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Static post-Newtonian limits in non-projectable Hořava-Lifshitz gravity with an extra U(1) symmetry

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In this paper, we study static post-Newtonian limits in non-projectable Hořava-Lifshitz gravity with an extra U(1) symmetry. After obtaining all static spherical solutions in the infrared, we apply them to the solar system tests, and obtain the Eddington-Robertson-Schiff parameters in terms of the coupling constants of the theory. These parameters are well consistent with observations for the physically viable coupling constants. In contrast to the projectable case, this consistence is achieved without taking the gauge field and Newtonian prepotential as part of the metric.

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I. INTRODUCTION

To quantize gravity in the framework of quantum field theory, recently Hořava proposed the Hořava-Lifshitz (HL) theory of gravity [1], in which the Arnowitt-Deser-Misner (ADM) variables [2] are taken as the fundamental quantities to describe gravity. By construction, the theory is power-counting renormalizable, which is realized by including high-order spatial derivative operators [up to six in (3+1)-dimensional spacetimes]. The exclusion of high-order time derivative operators, on the other hand, ensures that the theory is unitary, a problem that has been faced in high-order derivative theories of gravity for a long time [3]. Clearly, this inevitably breaks the general diffeomorphisms,

$$\delta x^{\mu} = -\zeta^{\mu}(t,x), (\mu = 0,1,2,3). \tag{1.1}$$

Although such a breaking in the gravitational sector is much less restricted by experiments/observations than that in the matter sector [4, 5], it is still a challenging question how to prevent the propagation of the Lorentz violations into the Standard Model of particle physics [6]. Hořava assumed that such a breaking only happens in the ultraviolet (UV) and down to the foliation-preserving diffeomorphism,

$$\delta t = -f(t), \quad \delta x^{i} = -\zeta^{i}(t,x), (i = 1,2,3), \tag{1.2}$$

often denoted by Diff(M, $\mathcal{F}$). In the infrared (IR), the low derivative operators take over, presumably providing a healthy low energy limit.

The breaking of the general diffeomorphisms immediately results in the appearance of spin-0 gravitons in the theory, in addition to the spin-2 ones, found in general relativity (GR). This is potentially dangerous, and leads to several problems, including instability, strong coupling and different speeds of (massless) particles [7]. To resolve these problems, various models have been proposed [7], including the healthy extension of the non-projectable HL theory [8]. In the healthy extension, the instability problem was fixed by the inclusion of the term $\beta_0 a_i a^i$ in the gravitational action, where β_0 must be in the range $0 < \beta_0 < 2$, and $a_i \equiv N_{,i}/N$ with N being the lapse function in the ADM decompositions [2]. The strong coupling problem is resolved by introducing a new energy scale M_*, so that $M_* \le \Lambda_\omega \equiv \sqrt{\beta_0}M_{pl}$, where M_* denotes the suppression energy of the high-order spatial operators, Λ_ω the would-be strong coupling energy scale, and M_{pl} the Planck mass. Clearly, in order for this mechanism to work, one must assume that $\beta_0 \neq 0$. Observational constraints requires 10^{10} GeV $\le M_* \le 10^{15}$ GeV [8]. The low bound was obtained by assuming that M_* also sets the suppression energy scale in matter fields, while the up bound was obtained from the preferred frame effects [9, 10]. In addition, to avoid the Cherenkov radiation, one must require that the speed of the spin-0 gravitons be superluminal [11] [1].

A more dramatical modification was proposed recently by Hořava and Melby-Thompson (HMT) [13], in which an extra local U(1) symmetry was introduced, so that the symmetry of the theory was enlarged to,

$$U(1) \ltimes \text{Diff}(M,\ \mathcal{F}). \tag{1.3}$$

Such an extra symmetry is realized by introducing a gauge field A and a Newtonian prepotential φ. Because of this extra symmetry, the spin-0 gravitons are eliminated [13, 14]. As a result, all the problems related to them, such as the instability, strong coupling and different speeds in the gravitational sector, as mentioned

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[1] This can be easily seen using the equivalence between the Einstein-aether theory and the nonprojectable HL theory in the IR limit [8, 12].

above, are automatically resolved. This was initially done with the projectability condition $N = N(t)$ and $\lambda = 1$ [13], where λ characterizes the IR deviation of the theory from GR. It was soon generalized to the case with any value of λ [15], in which the spin-0 gravitons are still eliminated [15, 16]. Although the strong coupling problem in the gravitational sector disappears, it still exists in the matter sector [16], but can be resolved also by introducing a new energy scale M_*, so that $M_* \leq |\lambda - 1|^{1/4} M_{pl}$ [17]. Cosmological applications of this model were considered in [18, 19], and found that it is consistent with current observations [20]. On the other hand, the studies of solar system tests recently showed that the theory is consistent with observations *if and only if* the gauge field A and the Newtonian prepotential φ are part of the metric [21] [2].

A non-trivial generalization of the enlarged symmetry (1.3) to the nonprojectable case $N = N(t, x)$ was recently worked out in [26, 27], and showed that the only degree of freedom of the model in the gravitational sector is the spin-2 massless gravitons, the same as that in GR. Because of the elimination of the spin-0 gravitons, the physically viable region of the coupling constants is considerably enlarged, in comparison with the healthy extension [8], where the extra U(1) symmetry is absent. In particular, the requirement $\beta_0 \neq 0$ now is dropped out. Furthermore, the number of independent coupling constants is also dramatically reduced, from about 100 down to 15. The consistence of the model with cosmology was showed recently in [27–29], and various remarkable features were found.

In this paper, we study the consistence of the model proposed in [26, 27] with the solar system tests. In particular, we investigate its static post-Newtonian limits. The paper is organize as follows: In the next section (Sect. II), we give a brief introduction to the model, and in Sec. III we find the spherical static solutions in the IR limit, from which we can see that the Birkhoff theorem is not applicable to the current theory, because of the breaking of the Lorentz symmetry. As a results, there are several classes of static asymptotically flat vacuum solutions. Then, in Sec. IV, we study the post-Newtonian limit for each class of these solutions, and find the Eddington-Robertson-Schiff parameters, γ and β, in terms of the coupling constants of the theory, where γ is related to the amount of spatial curvature generated by the spherical source, and β to the degree of non-linearity in the gravitational field. By doing so, we show that the observational constraints on γ and β can be easily satisfied within the physically viable region of the phase space of the coupling constants of the theory. This is true without taking the gauge field A and Newtonian prepotential φ as part of the metric, in contrast to the projectable case [21]. In Sec. V, we summarize our main results and present some discussions and remarks.

[2] When $\lambda = 1$, the theory is consistent with the solar system tests even without the gauge field and the Newtonian prepotential being part of the metric [22, 23], but the corresponding cosmology is quite different from the standard one [16]. Other considerations of this model can be found in [24, 25].

II. NON-PROJECTABLE HL THEORY WITH U(1) SYMMETRY

In this section, we shall give a brief review of the non-projectable HL theory with the enlarged symmetry (1.3) [26, 27]. The fundamental variables of the theory are $\big(N,\ N^i,\ g_{ij},\ A,\ \varphi\big)$, where N^i and g_{ij} are, respectively, the shift vector, and 3-metric of the leaves $t =$ Constant in the ADM decompositions [2]. Under the local $U(1)$ symmetry, the fields transform as

$$\begin{aligned} \delta_\alpha A &= \dot{\alpha} - N^i\nabla_i\alpha, \quad \delta_\alpha\varphi = -\alpha, \\ \delta_\alpha N_i &= N\nabla_i\alpha, \quad \delta_\alpha g_{ij} = 0 = \delta_\alpha N, \end{aligned} \tag{2.1}$$

where α is the generator of the local $U(1)$ gauge symmetry, $\dot{\alpha} \equiv \partial\alpha/\partial t$, and ∇_i the covariant derivative with respect to the 3-metric g_{ij}. Under the Diff($M,\ \mathcal{F}$), they transform as,

$$\begin{aligned} \delta N &= \zeta^k\nabla_k N + \dot{N} f + N\dot{f}, \\ \delta N_i &= N_k\nabla_i\zeta^k + \zeta^k\nabla_k N_i + g_{ik}\dot{\zeta}^k + \dot{N}_i f + N_i\dot{f}, \\ \delta g_{ij} &= \nabla_i\zeta_j + \nabla_j\zeta_i + f\dot{g}_{ij}, \\ \delta A &= \zeta^i\nabla_i A + \dot{f}A + f\dot{A}, \\ \delta\varphi &= f\dot{\varphi} + \zeta^i\nabla_i\varphi. \end{aligned} \tag{2.2}$$

The general action reads [26, 27],

$$\begin{aligned} S &= \zeta^2\int dt d^3x\sqrt{g}N\Big(\mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_A + \mathcal{L}_\varphi \\ &\qquad + \zeta^{-2}\mathcal{L}_M\Big), \end{aligned} \tag{2.3}$$

where $\zeta^2 = 1/(16\pi G)$ with G being the Newtonian constant, $\mathcal{L}_M$ describes matter fields, and

$$\begin{aligned} \mathcal{L}_K &= K_{ij}K^{ij} - \lambda K^2, \\ \mathcal{L}_V &= \mathcal{L}_V^R + \mathcal{L}_V^a, \\ \mathcal{L}_A &= \frac{A}{N}\left(2\Lambda_g - R\right), \\ \mathcal{L}_\varphi &= \varphi\mathcal{G}^{ij}\left(2K_{ij} + \nabla_i\nabla_j\varphi + a_i\nabla_j\varphi\right) \\ &\quad + (1-\lambda)\Big[\big(\Delta\varphi + a_i\nabla^i\varphi\big)^2 \\ &\qquad + 2\big(\Delta\varphi + a_i\nabla^i\varphi\big)K\Big] \\ &\quad + \frac{1}{3}\hat{\mathcal{G}}^{ijlk}\Big[4\left(\nabla_i\nabla_j\varphi\right)a_{(k}\nabla_{l)}\varphi \\ &\qquad + 5\left(a_{(i}\nabla_{j)}\varphi\right)a_{(k}\nabla_{l)}\varphi \\ &\quad + 2\left(\nabla_{(i}\varphi\right)a_{j)(k}\nabla_{l)}\varphi + 6K_{ij}a_{(l}\nabla_{k)}\varphi\Big], \end{aligned} \tag{2.4}$$

with $\Delta \equiv \nabla^2$, and

$$K_{ij} = \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right),$$

$$
\begin{aligned}
a_i &= \frac{N_{,i}}{N}, \quad a_{ij} = \nabla_j a_i, \\
\hat{\mathcal{G}}^{ijlk} &= g^{il}g^{jk} - g^{ij}g^{kl}, \\
\mathcal{G}_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R + \Lambda_g g_{ij}, \\
\mathcal{L}_V^R &= \gamma_0\zeta^2 + \gamma_1 R + \frac{\gamma_2 R^2 + \gamma_3 R_{ij}R^{ij}}{\zeta^2} + \frac{\gamma_5}{\zeta^4}C_{ij}C^{ij}, \\
\mathcal{L}_V^a &= -\beta_0 a_i a^i + \frac{1}{\zeta^2}\Bigg[\beta_1\left(a_i a^i\right)^2 + \beta_2\left(a^i{}_i\right)^2 \\
&\quad + \beta_3\left(a_i a^i\right)a^j{}_j + \beta_4 a^{ij}a_{ij} + \beta_5\left(a_i a^i\right)R \\
&\quad + \beta_6 a_i a_j R^{ij} + \beta_7 R a^i{}_i\Bigg] + \frac{1}{\zeta^4}\ \beta_8\left(\Delta a^i\right)^2. \quad (2.5)
\end{aligned}
$$

Here $R_{ij},\ R$ are, respectively the Ricci tensor and scalar of g_{ij}, and C_{ij} denotes the Cotton tensor, defined by

$$
C^{ij} = \frac{e^{ijk}}{\sqrt{g}}\nabla_k\Big(R^j_l - \frac{1}{4}R\delta^j_l\Big), \quad (2.6)
$$

with $e^{123} = 1$, etc. $\lambda, \gamma_n, \beta_s$ and Λ_g are the coupling constants of the theory. In terms of R_{ij}, we have [27],

$$
\begin{aligned}
C_{ij}C^{ij} &= \frac{1}{2}R^3 - \frac{5}{2}RR_{ij}R^{ij} + 3R^i_jR^j_kR^k_i + \frac{3}{8}R\Delta R \\
&\quad + \left(\nabla_i R_{jk}\right)\left(\nabla^i R^{jk}\right) + \nabla_k G^k, \quad (2.7)
\end{aligned}
$$

where

$$
G^k = \frac{1}{2}R^{jk}\nabla_j R - R_{ij}\nabla^j R^{ik} - \frac{3}{8}R\nabla^k R. \quad (2.8)
$$

The infrared limit requires that

$$
\Lambda = \frac{1}{2}\gamma_0\zeta^2, \quad \gamma_1 = -1, \quad (2.9)
$$

where Λ denotes the cosmological constant.

The variations of the action with respect to N, N^i, A, φ and g_{ij}, yield, respectively, the Hamiltonian, momentum, A-, φ- constraints, and dynamical equations [3], given explicitly in [27].

III. SPHERICAL STATIC VACUUM SOLUTIONS

In this paper, we consider the spherically symmetric static spacetimes, described by [23, 30],

$$
\begin{aligned}
&N = N(r), \quad N^i = h(r)\delta^i_r, \quad A = A(r), \\
&g_{ij}dx^idx^j = \frac{dr^2}{f(r)} + r^2d\Omega^2, \qquad \varphi = \varphi(r), \quad (3.1)
\end{aligned}
$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. Using the U(1) gauge freedom, without loss of the generality, we set

$$
\varphi = 0. \quad (3.2)
$$

Then, we find that $\mathcal{L}_\varphi = 0$.

In the IR, the spatial curvature is very small, and its high-order derivatives are negligible, so we can safely assume that $\mathcal{L}_V$ has only three non-vanishing terms, given by

$$
\mathcal{L}_V = 2\Lambda - R - \beta_0 a_i a^i. \quad (3.3)
$$

In the solar system, the effects of cosmological constant are negligible. In addition, the curvature is extremely small. So, without loss of generality, we can safely set $\Lambda = \Lambda_g = 0$. Then, we find that in the present case there are only four-independent equations, which can be taken as the Hamiltonian and momentum constraints, the A-constraint, obtained from the variation of the gauge field A, and the rr-component of the dynamical equations, given, respectively, by

$$
\begin{aligned}
&\frac{h^2}{4fN^2}\left[\left(\frac{f'}{f} - \frac{2h'}{h}\right)^2 - \lambda\left(\frac{f'}{f} - \frac{2h'}{h} - \frac{4}{r}\right)^2 + \frac{8}{r^2}\right] \\
&\quad + \beta_0\left[2\frac{N''}{N} - \left(\frac{N'}{N}\right)^2 + \left(\frac{f'}{f} + \frac{4}{r}\right)\frac{N'}{N}\right] = 0, \quad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
&(\lambda - 1)\Bigg\{2h'' - \left(\frac{f'}{f} + \frac{2N'}{N} - \frac{4}{r}\right)h' \\
&\quad - \left[\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{f'N'}{fN} + \frac{4}{r}\left(\frac{N'}{N} + \frac{1}{r}\right)\right]h\Bigg\} \\
&\quad + \frac{2h}{r}\left(\frac{f'}{f} - \frac{2N'}{N}\right) = 0, \quad (3.5)
\end{aligned}
$$

$$
(rf)' = 1, \quad (3.6)
$$

$$
\left(\frac{A}{f^{1/2}}\right)' = Q(r), \quad (3.7)
$$

where

$$
\begin{aligned}
Q(r) &= -\frac{(\lambda-1)rh^2}{16f^{3/2}N}\Bigg[4\left(\frac{f''}{f} - \frac{2h''}{h}\right) \\
&\quad - \frac{f'}{f}\left(\frac{3f'}{f} + \frac{4N'}{N} + \frac{8}{r}\right) \\
&\quad + \frac{4h'}{h}\left(\frac{h'}{h} + \frac{2N'}{N}\right) + \frac{16}{r}\left(\frac{N'}{N} + \frac{2}{r}\right)\Bigg] \\
&\quad + \frac{h^2}{f^{3/2}N}\left(\frac{f'}{f} - \frac{h'}{h} - \frac{N'}{N} - \frac{1}{2r}\right) \\
&\quad + \frac{N}{2f^{1/2}}\left(\frac{2N'}{N} - \frac{f'}{f}\right) + \frac{\beta_0 r}{4f^{1/2}N}N'^2. \quad (3.8)
\end{aligned}
$$

Therefore, in the present case, we have four independent ordinary differential equations for four unknowns, $N,\ f, h$

[3] In the expression of F^{ij}_a given by Eq.(3.22) in [27], the coefficients β_s should be replaced by $\hat{\beta}_s$, where $\hat{\beta}_s = (-\beta_0, \beta_n)$ with $n = 1, 2, ..., 8$, while $\hat{\gamma}_9$ defined in Eq.(3.24) should be defined as $\hat{\gamma}_9 = \gamma_5$, instead of $\gamma_5/2$.

and A. In particular, Eq.(3.6) has the general solution,

$$f(r) = 1 - \frac{2B}{r}, \tag{3.9}$$

where B is an integration constant. Once f is known, from Eqs.(3.4) and (3.5) one can find N and h, while Eq.(3.7) yields,

$$A(r) = \sqrt{1 - \frac{2B}{r}}\left(A_0 + \int Q(r)dr\right), \tag{3.10}$$

where A_0 is an integration constant. Therefore, the main task now reduces to solve Eqs.(3.4) and (3.5) for N and h. In the following, let us consider the two cases $h = 0$ and $h \neq 0$, separately.

A. $h = 0$

In this case, the momentum constraint is satisfied identically, while Eq.(3.4) reduces to

$$\beta_0\left[2\frac{N''}{N} + \left(\frac{f'}{f} - \frac{N'}{N} + \frac{4}{r}\right)\frac{N'}{N}\right] = 0. \tag{3.11}$$

When $\beta_0 = 0$, it is satisfied identically, while Eq.(3.10) yields,

$$N = N_0\sqrt{1 - \frac{2B}{r}} + A(r), \tag{3.12}$$

where N_0 is a constant, and $A(r)$ is undetermined. In particular, when $A = 0$, the above solutions reduce to the Schwarzschild solution,

$$f = N^2 = 1 - \frac{2B}{r}, \quad A(r) = 0. \tag{3.13}$$

Note that in writing the above expression, we had set $N_0 = 1$ by rescaling t. It is remarkable to note that the above solutions are independent of the coupling constant λ. That is, in contrast to all other models of the HL theory [7], the Schwarzschild solution (3.12) is a solution of the HL theory not only for $\lambda = 1$ but also for any value of λ in the IR limit. In addition, in the non-projectable HL theory without the U(1) symmetry, to solve the instability problem of the spin-0 gravitons, β_0 is necessarily non-zero, $\beta_0 \neq 0$. Otherwise, the spin-0 gravitons become unstable. However, in the current case the U(1) symmetry eliminates the spin-0 gravitons [27], so the instability problem is out of question, and the physically viable region of the phase space includes the point $\beta_0 = 0$. Furthermore, in the projectable case, we have $N = N(t)$, and the solutions with $h = 0$ take the Schwarzschild form only when the gauge field A and the Newtonian pre-potential φ are part of the metric [13, 21]. The above observations are important, when we consider the solar system tests.

When $\beta_0 \neq 0$, from Eq.(3.11) we find that

$$N(r) = \left(N_0 + N_1\sqrt{1 - \frac{2B}{r}}\right)^2, \tag{3.14}$$

where N_1 is another integration constant. Inserting it into Eqs.(3.8) and (3.10), we obtain

$$\begin{aligned} A(r) &= N_0^2 + A_0\sqrt{1 - \frac{2B}{r}} - N_1^2\frac{B}{r} \\ &\quad -(\beta_0 - 1)N_1^2\left(1 - \frac{B}{r}\right). \end{aligned} \tag{3.15}$$

B. $h \neq 0$

In this case, to solve Eqs.(3.4) and (3.5), we consider the cases $\lambda = 1$ and $\lambda \neq 1$, separately.

1. $\lambda = 1$

When $\lambda = 1$, Eq.(3.5) has the general solution,

$$N = N_0\sqrt{f}. \tag{3.16}$$

Then, Eq.(3.4) and Eq.(3.10) yield,

$$\begin{aligned} h(r) &= N_0\sqrt{\frac{r - 2B}{12r^2}}\left[C_1 + 3B\beta_0\ln\left(\frac{r}{r - 2B}\right)\right]^{1/2}, \\ A(r) &= -\frac{N_0\beta_0}{8r\sqrt{1 - \frac{2B}{r}}}\left[2B - (r - 2B)\ln\left(\frac{r}{r - 2B}\right)\right] \\ &\quad + A_0\sqrt{1 - \frac{2B}{r}}, \end{aligned} \tag{3.17}$$

where C_1 is a constant.

Note that, when $B = 0$, the above solutions reduce to

$$N = N_0, \quad h = \pm\sqrt{\frac{r_g}{r}}, \quad A = A_0, \tag{3.18}$$

where r_g is an integration constant. This is precisely the Schwarzschild solution, but written in the Painleve-Gullstrand coordinates [31].

2. $\lambda \neq 1$

In this case, let us consider the two cases, $B = 0$ and $B \neq 0$, separately.

Case (i) $B = 0$**:** Now Eqs. (3.5) and (3.4) reduce, respectively, to

$$\varepsilon h'' - \varepsilon\left(\frac{N'}{N} - \frac{2}{r}\right)h' - \frac{2}{r}\left[(1 + \varepsilon)\frac{N'}{N} + \frac{\varepsilon}{r}\right]h = 0,$$

$$\frac{h^2}{N^2}\left[\varepsilon r^2\frac{h'^2}{h^2}+4(1+\varepsilon)r\frac{h'}{h}+2(1+2\varepsilon)\right] \tag{3.19}$$

$$-\beta_0\left(2r^2\frac{N''}{N}-r^2\frac{N'^2}{N^2}+4r\frac{N'}{N}\right)=0, \tag{3.20}$$

where $\varepsilon \equiv \lambda - 1$.

When $\beta_0 = 0$, the solutions is given by

$$\begin{aligned} h(r) &= H_0 r^{\alpha_\pm}, \quad N(r) = N_0 r^{\beta_\pm},\\ A(r) &= \bar{A}_0 + E_1(r) + \varepsilon E_2(r), \end{aligned} \tag{3.21}$$

where $\bar{A}_0$ and H_0 are constant, and

$$\begin{aligned} \alpha_\pm &= -2-\frac{2\pm\sqrt{4+6\varepsilon}}{\varepsilon},\\ \beta_\pm &= \frac{-2+\alpha_\pm+\alpha_\pm^2}{2+(2+\alpha_\pm)\varepsilon}\varepsilon\\ E_1(r) &= \frac{1}{4N_0 r^{\beta_\pm}(2\alpha_\pm-\beta_\pm)}\Big[N_0^2 r^{2\beta_\pm}(2\alpha_\pm-\beta_\pm)\\ &\quad\times(4+\beta_\pm\beta_0)-2H_0^2 r^{2\alpha_\pm}(1+2\alpha_\pm+2\beta_\pm)\Big],\\ E_2(r) &= H_0^2\frac{(2+\alpha_\pm)(\alpha_\pm-4-2\beta_\pm)}{4N_0(2\alpha_\pm-\beta_\pm)}r^{2\alpha_\pm-\beta_\pm}. \end{aligned} \tag{3.22}$$

It is interesting to note that, taking the "-" sign in the above solutions, they reduce to the Schwarzschild solution when $\epsilon \to 0$. This is in contrast to the case with the projectability condition $N = N(t)$ [21], in which it was found that such relativistic limit does not exist.

When $\beta_0 \neq 0$, it is found difficult to obtain exact solutions. Instead, we consider the cases where ε is very small, as one would expect that physical viable solutions must be very close to that of GR where $\lambda_{GR} = 1$. Thus, expanding $h(r)$, $N(r)$ and $A(r)$ in terms of ϵ, we find that

$$\begin{aligned} h(r) &= \pm\sqrt{\frac{r_g}{r}}+\varepsilon\left[\frac{H_0}{\sqrt{r}}\mp\frac{9(N_0^2\beta_0 r+r_g\ln(r))}{16\sqrt{r_g r}}\right]\\ &\quad+\mathcal{O}(\varepsilon^2),\\ N(r) &= N_0+\varepsilon\left[N_1-\frac{9}{8}N_0\ln(r)\right]+\mathcal{O}(\varepsilon^2),\\ A(r) &= A_0+\varepsilon\frac{9}{16}N_0(\beta_0-2)\ln(r)+\mathcal{O}(\varepsilon^2). \end{aligned} \tag{3.23}$$

Hence,

$$h(r)\to\mp\frac{9}{16}N_0^2\beta_0\epsilon\sqrt{\frac{r}{r_g}}, \tag{3.24}$$

as $r \to \infty$. Thus, unless $\beta_0\epsilon = 0$, the solutions are not asymptotically flat. When $\beta_0 = 0$, it reduces to the last case, and the corresponding solutions are given by Eq.(3.21). When $\epsilon = 0$, the corresponding solutions are given by Eqs.(3.16) and (3.17). Therefore, the case $\epsilon\beta_0 \neq 0$ has no physically viable solutions, and in the following we shall not consider it any further.

Case (ii) $B \neq 0$**:** In this case, to obtain exact solutions is found also very difficult, and instead we expand $h(r)$, $N(r)$ and $A(r)$ in terms of ϵ, and find that

$$\begin{aligned} N(r) &= \hat{N}(r)+\varepsilon\bar{N}(r)+\mathcal{O}(\varepsilon^2),\\ h(r) &= \hat{h}(r)+\varepsilon\bar{h}(r)+\mathcal{O}(\varepsilon^2),\\ A(r) &= \hat{A}(r)+\varepsilon\sqrt{f}\int\bar{Q}(r)dr+\mathcal{O}(\varepsilon^2), \end{aligned} \tag{3.25}$$

where $\hat{h}(r)$, $\hat{N}(r)$ and $\hat{A}(r)$ are the solution for $\varepsilon = 0$, which are given by Eqs.(3.16) and (3.17), while

$$\begin{aligned} \bar{N}(r) &= \sqrt{f}\left(N_1+\int N_a(r)dr\right),\\ \bar{h}(r) &= \frac{f}{r\hat{h}(r)}\left(H_1+\int h_a(r)dr\right),\\ N_a(r) &= \frac{N_0 r}{2\hat{h}}\hat{h}''+N_0\frac{(3B-r)\hat{h}'}{(2B-r)\hat{h}}\\ &\quad-N_0\frac{5B^2-8Br+2r^2}{2r(r-2B)^2},\\ h_a(r) &= N_0\beta_0 r\sqrt{f}\left(\frac{r}{2}\bar{N}''+\bar{N}'\right)-\frac{r^2}{4f}(\hat{h}')^2\\ &\quad+\left[\frac{2f-1}{N_0 f^{\frac{5}{2}}}\bar{N}-\frac{(1-5f)^2}{16f^3}\right]\hat{h}^2\\ &\quad+\left[N_0(1-5f)+8\sqrt{f}\bar{N}\right]\frac{r\hat{h}\hat{h}'}{4N_0 f^2}\\ &\quad+\frac{\beta_0 N_0 B^2\bar{N}}{2r^2 f^{\frac{3}{2}}}\\ \bar{Q}(r) &= \frac{N_0(5-2f-35f^2)+8(\bar{N}-2r\bar{N}')f^{\frac{3}{2}}}{16N_0^2 r f^4}\hat{h}^2\\ &\quad+\frac{1}{N_0 f^2}\left(\frac{1-2f}{rf}\hat{h}\bar{h}-\hat{h}\bar{h}'+\frac{r}{2}\hat{h}''\hat{h}\right)\\ &\quad+\left(\frac{-B}{2N_0 r f^3}+\frac{\bar{N}}{N_0^2 f^{\frac{5}{2}}}\right)\hat{h}'\hat{h}+\frac{B\bar{N}}{rf^{\frac{3}{2}}}\\ &\quad+\frac{\bar{N}'}{\sqrt{f}}-\frac{\hat{h}'}{N_0 f^2}\left(\bar{h}+\frac{r}{4}\hat{h}'\right)\\ &\quad+\frac{\beta_0 B}{4r^2 f^{\frac{5}{2}}}\left(B\bar{N}+2rf\bar{N}'\right). \end{aligned} \tag{3.26}$$

From the above expressions, we can see that the solutions are also not asymptotically flat. In particular, we have $\bar{N}(r) \propto \ln(r)$. Therefore, this class of solutions is also physically discarded.

IV. SOLAR SYSTEM TESTS

In the solar system tests, the metric is usually cast in the diagonal form,

$$ds^2 = -e^{2\Psi(r)}d\tau^2 + e^{2\Phi(r)}dr^2 + r^2 d\Omega^2, \tag{4.1}$$

where Ψ and Φ of the gravitational field, produced by a point-like and motion-less particle with mass M, are parametrized in the forms,

$$\begin{aligned} e^{2\Psi} &= 1 - 2\chi + 2(\beta-\gamma)\chi^2 + ...,\\ e^{2\Phi} &= 1 + 2\gamma\chi + ..., \end{aligned} \tag{4.2}$$

where β and γ are the Eddington parameters. For the Schwarzschild solution, we have $\beta_{GR} = \gamma_{GR} = 1$.

In the solar system, we have $r_g \equiv GM_{\odot}/c^2 \simeq 1.5$ km, and its radius is $r_{\odot} \simeq 1.392 \times 10^6$ km. So, within the solar system the dimensionless quantity $\chi[\equiv GM/(rc^2)]$ in most cases is much less than one, $\chi \le r_g/r_{\odot} \le 10^{-6}$. The Shapiro delay of the Cassini probe [32], and the solar system ephemerides [33] yield, respectively, the bounds [34],

$$\begin{aligned} \gamma - 1 &= (2.1 \pm 2.3) \times 10^{-5},\\ \beta - 1 &= (-4.1 \pm 7.8) \times 10^{-5}. \end{aligned} \tag{4.3}$$

To fit the solutions found in the last section with above observations, let us consider the cases $h = 0$ and $h \neq 0$, separately.

A. $h = 0$

In this case, let us first consider the solutions given by Eqs.(3.9) and (3.12). Expanding $A(r)$ in the form,

$$A(r) = A_0 + A_1\chi + A_2\chi^2 + \mathcal{O}\left(\chi^3\right), \tag{4.4}$$

we find that

$$\begin{aligned} N^2(r) &= (N_0 + A_0)^2 \left[1 - 2\frac{N_0\sigma - A_1}{N_0 + A_0}\right]\chi\\ &\quad + \big(2N_0A_2 - 2\sigma N_0 A_1 + A_1^2 + 2A_0A_2\\ &\quad - N_0A_0\sigma^2\big)\chi^2 + \mathcal{O}\left(\chi^3\right),\\ f^{-1}(r) &= 1 + 2\sigma\chi + 4\sigma^2\chi^2 + \mathcal{O}\left(\chi^3\right), \end{aligned} \tag{4.5}$$

where $\sigma \equiv \frac{Bc^2}{GM}$. Note that in writing the above expressions, we can set $t \to (N_0 + A_0)t$. Comparing Eq.(4.5) with Eq.(4.2), we find that

$$\begin{aligned} A_1 &= N_0(\gamma - 1) - A_0,\\ A_2 &= (N_0 + A_0)(\beta - \gamma)\\ &\quad + \frac{N_0}{2}(\gamma+1)(\gamma-1) - \frac{A_0}{2},\\ \sigma &= \gamma. \end{aligned} \tag{4.6}$$

Without loss of generality, we can alway set $N_0 = 1$ and $A_0 = 0$, for which we find

$$\begin{aligned} A_1 &= \gamma - 1 \le 10^{-5},\\ A_2 &= (\beta - \gamma) + \frac{(\gamma+1)}{2}(\gamma - 1) \le 10^{-5},\\ \sigma &= \gamma \simeq 1 + (2.1 \pm 2.3) \times 10^{-5}. \end{aligned} \tag{4.7}$$

For the solutions given by Eqs.(3.9) and (3.14), we have

$$\begin{aligned} N^2(r) &= (N_0 + N_1)^4 - 4\sigma N_1 (N_0 + N_1)^3 \chi\\ &\quad + 2\sigma^2 N_1 (2N_1 - N_0)(N_0 + N_1)^2 \chi^2\\ &\quad + \mathcal{O}\left(\chi^3\right), \end{aligned} \tag{4.8}$$

where $f(r)$ is still given by Eq.(4.5). Then, comparing them with Eq.(4.2), we obtain

$$\begin{aligned} &2\beta - \gamma = \frac{3}{2}, \quad \gamma = \sigma,\\ &N_0 = \frac{2\gamma - 1}{2\gamma}, \quad N_1 = \frac{1}{2\gamma}. \end{aligned} \tag{4.9}$$

Clearly, this class of solutions does not satisfy the solar system tests, and must be discarded.

B. $h \neq 0$

To study the solar system tests for this class of solutions, we need first transform the experimental results of Eqs.(4.2) and (4.3) into the form,

$$ds^2 = -N^2(r)dt^2 + \frac{1}{f(r)}\Big(dr + h(r)dt\Big)^2 + r^2 d\Omega^2, \tag{4.10}$$

which can be done by the coordinate transformations,

$$\tau = t - \int^r e^{-\Psi}\sqrt{e^{2\Phi} - f^{-1}}dr, \tag{4.11}$$

where f is given by Eq.(3.9), and

$$N = e^{\Psi+\Phi}\sqrt{f}, \quad h = e^{\Psi}\sqrt{f^2 e^{2\Phi} - f}, \tag{4.12}$$

or inversely,

$$e^{2\Psi} = N^2 - \frac{h^2}{f}, \quad e^{2\Phi} = \left(N^2 - \frac{h^2}{f}\right)^{-1}\frac{N^2}{f}. \tag{4.13}$$

It should be noted that, although the coordinate transformations (4.11) are forbidden by the foliation-preserving diffeomorphisms, we assume that experimental results can be expressed freely in any coordinate systems.

Now let us first consider the solutions given by Eqs.(3.16) and (3.17), from which we find that

$$\begin{aligned} e^{2\Psi} &= 1 - \left(2 + \frac{C_1}{12B}\right)\sigma\chi - \beta_0\frac{\sigma^2}{2}\chi^2 + \mathcal{O}(\chi^3),\\ e^{2\Phi} &= 1 + \left(2 + \frac{C_1}{12B}\right)\sigma\chi + \mathcal{O}(\chi^2). \end{aligned} \tag{4.14}$$

Note that in writing the above expressions, we set $N_0 = 1$. Comparing the above expressions with Eq.(4.2), we find that

$$\begin{aligned} &\gamma = 1, \quad C_1 = 24\left(\frac{GM}{c^2} - B\right),\\ &\beta - 1 = -\left(\frac{c^2}{2GM}\right)^2 \beta_0 B^2. \end{aligned} \tag{4.15}$$

Clearly, by properly choosing β_0 and B, the conditions (4.3) can be easily satisfied for this class of solutions.

For the solutions (3.21), we find that

$$\begin{aligned} e^{2\Psi} &= N_0^2\left(\frac{B}{\sigma\chi}\right)^{2\beta_\pm} - H_0^2\left(\frac{B}{\sigma\chi}\right)^{2\alpha_\pm}\Bigg[1+2\sigma\chi \\ &\quad +4\sigma^2\chi^2+\mathcal{O}(\chi^3)\Bigg], \\ e^{2\Phi} &= N_0^2\left(\frac{B}{\sigma\chi}\right)^{2\beta_\pm}\Bigg[N_0^2\left(\frac{B}{\sigma\chi}\right)^{2\beta_\pm}(1-2\sigma\chi) \\ &\quad -H_0^2\left(\frac{B}{\sigma\chi}\right)^{2\alpha_\pm}\Bigg]^{-1}. \end{aligned} \tag{4.16}$$

Comparing the above with Eq.(4.2), we find that

$$\alpha_\pm = -\frac{1}{2}, \quad \beta_\pm = 0. \tag{4.17}$$

This means $\lambda = 1$. Thus, in this case the solutions are consistent with the solar system tests only when $\lambda = 1$.

V. CONCLUSIONS

In this paper, we have studied the observational constraints of the nonprojectable HL theory with the enlarged symmetry (1.3) within the solar system. In particular, we have first found the static spherical vacuum solutions of the theory in the IR limit, and then considered their post-Newtonian limits. Because of the breaking of the Lorentz symmetry of the theory, the Birkhoff theorem does not hold here, and there exist several families of vacuum solutions that are asymptotically flat. Among them, only two families of these solutions satisfy the observational constraints imposed on the Eddington-Robertson-Schiff parameters, γ and β, by properly choosing the free parameters of the solutions. One is given by Eqs.(3.9), (3.12) with an arbitrary function $A(r)$, subjected to the conditions (4.7). This class of solutions is diagonal and valid for any λ. The other is non-diagonal ($h \neq 0$), given by Eqs.(3.9), (3.16) and (3.17) with $\lambda = 1$, and subjected to the constraints (4.15).

It should be noted that, in contrast to the projectable HL theory with the enlarged symmetry (1.3) [13–16], the consistency is achieved without the gauge field and Newtonian prepotential being part of the metric [13, 21].

A remarkable feature is that the Schwarzschild solution (3.13) written in the Schwarzschild coordinates is also a physically viable solution of the nonprojectable HL theory with the enlarged symmetry (1.3) in the IR not only for $\lambda = 1$ but also for any value of λ. This is different from all other models of the HL theory proposed so far [7].

Applying the general results presented in [36] to the physically viable spherical vacuum solutions obtained in this paper, one can easily construct slowly rotating vacuum solutions of the nonprojectable HL theory with the enlarged symmetry (1.3). Some of such constructed solutions clearly represent slowly rotating black holes in the IR limit [35].

Post-Newtonian approximations are generally characterized by ten parameters [9]. The studies of observational constraints in the nonprojectable HL theory without the local U(1) symmetry [8] showed that the preferred frame effects imposed the most stringent constraints on the suppression energy M_*. The structures of the nonprojectable HL theory with and without the local U(1) symmetry are quite different. In particular, the spin-0 gravitons do not exist in the current model. Thus, it would be very interesting to see which kind of constraints the preferred frame effects can impose on M_* as well as on other coupling constants of the theory. We wish to come back to this issue soon in another occasion.

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[1] P. Horava, JHEP, **0903**, 020 (2009); Phys. Rev. D**79**, 084008 (2009); and Phys. Rev. Lett. **102**, 161301 (2009).
[2] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (W.H. Freeman and Company, San Francisco, 1973), pp.484-528.
[3] K.S. Stelle, Phys. Rev. D**16**, 953 (1977).
[4] D. Mattingly, Living Rev. Relativity, **8**, 5 (2005); S. Liberati and L. Maccione, Annu. Rev. Nucl. Part. Sci. **59**, 245 (2009).
[5] J. Polchinski, arXiv:1106.6346.
[6] M. Pospelov and Y. Shang, Phys. Rev. D**85**, 105001 (2012).
[7] S. Mukohyama, Class. Quant. Grav. **27**, 223101 (2010); P. Hořava,*ibid.*, **28**, 114012 (2011); T. Clifton, P.G. Ferreira, A. Padilla, and C. Skordis, Phys. Rept. **513**, 1 (2012).
[8] D. Blas, O. Pujolas, and S. Sibiryakov, Phys. Rev. Lett. **104**, 181302 (2010); Phys. Lett. B**688**, 350 (2010); JHEP, **1104**, 018 (2011).
[9] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, 1993).
[10] B.Z. Foster and T. Jacobson, Phys. Rev. D**73**, 064015 (2006).
[11] J.W. Elliott, G.D. Moore and H. Stoica, JHEP, **08**, 066

(2005).
[12] T. Jacobson, Phys. Rev. D**81**, 101502 (2010).
[13] P. Hořava and C.M. Melby-Thompson, Phys. Rev. D**82**, 064027 (2010).
[14] A. Wang and Y. Wu, Phys. Rev. D**83**, 044031 (2011).
[15] A.M. da Silva, Class. Quantum Grav. **28**, 055011 (2011) [arXiv:1009.4885].
[16] Y.-Q. Huang and A. Wang, Phys. Rev. D**83**, 104012 (2011).
[17] K. Lin, A. Wang, Q. Wu, and T. Zhu, Phys. Rev. D**84**, 044051 (2011).
[18] Y.-H. Huang, A. Wang, and Q. Wu, JCAP, **10**, 010 (2012).
[19] Y.-H. Huang and A. Wang, Phys. Rev. D**86**, 103523 (2012).
[20] E. Komatsu, *et al*, Astrophys. J. Suppl. **192**, 18 (2011).
[21] K. Lin, S. Mukohyama, and A. Wang, Phys. Rev. D**86**, 104024 (2012).
[22] J. Alexandre and P. Pasipoularides, Phys. Rev. D**83**, 084030 (2011).
[23] J. Greenwald, V.H. Satheeshkumar, and A. Wang, JCAP, **12**, 007 (2010).
[24] A. Borzou, K. Lin, and A. Wang, JCAP, **02**, 025 (2012).
[25] E. Abdalla, and A.M. da Silva, Phys. Lett. B**707**, 311 (2012).
[26] T. Zhu, Q. Wu, A. Wang, and F.-W. Shu, Phys. Rev. D**84**, 101502(R) (2011).
[27] T. Zhu, F.-W. Shu, Q. Wu, and A. Wang, Phys. Rev. D**85**, 044053 (2012).
[28] T. Zhu, Y.-Q. Huang, and A. Wang, JHEP *in press* (2012) [arXiv:1208.2491].
[29] A. Wang, Q. Wu, W. Zhao, and T. Zhu, "*Polarizing primordial gravitational waves by parity violation*," arXiv:1208.5490 (2012).
[30] J. Greenwald, A. Papazoglou, and A. Wang, Phys. Rev. D**81**, 084046 (2010).
[31] P. Painleve, C. R. Acad. Sci. (Paris) **173**, 677 (1921); A. Gullstrand, Arkiv. Mat. Astron. Fys. **16**, 1 (1922).
[32] B. Bertotti, L. Iess and P. Tortora, Nature, **425**, 374 (2003).
[33] A. Fienga, et al, Celestial Mechanics and Dynamical Astronomy, 3, 363 (2011).
[34] A. Hees, et al, arXiv:1110.0659 (2011).
[35] J. Greenwald, J. Lenells, J. X. Lu, V. H. Satheeshkumar, and A. Wang, Phys. Rev. D**84**, 084040 (2011).
[36] A. Wang, "*Stationary and slowly rotating spacetimes in Hořava-Lifshitz gravity*," arXiv:1212.1876 (2012).