Recursion relations for tree-level amplitudes in the SU(N) nonlinear sigma model

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Recursion Relations for Tree-level Amplitudes in the $SU(N)$ Non-linear Sigma Model

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It is well-known that the standard BCFW construction cannot be used for on-shell amplitudes in effective field theories due to bad behavior for large shifts. We show how to solve this problem in the case of the $SU(N)$ non-linear sigma model, i.e. non-renormalizable model with infinite number of interaction vertices, using scaling properties of the semi-on-shell currents, and we present new on-shell recursion relations for all on-shell tree-level amplitudes in this theory.

I. INTRODUCTION

Scattering amplitudes are physical observables that describe scattering processes of elementary particles. The standard perturbative expansion is based on the method of Feynman diagrams. In last two decades there has been a huge progress on alternative approaches, driven by the idea that the amplitude should be fully determined by the on-shell data with no need to access the off-shell physics. This effort has lead to amazing discoveries that have uncovered many surprising properties and dualities of amplitudes in gauge theories and gravity. One of the most important breakthroughs in this field was the discovery of the BCFW recursion relations [1, 2] that allow us to reconstruct the on-shell amplitudes recursively from most primitive amplitudes. They are applicable in many field theories, however, in some cases like effective field theories they can not be used. One particularly important example is the $SU(N)$ non-linear sigma model which describes the low-energy dynamics of the massless Goldstone bosons corresponding to the chiral symmetry breaking $SU(N)\times SU(N)\to SU(N)$.

The $SU(N)$ non-linear sigma model has played a crucial role in many developments of theoretical physics in the last almost fifty years. It has a broad range of applications from model building in particle phenomenology to string theories. For instance for $N=2$ it represents a low energy effective theory of QCD, describing the dynamics of pions [3, 4]. It is also a starting point for many extensions or alternatives of electroweak standard model.

In this note we find the recursion relations for all tree-level amplitudes of Goldstone bosons for the $SU(N)$ non-linear sigma model. The importance of this result is twofold: (i) It shows that the BCFW-like recursion relations can be applicable to much larger class of theories than expected before. This might also help to understand better the properties of the theory invisible otherwise. It also tells us that this model despite being an effective (and therefore for dim $d>2$ non-renormalizable) field theory behaves in some cases similar to renormalizable theories. (ii) It provides an effective tool for leading order (tree-level) calculations of amplitudes with many external Goldstone bosons which might be important for low energy particle phenomenology. More detailed description together with other results will be presented in [5].

II. BCFW RECURSION RELATIONS

Let us consider an n-pt on-shell scattering amplitude of massless particles in the adjoint representation of the symmetry group $SU(N)$, and denote t^a the generators of the corresponding Lie algebra. At tree-level each Feynman diagram carries a single trace $\mathrm{Tr}(t^{a_1}t^{a_2}\dots t^{a_n})$ and we can decompose the full amplitude $\mathcal{A}_n$ into sectors with the same group factor,

$$\mathcal{A}_n^{tree}=\sum_{\sigma/\mathbf{Z}_n} A_n(p_{\sigma(1)},\dots p_{\sigma(n)})\mathrm{Tr}(t^{a_{\sigma(1)}}\dots t^{a_{\sigma(n)}})\,, \quad (1)$$

where the sum is over all non-cyclic permutations. For each *stripped* amplitude A_n we have a natural ordering of momenta $p_{\sigma(1)},\dots p_{\sigma(n)}$ and a single term $A_n(p_1,p_2,\dots p_n)$ generates all the other by trivial relabeling. At the loop level we can define analogous object in the planar limit but in the general case this simple decomposition is not possible due to terms with multiple traces.

In 2004 Britto, Cachazo, Feng and Witten (BCFW) [1, 2] found a recursive construction of tree-level on-shell amplitudes. The stripped amplitude $A_n=A_n(p_1,\dots p_n)$ is a gauge invariant object and one can try to fully reconstruct it from its poles. Because of the ordering the only poles that can appear are of the form $P_{ab}^2=0$ where $P_{ab}=\sum_{k=a}^{b}p_k$ for some a,b. On the pole the amplitude factorizes into two pieces,

$$A_L(p_a,\dots p_b,-P_{ab})\frac{i}{P_{ab}^2}A_R(P_{ab},p_{b+1},\dots p_{a-1})\,. \quad (2)$$

Let us perform the following shift on the external data:

$$p_i(z)=p_i+zq, \qquad p_j(z)=p_j-zq\,, \quad (3)$$

where i and j are two randomly chosen indices, z is a complex parameter and q is a fixed null vector which is also orthogonal to p_i and p_j, $q^2=(q\cdot p_i)=(q\cdot p_j)=0$ (such a q exists only for dimension $d\geq 4$). Note that the shifted momenta remain on-shell and still satisfy momentum conservation. The original amplitude A_n becomes a meromorphic function $A_n(z)$ with only simple poles. If it vanishes for $z\to\infty$ we can use Cauchy theorem to reconstruct it,

$$A_n(z)=\sum_i\frac{\mathrm{Res}(A_n,z_i)}{z-z_i}\,, \quad (4)$$

where z_i are poles of $A_n(z)$ determined by

$$P_{ab}(z)^2 = (p_a + \cdots + p_i(z) + \ldots p_b)^2 = 0 \tag{5}$$

and located in $z_{ab} = -P_{ab}^2/2(q \cdot P_{ab})$. Note that $A_n(z)$ has a pole only if $i \in (a, \ldots b)$ or $j \in (a, \ldots b)$ (not both or none). There exists a convenient choice $j = i+1$ which minimizes a number of terms in (4). According to (2) $\mathrm{Res}(A_n, z_i)$ is a product of two lower point amplitudes with shifted momenta and the Cauchy theorem (4) can be rewritten as

$$A_n(z) = \sum_{a,b} A_L(z_{ab}) \frac{i}{P_{ab}(z)^2} A_R(z_{ab})\,, \tag{6}$$

where the sum is over all poles $P_{ab}(z)^2 = 0$ and

$$A_L(z) = A_L(p_a, \ldots, p_i(z), \ldots p_b, P_{ab}(z)), \tag{7}$$

$$A_R(z) = A_R(-P_{ab}(z), p_{b+1}, \ldots, p_j(z), \ldots p_{a-1}). \tag{8}$$

In the physical case we set $z = 0$. A_L and A_R in (6) are lower point amplitudes, $n_R, n_L < n$ and therefore we can reconstruct $A_n(z)$ recursively from simple on-shell amplitudes not using the off-shell physics at any step. BCFW recursion relations were originally found for Yang-Mills theory [1, 2], and proven to work in gravity [6, 7]. There are many works showing validity in other theories (e.g. for coupling to matter see [8]).

If the amplitude $A_n(z)$ is constant or grows for large z, the prescription (4) cannot be used directly. The constant behavior was studied e.g. in [9] on the cases of $\lambda\phi^4$ and Yukawa theory. In the generic situation of a power behavior $A_n(k) \approx z^k$ for $z \to \infty$ we can use the following formula [5]

$$A_n(z) = \sum_{i=1}^{n} \frac{\mathrm{Res}\,(A_n; z_i)}{z - z_i} \prod_{j=1}^{k+1} \frac{z - a_j}{z_i - a_j} + \sum_{j=1}^{k+1} A_n(a_j) \prod_{l=1, l\neq j}^{k+1} \frac{z - a_l}{a_j - a_l}\,, \tag{9}$$

which reconstructs the amplitude in terms of its residues and its values at additional points a_i different from z_i. This is a generalization of formula first written in this context in [10] and further discussed in [11] where a_i are chosen to be roots of $A_n(z)$.

The other option is to use the all-line shift, i.e. deforming all external momenta. This was inspired by the work by Risager [12] and recently used for studying the on-shell constructibility of generic renormalizable theories in [13]. This approach will be useful for our purpose.

III. SEMI-ON-SHELL AMPLITUDES

The Lagrangian of the $SU(N)$ non-linear sigma model in d dimensions can be written as

$$\mathcal{L} = \frac{F^2}{4} \mathrm{Tr}\left(\partial_\mu U\, \partial^\mu U^\dagger\right)\,, \tag{10}$$

where F is a constant with canonical dimension $d/2 - 1$ and $U \in SU(N)$ is dimensionless. In the most common exponential parametrization $U = \exp(i\phi/F)$ where $\phi = \sqrt{2}\phi^a t^a$. The t^as are generators of $SU(N)$ Lie algebra normalized according to $\mathrm{Tr}(t^a t^b) = \delta^{ab}$. Note that for $N = 2$ and $d = 4$, (10) is a leading $\mathcal{O}(p^2)$ term in the Lagrangian for the Chiral Perturbation Theory [4], which provides a systematic effective field theory description for low energy QCD with two massless quarks. In this case ϕ^a represent the pion triplet. In what follows neither N nor d are restricted.

For calculations of on-shell scattering amplitudes within this model we use stripped amplitudes $A_n(p_1, \ldots p_n)$. The Lagrangian (10) contains only terms with the even number of ϕ, therefore $A_{2n+1} = 0$ and only A_{2n} are non-vanishing. It is easy to show that it makes no difference whether we use $SU(N)$ or $U(N)$ symmetry group because the $U(1)$ piece decouples [5]. For our purpose it is convenient to use Cayley parametrization of $U(N)$ non-linear sigma model,

$$U = \frac{1 + \frac{i}{2F}\phi}{1 - \frac{i}{2F}\phi} = 1 + 2\sum_{n=1}^{\infty} \left(\frac{i}{2F}\phi\right)^n. \tag{11}$$

Plugging for U into (10) we get an infinite tower of terms with two derivatives and an arbitrary number of ϕ. This is common for any parametrization, however, in this parametrization, the stripped Feynman rule for the interaction vertex is particularly simple,

$$V_{2n+1} = 0, \quad V_{2n+2} = \left(\frac{-1}{2F^2}\right)^n \left(\sum_{i=0}^{n} p_{2i+1}\right)^2. \tag{12}$$

It is easy to see that the shifted amplitudes $A_n(z) \approx z$ for $z \to \infty$. Without additional information on the values at two points a_i the relation (9) cannot be used. Therefore, we will follow different strategy to determine $A_n(z)$ recursively.

Let us define a semi-on-shell current

$$J_n^{a,a_1,a_2,\ldots a_n}(p_1, \ldots p_n) = \langle 0|\phi^a(0)|\pi^{a_1}(p_1)\ldots\pi^{a_n}(p_n)\rangle \tag{13}$$

as a matrix element of the field $\phi^a(0)$ between vacuum and the n-particle state $|\pi^{a_1}(p_1)\ldots\pi^{a_n}(p_n)\rangle$. The momentum p_{n+1} attached to $\phi^a(0)$ is off-shell satisfying $p_{n+1} = -\sum_{j=1}^{n} p_j = -P_{1n}$. At the tree-level the current can be written in terms of stripped currents

$$J_n^{a,a_1,a_2,\ldots a_n}(p_1, \ldots p_n) = \sum_{\sigma/\mathbf{Z}_n} \mathrm{Tr}(t^a t^{a_{\sigma(1)}} \ldots t^{a_{\sigma(n)}}) J_n(p_{\sigma(1)} \ldots p_{\sigma(n)})\,. \tag{14}$$

The on-shell amplitude $A_{n+1}(p_1, \ldots p_{n+1})$ can be extracted from $J_n(p_1, \ldots p_n)$ by means of the LSZ formulas

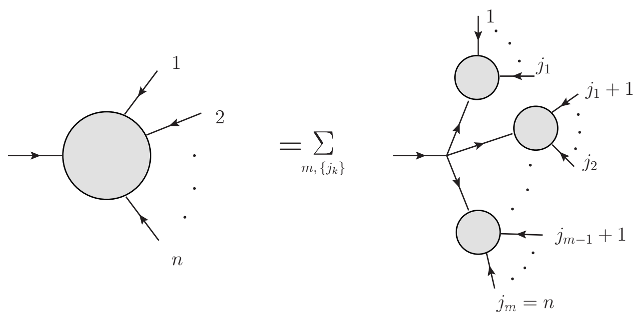
$$A_{n+1}(p_1, \ldots p_{n+1}) = -\lim_{p_{n+1}^2 \to 0} p_{n+1}^2 J_n(p_1, \ldots p_n). \tag{15}$$

The one particle states are normalized according to $J_1(p) = 1$. Note that $J_{2n} = 0$ in agreement with

$A(p_1 \dots p_{2n+1}) = 0$ via (15). For currents $J(1, \dots, n) \equiv J_n(p_1, \dots p_n)$ we can write generalized Berends-Giele recursion relations [14] (n.b. $P_{ab} = \sum_{k=a}^{b} p_k$),

$$J(1, \dots n) = \frac{i}{p_{n+1}^2} \sum_{m=3}^{n} \sum_{j_0 < j_1 < \dots < j_m} iV_{m+1}(P_{j_0 j_1}, \dots, -P_{1n}) \times \prod_{k=0}^{m-1} J(j_k{+}1, \dots, j_{k+1}) \,, \tag{16}$$

where $j_0 = 0$ and $j_m = n$. This equation can be equivalently graphically represented as



The right hand side is a sum of products of lower point currents with Feynman vertices (12). The current J_n is obviously a homogeneous function of momenta of degree 0. It is not cyclic because there is a special off-shell momentum p_{n+1}. Note, however, J_n is unphysical object and can be different in different parametrizations. From now on we will use only Cayley parametrization where it has interesting properties under the re-scaling of all even or all odd on-shell momenta. Namely for $t \to 0$:

$$J_{2n+1}(tp_1, p_2, tp_3, \dots p_{2n}, tp_{2n+1}) = O(t^2), \tag{17}$$

$$J_{2n+1}(p_1, tp_2, p_3, \dots tp_{2n}, p_{2n+1}) \to \frac{1}{(2F^2)^n}. \tag{18}$$

We postpone the detailed discussion to [5]. The proof is by induction using Berends-Giele recursion relations [14] which are more suitable for this purpose than the analysis of Feynman diagrams used to show scaling properties of Yang-Mills theory and gravity in [15].

IV. NEW RECURSION RELATIONS

The scaling properties (17) and (18) are our guide for finding recursion relations for J_{2n+1}. Let us define the complex deformation of the current $J_{2n+1}(z)$:

$$J_{2n+1}(z) \equiv J_{2n+1}(p_1, zp_2, \dots, zp_{2n}, p_{2n+1}) \,, \tag{19}$$

i.e. the momenta are shifted according to

$$p_{2k}(z) = zp_{2k} \,, \quad p_{2k+1}(z) = p_{2k+1} \,. \tag{20}$$

This deformation is possible for general d. Note that the momentum conservation is hold because the off-shell momentum $p_{2n+2} = -\sum_{k=1}^{2n+1} p_k$ becomes also shifted. In the limit $z \to 0$ using (18) we get

$$\lim_{z \to 0} J_{2n+1}(z) = \frac{1}{(2F^2)^n}. \tag{21}$$

On the other hand for $z \to \infty$ as a consequence of homogeneity and (17) the current $J_{2n+1}(z)$ vanishes like

$$J_{2n+1}(z) = O\left(\frac{1}{z^2}\right) \tag{22}$$

and we can use the standard BCFW recursion relations to reconstruct it from its poles. The singularities of the physical current $J_{2n+1}(1)$ are determined by condition $P_{ij}^2 = 0$ which implies the following condition for the poles of $J_{2n+1}(z)$

$$P_{ij}^2(z) = (zp_{ij} + q_{ij})^2 = 0 \,, \tag{23}$$

where $j-i$ is even and we have decomposed $P_{ij} = p_{ij} + q_{ij}$ where p_{ij} and q_{ij} is the sum of even and odd momenta respectively between i and j,

$$p_{ij} = \sum_{i \le 2k \le j} p_{2k}, \qquad q_{ij} = \sum_{i \le 2k+1 \le j} p_{2k+1}. \tag{24}$$

For $j - i > 2$ we find two solutions of (23), namely

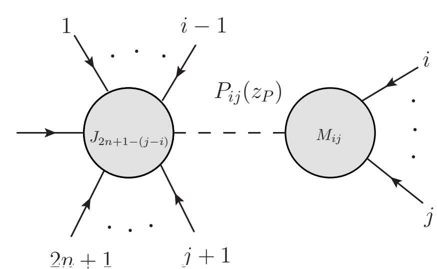
$$z_{ij}^{\pm} = \frac{-(p_{ij} \cdot q_{ij}) \pm \sqrt{(p_{ij} \cdot q_{ij})^2 - p_{ij}^2 q_{ij}^2}}{p_{ij}^2}. \tag{25}$$

For the special case of three-particle pole, $j-i = 2$, either $q_{ij}^2 = 0$ or $p_{ij}^2 = 0$. For the first case $z_{ij}^+ = 0$ and the corresponding residue does vanish, $\mathrm{Res}(J_{2n+1}, z_{ij}^+) = 0$, while $z_{ij}^- = -2(p_{ij} \cdot q_{ij})/p_{ij}^2$. In the second case there is only one solution of (23) $z_{ij} = -q_{ij}^2/2(p_{ij} \cdot q_{ij})$.

Let us denote a generic solution of (23) by z_P. Then the internal momentum $P_{ij}(z_P)$ is on-shell, therefore the current $J_{2n+1}(z)$ factorizes into the product of lower-point semi-on-shell current J_{m_1} and the on-shell amplitude M_{m_2}. Residues at the poles $z_{ij}^{\pm}$ are given by

$$\mathrm{Res}(J_{2n+1}, z_{ij}^{\pm}) = \mp [p_{ij}^2(z_{ij}^+ - z_{ij}^-)]^{-1} M_{ij}(z_{ij}^{\pm}) \times J_{2n-j+i+1}(p_1(z_{ij}^{\pm}), \dots, P_{ij}(z_{ij}^{\pm}), \dots, p_{2n+1}(z_{ij}^{\pm})) \tag{26}$$

or graphically by



In this formula $M_{ij}(z) = P_{ij}^2(z)J_{j-i+1}(p_i(z),\ldots p_j(z))$. In the case of single solution z_{ij} the residue is given by the similar formula where $\mp[p_{ij}^2(z_{ij}^+ - z_{ij}^-)]^{-1}$ is replaced by $[2(p_{ij}\cdot q_{ij})]^{-1}$. Because of (22) we can write

$$J_{2n+1}(z) = \sum_{z_P} \frac{\mathrm{Res}(J_{2n+1}, z_P)}{z - z_P}. \tag{27}$$

The residues $\mathrm{Res}(J_{2n+1}, z_P)$ can be determined recursively from (26) as in the case of BCFW recursion relations. However, there is one difficulty. In the boundary case $i = 1$, $j = 2n+1$ the equation (26) for residue $\mathrm{Res}(J_{2n+1}, z_{1,2n+1}^{\pm})$ contains a current J_{2n+1} on the right hand side and therefore we can not express it using lower point currents. The solution to this problem is to use two extra relations. The first is the residue theorem: because of the asymptotic behavior (22) the residue at infinity vanishes and the sum of all residues is zero,

$$\sum_{z_P} \mathrm{Res}(J_{2n+1}, z_P) = 0\,, \tag{28}$$

while the second one is the scaling property (21) for $z \to 0$ together with (27)

$$\sum_{z_P} \frac{\mathrm{Res}(J_{2n+1}, z_P)}{z_P} = -\frac{1}{(2F^2)^n}. \tag{29}$$

Let us note that the relation (28) is an analogue of the so-called bonus relations for the on-shell amplitudes, investigated e.g. in [16].

Denoting $z_\pm = z_{1\,2n+1}^{\pm}$ and solving for $\mathrm{Res}(J_{2n+1}, z_\pm)$ from (28) and (29) in terms of all other residues we can rewrite (27) in the form

$$J_{2n+1}(z) = \frac{q_{1,2n+1}^2}{P_{1,2n+1}(z)^2}\frac{1}{(2F^2)^n} + {\sum_{z_P}}' \Bigg[\frac{\mathrm{Res}(J_n, z_P)}{z - z_P} + \frac{q_{1,2n+1}^2}{P_{1,2n+1}(z)^2}\frac{\mathrm{Res}(J_n, z_P)}{z_P}\Big(1 - z\frac{p_{1,2n+1}^2}{q_{1,2n+1}^2} z_P\Big)\Bigg], \tag{30}$$

where the sum is over all solutions of (23) with the exception of $z_\pm$. The residues on the right-hand side depend only on lower point currents via (26). The physical case is $z = 1$ and the on-shell amplitude $A_n(p_1,\ldots p_n)$ can be obtained from $J_n(1)$ using the limit (15). Interestingly, even the fundamental 4pt case, i.e. the current J_3 is included in the equation (30) (here the sum is empty). As the explicit forms of the amplitudes are rather lengthy we postpone the examples of the calculation to [5], where we also discuss further applications including the formula for the double soft-limit and the proof of Adler's zeroes for stripped amplitudes.

Notice a very important difference between our recursion relations and the original Berends-Giele formula (16): we construct the amplitude recursively from the 4pt formula via BCFW while (16) uses critically the Lagrangian and the infinite tower of terms in the expansion of (10). For this reason the Berends-Giele relations for effective theories are also much less efficient than in the renormalizable case due to exponential growth of the number of terms with increasing n. On the other hand, the number of terms in our BCFW-like relations is dictated by number of factorization channels. From this point of view there is no quantitative difference between our recursive procedure and all-line shift BCFW reconstruction of renormalizable model.

CONCLUSION AND OUTLOOK

We found the recursion relations for on-shell scattering amplitudes of Goldstone bosons in the $SU(N)$ non-linear sigma model. We defined a semi-on-shell current J_n and used the Berends-Giele recursion relations to prove its special scaling properties. This enables to define the alternative all-line BCFW-like deformation of the external momenta which allows to recursively construct J_n from its poles. The proposed deformation is not restricted to $d \geq 4$ dimensions and therefore our recursive construction can be used in any dimension in contrast to the standard BCFW one. Another benefit of the reconstruction is in the efficiency of the actual calculation which is now comparable with those for renormalizable models.

The existence of such recursion relations for effective theory gives an evidence that on-shell methods can be used for much larger classes of theories than has been considered so far. It also shows that this theory is very special and deeper understanding of all its properties is still missing. For future directions, it would be interesting to see if the construction can be re-formulated purely in terms of on-shell scattering amplitudes not using the semi-on-shell current. Next possibility is to focus on loop amplitudes. As was shown e.g. in [17, 18] the loop integrand can be also in certain cases constructed using BCFW recursion relations, it would be spectacular if the similar construction can be applied for effective field theories.

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