Phase with no mass gap in nonperturbatively gauge-fixed Yang-Mills theory

Maarten Golterman and Yigal Shamir

Phase with no mass gap in non-perturbatively gauge-fixed Yang–Mills theory

Maarten Golterman[a][†]   and   Yigal Shamir[b]

[a]*Institut de Física d'Altes Energies, Universitat Autònoma de Barcelona,*
E-08193 Bellaterra, Barcelona, Spain

[b]*School of Physics and Astronomy*
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University, Ramat Aviv, 69978 ISRAEL

ABSTRACT

An equivariantly gauge-fixed non-abelian gauge theory is a theory in which a coset of the gauge group, not containing the maximal abelian subgroup, is gauge fixed. Such theories are non-perturbatively well-defined. In a finite volume, the equivariant BRST symmetry guarantees that expectation values of gauge-invariant operators are equal to their values in the unfixed theory. However, turning on a small breaking of this symmetry, and turning it off after the thermodynamic limit has been taken, can in principle reveal new phases. In this paper we use a combination of strong-coupling and mean-field techniques to study an $SU(2)$ Yang–Mills theory equivariantly gauge fixed to a $U(1)$ subgroup. We find evidence for the existence of a new phase in which two of the gluons becomes massive while the third one stays massless, resembling the broken phase of an $SU(2)$ theory with an adjoint Higgs field. The difference is that here this phase occurs in an asymptotically-free theory.

[†] Permanent address: Department of Physics and Astronomy, San Francisco State University, San Francisco, CA 94132, USA

I. INTRODUCTION

Some years ago, we proposed a new approach to discretizing non-abelian chiral gauge theories, in order to make them accessible to the methods of lattice gauge theory [1]. An essential ingredient in this new approach is the inclusion of a gauge-fixing action at the level of the path integral defining the theory. Of course, whatever gauge-fixing action one chooses, the path integral has to be well-defined non-perturbatively. It was shown in Ref. [2] that this is impossible if one insists on maintaining BRST invariance: In a fully gauge-fixed lattice gauge theory the BRST symmetry causes the partition function, as well as all (un-normalized) expectation values of gauge-invariant operators, to vanish.

The problem can be circumvented if, instead of the full non-abelian group G, one gauge fixes only a coset G/H, where H is a subgroup containing the maximal abelian subgroup of G. Building on earlier work by Schaden [3] for the group $SU(2)$, we showed that for $G = SU(N)$ and H the maximal abelian subgroup itself, partially ("equivariantly") gauge-fixed Yang–Mills theories can be constructed on the lattice satisfying an invariance theorem: In such theories, expectation values of gauge-invariant operators do not vanish, and are exactly equal to those in the unfixed theory, in any finite volume [1]. Since only the coset G/H is gauge-fixed, ghosts are only introduced for the coset generators. Such theories are then invariant under an equivariant BRST (eBRST) symmetry, and necessarily contain four-ghost interaction terms. Both eBRST symmetry and the appearance of four-ghost interactions are key ingredients for the proof of the invariance theorem. This eBRST symmetry is nilpotent on operators which are gauge invariant under the subgroup H, guaranteeing perturbative unitarity to all orders in much the same way that standard BRST symmetry does in the usual case [1]. The key difference is that, contrary to the usual gauge-fixing procedure which is inherently perturbative [2], the equivariantly gauge-fixed theory is well defined non-perturbatively.[1]

Equivariantly gauge-fixed Yang–Mills theories have two independent couplings, the gauge coupling g and the gauge-fixing parameter ξ. The latter can be traded for the "longitudinal" coupling $\tilde{g}$ defined by $\tilde{g}^2 \equiv \xi g^2$. The couplings g and $\tilde{g}$ are both asymptotically free [5]. In this article we will argue that the phase diagram in the g–$\tilde{g}$ plane is non-trivial, with

---

[1] For an interesting perspective on the possible role of $U(1)$ (sub)groups in this context, see Ref. [4].

potentially interesting physical consequences, despite the invariance theorem.

We will limit ourselves to a study of $SU(2)$ Yang–Mills theory, equivariantly gauge-fixed to a $U(1)$ gauge theory. In the framework of equivariant gauge fixing, $SU(2)$ gauge transformations divide into the gauge transformations contained in the $U(1)$ subgroup, which will remain unfixed, and the remainder, which live in the $SU(2)/U(1)$ coset, and which will be gauge-fixed such that the resulting gauge theory is invariant under eBRST symmetry.

While both g and $\tilde{g}$ grow towards the infrared because of asymptotic freedom, the renormalization-group equation for $\tilde{g}$ depends on g in such a way that $\tilde{g}$ has to become strong in the infrared along with g. This can happen in two different ways. One scenario is that both couplings become strong simultaneously, going into the infrared. The other scenario is that $\tilde{g}$ becomes strong first, while g remains relatively weak. A third scenario, in which $\tilde{g}$ remains weak while g becomes strong is excluded by the renormalization-group analysis.

Which of the two possible scenarios is realized depends on the relative size of the couplings at the cutoff. Based as it is on solving the one-loop renormalization-group equations, we do not know what actually happens when one or both couplings have become strong. However, in the case of the second scenario, we do know that the physics of the longitudinal sector, which is governed by $\tilde{g}$, becomes strongly coupled first, while g still remains weak, keeping the transverse sector still perturbative.

A natural framework for studying the second scenario is the so-called reduced model, which corresponds to the $g = 0$ boundary of the phase diagram. The reduced model is constructed as follows. First, we perform a gauge transformation, so that the gauge transformations become explicit as an $SU(2)$-valued scalar field in the path integral defining the theory. The action depends on this field because the gauge-fixing and ghost terms are not invariant under $SU(2)/U(1)$ gauge transformations. Then, we turn off the transverse gauge fields by setting $g = 0$. The result is a theory of an $SU(2)$-valued scalar field coupled to the ghosts. The reduced model has one coupling, $\tilde{g}$, which is asymptotically free. Since only the coset $SU(2)/U(1)$ was gauge-fixed, the remaining $U(1)$ remains a gauge symmetry, also in the reduced model. The growth of $\tilde{g}$ towards the infrared suggests that dynamical symmetry breaking might take place at low energy in the reduced model, and it is this question, along with its consequences, that we investigate in this article.

The reduced model is invariant under $U(1)_L \times SU(2)_R$ transformations, where $U(1)_L$ is a

local symmetry and $SU(2)_R$ is a global symmetry. The spontaneous breaking of the global $SU(2)_R$ symmetry can be studied within the reduced model by standard methods. Using a combination of strong-coupling and mean-field techniques, we will find that a phase in which $SU(2)_R$ is broken to an abelian subgroup $U(1)_R$ may appear at large $\tilde{g}$.

While the existence of the phase with broken symmetry will have to be studied in the future using more reliable methods than mean field, it is interesting to address the consequences of such a phase for the full theory. Since the reduced model corresponds to the $g = 0$ boundary of the phase diagram, we recover the full theory by turning g back on. Our aim is then to see whether any dynamical symmetry breaking that occurs in the reduced model might have consequences for the physics of the transverse degrees of freedom.

Naively, one expects this not to be the case. According to the standard paradigm, physics in the transverse sector should be independent of the choice of gauge; therefore it should be blind to any symmetry breaking in the longitudinal sector that may have occurred in the reduced model. Indeed, in a finite volume, this is true: The invariance theorem introduced above guarantees that gauge invariant correlation functions in the full theory are independent of $\tilde{g}$, and equal to those in the unfixed theory [1].[2]

However, this may not be the whole story. Let us introduce a small breaking of eBRST symmetry (which we will generically refer to as a "seed") into the theory. We take the infinite-volume limit first, and then turn off the seed. With this order of limits, the invariance theorem ceases to hold. There are two possible outcomes. The first possibility is that the consequences of the theorem are recovered when the seed is eventually taken to zero. Then the transverse sector of the full theory has the same physics as that of the unfixed gauge theory. Another possibility is that the dynamics of the longitudinal sector carries over to the transverse sector, thus uncovering a new phase in the full theory with physical consequences.

In the reduced model there are local order parameters that signal the spontaneous breaking of the global $SU(2)_R$ symmetry. Once we turn back on the transverse gauge fields, which is equivalent to gauging the $SU(2)_R$ symmetry, no local order parameter is available any more to monitor its breaking [6]. Nevertheless, the phases of the theory can still be rigorously distinguished. The familiar confining phase of an unfixed $SU(2)$ Yang–Mills theory is characterized by a mass gap. By contrast, if the spontaneous breaking of $SU(2)_R$ in the

_____

[2] The latter, of course, is well-defined in the compact formulation on the lattice.

4

longitudinal sector carries over to the transverse sector, the gauge fields in the $SU(2)/U(1)$ coset, which we will refer to as the W fields,[3] become massive. The third gauge field stays massless, and the long-distance physics is that of an abelian theory. Therefore, phases corresponding to the two different possibilities can be distinguished by the presence or absence of a mass gap [7, 8].

As in any asymptotically-free theory, the infrared physics is non-perturbative. The only proven method for the investigation of this non-perturbative dynamics is numerical, through Monte-Carlo evaluation of correlation functions in the lattice-regularized version of the theory. However, a number of analytic techniques is available which, although heuristic and/or generally not applicable near the continuum limit, can help in gaining insight into the nature of the phase diagram. Here, we begin with an analytic study of parts of the phase diagram through a combination of strong-coupling and mean-field techniques. The motivation for this exploratory study is two-fold: First, it is interesting to use these techniques to form an idea of what the physics of equivariantly gauge-fixed Yang–Mills theories might look like. At a more concrete level, the results of such an analytic study provide a framework for numerical studies, by identifying the important observables as well as the properties of a phase in which the (consequences of the) invariance theorem are avoided.

In Sec. II we review the definition of the $SU(2)/U(1)$ equivariantly gauge-fixed Yang–Mills theory, listing all symmetries, and including a definition of the reduced model. This is followed in Sec. III by a discussion of possible patterns of symmetry breaking. As a corollary of the invariance theorem one can show that the reduced model is a topological field theory, with a partition function that is a pure number, independent of the parameters of the action. We demonstrate, through a toy model, that a topological field theory can nevertheless accommodate a non-trivial effective potential for an order parameter, and phases with a broken symmetry. In Sec. IV we derive the effective action for the reduced model to order $1/\tilde{g}^2$ by integrating out the ghosts in a strong-coupling expansion, to which we then apply mean-field techniques in Sec. V. We find that spontaneous symmetry breaking from $SU(2)_R$ to $U(1)_R$ occurs in mean field. Turning on a weakly-coupled transversal field, we derive an expression for the massive W propagator. We then use these results, augmented by additional considerations, to discuss the possible phase diagram of the theory in Sec. VI,

---

[3] In more physical terms, the gauge fields which are charged under the unfixed $U(1)$.

and conclude in Sec. VII. There are several appendices. In App. A we generalize the invariance theorem to the case where $G = SU(N)$ and H is any maximal subgroup of G. In App. B we prove a result pertaining to the toy model of Sec. III. Appendix C summarizes some group integrals. Appendix D discusses an alternative mean-field analysis, and App. E proves a perturbative result referred to in the main text. Throughout this article we will set the lattice spacing equal to one.

II.   EQUIVARIANTLY GAUGE-FIXED YANG–MILLS THEORY

In this section we define the theory of interest, the $SU(2)$ equivariantly gauge-fixed Yang–Mills theory. In this case, the only non-trivial subgroup is $H = U(1)$. We will be brief in this section; for a general discussion of the construction of equivariantly gauge-fixed theories, we refer to Ref. [1].

A.   Vector picture

We begin with the field content of our theory. The lattice gauge field consists of $SU(2)$-valued link variables $U_{x,\mu}$, and can be written in terms of a Lie-algebra valued field $V_{x,\mu}$ through

$$U_{x,\mu} = e^{iV_{x,\mu}} \ , \qquad V_{x,\mu} = V_{x,\mu,i}\frac{\tau_i}{2} \ , \tag{2.1}$$

with τ_i, $i = 1, 2, 3$, the Pauli matrices. The gauge-fixing action depends on ghost and anti-ghost fields, C_x and $\overline{C}_x$, and on a real auxiliary field b_x. Since we are only fixing the coset $SU(2)/U(1)$, these fields live in the corresponding coset of the Lie algebra. In components, the ghost field thus reads

$$C_x = C_{x,1}\frac{\tau_1}{2} + C_{x,2}\frac{\tau_2}{2} \ , \tag{2.2}$$

with similar expansions for the anti-ghost and auxiliary fields.

The lattice action defining the theory is given by

$$S = S_{\text{gauge}} + S_{\text{gf}} , \tag{2.3a}$$

$$S_{\text{gf}} = \text{tr} \sum_x \left(\tilde{g}^2 b_x^2 - 2ib_x D_\mu^- \mathcal{W}_{x,\mu} , \right.$$

$$+ \frac{1}{2} [\tau_3, D_\mu^+ C_x][U_{x,\mu} \tau_3 U_{x,\mu}^\dagger, D_\mu^+ \overline{C}_x]$$

$$\left. + i \, \mathcal{W}_{x,\mu} \{\overline{C}_x, D_\mu^+ C_x\} + \tilde{g}^2 \{C_x, \overline{C}_x\}^2 \right) . \tag{2.3b}$$

Here S_{gauge} is any gauge-invariant lattice action constructed from the link variables $U_{x,\mu}$ that reduces in the classical continuum limit to $\frac{1}{2g^2} \int d^4x \, \text{tr} \, (F_{\mu\nu}^2)$, where

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu] , \tag{2.4}$$

is the corresponding Lie-algebra valued continuum field strength, and we have used the same notation for the continuum gauge field as in Eq. (2.1). The lattice vector potential can be decomposed as

$$V_{x,\mu} = W_{x,\mu} + A_{x,\mu} \frac{\tau_3}{2} , \tag{2.5}$$

$$W_{x,\mu} = W_{x,\mu,1} \frac{\tau_1}{2} + W_{x,\mu,2} \frac{\tau_2}{2} ,$$

with $W_{\mu,1}$ and $W_{\mu,2}$ the coset fields, and A_μ the $U(1)$ field. In Eq. (2.3b), we have used the definitions

$$\mathcal{W}_{x,\mu} = -\frac{i}{4} [U_{x,\mu} \tau_3 U_{x,\mu}^\dagger, \tau_3] , \tag{2.6a}$$

$$D_\mu^+ \Phi_x = U_{x,\mu} \Phi_{x+\hat{\mu}} U_{x,\mu}^\dagger - \Phi_x , \tag{2.6b}$$

$$D_\mu^- \Phi_x = \Phi_x - U_{x-\hat{\mu},\mu}^\dagger \Phi_{x-\mu} U_{x-\hat{\mu},\mu} , \tag{2.6c}$$

where Φ is any Lie-algebra valued field. It follows from Eq. (2.6a) that also $\mathcal{W}_{x,\mu}$ has components only in the coset. In the classical continuum limit $\mathcal{W}_{x,\mu} \to W_{x,\mu}$, and $D_\mu^- \mathcal{W}_{x,\mu} \to D_\mu(A) W_\mu$, with $D_\mu(A)$ the $U(1)$ covariant derivative. Indeed, we see that this is a gauge-fixing condition for the coset gauge field W, which leaves the $U(1)$ gauge invariance intact. A gauge-fixing parameter ξ may be defined by identifying

$$\tilde{g}^2 \equiv \xi g^2 . \tag{2.7}$$

Under $U(1)$ gauge transformations, $h_x = e^{i\theta_x \tau_3/2}$, all coset fields $(\mathcal{W}_{x,\mu}, C_x, \overline{C}_x$ and $b_x)$ transform as

$$\Phi_x \rightarrow h_x \Phi_x h_x^\dagger \ , \tag{2.8}$$

and the action is thus $U(1)$ gauge invariant. The remaining $SU(2)$ gauge transformations are broken by $S_{\rm gf}$. Instead, the action is invariant under the eBRST transformations

$$
\begin{aligned}
sU_{x,\mu} &= i(D_\mu^+ C)_x U_{x,\mu} = i\left(U_{x,\mu} C_{x+\hat{\mu}} - C_x U_{x,\mu}\right) \ , \\
sC_x &= 0 \ , \\
s\overline{C}_x &= -ib_x \ , \\
sb_x &= i[C_x^2, \overline{C}_x] \ .
\end{aligned}
\tag{2.9}
$$

Normally, one would expect the ghost field to transform as $sC = -iC^2$, but since C^2 points in the τ_3 direction, this is not allowed. This change in the transformation rule of C_x modifies the nilpotency property of s: Instead of $s^2 = 0$, we now have

$$s^2(\text{field}) = \delta_X(\text{field}) \ , \qquad X = iC^2 \ , \tag{2.10}$$

where δ_X is an infinitesimal $U(1)$ gauge transformation with parameter X. It follows that s is nilpotent when acting on operators which are invariant under $U(1)$ gauge transformations.

The action breaks not only the local, but also the global $SU(2)$ symmetry. Apart from the $U(1)$ subgroup, the action is invariant under a discrete subgroup $\tilde{\mathcal{S}}_2$ generated by

$$\tilde{P}_\alpha = e^{i\pi\tau_\alpha/2} = i\tau_\alpha \ , \qquad \alpha = 1, 2 \ , \tag{2.11}$$

under which

$$U_{x,\mu} \rightarrow \tilde{P}_\alpha U_{x,\mu} \tilde{P}_\alpha^\dagger \ , \tag{2.12}$$

with the other fields all transforming in the same way.

In addition, the action is invariant under a symmetry of the ghost sector that we will refer to as Schaden symmetry [9], generated by

$$\Pi_+ = C_\alpha \frac{\delta}{\delta \overline{C}_\alpha} \ , \qquad \Pi_- = \overline{C}_\alpha \frac{\delta}{\delta C_\alpha} \ , \tag{2.13}$$

and

$$\Pi_3 = \frac{1}{2} \left(C_\alpha \frac{\delta}{\delta C_\alpha} - \overline{C}_\alpha \frac{\delta}{\delta \overline{C}_\alpha} \right) \ , \tag{2.14}$$

where the latter generates the $U(1)$ ghost-number symmetry. These three generators together generate the group $SL(2, R)$.

The construction described above applies to a more general class of theories, in which $G = SU(N)$, and H is any maximal subgroup of $SU(N)$. This class of theories is introduced in App. A, where we also observe that the invariance theorem applies to all theories in this class. For such theories the gauge-fixing action, generalizing Eq. (2.3b), and defined in App. A, is the most general possible form. In particular, all such theories have only two couplings, g and $\tilde{g}$.

Finally, for theories in this class "flip" symmetry, defined in Ref. [1], takes a simple form. It transforms only the ghost and anti-ghost fields, according to $C \to \overline{C}$, $\overline{C} \to -C$, while leaving all other fields invariant. Applying flip symmetry to Eq. (2.9), it follows that the action is also invariant under an anti-eBRST symmetry $\bar{s}$.[4] It is then straightforward to show that

$$\bar{s}^2(\text{field}) = \delta_{\overline{X}}(\text{field}) , \qquad \overline{X} = i\overline{C}^2 , \qquad (2.15)$$

$$\{s, \bar{s}\}(\text{field}) = \delta_{\widetilde{X}}(\text{field}) , \qquad \widetilde{X} = i\{C, \overline{C}\} ,$$

analogous to Eq. (2.10).

B. Higgs picture and reduced model

The action (2.3) is not gauge invariant (except under the $U(1)$ subgroup), and the longitudinal part of the gauge field, or equivalently, the gauge degrees of freedom thus couple to the other fields in the theory. The standard paradigm is that $SU(2)$ gauge invariant correlators remain the same as in the unfixed theory; but it is precisely the aim of this article to investigate whether this is indeed true everywhere in the phase diagram.

The gauge degrees of freedom can be exposed by carrying out a gauge transformation on the theory defined by Eq. (2.3). In technical terms, the "Higgs–Stückelberg" picture, or, for short, the "Higgs" picture of our theory is obtained by making the replacement

$$U_{x,\mu} \to U_{x,\mu}^{\phi} \equiv \phi_x U_{x,\mu} \phi_{x+\hat{\mu}}^{\dagger} \qquad (2.16)$$

---

[4] While other equivariantly gauge-fixed theories may have flip symmetry and anti-eBRST symmetry, these symmetries, as well as the eBRST symmetry (2.9), take a more complicated form in general [1].

in Eq. (2.3), and integrating over the $SU(2)$ valued field ϕ_x. Explicitly,

$$Z = \int \mathcal{D}U\mathcal{D}b\mathcal{D}C\mathcal{D}\overline{C} \, \exp\left(-S_{\mathrm{gauge}}(U) - S_{\mathrm{gf}}(U, b, C, \overline{C})\right) \tag{2.17a}$$

$$= \int \mathcal{D}U \, \exp\left(-S_{\mathrm{gauge}}(U)\right) \widetilde{Z}(U), \tag{2.17b}$$

where

$$\widetilde{Z}(U) = \int \mathcal{D}\phi\mathcal{D}b\mathcal{D}C\mathcal{D}\overline{C} \, \exp\left(-S_{\mathrm{gf}}(U^\phi, b, C, \overline{C})\right). \tag{2.18}$$

Equation (2.17a) is the partition function in the vector picture, introduced in the previous subsection. Equation (2.17b), together with Eq. (2.18), gives the Higgs-picture partition function. Except for the integration over the gauge field itself, we have absorbed the integration over all other fields, including the gauge transformations described by ϕ_x, into Eq. (2.18). The motivation for doing so will become clear shortly. Since S_{gauge} is gauge invariant, only S_{gf} is effected by the gauge transformation. One can always go back to the vector picture by making the field redefinition $U_{x,\mu} \to \phi_x^\dagger U_{x,\mu}\phi_{x+\hat{\mu}}$ which, when substituted into the right-hand side of Eq. (2.18) eliminates all dependence on ϕ_x from the action.

The symmetry structure of the action is modified in the Higgs picture. Equivariant BRST transformations are as in Eq. (2.9), except that now $U_{x,\mu}$ is invariant, while the Higgs–Stückelberg field ϕ_x transforms as

$$s\phi_x = -iC_x\phi_x . \tag{2.19}$$

This makes the combination (2.16) transform as in the first line of Eq. (2.9), thus maintaining the eBRST invariance of the action.

An inspection of Eq. (2.16) shows that the theory is now invariant under an extra $SU(2)$ gauge symmetry,

$$U_{x,\mu} \to g_x U_{x,\mu} g_{x+\hat{\mu}}^\dagger , \tag{2.20}$$

$$\phi_x \to h_x \phi_x g_x^\dagger ,$$

in which $g_x \in SU(2)$. In Eq. (2.20) we also show how these fields transform under the $U(1)$ gauge symmetry (2.8) in the Higgs picture. The full gauge symmetry of the action in the Higgs picture is thus the group $U(1)_L \times SU(2)_R$, where we used the labels L and R to indicate on which side of the field ϕ_x these transformations act.

The action S_{gauge}, with gauge coupling g, participates only in the dynamics of the transverse part of the gauge field, while S_{gf}, with coupling $\tilde{g}$, also governs the dynamics of the

rest of the degrees of freedom. Here, we are interested in possible strong $\tilde{g}$-dynamics while keeping the gauge coupling g perturbative. It then makes sense to first consider the limiting case $g = 0$, which freezes out the transverse part of the gauge field described in the Higgs picture by $U_{x,\mu}$. In this limit, the theory is thus defined by S_{gf}, in which first the replacement (2.16) has been made, followed by setting $U_{x,\mu} = I$.

We will refer to this simplified version of the theory as the reduced model. In short, the reduced model's action is given by Eq. (2.3b), with the replacement $U_{x,\mu} \to \phi_x \phi_{x+\hat{\mu}}^\dagger$. Its partition function is $Z_{\text{red}} = \widetilde{Z}(I)$. This theory is invariant under eBRST with the rule (2.19), as well as under the group $U(1)_L \times SU(2)_R$, where now $SU(2)_R$ has become a global symmetry.[5] The role of the local $U(1)_L$ symmetry is to effectively make ϕ_x take values in the coset $SU(2)/U(1)$, with two fields ϕ_x and ϕ_x' being in the same equivalence class if they differ only by a $U(1)_L$ gauge transformation.

A key question we will address in this article is how the global $SU(2)_R$ symmetry of the reduced model is realized as a function of $\tilde{g}$.

III.   PATTERNS OF SYMMETRY BREAKING

The invariance theorem states that finite-volume expectation values of gauge-invariant operators in the eBRST gauge-fixed theory are exactly equal to their values in the unfixed theory [1]. The partition function Z is equal to that of the unfixed theory up to a non-vanishing multiplicative constant. Briefly, the proof works by noting that S_{gf} can be written as an eBRST variation, *cf.* Eq. (A3). Multiplying the first term on the right-hand side of Eq. (A3) by a parameter t, it follows that $\partial Z/\partial t$ vanishes, because it is the expectation value of an eBRST variation. This conclusion generalizes to expectation values of gauge invariant operators. In the limit $t \to 0$, the first term on the right-hand side of Eq. (A3) drops out, while the remaining terms depend only on the ghost-sector fields. The partition function Z of Eq. (2.17) is therefore a product of the original gauge-invariant partition function, and a decoupled partition function for the ghost-sector fields. Moreover, the latter is a product of decoupled integrals at each lattice site, which can be shown to yield a non-zero constant [1].

In the Higgs picture, the invariance theorem applies even if we keep the gauge field $U_{x,\mu}$

---

[5] The full theory, in the Higgs picture, can be recovered from the reduced model by introducing an $SU(2)$ gauge field and promoting $SU(2)_R$ to a local symmetry.

11

external, because $U_{x,\mu}$ does not transform under eBRST in this case. It follows that the partition function $\widetilde{Z}(U)$ of Eq. (2.18) defines a topological field theory: While the action $S_{\rm gf}$ depends on $\tilde{g}$, the partition function $\widetilde{Z}(U)$ does not, and it is also independent of the external gauge field $U_{x,\mu}$. It is just a pure number.

The existence of the invariance theorem would lead one to expect that if any spontaneous symmetry breaking took place in the reduced model, this could not possibly have any effect on the physics of the transverse gauge fields. This state of affairs would be in accordance with the standard paradigm that, whatever gauge-fixing procedure one applies to a Yang–Mills theory, the transverse dynamics is unchanged. Indeed, to the extent that exact eBRST invariance is maintained in any finite volume, the invariance theorem applies, with the anticipated consequences.

Furthermore, one can question the proposition that the reduced model could have any non-trivial phase structure at all. A phase transition occurs when the effective potential for some order parameter develops a new global minimum, as a function of the couplings of the theory. For this to happen, the effective potential should depend on the couplings (here only $\tilde{g}$) in a non-trivial way in the first place. But the reduced model is a topological field theory, with a partition function that is independent of the parameter $\tilde{g}$. The question arises whether such a partition function can nevertheless accommodate a dynamics that generates a non-trivial effective potential and gives rise to spontaneous symmetry breaking.

In the following subsection we will illustrate and address these questions in a toy model that shares the most relevant features of the reduced model. The toy model is a zero-dimensional "field theory," *i.e.*, an ordinary integral. This exercise will show that a topological (field) theory can in principle break a symmetry spontaneously. Encouraged by this result we return in Sec. III B to the equivariantly gauge-fixed theory itself, and study patterns of spontaneous symmetry breaking that might be triggered by strong $\tilde{g}$ dynamics.

A. Toy model

In this subsection we demonstrate through a toy model that global symmetries can, in principle, break spontaneously in a theory satisfying a similar invariance theorem. Our toy model is a zero-dimensional field theory with an exact BRST-type invariance, as well as a discrete Z_2 symmetry. In a sense that will be clarified below, the Z_2 symmetry gets broken

spontaneously, whereas BRST symmetry remains unbroken.

The model contains two real degrees of freedom, ϕ and b, and a ghost, anti-ghost pair c, $\bar{c}$. The nilpotent BRST transformation s acts as follows

$$s\phi = c \, , \tag{3.1}$$
$$sc = 0 \, ,$$
$$s\bar{c} = ib \, ,$$
$$sb = 0 \, .$$

We choose the action to be[6]

$$S = s\left(-i\bar{c}b + \bar{c}f(\phi)\right) = b^2 + ibf(\phi) - \bar{c}f'(\phi)c \, , \tag{3.2}$$

where $f'(\phi) = \partial f/\partial\phi$. We assume that, as ϕ ranges from $-\infty$ to $+\infty$, so does $f(\phi)$.

BRST invariance of the toy model, together with the fact that the action is a BRST variation, leads to an invariance theorem: The partition function Z is independent of any parameters inside f. Indeed,

$$Z \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\phi \int_{-\infty}^{\infty} db \int dcd\bar{c}\, e^{-\left(b^2 + ibf(\phi) - \bar{c}f'(\phi)c\right)} \tag{3.3a}$$
$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} d\phi\, f'(\phi)\, e^{-f^2(\phi)/4} = 1 \, . \tag{3.3b}$$

The only reservation is that the assumption we have made about the asymptotic behavior of $f(\phi)$ must be respected. Therefore, if $f(\phi)$ is a polynomial, it must be of odd degree, and the coefficient of the highest power must be positive. The coefficients of the rest of the polynomial can then be changed at will, without altering the value of Z.

Now, let us consider a concrete example. We choose

$$f(\phi) = \frac{1}{\lambda}\left(\phi^3 - v^2\phi\right) \, , \tag{3.4}$$

with λ and v^2 two real parameters. In addition to the BRST symmetry (3.1), this choice leads to invariance under a discrete Z_2 symmetry that flips the signs of ϕ and b.

The "classical potential" $f^2(\phi)/4$ has a single minimum at $\phi = 0$ when $v^2 < 0$. But, for $v^2 > 0$, additional degenerate minima appear at $\phi = \pm v$. These new minima are not

---

[6] This action is reminiscent of the supersymmetric Wess–Zumino model, except here we can use real bosonic fields.

13

invariant under the Z_2 discrete symmetry. Were we dealing with a true field theory, we would observe a spontaneous breaking of the Z_2 symmetry by introducing a "seed" that breaks this symmetry explicitly into the finite-volume action, and turning it off after the thermodynamic limit has been taken.[7] Choosing the seed to be

$$S_{\text{seed}} = -\epsilon\phi \, , \tag{3.5}$$

the minimum $\phi = v > 0$ would be selected for $\epsilon > 0$, while $\phi = -v$ would be selected for $\epsilon < 0$.

Being a zero-dimensional field theory, in the toy model there is no thermodynamic limit to be taken. Strictly speaking, we must always keep ϵ non-zero to maintain a preference for a particular saddle point with broken symmetry. The situation is more favorable in perturbation theory. As usual, when the coupling constant λ is small, we may select any one of the saddle points and develop a perturbative expansion around it.[8] Within this framework, a seed is not introduced. We will denote perturbatively calculated quantities obtained this way with a subscript that identifies the saddle point.

We first consider the perturbative expansion around the minimum at $\phi = 0$. Setting $\phi = \lambda\psi$ and integrating over ψ we find

$$Z_0 = -1 \, , \qquad \text{to all orders,} \tag{3.6a}$$

$$\langle\phi\rangle_0 = 0 \, , \qquad \text{to all orders.} \tag{3.6b}$$

An explicit calculation of Z_0 can be found in App. B. It is easily seen that the vanishing result for $\langle\phi\rangle_0$ generalizes to every order parameter for the Z_2 symmetry, and therefore this symmetry is not broken spontaneously at the $\phi = 0$ saddle point.

For the other two saddle points we expand $\phi = \pm v + \lambda\psi$. We find

$$Z_{\pm v} = 1 \, , \qquad \text{to all orders,} \tag{3.7a}$$

$$\langle\phi\rangle_{\pm v} = \pm v \left(1 - \frac{3}{4}\frac{\lambda^2}{v^6} + O\left((\lambda^2/v^6)^2\right) \right) \, , \tag{3.7b}$$

showing that indeed the Z_2 symmetry is broken spontaneously at these saddle points. Of course, summing over all the saddle points (for vanishing seed) will restore the symmetry. We will return to the proof of Eq. (3.7a) below.

---

[7] As an example, a similar discrete symmetry undergoes spontaneous breaking in the Ising model in $d \geq 2$ euclidean dimensions.

[8] As usual, the expansion in λ is an asymptotic expansion.

Let us now consider how BRST symmetry is realized at the $\phi = \pm v$ saddle points. While ϕ is not invariant under BRST, an expectation value $\langle \phi \rangle_{\pm v} \neq 0$ does not imply the breaking of BRST symmetry, because ϕ is not the BRST variation of anything; spontaneous BRST symmetry breaking would be signaled by $\langle sX \rangle_{\pm v} \neq 0$ for some X. We can in fact prove that BRST symmetry is not broken within the perturbative expansion around any saddle point. Taking for example the saddle point $\phi = v$ we first expand $\phi = v + \lambda\psi$, and then define the BRST rule to be $s\psi = c/\lambda$. This rule is clearly consistent with Eq. (3.1). It follows that the perturbative expansion around the saddle point respects BRST symmetry. We may derive BRST Ward identities in the usual way, obtaining that $\langle sX \rangle_v = 0$ for any X. The same reasoning applies at the other saddle points.

An interesting corollary is that the invariance theorem applies within the perturbative expansion around each saddle point of the toy model. The saddle-point partition functions Z_0 and $Z_{\pm v}$ must therefore be independent, in particular, of the coupling constant λ. Since these partition functions are intrinsically defined within a perturbative expansion in λ, it follows that their tree-level values receive no corrections to all orders in λ. The tree-level values can easily be found, leading to Eqs. (3.6a) and (3.7a).[9]

While the saddle-point partition functions are independent of λ and v, our explicit calculation shows that observables, such as $\langle \phi \rangle_{\pm v}$, depend on these parameters in a non-trivial way. The technical reason is easily understood. We may calculate $\langle \phi \rangle_{\pm v}$ by augmenting the action with a source term, $-J\psi$, and taking the derivative of $W_{\pm v}(J) = -\log Z_{\pm v}(J)$ with respect to J. The source term is recognized as nothing but the seed, Eq. (3.5), with $\epsilon \to J$. This term is not invariant under BRST, leading to the failure of the invariance theorem. Hence, $W_{\pm v}(J)$ can depend non-trivially on λ and v. The same conclusion applies to the effective action $\Gamma_{\pm v}(\phi_{\text{eff}})$ obtained through a Legendre transformation, which, in the case of the toy model, coincides with the effective potential for the order parameter $\langle \phi \rangle_{\pm v}$.

The main lesson we have learned is that topological (field) theories can accommodate non-trivial effective potentials for order parameters, as well as symmetry-breaking saddle points. Later on we will find evidence that similar conclusions hold in the eBRST gauge-fixed theory that is the main subject of this paper.

---

[9] Note that the sum over the three saddle points $Z_{+v} + Z_{-v} + Z_0$ reproduces the exact partition function (3.3). As already mentioned, an explicit verification of Eq. (3.6a) to all orders is given in App. B.

B. Symmetry breaking in the equivariantly gauge-fixed theory

In this subsection we consider a symmetry breaking pattern that might be triggered by strong $\tilde{g}$ dynamics in the reduced model, and address its implications for the full theory. We are encouraged by the results of the previous subsection, which show that spontaneous symmetry breaking can in principle take place in a topological field theory. Even though the reduced model's partition function is independent of the coupling constant $\tilde{g}$ (and of the external gauge field, $cf.$ Eq. (2.18)), this does not preclude the existence of an effective potential with non-trivial minima for order parameters.

We will be interested in a symmetry breaking pattern where the global $SU(2)_R$ is broken down to an abelian subgroup $U(1)_R$, while eBRST symmetry remains unbroken. A natural order parameter for this symmetry breaking is the expectation value of

$$\tilde{A}_x = \phi_x^\dagger \tau_3 \phi_x \ . \tag{3.8}$$

This operator is invariant under the local $U(1)_L$ symmetry, as it should be. It transforms in the adjoint representation of $SU(2)_R$, and therefore a non-zero expectation value, $\langle \tilde{A} \rangle \neq 0$, signals the symmetry breaking $SU(2)_R \to U(1)_R$.

While the operator $\tilde{A}$ is not invariant under eBRST, its expectation value cannot serve as an order parameter for eBRST symmetry breaking. Again the reason is that $\tilde{A}$ is not the eBRST variation of any other operator, and therefore it will not occur in any Ward identity for eBRST symmetry.

Following standard practice, in order to study the anticipated symmetry breaking pattern we add to the finite-volume action the term

$$S_{\text{seed}} = -h \sum_x \text{tr} \left(\tau_3 \tilde{A}_x \right) \ , \tag{3.9}$$

choosing $h > 0$. The limit $h \to 0$ is taken after the thermodynamic limit. The seed tilts the effective potential for the order parameter, and, in the broken phase, selects the vacuum state where $\langle \tilde{A} \rangle$ points in the τ_3 direction. Once the infinite-volume limit has been taken, transitions between different vacua are kinematically suppressed. Hence the vacuum state will remain that selected by the seed when the latter is turned off.

As usual, we may obtain the order parameter $\langle \tilde{A} \rangle$ from the logarithmic derivative of the partition function with respect to the external magnetic field h. Once again, the seed (3.9) is

not invariant under eBRST transformations, leading to the failure of the invariance theorem for non-zero h. Much like the toy model of Sec. III A, this allows $\langle \tilde{A} \rangle$ to depend non-trivially on the coupling $\tilde{g}$.

Next, we turn to the full theory, which corresponds to moving away from the $g = 0$ boundary into the $g > 0$ phase diagram. We will assume that the symmetry breaking discussed above takes place in the reduced model. For definiteness, anticipating our analysis in subsequent sections, we assume the existence of a critical value $\tilde{g}_c$ such that, if we start from the strong-coupling limit, the reduced model is in a symmetric phase for $\tilde{g}_c < \tilde{g} < \infty$, while for $\tilde{g} < \tilde{g}_c$ we enter the phase in which $SU(2)_R$ is broken down to $U(1)_R$.

In lattice Higgs models such as Ref. [8], no symmetry-breaking seed is needed once the global symmetry is promoted to a local symmetry. Here, too, a seed that breaks the $SU(2)_R$ symmetry is not needed for $g > 0$. But, as explained above, in the equivariantly gauge-fixed theory it is necessary to turn on a seed that breaks the eBRST symmetry in order to probe any non-trivial phase structure. Because of the invariance theorem, we are bound to recover the usual confining physics everywhere in the $g > 0$ phase diagram if we did not introduce an eBRST-breaking seed.

Let us first consider what happens when we move into the phase diagram starting from some $\tilde{g} > \tilde{g}_c$ in the symmetric phase. The presence of the eBRST-breaking seed, and the ensuing failure of the invariance theorem, imply that gauge invariant observables will be somewhat distorted relative to the un-gauge fixed Yang–Mills theory. However, in this region of the phase diagram we expect to recover the consequences of the theorem in the thermodynamic limit, and thus, that the theory is in the usual confining phase.

If, instead, we move into the $g > 0$ phase diagram starting at a point inside the broken phase of the reduced model, it is in fact very natural to find the full theory in a different phase resembling the broken phase of the $SU(2)$ theory with an adjoint Higgs field [7, 8]. In the reduced model, the symmetry breaking of $SU(2)_R$ down to $U(1)_R$ produces two Goldstone bosons. In the full theory, $SU(2)_R$ is promoted to a gauge symmetry (see Sec. II B). Therefore, we would expect a Higgs-like phenomenon: the two Goldstone bosons will turn into the longitudinal modes of the W fields which, in turn, will become massive. Since the remaining unbroken gauge symmetry is abelian, the gauge coupling will stop growing on larger distance scales, and the third gauge field will stay massless. We will refer to such a phase, if it exists, as a Coulomb phase.

As it turns out, the emergence of a W-boson mass term is consistent with the preservation of eBRST symmetry in the limit of vanishing seed, provided that the ghost fields acquire the same mass. Indeed, if we consider the (continuum) action

$$S_{\mathrm{m}} = m^2 \int d^4x \, \mathrm{tr} \left(W_\mu^2/2 + \overline{C}C \right) , \qquad (3.10)$$

we find that it is eBRST invariant on-shell,

$$
\begin{aligned}
sS_{\mathrm{m}} &= m^2 \int d^4x \, \mathrm{tr} \left(W_\mu D_\mu(A)C - ibC \right) \qquad (3.11) \\
&= -m^2 \int d^4x \, \mathrm{tr} \left((D_\mu(A)W_\mu + ib)C \right) = 0 ,
\end{aligned}
$$

where we have integrated by parts and, in the last step, used the auxiliary field's equation of motion.[10]

Finally, we recall that the couplings g and $\tilde{g}$ are both asymptotically free. If a Coulomb phase indeed exists in the phase diagram of the full theory, the most interesting question is whether this phase extends to the gaussian fixed point $(g, \tilde{g}) = (0, 0)$, where the correlation lengths associated with both the transverse and longitudinal sectors diverge in lattice units. While we will not be able to answer this question in this article, we will return to a further discussion of this possibility, as well as a discussion of the renormalization-group evolution, in Sec. VI.

IV. EFFECTIVE THEORY AT LARGE $\tilde{g}$: INTEGRATING OUT THE GHOSTS

In the next two sections, we will use a combination of strong-coupling and mean-field techniques in order to investigate the phase diagram of the reduced model. The reduced model has only one coupling, $\tilde{g}$, and its phase diagram is thus one-dimensional. This phase diagram corresponds to the $g = 0$ boundary of the two-dimensional phase diagram of the full theory.

In this section, we will integrate over the ghost and anti-ghost fields in a $1/\tilde{g}^2$ expansion, thus deriving an effective action for the gauge field $U_{x,\mu}$, to leading order in this expansion. In the next section, we will first restrict ourselves to the reduced model by setting $U_{x,\mu} =$

---

[10] In view of the eBRST invariance of S_{m}, the question arises whether this mass term could be induced in perturbation theory. In App. E we prove that this is not the case.

$\phi_x \phi^\dagger_{x+\hat\mu}$, and look for phase transitions in the reduced model by applying mean-field methods to this effective action. We will then investigate the implications of the mean-field results for the full theory.

Integrating over the auxiliary field b_x in Eq. (2.3b), we obtain the on-shell version of the gauge-fixing action,

$$S^{\text{on}}_{\text{gf}} = \frac{1}{\tilde g^2} \operatorname{tr} \sum_{x,\mu} (D^-_\mu \mathcal{W}_{x,\mu})^2 + \tilde g^2 \operatorname{tr} \sum_x \{C_x, \overline{C}_x\}^2 + \sum_{xy,\alpha\beta} \overline{C}_{x,\alpha} \Omega_{x,\alpha;y,\beta} C_{y,\beta} \ . \tag{4.1}$$

The matrix Ω, defining the bilinear ghost action, has the following non-zero components:

$$\Omega_{x,\alpha;x,\beta} = \frac{1}{2} \delta_{\alpha\beta} \operatorname{tr} \left(\tau_3 U_{x-\hat\mu,\mu} \tau_3 U^\dagger_{x-\hat\mu,\mu} + \tau_3 U_{x,\mu} \tau_3 U^\dagger_{x,\mu} \right) \equiv \frac{1}{2} \delta_{\alpha\beta} \hat\Omega_{xx} \ , \tag{4.2}$$

$$\Omega_{x,\alpha;x+\hat\mu,\beta} = \frac{1}{2} \left(\operatorname{tr} \left(U_{x,\mu} \tau_\alpha U^\dagger_{x,\mu} \tau_\beta \right) - \delta_{\alpha\beta} \sum_\gamma \operatorname{tr} \left(U_{x,\mu} \tau_\gamma U^\dagger_{x,\mu} \tau_\gamma \right) \right) \ ,$$

$$\Omega_{x+\hat\mu,\alpha;x,\beta} = \frac{1}{2} \left(\operatorname{tr} \left(\tau_\alpha U_{x,\mu} \tau_\beta U^\dagger_{x,\mu} \right) - \delta_{\alpha\beta} \sum_\gamma \operatorname{tr} \left(U_{x,\mu} \tau_\gamma U^\dagger_{x,\mu} \tau_\gamma \right) \right) \ ,$$

with coset indices $\alpha, \beta, \gamma = 1, 2$, and a sum over μ implied in the first line. It can be checked that Ω is symmetric and real.

We may now integrate over the ghost fields in a strong-coupling expansion in $\tilde g$, using that for large $\tilde g$ the dominant term in Eq. (4.1) is the single-site four-ghost term. We define the normalization of single-site ghost integrals by requiring that

$$\int [dC d\overline{C}] C_\alpha \overline{C}_\beta C_\gamma \overline{C}_\delta = \epsilon_{\alpha\gamma} \epsilon_{\beta\delta} \ , \tag{4.3}$$

where $\epsilon_{\alpha\beta}$ is the two-dimensional epsilon tensor. It is then straightforward to calculate the effective action for $U_{x,\mu}$ to order $1/\tilde g^2$, and we find

$$S_{\text{eff}} = \frac{1}{\tilde g^2} \sum_x \left(\operatorname{tr} \left(D^-_\mu \mathcal{W}_{x,\mu} \right)^2 - \frac{1}{4} \hat\Omega^2_{xx} \right) + O\left(\frac{1}{\tilde g^4} \right) \ . \tag{4.4}$$

At this order, S_{eff} only depends on the diagonal elements of Ω.

V. DYNAMICAL SYMMETRY BREAKING: MEAN-FIELD STUDY

Our next step is to transition to the reduced model by substituting

$$U_{x,\mu} = \phi_x e^{igV_{x,\mu}} \phi^\dagger_{x+\hat\mu} \tag{5.1}$$

into Eq. (4.4), keeping the transverse gauge field $V_{x,\mu}$ external. This turns S_{eff} into a scalar theory, to which we will apply a mean-field analysis. First, in Sec. V A, we set $V_{x,\mu} = 0$ and study the phase diagram of the reduced model as a function of $\tilde{\beta} = 1/\tilde{g}^2$. Starting from $\tilde{\beta} = 0$, as we increase $\tilde{\beta}$ the mean-field analysis predicts a first-order transition into a phase where $SU(2)_R$ is spontaneously broken to $U(1)_R$. Then, in Sec. V B, in order to probe the consequences of the spontaneous symmetry breaking in the reduced model for the transverse gauge fields, we expand the full theory around the mean-field solution to order g^2. We find that in the broken phase the W fields pick up a mass, in accordance with our general remarks in Sec. III. Some further remarks on the mean-field analysis that put it into a broader context are collected in Sec. V C.

A.   The reduced model

The reduced model is invariant under a local $U(1)$ symmetry and a global $SU(2)$ symmetry,

$$\phi_x \to h_x \phi_x g^\dagger , \tag{5.2}$$

with $h_x \in U(1)_L$ and $g \in SU(2)_R$, *cf.* Eq. (2.20). We repeat the definition of the order parameter introduced in Sec. III,

$$\tilde{A}(\phi) = \phi^\dagger \tau_3 \phi . \tag{5.3}$$

This operator is invariant under the local $U(1)_L$. A non-zero expectation value for this composite field signals the symmetry breaking of $SU(2)_R$ down to $U(1)_R$. Indeed, we may define a three-dimensional vector $\tilde{A}_i(\phi)$ through

$$\tilde{A}(\phi) = \tilde{A}_i(\phi)\tau_i \quad \Rightarrow \quad \tilde{A}_i(\phi) = \frac{1}{2} \operatorname{tr}\left(\tilde{A}(\phi)\tau_i\right) . \tag{5.4}$$

This unit vector transforms in the adjoint representation of $SU(2)_R$. If it acquires a non-zero expectation value this breaks $SU(2)_R$, while leaving unbroken the $U(1)_R$ subgroup corresponding to rotations around this vector.

At order $\tilde{\beta}$, the effective action of the reduced model is obtained by substituting Eq. (5.1) with $V_{x,\mu} = 0$ into Eq. (4.4). As it turns out, this effective action can be expressed solely in terms of $\tilde{A}(\phi)$.[11] We may thus rewrite the partition function of the effective theory for the

---

[11] For some details on the determination of $S_{\text{eff}}(\tilde{A})$ from $S_{\text{eff}}(\phi)$, see App. D.

reduced model as

$$Z = \int \prod_x d_H\phi_x \, e^{-S_{\text{eff}}(\phi)} \tag{5.5}$$

$$= \int \prod_x d_H\phi_x \, e^{-S_{\text{eff}}(\phi)} \prod_x \int da_{i,x} \int_{-i\infty}^{i\infty} dh_{i,x} \, e^{-\frac{1}{2}\sum_x \text{tr}\,(H_x(A_x - \tilde{A}(\phi_x)))}$$

$$= \prod_x \int da_{i,x} \int dh_{i,x} \, e^{-\sum_x \left(\frac{1}{2}\text{tr}\,(H_x A_x) + S_{\text{eff}}(A_x) - u(H_x)\right)} \, ,$$

where $d_H\phi$ is the Haar measure on $SU(2)$,

$$H = h_i\tau_i \, , \qquad A = a_i\tau_i \, , \tag{5.6}$$

are new fields with unconstrained real components h_i and a_i, and

$$e^{u(H)} = \int d_H\phi \, e^{\frac{1}{2}\,\text{tr}\,(H\phi^\dagger \tau_3 \phi)} \, . \tag{5.7}$$

The latter integral can be calculated by making use of the invariance under $SU(2)_R$. We find

$$e^{u(H)} = \frac{\sinh h}{h} \, . \tag{5.8}$$

where $h = |\vec{h}|$ is the length of the vector. For some steps of the calculation of this integral, we refer to App. C.

So far, our manipulations were exact. The mean-field solution is obtained by calculating the partition function in the saddle-point approximation [10]. To this end we minimize the free energy

$$f(H, A) = h_i a_i + s_{\text{eff}}(A) - u(H) \, . \tag{5.9}$$

We assume that the mean-field solution is translationally invariant, so that A and H are x-independent. This leads to the mean-field equations

$$a_i = \frac{\partial u}{\partial h_i} = \frac{\partial u}{\partial h} \frac{h_i}{h} \, , \tag{5.10a}$$

$$h_i = -\frac{\partial s_{\text{eff}}}{\partial a_i} \, , \tag{5.10b}$$

with

$$s_{\text{eff}} = \frac{1}{V} S_{\text{eff}} = -\tilde{\beta}\left(\frac{d}{4}(1 - 2d)\,\text{tr}\,A^2 + d^2\left(\text{tr}\,A^2\right)^2 + \frac{d^2}{2}\,\text{tr}\,A^4\right) \, , \tag{5.11}$$

where we wrote the mean-field action in d rather than four dimensions. Equation (5.10a) implies that the orientation of the vector $\vec{a}$ will follow that of $\vec{h}$. We may thus choose $h_1 = h_2 = a_1 = a_2 = 0$ and write $h_3 = h$ and $a_3 = a$.
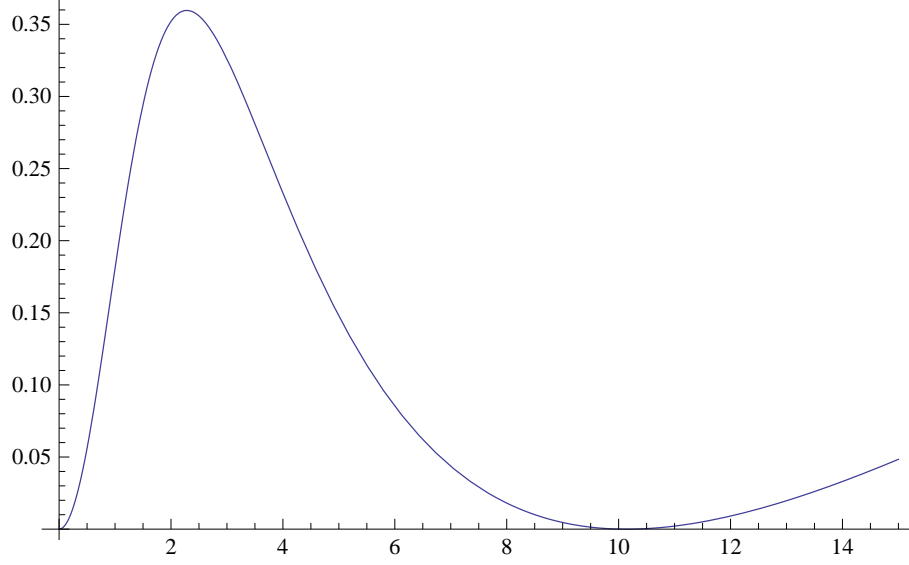
21

FIG. 1: The free energy (5.9) as a function of h, with $d = 4$ and $\tilde{\beta} = 0.04850$.

Solving for a in terms of h, and substituting into Eq. (5.9), we may plot the free energy as a function of h. The free energy always has a zero at $h = 0$. For small h, the $d = 4$ free energy may be expanded as

$$f(h) = \left(\frac{1}{6} + \frac{14}{9} \tilde{\beta} \right) h^2 + O(h^4) . \tag{5.12}$$

showing that the curvature at the origin is always positive. This rules out a continuous phase transition.

For very small $\tilde{\beta}$, the free energy is positive for $h > 0$, and its absolute minimum is at $h = 0$. When $\tilde{\beta}$ increases, a new local minimum, $h_1(\tilde{\beta})$, develops. This minimum becomes degenerate with the global minimum at $h = 0$ for $\tilde{\beta} = \tilde{\beta}_c = 0.04850$. The free energy for $\tilde{\beta} = \tilde{\beta}_c$ is shown in Fig. 1. The non-trivial minimum is at $h = h_1(\tilde{\beta}_c) = 10.084$, and the order parameter at this minimum is $a(h_1(\tilde{\beta}_c)) = 0.90083$. For $\tilde{\beta} > \tilde{\beta}_c$ the minimum $h_1(\tilde{\beta})$ becomes negative, and thus the absolute minimum of the free energy. This implies that there is a first-order phase transition at $\tilde{\beta} = \tilde{\beta}_c$, with $\langle \tilde{A} \rangle = 0$ for $\tilde{\beta} < \tilde{\beta}_c$, and $\langle \tilde{A} \rangle \neq 0$ for $\tilde{\beta} > \tilde{\beta}_c$, with $SU(2)_R$ broken down to $U(1)_R$ in the latter case.

Our mean-field approximation depends, in particular, on the choice of the mean field. In order to test the robustness of our conclusions we performed a similar analysis, but now starting from a mean field for ϕ itself, rather than for $\phi^\dagger \tau_3 \phi$. The details are summarized

22

in App. D. Once again we find a first-order transition to a phase with the same pattern of spontaneous symmetry breaking, except with a different estimate for the value of $\tilde{\beta}_c$.

B.   The W mass

In Sec. III we argued that, much like in the adjoint-Higgs theory [7, 8], the spontaneous symmetry breaking pattern $SU(2)_R \to U(1)_R$ in the reduced model will cause the W fields to acquire a mass in the full theory. Using the results from the previous subsection, we can now check this within the mean-field approximation. To this end, we restore the external gauge field $e^{igV_{x,\mu}}$ of Eq. (5.1), and expand the action to order g^2. The ϕ dependence is still through the composite operator $\tilde{A}(\phi)$ of Eq. (5.3), and upon substituting its mean-field value we obtain the quadratic part of the effective action for the gauge fields in the broken phase.

We find that the terms linear in $V_{x,\mu}$ vanish, and those quadratic in $V_{x,\mu}$ are given by

$$\frac{1}{\xi} a^2 \sum_x \left(2d(5a^2 - 1) \operatorname{tr}(W^2_{x,\mu}) + \frac{1}{2}(a^2 + 1) \operatorname{tr}\left((\partial^-_\mu W_{x,\mu})^2 \right) \right) , \qquad (5.13)$$

where ∂^-_μ is the backward derivative. In this subsection, $a = a(h_{\min})$ is the mean-field value of the order parameter at the global minimum $h_{\min}$ of the free energy. Indeed, a mass term for the W fields is generated inside the broken phase, where $a \neq 0$. The $U(1)_R$ gauge invariance implies that, if higher orders in g would be included, the derivative ∂^-_μ would change into the derivative D^-_μ covariant with respect to $U(1)_R$. Likewise, we observe that no mass term for the $U(1)_R$ gauge field A_μ is generated.[12]

The field $\tilde{A}(\phi)$ fluctuates around its mean-field value, and to go beyond mean-field, this would have to be taken into account. Some of the fluctuations in $\tilde{A}(\phi)$ correspond to the Goldstone bosons resulting from the symmetry breaking. Other fluctuations, usually referred to as radial ones, may be generated dynamically. In general, it is difficult to systematically investigate all possible fluctuations, because the mean-field approximation does not correspond to the leading order in a systematic expansion in these fluctuations. However, fluctuations of the field $\tilde{A}(\phi)$ that follow from performing an $SU(2)_R$ gauge transformation

---

[12] The role of $U(1)_L$ symmetry is only to turn the field ϕ into an $SU(2)/U(1)$ coset-valued field, as discussed in Sec. II B.

on ϕ can be removed by returning to the vector picture, as explained in Sec. II B. In physical terms, this is how the Goldstone bosons are being "eaten" by the massive gauge fields.

Combining Eq. (5.13) with the transverse kinetic term for the W fields, one finds the W propagator in momentum space

$$\langle W_{\mu,\alpha}(p)W_{\nu,\beta}(q)\rangle = \delta(p+q)\delta_{\alpha\beta}\left[\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2+m_W^2}+\frac{p_\mu p_\nu}{p^2}\frac{\eta}{p^2+\eta m_W^2}\right] , \quad (5.14)$$

with

$$\eta = \frac{2\xi}{a^2(a^2+1)} , \quad (5.15)$$

and where the W mass is given by

$$m_W^2 = 2d(5a^2-1)a^2/\xi . \quad (5.16)$$

The question arises whether or not this is positive in the broken phase. At the first-order phase transition, the order parameter $a = a(h_{\min})$ jumps discontinuously from zero to a different value, as does the global minimum $h_{\min}$ itself. The new value, which is $a = 0.90083$ at the transition, increases to one for increasing $\tilde{\beta}$. Thus, m_W^2 is positive everywhere in the broken phase.[13]

The W mass vanishes in the symmetric phase where $a = 0$. In fact, all of Eq. (5.13) vanishes in that phase. In other words, within our mean-field approximation, an effective potential for the gauge field is generated only in the broken phase, in accordance with the physical picture advocated in Sec. III.

C. **Discussion of the mean-field solution**

The main weakness of our analysis is that a mean-field approach does not provide a controlled approximation. Nevertheless, we believe that the mean-field calculation is a useful exercise, since it gives a concrete description of how symmetry breaking might take place in the equivariantly gauge-fixed theory. This provides guidance for future numerical simulations that are the only way to convincingly determine the phase diagram.

Notwithstanding the basic limitations of the mean-field solution, we believe that it passes a number of consistency checks, which are discussed in the rest of this subsection.

---

[13] The same holds in the mean-field approximation considered in App. D.

As a preparatory step to the mean-field analysis, we have computed the effective action (4.4) to leading non-trivial order in $\tilde{\beta}$. The question arises whether neglecting higher orders is justified. Put differently, the question is when different orders in the $\tilde{\beta}$-expansion might start competing with each other.

In order to address this question, we have calculated the $\overline{C}C$ bound-state propagator within the strong-coupling expansion in the reduced model by employing the techniques developed in Ref. [11]. To lowest order in $\tilde{\beta}$ we find for the bound-state propagator

$$D(p) = \frac{2}{1 + \frac{4\tilde{\beta}}{3}\left(\sum_\mu \cos p_\mu + 2\right)} \ . \tag{5.17}$$

The smallest $\tilde{\beta}$ for which the denominator vanishes is found by setting all the momentum components to $p_\mu = \pi$. This happens at $\tilde{\beta} = 3/8$. We take this value as a rough estimate of the place where successive orders in the $\tilde{\beta}$-expansion become comparable. Since the point $\tilde{\beta} = 3/8$ lies deep inside the broken phase we found in mean field,[14] this supports the (self-)consistency of applying a mean-field approximation to Eq. (4.4).

For technical convenience, we have studied in this section the free energy $f = f(h, a(h))$ as a function of h. Since the saddle-point condition (5.10a) defines a one-to-one mapping between h and a, we may regard the free energy instead as a function of a, $f = f(h(a), a)$. This function is interpreted as the mean-field value of the effective potential for the order parameter a. Thus, even though the reduced model is a topological field theory, we find that the effective potential has non-trivial minima. Given our findings for the toy model of Sec. III A, this should not come as a surprise. While the mean-field estimate of the effective potential may or may not be correct, the topological nature of the reduced model does not prevent the effective potential from depending non-trivially on $\tilde{\beta}$.

We have assumed that eBRST symmetry is restored in the thermodynamic limit of the equivariantly gauge-fixed theory, in both the symmetric and the broken phases of the reduced model, as well as everywhere in the phase diagram of the full theory. Unlike in the toy model, confirming this result is much more difficult in the field theory case, and goes beyond the scope of the present paper. However, it is important to note that the assumption about eBRST symmetry restoration is not in conflict with anything we know about the broken

---

[14] Both mean-field estimates of the critical value $\tilde{\beta}_c$, that of Sec. V A and that of App. D, are much smaller than 3/8.

phase of the reduced model. We have already noted in Sec. III B, that, first, $\langle \tilde{A} \rangle$ is not an order parameter for eBRST symmetry breaking, and second, that a W mass is compatible with unbroken eBRST, provided that the ghosts acquire a mass equal to the W mass.

In the toy model, we have used BRST invariance of the saddle-point expansion to prove an invariance theorem for $Z_{\pm v}$, from which it follows that $Z_{\pm v} = 1$ to all orders. This can be interpreted as the statement that the effective potential for ϕ vanishes at the non-trivial minima $\phi = \pm v$ of the toy model.

Turning to the reduced model, since we are assuming that eBRST symmetry is not broken spontaneously, the question arises whether the global minimum of the effective potential for the order parameter a should always vanish too. The answer provided by our mean-field solution is negative. As explained above, the effective potential is identified with $f(h(a), a)$, whose global minimum becomes negative in the broken phase. In order to make sure that there is no conflict here, it is useful to think in terms of an effective low-energy lagrangian for the broken phase. Much like the chiral lagrangian of QCD, the effective lagrangian $\mathcal{L}_{\text{eff}}$ will be a non-linear model realizing the symmetry breaking pattern $SU(2) \to U(1)$. In addition, it should have some type of eBRST symmetry $\hat{s}$ which is inherited from the underlying theory, the reduced model. Now, in order for the effective theory to satisfy an invariance theorem, $\mathcal{L}_{\text{eff}}$ would have to be cohomologically exact, namely, it should have the form of $\hat{s}X$ for some X. In reality, there is no reason why this should be true. Indeed the reduced-model version of the mass term (3.10) provides an example of a term which is eBRST invariant, (or, eBRST-closed), and yet it is not eBRST-exact.

VI.   DISCUSSION OF THE PHASE DIAGRAM

In this section we will discuss what the two-dimensional phase diagram in the plane spanned by g and $\tilde{g}$ may look like. We rely on what we have learned from the strong-coupling plus mean-field analysis of the previous sections, augmented by further general considerations. It should be kept in mind that there is no guarantee that the mean-field results are correct, but in this section we will assume that they are.

In the reduced model, $i.e.$, on the $g = 0$ boundary, our main result is that a first-order phase transition occurs going from a symmetric phase at small $\tilde{\beta}$ to a phase in which the global $SU(2)_R$ symmetry breaks spontaneously to $U(1)_R$. We are assuming that eBRST

symmetry is not broken spontaneously in that phase (nor anywhere else in the phase diagram). Some considerations supporting this assumption were presented in Sec. III B and Sec. V C.

Since the effective action to which we applied mean-field techniques was derived in a strong-coupling expansion, our analysis has nothing to say on what happens near $\tilde{g} = 0$ (i.e., $\tilde{\beta} = \infty$). In particular, we do not know whether or not the broken phase extends all the way to $\tilde{g} = 0$. We will return to this point below.

An important consequence of Eq. (5.13) is that the first-order phase transition we found on the $g = 0$ boundary extends into the two-dimensional phase diagram.[15] The symmetric phase of the reduced model at small $\tilde{g}$ is the boundary of the familiar confining phase of the full theory. When, for $g > 0$, the phase-transition line is crossed towards larger $\tilde{\beta}$ the W fields become massive, whereas the A field (the "photon") stays massless. At large distances the W fields decouple, leaving us with an effective abelian theory, which is why we have referred to this phase as a Coulomb phase. We conclude that the two phases we found in the reduced model can be unambiguously differentiated in the full phase diagram as well. Either the theory is confining with a non-vanishing mass gap; or there exists a massless photon.

In order to map out possibilities for the full phase diagram, let us consider the other boundaries, starting with the boundary at $\tilde{g} = 0$. Near this boundary it is convenient to rescale $\mathcal{W} \to g\mathcal{W}$ in Eq. (2.3). The longitudinal kinetic term then has a prefactor $1/\xi = g^2/\tilde{g}^2$, which still goes to infinity when $\tilde{g} \to 0$ at fixed non-zero g. The four-ghost coupling in Eq. (2.3b) goes to zero, while the remaining terms, which are bilinear in the ghost fields, provide the Faddeev–Popov determinant appropriate for a maximal abelian gauge (see for example Ref. [12]). Near the $\tilde{g} = 0$ boundary the theory is thus an $SU(2)$ Yang–Mills theory in a maximal abelian gauge. We thus expect gauge-invariant observables near the $\tilde{g} = 0$ boundary to be the same as in the $SU(2)$ Yang–Mills theory without gauge fixing. In particular, the theory should be confining, and possess a mass gap equal to the lowest glueball mass.

Near the $\tilde{\beta} = 1/\tilde{g}^2 = 0$ boundary the gauge-fixing sector decouples. In order to see this,

---

[15] As explained in Sec. III, for $g > 0$ we take the thermodynamic limit with an eBRST-breaking seed so as to avoid the invariance theorem.

consider the on-shell version of S_{gf}, Eq. (4.1), obtained by integrating over the auxiliary field b. Rescaling the ghost and anti-ghost fields as $C \to C/\sqrt{\tilde{g}}$, $\overline{C} \to \overline{C}/\sqrt{\tilde{g}}$, and taking $\tilde{g} \to \infty$, only the four-ghost term survives. The gauge-fixing sector decouples, and the theory is again in the confining phase.

Finally, we consider the fourth boundary of the phase diagram, the one at $\beta = 0$. Working to leading order in both $\tilde{\beta}$ and β, the effective action for ϕ is obtained by integrating over the link variables $U_{x,\mu}$ in Eq. (4.4). For small β the link variables are randomly distributed, and the integrals reduce to simple group integrals. Carrying out these integrals we find that S_{eff} reduces to a constant. This suggests that there is no phase transition near the $\beta = 0$ boundary, and that the theory is again in the confining phase everywhere near this boundary.

This conclusion is supported by the fact that the compact $U(1)$ lattice theory has a phase transition from a Coulomb phase at weak coupling to a confining phase at strong coupling. Below the symmetry-breaking scale originating from the longitudinal dynamics we have an effective $U(1)$ theory, and thus we should expect a similar phase transition going towards strong (transversal) coupling. Once again, the conclusion is that the Coulomb phase of the full theory does not extend to the $\beta = 0$ boundary.

Putting together the information about all four boundaries, we draw two possible phase diagrams in Fig. 2. What is common to both panels is the existence of a single confining phase embracing the Coulomb phase.

The most interesting question concerns the precise structure of the phase diagram near the gaussian critical point at $(g, \tilde{g}) = (0, 0)$. Assuming that the Coulomb phase, predicted by our mean-field approximation, does indeed exist, the two panels of Fig. 2 show the two possibilities.

The panel on the left shows a scenario in which the broken phase ends at some non-zero $\tilde{g}$ for $g \to 0$. If this is the case, the Coulomb phase is a lattice artifact. A more detailed knowledge of its location and properties is then only important in order to guide numerical simulations of the theory.

The panel on the right shows a more intriguing scenario, where both the confining and the Coulomb phases extend to the asymptotically-free critical point. The nature of the continuum limit then depends on how it is taken: from inside the confining phase, or from inside the Coulomb phase. The first choice will recover the standard $SU(2)$ Yang–Mills con-
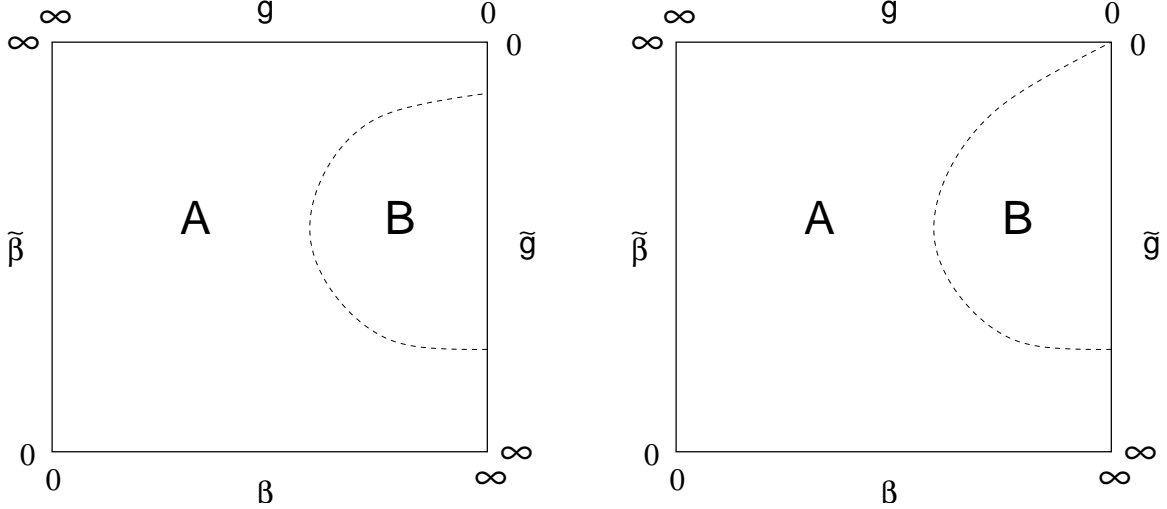
FIG. 2: Two scenarios for the phase diagram. The confining phase A has a mass gap, while the Coulomb (or Higgs) phase B has a massless photon. Left panel: the Coulomb phase ends at some non-zero $\tilde{g}$ for $g \to 0$. Right panel: the Coulomb phase extends to the critical point at $g = \tilde{g} = 0$.

tinuum theory, characterized by confinement and a mass gap. By contrast, if the continuum limit is taken from inside the Coulomb phase, the continuum theory will have massive W gauge bosons and a massless photon, resembling the broken phase of the $SU(2)$ theory with a Higgs field in the adjoint representation [7]. A novel feature of this scenario is that the Higgs-like behavior would emerge in a theory in which all couplings are asymptotically free, and there is no triviality problem.

This speculative scenario raises two main questions. While answering them is outside the scope of this article, we offer our current perspective.

The first question is whether there is any evidence supporting this scenario. In fact, we believe there is. In Ref. [5] we derived the one-loop beta function for $\tilde{g}$, finding that it is asymptotically free just like the gauge coupling g. Of course, the familiar beta function for g does not depend on $\tilde{g}$, whereas the beta function for $\tilde{g}$ does depend on both g and $\tilde{g}$. Integrating the renormalization-group equations simultaneously one encounters dimensional transmutation for both couplings. We will denote the dynamically generated scales by Λ and $\tilde{\Lambda}$, respectively. We choose to define them as the scale where the relevant coupling becomes equal to one, according to the one-loop running. In physical terms, Λ and $\tilde{\Lambda}$ are estimates of the energy scales where the couplings g and $\tilde{g}$ become strong. We remark that the gaussian

critical point at $(g, \tilde{g}) = (0, 0)$ is the unique place in the phase diagram where both of the dynamically generated scales, Λ and $\tilde{\Lambda}$, tend to zero in lattice units.

The g-dependence of the beta function for $\tilde{g}$ turns out to have a non-trivial consequence: When g becomes strong, $\tilde{g}$ has to become strong, too [5]. In other words, we can have $\tilde{\Lambda} \gg \Lambda$ or $\tilde{\Lambda} \sim \Lambda$, but not $\tilde{\Lambda} \ll \Lambda$. The exclusion of the latter option leaves us with just two possibilities. It is thus natural to identify $\tilde{\Lambda} \sim \Lambda$ with the confining phase, and $\tilde{\Lambda} \gg \Lambda$ with the Coulomb phase.

An obvious caveat is that the notion of the Λ parameters is rather elusive, both due to the freedom in selecting a criterion to define them, and since we solve the renormalization-group equations in the one-loop approximation. With this cautionary remark we proceed to discuss the tentative connection to the phase diagram.

When $\tilde{\Lambda} \sim \Lambda$, we would not expect the longitudinal dynamics governed by $\tilde{g}$ to qualitatively alter the dynamics of the transverse degrees of freedom governed by the coupling g. Therefore, if we approach the continuum limit along a trajectory where the bare $\tilde{g}$ is small enough relative to the bare g such that indeed $\tilde{\Lambda} \sim \Lambda$, we expect to be in the confining phase. This is consistent with our discussion of the phase diagram near the boundary $\tilde{g} = 0$. Indeed when initially the bare $\tilde{g}$ is very small, its running is primarily driven by that of g, leading to $\tilde{\Lambda} \sim \Lambda$ [5].

The Coulomb phase would then correspond to choices of bare couplings such that $\tilde{\Lambda} \gg \Lambda$. This hierarchy of scales is natural for the broken-symmetry phase: Already at energies large compared to Λ, where g is still small, the longitudinal sector becomes strongly interacting and, presumably, drives the spontaneous symmetry breaking we have found in mean field.

An important observation is that, while the longitudinal dynamics that drives the symmetry breaking must be studied by non-perturbative methods, the smallness of g allows for a perturbative treatment of the transverse sector. In particular, the running of g is still governed by the one-loop beta function. However, since the W fields have acquired a non-zero mass, a mass-independent scheme would be clearly inappropriate. A physically sensible definition of a running coupling would take the decoupling of the massive W's into account. For instance, above the W mass one can take the running to be defined in a mass-independent scheme in the full $SU(2)$ gauge theory, while below the W mass it makes more sense to define the running of the coupling in the surviving effective abelian theory, with matching of the two couplings at the W mass.

The second question is whether there is any reason to expect that the new continuum limit taken inside the Coulomb phase would respect the rules of quantum mechanics and relativity including, in particular, unitarity; or whether it would just be a curiosity of a statistical system.

At this stage, we can say even less about this second question. We note that at short distance or high energy unitarity can be investigated perturbatively, because our theory is asymptotically free in both couplings. For a detailed discussion we refer to Ref. [1], where we argued that the theory is indeed unitary in perturbation theory. However, this does not probe physics in the infrared, which is where the properties of the two phases are different.

Non-perturbatively, we expect the theory to be unitary in the confining phase, simply because the physics coincides with that of unfixed $SU(2)$ Yang–Mills theory, which is unitary.

It is much harder to address the same question in the Coulomb phase. Within our mean-field approximation, it is encouraging that both terms in Eq. (5.13) are positive everywhere in this phase, leading to a standard massive W propagator, *cf.* Eq. (5.14). According to Refs. [13, 14] this could therefore lead to a unitary effective low-energy theory if it is accompanied by a dynamically-generated Higgs field. What is also likely to be relevant to this question is our expectation that eBRST symmetry remains unbroken in the Coulomb phase. This raises the possibility that just as in a "standard" Higgs model, also in the Coulomb phase eBRST symmetry provides the tool to define a projection onto a physical subspace with a unitary S-matrix.

We will postpone further investigations of these questions to the future. Clearly, first the existence of the Coulomb phase will have to be established more firmly than is possible with the techniques we employed in this article.

VII. CONCLUSION AND OUTLOOK

In this article, we started an investigation of the phase diagram of equivariantly gauge-fixed $SU(2)$ Yang–Mills theory. While originally we considered such theories in the context of a lattice construction of chiral gauge theories, we believe that they are interesting in their own right, even without the addition of any fermion fields. On the one hand, equivariantly gauge-fixed Yang–Mills theories are well-defined non-perturbatively. On the other hand, the transverse and longitudinal gauge couplings, g and $\tilde{g}$, are both asymptotically free, so that

at least one of them is expected to become strong towards the infra-red. With standard BRST symmetry, these two couplings are also asymptotically free in common gauges such as Lorenz gauge or maximal abelian gauge.[16] The key difference is that Yang–Mills theory with standard BRST gauge-fixing action is not defined outside of perturbation theory [2].

Since the goal is to understand this new class of gauge-fixed Yang–Mills theories non-perturbatively, whether one is ultimately interested in the application to chiral gauge theories or not, the first order of business is to explore the phase diagram. Naively, one does not expect the gauge-fixing sector to alter the physics of the unitary sector of the theory, and therefore one might expect the whole phase diagram to consist of a single phase—the usual confining phase. This expectation is amplified by the eBRST-based invariance theorem that was proven in Ref. [1], extended here in App. A, and reviewed in Sec. III.

However, as we also discussed in Sec. III, there is a loophole. A richer phase diagram, with potentially physical consequences, could be uncovered by following a procedure familiar in the study of spontaneous symmetry breaking: in order to evade the invariance theorem, a small breaking of eBRST symmetry is introduced into the finite-volume system, and is turned off after the thermodynamic limit has been taken.

Our central result concerns the phase diagram of the reduced model. The latter corresponds to the $g = 0$ boundary of the phase diagram, and inherits from the gauge-fixed Yang–Mills theory a global symmetry, $SU(2)_R$. We find that the reduced model has a phase in which $SU(2)_R$ is broken down to an abelian symmetry. We discussed circumstantial evidence in support of the assumption that eBRST symmetry is restored in the limit of vanishing seed.

Furthermore, as we move into the $g > 0$ phase diagram, it is very natural for the broken phase of the reduced model to become the boundary of a novel phase of the equivariantly gauge-fixed theory: Two gauge bosons, the W's, "eat" the Goldstone bosons arising from the symmetry breaking, and become massive. The third gauge boson, the "photon," stays massless. Hence the long-distance physics is that of a Coulomb phase. We stress that the transverse sector can be treated perturbatively in g, and so it is hard to avoid this conclusion if indeed the reduced model has the $SU(2)_R \to U(1)_R$ broken phase.

---

[16] The $\tilde{g}$ beta function for standard BRST gauge fixing in maximal abelian gauge [12] is the same as for equivariant BRST [5].

We have discussed the shortcoming of our analysis. The main one is that it involves a mean-field study that does not provide a controlled approximation. Nevertheless, the toy model of Sec. III A teaches us that a topological field theory, such as the reduced model, can have a non-trivial effective potential, and thus, a non-trivial phase diagram. We regard the mean-field calculation as a means to gain insight into what that phase diagram might be. Ultimately, the only reliable method for mapping out the phase diagram is through numerical lattice computations; the results presented in this paper provide guidance for initiating a numerical investigation.

Because of the invariance theorem, we know that turning on an eBRST-breaking seed is a necessary condition for the unveiling of the phase diagram. This is a novel feature. In the adjoint-Higgs model, for example, no "seed" is needed in order to probe the Coulomb phase [8]. It remains an open question precisely how the presence of the seed can alter the long-distance dynamics of the equivariantly gauge-fixed theory. We observe that the invariance theorem is, ultimately, a statement about cancellations among Gribov copies. Standard arguments show that Gribov copies can contribute to the partition function with both signs [2, 15]. Consequently, the measure of the eBRST gauge-fixed theory can be both positive and negative. We conjecture that this fact is relevant for the dynamical role of the eBRST-breaking seed as well. We hope that future numerical studies will shed light on this question.

If the new Coulomb phase predicted by the mean-field analysis truly exists, the most conservative view would be to expect it to be a lattice artifact, disconnected from the continuum limit defined near the gaussian critical point $(g, \tilde{g}) = (0, 0)$. But, based on our earlier work on the one-loop renormalization-group flow near the critical point [5], we pointed out in Sec. VI that another possibility exists: the critical point may lie on the boundary separating the two phases. Which phase is actually realized in the continuum limit would then depend on how this limit is taken in the $(g, \tilde{g})$ plane, as we have described in some detail in Sec. VI.

While the possibility that the new phase is connected to the gaussian critical point is quite speculative, it is also a very exciting scenario. If indeed this happens, and if moreover it could be shown that the continuum limit taken inside the Coulomb phase is unitary, this would provide us with a novel type of theory in which all couplings are asymptotically free, and yet it exhibits the physics of $SU(2) \rightarrow U(1)$ gauge symmetry breaking at low energy.

Clearly, much work remains to be done to establish the existence of a Coulomb phase, and, then, to investigate its properties. The results presented in this article provide us with a framework for setting up a numerical investigation which we hope to report on in future work. In addition, we also plan to investigate whether semi-classical methods can provide us with more insight by compactifying the theory on one or more spacetime directions and reducing the size of the system in those directions. Such an approach could be helpful, in particular, in order to find out whether the Coulomb phase, if it exists, extends to the gaussian fixed point.

We conclude with a few more thoughts on the speculation of a possible continuum limit with massive W's and a massless photon in the equivariantly gauge-fixed $SU(2)$ Yang–Mills theory.

First, since in general gauge fixing is not unique, if such a continuum limit does exist, it appears to open a Pandora's box of possibilities, making our scenario less attractive from the point of view of universality.

In fact, our construction of equivariantly gauge-fixed theories is rather unique. Consider a (continuum) Yang–Mills theory based on some gauge group G, gauge fixed on the coset space G/H, with H a proper subgroup of G. Since the gauge-fixing action is an integral part of the non-perturbative definition of the theory, we demand that it will be Lorentz invariant. Furthermore, it has to provide a kinetic term for the longitudinal component of the coset gauge fields, which in turn should be gauge invariant under the unfixed subgroup H. Together, this implies that the (on-shell) gauge-fixing action should contain the term $\mathrm{tr}\,(D_\mu W_\mu)^2$, with D_μ the covariant derivative with respect to the subgroup H. This uniquely fixes the full gauge-fixing action for the class of theories considered in App. A, where $G = SU(N)$ and H is a maximal subgroup. In this case, the non-perturbative construction of the equivariantly gauge-fixed Yang–Mills theory is thus unique, up to the usual freedom of changing irrelevant couplings on the lattice.

Finally, we comment on the potential relevance for model building. At this point, we do not know whether it is possible to extend the non-perturbative framework to include the case $G = SU(2)_L \times U(1)_Y$, $H = U(1)_{\mathrm{em}}$. What does fit into the framework of App. A is the choice $G = SU(5)$, $H = SU(3) \times SU(2) \times U(1)$. This raises the interesting possibility that an $SU(5)$ Grand Unified Theory with symmetry breaking down to $SU(3)_{\mathrm{color}} \times SU(2)_L \times U(1)_Y$ might exist without the need to introduce a Higgs field.

34

Acknowledgments

Appendix A: Class of equivariantly gauge-fixed Yang–Mills theories

In Ref. [1] we discussed non-abelian theories with gauge group $G = SU(N)$, which are equivariantly gauge fixed to a subgroup H, such that the gauge-fixed theory has both eBRST and anti-eBRST symmetry. In the continuum, the gauge-fixing lagrangian is given by

$$\mathcal{L}_{\text{gf}} = \bar{s}s \operatorname{tr} \left(W^2 + \tilde{g}^2 \overline{C}C \right) , \tag{A1}$$

where W_μ is the restriction of the $SU(N)$ vector potential to the coset, and likewise the ghost fields take values in the coset. On the lattice, the same definition can be used except that one has to provide some transcription of the W^2 term. In Ref. [1] this was done for the case that H is the Cartan subgroup.

Here we construct a lattice gauge-fixing action for all cases where H is a maximal subgroup of $SU(N)$. A maximal subgroup is uniquely defined by introducing a diagonal matrix

$$\tilde{T}_0 = \frac{1}{2} \operatorname{diag}(\underbrace{1,1,\cdots,1}_{N-M \text{ times}} , \underbrace{-1,-1,\cdots,-1}_{M \text{ times}}) . \tag{A2}$$

The maximal subgroup is the subgroup whose generators commute with $\tilde{T}_0$.[17] It is $SU(N-M) \times SU(M) \times U(1)$ for $M > 1$, and $SU(N-1) \times U(1)$ for $M = 1$. The lattice gauge-fixing action is

$$S_{\text{gf}} = \bar{s}s \sum_x \operatorname{tr} \left(-2U_{x,\mu} \tilde{T}_0 U_{x,\mu}^\dagger \tilde{T}_0 + \tilde{g}^2 \overline{C}_x C_x \right) . \tag{A3}$$

---

[17] For $N > 2$, $\tilde{T}_0$ is a linear combination of the generators of the Cartan subgroup and of the identity matrix. The part proportional to the identity matrix is introduced merely for convenience, to obtain the suggestive form in Eq. (A2).

The eBRST transformation rules retain the simple form (2.9), and the anti-eBRST rules again follow via flip symmetry, as discussed in Sec. II A. It can be checked that Eq. (A3) reduces to Eq. (A1) in the classical continuum limit. The proof of the invariance theorem, given in Ref. [1] for the case that H is the Cartan subgroup, generalizes easily to the case at hand by noting that the Cartan subgroup is also a subgroup of any maximal subgroup of $SU(N)$.

Appendix B: Proof of Eq. (3.6a)

In order to calculate Z_0 we rescale $\phi = \lambda\psi$, finding

$$
\begin{aligned}
Z_0 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} d\psi \left(-v^2 + 3\lambda^2\psi^2\right) e^{-b^2 + iv^2 b\psi} e^{-i\lambda^2 b\psi^3} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} db \sum_{n=0}^{\infty} \frac{1}{n!} e^{-b^2} \left(\frac{b\lambda^2}{v^6}\right)^n \left(-v^2 - 3\frac{\lambda^2}{v^4}\frac{\partial^2}{\partial^2 b}\right) \frac{\partial^{3n}}{\partial^{3n} b} \int_{-\infty}^{\infty} d\psi \, e^{iv^2 b\psi} \\
&= \int_{-\infty}^{\infty} db \sum_{n=0}^{\infty} \frac{1}{n!} e^{-b^2} \left(\frac{b\lambda^2}{v^6}\right)^n \left(-v^2 - 3\frac{\lambda^2}{v^4}\frac{\partial^2}{\partial^2 b}\right) \frac{\partial^{3n}}{\partial^{3n} b} \delta(v^2 b) \ .
\end{aligned}
\tag{B1}
$$

In the transition from the first to the second line we have traded ψ in the interaction term $\exp(-i\lambda^2 b\psi^3)$, as well as in the "measure" term $3\lambda^2\psi^2$, with a derivative with respect to b. The integral over b in the last line is evaluated by integration by parts, using

$$
\left.\frac{\partial^m}{\partial^m b}\left(b^n e^{-b^2}\right)\right|_{b=0} = (-1)^{\frac{m-n}{2}} \frac{m!}{\left(\frac{m-n}{2}\right)!} \ ,
\tag{B2}
$$

which is true for even $m - n \geq 0$ (otherwise the result is zero). We find

$$
Z_0 = -\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^2} \left(\frac{\lambda^2}{v^6}\right)^n + \sum_{n=0}^{\infty} \frac{(3(n+1))!}{((n+1)!)^2} \left(\frac{\lambda^2}{v^6}\right)^{n+1} = -1 \ ,
\tag{B3}
$$

where the first and second sums come from the terms with $3n$ and $3n+2$ derivatives respectively. The final result comes from the $n = 0$ term in the first sum. For all higher orders in λ there is a cancellation between the terms in the first and second sum. This proves Eq. (3.6a) to all orders in λ.

36

Appendix C: Group integrals

In this Appendix, we collect a few technical details about the calculation of the integrals in Eqs. (5.7) and (D2). First, consider Eq. (5.7). Parametrizing the $SU(2)$ matrix

$$\phi = x_0 + ix_i\tau_i , \qquad x^2 \equiv x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 , \tag{C1}$$

Eq. (5.7) takes the form

$$\frac{1}{\pi^2} \int d^4x \, \delta(x^2 - 1) \, e^{h(x_0^2 + x_3^2 - x_1^2 - x_2^2)} . \tag{C2}$$

We introduce new variables

$$x_0 = \sqrt{u} \cos\chi , \qquad x_3 = \sqrt{u} \sin\chi , \tag{C3}$$
$$x_1 = \sqrt{v} \cos\psi , \qquad x_2 = \sqrt{v} \sin\psi ,$$

which transforms the integral into

$$\int_0^1 du \int_0^1 dv \, \delta(u + v - 1) \, e^{h(u-v)} , \tag{C4}$$

where the boundaries are a consequence of the delta function. This integral is easily calculated, and yields the result (5.8).

Next, we also need the integral in Eq. (D2). In order to calculate this integral, we first simplify the form of the 2×2 complex matrix H. Using that $H^\dagger H \geq 0$, we can write

$$H^\dagger H = \Lambda^2 , \qquad \Lambda = R \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R^\dagger , \tag{C5}$$

with real $\lambda_{1,2} \geq 0$, and $R \in SU(2)$. Therefore, H can be written as

$$H = UR \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R^\dagger , \tag{C6}$$

with U unitary. Since the Haar measure is both left- and right-invariant, we can drop $R^\dagger$ on the right and the $SU(2)$ part of UR on the left, so that

$$H \rightarrow e^{i\omega} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \equiv \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \tag{C7}$$

and Eq. (D2) simplifies to

$$e^{u(H)} = \frac{1}{\pi^2} \int d^4x \, \delta(x^2 - 1) \, e^{(z+z^*)x_0 + i(z-z^*)x_3} \ , \tag{C8}$$

with $z = (z_1^* + z_2)/2$, and where we parametrized ϕ as in Eq. (C1). Using polar coordinates in four dimensions, with $x_0 = r \cos \eta$ and $x_3 = r \cos \eta \sin \chi$, and performing the integral over all variables except η, the integral becomes

$$\begin{aligned} e^{u(H)} &= \frac{1}{i\pi y} \int_0^\pi d\eta \, \sin \eta \, e^{2x \cos \eta} \, \sin(2iy \sin \eta) \\ &= \frac{1}{\pi x} \int_0^\pi d\eta \, \cos \eta \, e^{2x \cos \eta} \, \cos(2iy \sin \eta) \\ &= \frac{1}{4\pi x} \frac{\partial}{\partial x} \int_0^{2\pi} d\eta \, e^{2x \cos \eta} \, \cosh(2y \sin \eta) \ , \end{aligned} \tag{C9}$$

where we wrote $z = x + iy$. The second line follows from an integration by parts, and in the last line we used that the integral over the interval $[0, \pi]$ is equal to that over the interval $[\pi, 2\pi]$. Finally, switching to planar polar coordinates for x and y and using periodicity of the integral over η, we find that

$$e^{u(H)} = \frac{1}{2h} \frac{\partial}{\partial h} I_0(2h) \Big|_{h=\sqrt{x^2+y^2}} = \frac{1}{h} I_1(2h) \ . \tag{C10}$$

Tracing back, we may identify h with a combination of invariants of the matrix H. Writing

$$H = \sum_\mu (h_\mu + ig_\mu)\tau_\mu \ , \qquad \tau_\mu = (1, i\tau_i) \ , \tag{C11}$$

with h_μ and g_μ real, the combination $x^2 + y^2$ in Eq. (C10) can be written as

$$x^2 + y^2 = \frac{1}{4} \left(\text{tr}\,(H^\dagger H) + 2\,\text{Re}\,\det H \right) = \sum_\mu h_\mu^2 \equiv h^2 \ . \tag{C12}$$

Appendix D: Alternative mean field analysis

Instead of the composite field of Eq. (5.3), we may introduce a mean field for the field ϕ. Analogous to Eq. (5.5), we write

$$\begin{aligned} Z &= \int \prod_x d_H \phi_x \, e^{-S_{\text{eff}}(\phi)} \\ &= \int \prod_x d_H \phi_x \, e^{-S_{\text{eff}}(\phi)} \prod_x \int dV_x \int_{-i\infty}^{i\infty} dH_x \, e^{-\sum_x \text{Re}\,\text{tr}\,(H_x^\dagger (V_x - \phi_x))} \\ &= \prod_x \int dV_x \int dH_x \, e^{-\sum_x \left(\text{Re}\,\text{tr}\,(H_x^\dagger V_x) + S_{\text{eff}}(V_x) - u(H_x) \right)} \ , \end{aligned} \tag{D1}$$

with

$$e^{u(H)} = \int d_H \phi \, e^{\mathrm{Re\,tr}\,(H^\dagger \phi)} \ . \tag{D2}$$

This integral is calculated in App. C, and the result is given in Eq. (C10). As in Sec. V, the mean-field approximation is again obtained by taking the fields H and V constant,[18] and evaluating Eq. (D1) in the saddle-point approximation, which corresponds to minimizing the free energy density

$$f(H,V) = \mathrm{Re\,tr}\,(H^\dagger V) + s_{\mathrm{eff}}(V) - u(H) \ . \tag{D3}$$

The action term in f is obtained as follows. Upon making the replacement (5.1) in Eq. (4.4), one encounters the combination $\phi_x \phi_x^\dagger$ at several places, for which we will always substitute the unit matrix. In all the remaining occurrences we then make the replacement $\phi \to V$. We find that $s_{\mathrm{eff}}(V)$ is given by Eq. (5.11), where now $A = V^\dagger \tau_3 V$.

Using Eq. (C11) and, likewise, parametrizing

$$V = \sum_\mu (v_\mu + i w_\mu)\tau_\mu \ , \tag{D4}$$

with v_μ and w_μ real, Eq. (D3) leads to the saddle-point equations

$$v_\mu = \frac{1}{2}\frac{\partial u(h)}{\partial h_\mu} = \frac{1}{2}\frac{h_\mu}{h}\frac{\partial u(h)}{\partial h} \ , \qquad w_\mu = \frac{1}{2}\frac{\partial u(h)}{\partial g_\mu} = 0 \ , \tag{D5}$$

$$h_\mu = -\frac{1}{2}\frac{\partial s_{\mathrm{eff}}(V)}{\partial v_\mu} \ , \qquad g_\mu = -\frac{1}{2}\frac{\partial s_{\mathrm{eff}}(V)}{\partial w_\mu} \ ,$$

where we used that $u(H)$ is independent of g_μ, cf. Eq. (C10). If we set $w_\mu = 0$ in Eq. (D3), the free energy becomes independent of g_μ, and we can thus set $g_\mu = 0$ as well.

The first equation of Eq. (D5) shows that the direction of v_μ is the same as that of h_μ. Multiplying both sides of this equation by τ_μ, and writing $h_\mu \tau_\mu = hU_h$ and $v_\mu \tau_\mu = vU_v$ where the $U_{h,v}$ are unitary, it follows that $U_h = U_v$, and we end up with the simpler equations

$$v = \frac{1}{2}\frac{\partial u(h)}{\partial h} = \frac{I_2(2h)}{I_1(2h)} \ , \qquad h = -\frac{1}{2}\frac{\partial s_{\mathrm{eff}}(v)}{\partial v} \ . \tag{D6}$$

We have used that s_{eff} is independent of U_v thanks of the $SU(2)_R$ invariance of the reduced model. Note that for $h \to \infty$, $v(h) \to 1$ from below, increasing monotonically from $v = 0$ at $h = 0$, reflecting the fact that the original field ϕ_x is compact.

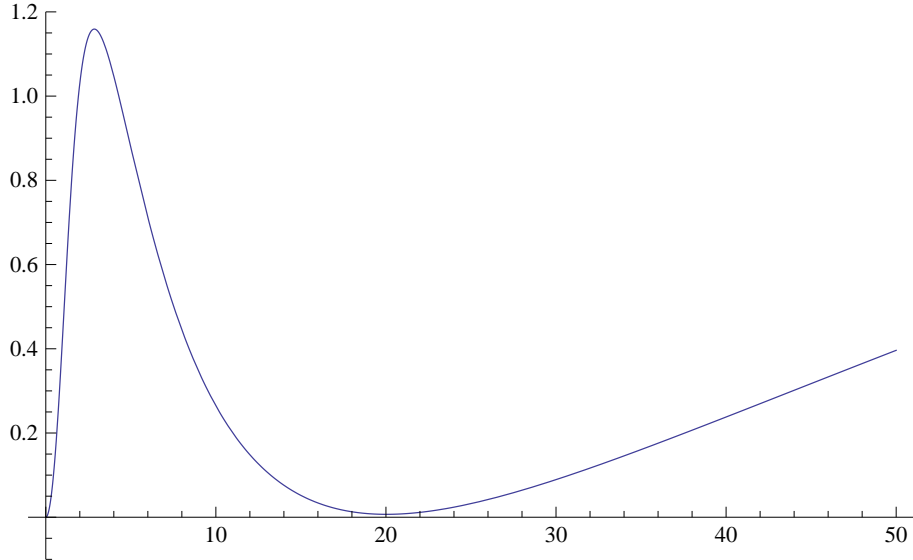[18] In this Appendix, V always denotes the constant mean field.

FIG. 3: Free energy (D3) as a function of h, with $d = 4$ and $\tilde{\beta} = 0.09085$.

We show the free energy $f(h, v(h))$ as a function of h in Fig. 3. Just as in Sec. V A, there is a first-order transition at a critical value of the coupling $\tilde{\beta} = \tilde{\beta}_c$, which in this case turns out to be $\tilde{\beta}_c = 0.09085$. At the transition, v jumps from zero to 0.963, increasing to one with increasing $\tilde{\beta}$. Again this leads to a positive value for m_W^2 everywhere in the broken phase.

The method we followed in this Appendix violates Elitzur's theorem [6] with respect to the $U(1)_L$ gauge group, under which ϕ_x is not invariant. For a detailed discussion of the recovery of gauge invariance in the presence of a non-zero expectation value for a gauge non-invariant operator, we refer to Ref. [10]. In any event, this does not affect the combination $A = V^\dagger \tau_3 V$, which is invariant under $U(1)_L$. Of course, one can also resort directly to the mean-field treatment we presented in Sec. V A.

Appendix E: Perturbative equivalence to standard gauge fixing

In this Appendix we will prove that an equivariantly gauge-fixed Yang–Mills theory [1] is equivalent to a Yang–Mills theory in a standard Lorenz gauge at the level of weak-coupling perturbation theory. With "equivalent" we mean that all correlation functions of gauge-invariant operators are the same between the two theories. Of course, since Yang–Mills

40

theory in standard Lorenz gauge is not well defined outside perturbation theory [2], here the equivalence is necessarily restricted to perturbation theory. This is sufficient for our purpose, which is to prove that the mass term of Eq. (3.10) is not generated in perturbation theory, despite the fact that it is invariant under (on-shell) eBRST symmetry when H is a maximal subgroup of $G = SU(N)$. This corollary is confirmed by an explicit calculation showing that no ghost mass is generated at one loop [16].

Using the lattice as a regulator, we start from the action

$$S = S_{\text{gauge}}(U) + S_{G/H}(U, C, \overline{C}, b) , \tag{E1}$$

in which $S_{\text{gauge}}(U)$ is a lattice discretization of $\frac{1}{2g^2} \int d^4x \, \text{tr} \, (F_{\mu\nu}^2)$, and $S_{G/H}$ is a discretization of the eBRST gauge-fixing action (A1).[19] In the Higgs picture, the action takes the form

$$S = S_{\text{gauge}}(U) + S_{G/H}(U^\phi, C, \overline{C}, b) , \tag{E2}$$

where $U_{x,\mu}^\phi = \phi_x U_{x,\mu} \phi_{x+\mu}^\dagger$ (cf. Eq. (2.16)). We recall that in this picture, $U_{x,\mu}$ is invariant under eBRST transformations, while ϕ_x transforms as given in Eq. (2.19). The local symmetry of the Higgs picture is $H_L \times G_R$. The transformation rules are given by Eq. (2.20) generalized to $g_x \in G_R$ and $h_x \in H_L$, where we have added the the subscripts R, L to indicate from which side the transformation acts on the ϕ_x field.

The Higgs picture has an extra copy G_R of the original local gauge group G, which we will gauge-fix to a standard Lorenz gauge by adding yet another gauge-fixing term. Perturbation theory in the Higgs picture is thus developed from the action

$$S = S_{\text{gauge}}(U) + S_{G/H}(U^\phi, C, \overline{C}, b) + S_G(U, \zeta, \overline{\zeta}, \eta) , \tag{E3}$$

in which the coset-valued field is expanded as

$$\phi_x = \exp\left(i\sqrt{\xi} g \, \theta_{x,\alpha} T_\alpha\right) . \tag{E4}$$

The restriction of the index α to the coset generators eliminates the local H_L invariance. In Eq. (E3), ζ and $\overline{\zeta}$ are new ghost and anti-ghost fields, and η a new auxiliary field, while

$$S_G = s_G \mathcal{K}_G(U, \zeta, \overline{\zeta}, \eta) , \tag{E5}$$

---

[19] For $G = SU(N)$ and H a maximal subgroup, a discretization that is valid non-perturbatively is given in App. A. Here, however, we are only interested in perturbation theory, and so any discretization with the correct classical continuum limit will do.

where s_G is a standard BRST transformation defined for the gauge group G_R in terms of the new ghost and auxiliary fields. Furthermore, we require that s annihilates ζ, $\overline{\zeta}$ and η, and that s_G annihilates C, $\overline{C}$ and b. It follows that $sS_G = 0$, because U is invariant under s in the Higgs picture, *cf.* Eqs. (2.19) and (2.20). In addition, $s_G S_{G/H} = 0$, because $S_{G/H}$ depends only on the combination U^ϕ.

In Eq. (E4) we have reintroduced the gauge-fixing parameter ξ. Our next step is to examine the dependence of (un-normalized) expectation values on this parameter:

$$\frac{1}{g^2}\frac{d}{d\xi}\langle\mathcal{O}\rangle = \left\langle \mathcal{O}\, s\bar{s}\sum_x \text{tr}\left(\overline{C}C\right)\right\rangle \tag{E6}$$

$$= -\left\langle (s\mathcal{O})\,\bar{s}\sum_x \text{tr}\left(\overline{C}C\right)\right\rangle,$$

where in the last line we used that $sS_G = 0$. It follows that $\langle\mathcal{O}\rangle$ is ξ-independent provided that $s\mathcal{O} = 0$, which is true, in particular, when $\mathcal{O}$ is gauge invariant.

The final step is to observe that, since $\langle\mathcal{O}\rangle$ does not depend on ξ, we may obtain this expectation value order by order in perturbation theory by considering the $\xi \to \infty$ limit. It is easy to see from Eq. (A1) that, in this limit, the fields ϕ, C, $\overline{C}$ and b decouple from the rest. The gauge field is controlled by the action

$$S = S_{\text{gauge}}(U) + S_G(U,\zeta,\overline{\zeta},\eta)\,, \tag{E7}$$

which is recognized as (lattice discretized) Yang–Mills theory in standard Lorenz gauge. The partition function of the decoupled sector containing the fields ϕ, C, $\overline{C}$ and b collapses to a non-zero constant.

We comment that, while the argument based on Eq. (E6) closely resembles a key step of the proof of the invariance theorem [1], the discussion in this appendix is restricted to perturbation theory only. Indeed this is why we may invoke the Lorenz gauge in the first place, and there is no conflict with the inability to define this gauge non-perturbatively [2].

We conclude that expectation values of gauge-invariant operators are equal, order by order in perturbation theory, in the eBRST gauge-fixed theory defined by Eq. (E1) and in the Lorenz gauge-fixed theory defined by Eq. (E7). In the latter theory, it is well known that no mass term is generated perturbatively. But, had a mass term for (some of) the gluons been generated in the eBRST gauge-fixed theory (E1), evidently this would have altered

gauge-invariant correlation functions.[20] As a corollary, it follows that the mass term (3.10) cannot be generated perturbatively in the theory defined by Eq. (E1) either.

---

[1] M. Golterman and Y. Shamir, *SU(N) chiral gauge theories on the lattice,* Phys. Rev. D **70**, 094506 (2004) [hep-lat/0404011].

[2] H. Neuberger, *Nonperturbative BRS invariance and the Gribov problem,* Phys. Lett. B **183**, 337 (1987).

[3] M. Schaden, *Equivariant gauge fixing of SU(2) lattice gauge theory,* Phys. Rev. D **59**, 014508 (1999) [hep-lat/9805020].

[4] M. Testa, *Lattice gauge fixing, Gribov copies and BRST symmetry,* Phys. Lett. B **429**, 349 (1998) [hep-lat/9803025].

[5] M. Golterman and Y. Shamir, *Running couplings in equivariantly gauge-fixed SU(N) Yang-Mills theories,* Phys. Rev. D **73**, 014510 (2006) [hep-lat/0511042].

[6] S. Elitzur, *Impossibility of spontaneously breaking local symmetries,* Phys. Rev. D **12**, 3978 (1975).

[7] H. Georgi and S. L. Glashow, *Unified weak and electromagnetic interactions without neutral currents,* Phys. Rev. Lett. **28**, 1494 (1972).

[8] I-H. Lee and J. Shigemitsu, *Spectrum calculations in the lattice Georgi-glashow model,* Nucl. Phys. B **263**, 280 (1986).

[9] M. Schaden, *Mass generation in continuum SU(2) gauge theory in covariant abelian gauges,* hep-th/9909011.

[10] For a review, see for instance J.-M. Drouffe and J.-B. Zuber, *Strong coupling and mean-field methods in lattice gauge theories,* Phys. Rept. **102**, 1 (1983).

[11] E. Eichten and J. Preskill, *Chiral gauge theories on the lattice,* Nucl. Phys. B **268**, 179 (1986); M. F. L. Golterman, D. N. Petcher and E. Rivas, *Absence of chiral fermions in the Eichten-Preskill model,* Nucl. Phys. B **395**, 596 (1993) [hep-lat/9206010].

[12] D. Dudal, J. A. Gracey, V. E. R. Lemes, M. S. Sarandy, R. F. Sobreiro, S. P. Sorella and

---

[20] For example, the long-range behavior of $\left\langle F^2_{\mu\nu}(x) F^2_{\lambda\rho}(y) \right\rangle$ is sensitive to the number of massless gluons.

H. Verschelde, *An Analytic study of the off-diagonal mass generation for Yang-Mills theories in the maximal Abelian gauge,* Phys. Rev. D **70**, 114038 (2004) [hep-th/0406132]. M. A. L. Capri, D. Dudal, J. A. Gracey, S. P. Sorella and H. Verschelde, *Nonperturbative ghost dynamics in the maximal Abelian gauge,* JHEP **0801**, 006 (2008) [arXiv:0708.4303 [hep-th]].

[13] J.M. Cornwall, D.N. Levin and G. Tiktopoulos, *Derivation of gauge invariance from high-energy unitarity bounds on the S-matrix,* Phys. Rev. D **10**, 1145 (1974) [Erratum-ibid. D **11**, 972 (1975)].

[14] B.W. Lee, C. Quigg and H.B. Thacker, *Weak interactions at very high energies: the role of the Higgs boson mass,* Phys. Rev. D **16**, 1519 (1977);
The strength of weak interactions at very high energies and the Higgs boson mass, Phys. Rev. Lett. **38**, 883 (1977).

[15] P. Hirschfeld, *Strong Evidence That Gribov Copying Does Not Affect Gauge Theory Functional Integral,* Nucl. Phys. B **157**, 37 (1979).

[16] M. Golterman and L. Zimmerman, *Masslessness of ghosts in equivariantly gauge-fixed Yang-Mills theories,* Phys. Rev. D **71**, 117502 (2005) [hep-lat/0504023].