-2}. Thus taking $C \ll m$ ensures that this limit is satisfied for all time.

In this $C \ll m$ and $a = e^{Ht}$ case, to leading order $\delta\Gamma$ obeys

$$2ar^2 C^2 \frac{\delta\dot{\Gamma}}{H} + r\delta\Gamma' = -2\delta\Gamma, \tag{69}$$

which is a stiff equation as $r \to 0$ with an equilibrium static solution of $\delta\Gamma \propto 1/r^2$. The stress components

$$\begin{aligned} \delta\rho &= -\frac{r^2 C^3}{H} \frac{x_0 P_1' \delta\Gamma}{\sqrt{a} r}, \quad \delta q = -\frac{H}{2arC^2}\delta\rho, \\ \delta p &= \frac{1}{3}\delta\rho, \quad \delta T^r{}_r = \frac{H^2}{4r^2 C^4}\delta\rho. \end{aligned} \tag{70}$$

Since $C \ll H$, $\delta T^r{}_r \gg \delta p$ and again involves a delicate balance of anisotropic stresses. This case is very similar to the $C \to 0$ case of the previous section but has the benefit that $\mu = 1$ everywhere and so C can be set to a finite number.

V. DISCUSSION

We have presented a general analysis of isotropic Stückelberg field perturbations around the full class of self-accelerating solutions [13] of the massive gravity theory. These background solutions are defined by two fields, one spatial $g(r,t)$ and one temporal $f(r,t)$, where only the spatial one is responsible for the stress-energy of the effective cosmological constant.

Likewise Stückelberg field perturbations come in two classes, spatial ($\delta\Gamma$) and temporal (δf). Spatial perturbations can be consistently set to zero by a choice of initial conditions. They are not generated by any matter-induced metric perturbations as was already apparent in that the self-accelerating solution is exact and non-perturbative for isotropic metrics [13]. With an arbitrary choice of initial conditions, spatial Stückelberg fluctuations generically possess stress energy. This stress energy produces metric fluctuations to which the matter responds. Temporal Stückelberg fluctuations carry no stress energy and have no effect on metric fluctuations at this order in perturbation theory.

Importantly, the dynamics of spatial Stückelberg fluctuations make energy density deviations from the constant background always decay with the expansion in cosmological solutions, implying that the background solution is stable. There are special choices where the Stückelberg fields perturbations are static, but in those cases the fields carry no energy density. In the general case, the decay rate depends on the equation of state in the background.

Different behaviors of the spatial Stückelberg fluctuation are related to the background temporal Stückelberg field. This difference is the source of apparently conflicting claims in the literature regarding Stückelberg perturbation dynamics. The temporal Stückelberg field in the background is itself the choice of the time coordinate in which the massive gravity theory appears locally as the linearized Fierz-Pauli theory, i.e. it is a choice of unitary gauge for the covariant theory. For unitary time coinciding locally with isotropic time, the single propagating degree of freedom found in the decoupling limit [6, 17] appears in the exact theory as well. For unitary time that scales with the spatial scale factor, there are no Stückelberg dynamics, consistent with the open universe solutions [20]. Furthermore these solutions remain static not by possessing a vanishing stress tensor perturbation but rather by a potentially unstable balance of anisotropic stresses.

The sign of the energy density carried by the Stückelberg field $\delta\Gamma$, as well as the sign of the prefactor of the term in the quadratic action that generates the $\delta\Gamma$ dynamics, are both determined by the quantity P_1', a constant determined by the parameters of the theory (α_3 and α_4, see Eq. 30). $P_1' = -4$ when $\alpha_3 = \alpha_4 = 0$, but can be either positive or negative for general α_3 and α_4. This is consistent with decoupling limit findings that perturbations around self-accelerating solutions generically are ghost-like for $\alpha_3 = \alpha_4 = 0$, but can be made healthy in more general circumstances [6, 17]. P_1' can be also be set to zero by a special choice. These models have no quadratic Stückelberg contributions to the action and no linearized stress energy perturbations, consistent with the findings of Ref. [9, 10], who also made this parameter choice.

In the exact theory, the dynamical system represented by the Stückelberg fields exhibits some peculiar properties that are obscured in decoupling limit analyses. The two Stückelberg field perturbations require two sets of initial data – $\delta f(r, 0)$ and $\delta\Gamma(r, 0)$ – and each obey coupled first order differential equations of motion. This is the amount of initial data and dynamics that we would expect for a single propagating degree of freedom. This situation might have been anticipated at the outset from a counting argument. Starting with 4 Stückelberg scalar degrees of freedom, we remove two by restriction to spherical symmetry, and a third (the Boulware-Deser ghost) is removed by construction in this theory.

However, these two “half” degrees of freedom are unusual in their interrelation. As indicated above, the $\delta\Gamma$ field has a first order equation of motion that is independent of the δf field. The δf dynamics do depend on the $\delta\Gamma$ field, but they carry no stress energy at this order in perturbation theory. In addition to this, we find that the time derivatives of $\delta\Gamma$ do not actually contribute energy or momentum density either, so that there is no obvious classical instability associated with these fields even when they are apparently ghost-like.

Finally, we uncovered a potential physical problem with certain background configurations for the temporal Stückelberg field. The structure of the theory itself does not prevent the determinant of the fiducial metric from vanishing when solving the temporal or spatial boundary value problem. Some solutions for f pass through a zero determinant at finite radius (e.g. [11]) or time from well-posed initial values. At this point, the mapping between unitary and isotropic coordinates becomes singular.

Since such a choice may have physical pathologies, we also constructed a new solution for f that passes from radiation domination through matter domination to self-acceleration without exhibiting this potential problem or resorting to static fields. We leave the larger question of the theoretical implications for the existence of potentially pathological solutions to a future work.

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Appendix A: Second Order Action

The quadratic Lagrangian for the perturbations ($\delta\Gamma$, δf, δa, δb) can be separated out as

$$\delta^2 \mathcal{U}_g = \delta^2_{SS}\mathcal{U}_g + \delta^2_{ST}\mathcal{U}_g + \delta^2_{TT}\mathcal{U}_g, \tag{A1}$$

where $\mathcal{U}_g = \sqrt{-g}\mathcal{U}/(4\sin\theta) = a^3br^2\mathcal{U}/4$ carries the relevant pieces of the metric determinant, S refers to pieces involving Stückelberg fluctuations and T those involving metric fluctuations. The Stückelberg-Stückelberg terms are

$$\begin{aligned}\delta^2_{SS}\mathcal{U}_g =& \frac{1}{2}\delta\Gamma^2 ab\left(P_0'' + P_1''\sqrt{X} + P_2''W\right) + a^2brP_2'\delta\Gamma\left(\delta\Gamma'\partial_{g'}W + \delta\dot{\Gamma}\partial_{\dot{g}}W\right)\\ &+a^2brP_1'\delta\Gamma\left[\delta\Gamma'\partial_{g'} + \delta\dot{\Gamma}\partial_{\dot{g}} + \delta\dot{f}\partial_{\dot{f}} + \delta f'\partial_{f'}\right]\sqrt{X} + \mu ar(2P_2 - x_0P_2')\delta f(\delta\dot{\Gamma}\partial_r - \delta\Gamma'\partial_t)(ar)\\ =& P_1'\left[-\delta f\delta\Gamma_{\rm eom} + \frac{ab}{2}\delta\Gamma^2 + a^2br\delta\Gamma(\delta\Gamma'\partial_{g'} + \delta\dot{\Gamma}\partial_{\dot{g}})\sqrt{X}\right],\end{aligned} \tag{A2}$$

where we have used the background solution $g = x_0ar$, the polynomial recursion Eq. (12), and the relationship between W and $\sqrt{X}$ on the background from Eq. (16). Note that equality here means equality up to total derivative terms. Here

$$\delta\Gamma_{\rm eom} \equiv \partial_t[a^2rb\delta\Gamma\partial_{\dot{f}}\sqrt{X}] + \partial_r[a^2rb\delta\Gamma\partial_{f'}\sqrt{X}] - \mu ar[(ar)'\delta\dot{\Gamma} - \dot{a}r\delta\Gamma']. \tag{A3}$$

and its correspondence to Eq. (20) can be established by noting

$$\partial_{\dot{f}}\sqrt{X} = \frac{1}{b\sqrt{X}}\left(\frac{\dot{f}}{b} + \mu\frac{g'}{a}\right), \qquad \partial_{f'}\sqrt{X} = -\frac{1}{a\sqrt{X}}\left(\mu\frac{\dot{g}}{b} + \frac{f'}{a}\right). \tag{A4}$$

Next, the mixing between the Stückelberg fields and the metric fluctuations is given by

$$\begin{aligned}\delta^2_{ST}\mathcal{U}_g =& abr\left[2P_0' + P_1'(2 + a\partial_a + ax_0\partial_{g'})\sqrt{X} + P_2'(1 + ax_0\partial_{g'})W\right]\delta a\delta\Gamma\\ &+a^2br^2x_0\left[(P_2'\partial_{\dot{g}}W + P_1'\partial_{\dot{g}}\sqrt{X})\dot{\delta a} + (P_2'\partial_{g'}W + P_1'\partial_{g'}\sqrt{X})\delta a'\right]\delta\Gamma\\ &+2a^2br^2P_2\delta a(\delta\dot{\Gamma}\partial_{\dot{g}} + \delta\Gamma'\partial_{g'})W + a^2r\left[P_0' + P_1'(1 + b\partial_b)\sqrt{X}\right]\delta b\delta\Gamma\\ =& a^2brP_1'\left\{\left[(1 + a\partial_a)\sqrt{X} - x_0\right]\frac{\delta a}{a} + \left[(\delta a\, r)'\partial_{g'} + \dot{\delta a}\, r\partial_{\dot{g}}\right](x_0\sqrt{X} - W) + \left[(1 + b\partial_b)\sqrt{X} - x_0\right]\frac{\delta b}{b}\right\}\delta\Gamma.\end{aligned} \tag{A5}$$

Both the SS and ST pieces are directly proportional to P_1'. Note that

$$(1 + a\partial_a)\sqrt{X} = \frac{\dot{f}^2 - \dot{g}^2 + b^2W}{b^2\sqrt{X}}, \qquad (1 + b\partial_b)\sqrt{X} = \frac{g'^2 - f'^2 + a^2W}{a^2\sqrt{X}}, \tag{A6}$$

and these factors appear also in the perturbed energy and radial pressure relations, Eqs. (21) and (23), since functional variation of the action with the metric must return these components. Note that there is no mixing between the δf and metric fluctuations, as anticipated from our direct calculation of the stress energy tensor. On the other hand, variation of the action with respect to $\delta\Gamma$ yields an equation of motion for δf that depends on all of the other perturbations.

Finally, the metric-metric part is

$$\delta^2_{TT}\mathcal{U}_g = 3ar^2P_0\left(b\delta a^2 + a\delta a\delta b\right). \tag{A7}$$

As expected, it represents the perturbation to $\sqrt{-g}$ multiplying the effective background cosmological constant.

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