Lepton flavor mixing in the Wolfenstein scheme

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Abstract

We mimic the Wolfenstein scheme of quark flavor mixing to describe neutrino oscillations in the standard model. We identify the parameter λ with the experimentally measured value of the mixing element responsible for atmospheric neutrino oscillations. The matrix elements responsible for solar neutrino oscillations and the Chooz angle are taken to be proportional to λ^2 and λ^3, respectively. Using present world average data on neutrinos, we derive bounds on the other parameters A, ρ, and η, of the new scheme.

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Activities in neutrino physics over the past several years have accumulated substantial data to establish a near robust picture of neutrino properties. All experiments confirm that neutrinos have *tiny* masses and *oscillate*. The appropriate squared mass differences Δm_{ij}^2 and the mixing angles θ_{ij} ($i, j = 1, 2, 3; i < j$) are probed for solar neutrinos [1–5], atmospheric neutrinos [6, 7], reactor neutrinos [8, 9, 11], and accelerator neutrinos [12–17]. At present, empirical information on whether neutrinos are massive Dirac or Majorana particles is still lacking. Also, the absolute values of neutrino masses are unknown. Although the complete picture seems a long way in the future, enough information exists to create a neutrino paradigm on the broader aspects of neutrino properties. Hopefully, to complete the picture, future experiments will unravel more information on neutrinos pertaining to their absolute masses, the phenomenon of neutrino CP violation and any symmetries, discrete or continuous, lurking in lepton flavor mixing. In the following we will assume that there are only three neutrino species.

Neutrinos (ν_e, ν_μ, ν_τ), emitted in weak decays, are admixtures of different mass eigenstates (ν_1, ν_2, ν_3). This leads to leptonic flavors mixing that is responsible for the presently observed neutrino oscillations. The lepton flavor mixings originate in the charged current weak interactions Lagrangian $\mathcal{L}_{\rm cc}$. It is described by the 3×3 unitary matrix U, the PMNS mixing matrix [19, 20], analogous to the CKM mixing matrix for quarks [21, 22]:

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \left(\overline{e} \ \overline{\mu} \ \overline{\tau} \right)_{\rm L} \gamma^\mu \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} W_\mu^- + {\rm h.c.} \qquad (1)$$

The neutrino flavor eigenstates are related to the corresponding mass eigenstates as

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_{\rm L} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} \qquad (2)$$

The elements of the matrix U can be complex and probe directly the mixing between the three neutrinos. This parameterization will be relevant when we discuss the Wolfenstein scheme for neutrinos. However, the standard parameterization of U is in terms of three angles (θ_{12}, θ_{23}, θ_{13}) and three phases (δ, ρ, σ). Explicitly,

$$U = R_{23}(\theta_{23})\, P_\delta^\dagger\, R_{13}(\theta_{13})\, P_\delta\, R_{12}(\theta_{12}) P_\nu \tag{3}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 1 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} P_\nu$$

$$= \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} P_\nu,$$

where $R_{jk}(\theta_{jk})$ describes a rotation in the jk-plane through angle θ_{jk} and $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ (for $ij = 12, 13, 23$). The phase matrices P_δ and P_ν are defined as

$$P_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta} \end{pmatrix}, \quad P_\nu = \begin{pmatrix} e^{i\rho} & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

where P_ν is relevant only if the neutrinos are massive Majorana particles while the Dirac phase δ is always present irrespective of the nature of neutrinos. Since information on absolute masses for neutrinos is lacking, the phase matrix P_ν will not be considered further.

In the standard parameterization of U, the oscillation parameters are the mixing angles (θ_{12}, θ_{23}, θ_{13}) and the mass squared differences Δm_{12}^2 and Δm_{23}^2. The parameters Δm_{12}^2 and θ_{12} describe solar neutrinos [1–5] and the reactor experiment KamLAND [8, 10]; Δm_{23}^2 and θ_{23} describe atmospheric neutrino oscillations [6, 7] and the K2K accelerator experiment [12]; while θ_{13} and Δm_{13}^2 describe neutrino oscillations in the T2K, MINOS, Double Chooz, Daya Bay and RENO experiments [12–18]. For normal neutrino mass hierarchy ($m_3 \geq m_2 \geq m_1$) the sum total of the present knowledge on these parameters, averaged over all experiments, at various statistical significance levels is collected in Table I.

Neutrinoless double beta decay and cosmology provide direct information on the mass parameters that are complementary to the oscillation parameters. The tritium experiments [23] provide an upper bound on the absolute value of neutrino mass

$$m_i \leq 2.2\,\text{eV}. \tag{5}$$

A more strict bound

$$m_i \leq 0.6\,\text{eV}, \tag{6}$$

follows from the analysis of the cosmological data [24].

The data in Table I indicates that neutrino oscillations are described by two large mixing angles: $\theta_{12} \simeq 34°$ and $\theta_{23} \simeq 45°$. The third mixing angle θ_{13} is much smaller. The results of the experiments T2K [14], MINOS [15], Double Chooz [16], DAYA BAY [17], and RENO [18], imply that this angle is non-zero and about $9°$.

In our analysis, for normal neutrino mass hierarchy, we will use the neutrino oscillation parameters spanning over the 3σ range [25]:

$$
\begin{aligned}
0.259 \leq\ &\sin^2\theta_{12}\ \leq 0.359, \\
0.331 \leq\ &\sin^2\theta_{23}\ \leq 0.637, \\
0.017 \leq\ &\sin^2\theta_{13}\ \leq 0.031,
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
6.99 \times 10^{-5}\ \text{eV}^2 \leq\ &\Delta m_{12}^2\ \leq 8.18 \times 10^{-5}\ \text{eV}^2, \\
2.19 \times 10^{-3}\ \text{eV}^2 \leq\ &\Delta m_{23}^2\ \leq 2.62 \times 10^{-3}\ \text{eV}^2.
\end{aligned}
\tag{8}
$$

One can derive the following 3σ CL ranges on the magnitude of the elements of the leptonic mixing matrix U,

$$
|U| = \begin{pmatrix}
0.795 \rightarrow 0.846 & 0.513 \rightarrow 0.585 & 0.126 \rightarrow 0.178 \\
0.205 \rightarrow 0.543 & 0.416 \rightarrow 0.730 & 0.579 \rightarrow 0.808 \\
0.215 \rightarrow 0.548 & 0.409 \rightarrow 0.725 & 0.567 \rightarrow 0.800
\end{pmatrix}. \tag{9}
$$

The ranges in the different entries of the matrix U are correlated due to the constraints imposed by unitarity.

In 1983, Wolfenstein [26] introduced an elegant scheme for describing the quark mixing [21, 22] matrix V_{CKM}. In this scheme the three angles (θ_{12}, θ_{23}, θ_{31}) and the CP-violating phase δ of V_{CKM} are replaced by four new parameters (λ, A, ρ, η). The parameter $\lambda = \sin\theta_C = 0.22$ represents the Cabibbo angle, a well determined quantity experimentally, A represents the "sizing" parameter and ρ and η are CP violating parameters. In terms of the new parameters V_{CKM} takes an elegant form,

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \tag{10}$$

This matrix gives a good description of data on quark mixing. In Wolfenstein's parameterization the parameter λ is identified with the experimentally best known angle, the Cabibbo angle, and the other elements are identified with quantities of order λ^2 and λ^3.

In the case of neutrinos [19, 20] we adopt the same philosophy. We identify the parameter λ with the element $U_{\mu 3}$ describing atmospheric neutrino oscillations. The matrix element U_{e2} describing solar neutrino oscillations is assigned order λ^2 and the matrix element U_{e3} is assigned order λ^3 which gets identified with the mixing Chooz angle θ_{13}. Explicitly,

$$\begin{aligned} U_{\mu 3} &= \lambda, \\ U_{e2} &= A\lambda^2, \\ U_{e3} &= A\lambda^3(\rho - i\eta). \end{aligned} \tag{11}$$

It follows that,

$$\begin{aligned} \sin\theta_{13} &= A\lambda^3\sqrt{\rho^2 + \eta^2}, \\ \sin\theta_{12} &= \frac{A\lambda^2}{\sqrt{1 - A^2\lambda^6(\rho^2 + \eta^2)}}, \\ \sin\theta_{23} &= \frac{\lambda}{\sqrt{1 - A^2\lambda^6(\rho^2 + \eta^2)}}, \\ \tan\delta &= \frac{\eta}{\rho}. \end{aligned} \tag{12}$$

Unitarity of the PMNS matrix U determines the remaining elements:

$$U_{e1} = \sqrt{1 - A^2\lambda^4 - A^2\lambda^6(\rho^2 + \eta^2)}$$

$$U_{\mu 1} = \frac{-A\lambda^2\sqrt{1 - \lambda^2 - A^2\lambda^6(\rho^2 + \eta^2)} - A\lambda^4\sqrt{\rho^2 + \eta^2}\sqrt{1 - A^2\lambda^4 - A^2\lambda^6(\rho^2 + \eta^2)}\,e^{i\delta}}{1 - A^2\lambda^6(\rho^2 + \eta^2)}$$

$$U_{\mu 2} = \frac{\sqrt{1 - A^2\lambda^4 - A^2\lambda^6(\rho^2 + \eta^2)}\sqrt{1 - \lambda^2 - A^2\lambda^6(\rho^2 + \eta^2)} - A^2\lambda^6\sqrt{\rho^2 + \eta^2}\,e^{i\delta}}{1 - A^2\lambda^6(\rho^2 + \eta^2)} \qquad (13)$$

$$U_{\tau 1} = \frac{-A\lambda^3\sqrt{\rho^2 + \eta^2}\,\sqrt{1 - A^2\lambda^4 - A^2\lambda^6(\rho^2 + \eta^2)}\,\sqrt{1 - \lambda^2 - A^2\lambda^6(\rho^2 + \eta^2)}\,e^{i\delta} + A\lambda^3}{1 - A^2\lambda^6(\rho^2 + \eta^2)}$$

$$U_{\tau 2} = \frac{-A^2\lambda^5\sqrt{\rho^2 + \eta^2}\,\sqrt{1 - \lambda^2 - A^2\lambda^6(\rho^2 + \eta^2)}\,e^{i\delta} - \lambda\sqrt{1 - A^2\lambda^4 - A^2\lambda^6(\rho^2 + \eta^2)}}{1 - A^2\lambda^6(\rho^2 + \eta^2)}$$

$$U_{\tau 3} = \sqrt{1 - \lambda^2 - A^2\lambda^6(\rho^2 + \eta^2)}. \qquad (14)$$

Notice that, unlike the case of Wolfenstein scheme for quarks where flavor mixing matrix elements are expanded in terms of the parameter λ, we provide exact results for the matrix elements of the PMNS matrix U in terms of our parameter λ for neutrinos.

Next, we determine the constraints on the remaining parameters A, ρ, and η from present data on neutrino oscillations. We take, from Eq. (9), the following range of values for $U_{\mu 3}$, U_{e2}, and U_{e3}, to stretch over the 3σ CL:

$$\begin{aligned} 0.808 &\geq U_{\mu 3} \geq 0.579, \\ 0.585 &\geq U_{e2} \geq 0.513, \qquad (15) \\ 0.178 &\geq U_{e3} \geq 0.126; \end{aligned}$$

with this as input, we determine the following constraints on the "sizing" parameter A and the "CP" parameters ρ and η:

$$1.530 \geq A \geq 0.896, \qquad 0.424 \geq \sqrt{\rho^2 + \eta^2} \geq 0.377. \qquad (16)$$

The Jarlskog [27] parameter J that provides a measure of CP violation is given by

$$J = \frac{\sqrt{[1 - \lambda^2 - A^2\lambda^6(\rho^2 + \eta^2)][1 - A^2\lambda^4 - A^2\lambda^6(\rho^2 + \eta^2)]}}{\sqrt{\rho^2 + \eta^2}} A^2\lambda^6\eta. \quad (17)$$

Maximal CP violation corresponds to $\rho = 0$ while CP conservation corresponds to $\eta = 0$. Presently, extracting the CP-violating phase δ from neutrino experiment is a challenging task. However, the situation looks promising because δ is tied to the mixing angle θ_{13} which itself is as large as the Cabibbo angle. Additionally, even if $\rho^2 + \eta^2$ is accessed phenomenologically, there will still remain the ambiguity of knowing to which of the four quadrants δ belongs.

Acknowledgments

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TABLE I: The latest global-fit results of three neutrino mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ and two neutrino mass-squared differences $\Delta m_{12}^2 \equiv |m_2^2 - m_1^2|$ and $\Delta m_{23}^2 \equiv |m_3^2 - m_2^2|$ in the case of normal neutrino mass hierarchy [25].

	Parameter				
	Δm_{12}^2 $(10^{-5}\ \mathrm{eV}^2)$	Δm_{23}^2 $(10^{-3}\ \mathrm{eV}^2)$	θ_{12}	θ_{23}	θ_{13}
Best fit	7.54	2.43	$33.6°$	$38.4°$	$8.9°$
1σ range	$[7.32, 7.80]$	$[2.33, 2.49]$	$[32.6°, 34.8°]$	$[37.2°, 40.0°]$	$[8.5°, 9.4°]$
2σ range	$[7.15, 8.00]$	$[2.27, 2.55]$	$[31.6°, 35.8°]$	$[36.2°, 42.0°]$	$[8.0°, 9.8°]$
3σ range	$[6.99, 8.18]$	$[2.19, 2.62]$	$[30.6°, 36.8°]$	$[35.1°, 53.0°]$	$[7.5°, 10.2°]$