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Properties of a Discrete Quantum Field Theory

Micheal S. Berger^{*} and Naoki Yamatsu[†]

Physics Department, Indiana University, Bloomington, IN 47405, U.S.A.

A scalar quantum field theory defined on a discrete spatial coordinate is examined. The renormalization of the lattice propagator is discussed with an emphasis on the periodic nature of the associated momentum coordinate. The analytic properties of the scattering amplitudes indicate the development of a second branch point on which the branch cut from the optical theorem terminates.

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I. INTRODUCTION

Quantum mechanics can be and has been formulated on compact manifolds. The fact that the momenta becomes quantized is well-known. The usual Fourier transform which connects the position space of the compact manifold to momentum space becomes a discrete Fourier transform. Topological features can arise when quantizing a theory on a manifold. Some of the interesting global aspects arise even in the simple case where the configurations space is a circle (S^1) and the phase space is its cotangent bundle. See Ref. [1] for a review and discussion of the mathematical details. Since the phase space is a symplectic manifold, one can also interpret a quantum theory where the roles of the configuration space and the momentum space are reversed. While the interpretation of the theory as the quantization of a classical system may be absent, the theory is nevertheless a welldefined quantum system. One straightforward technique for producing a well-defined quantum mechanics on a discrete configuration space is to consider momentum space as being compact, in which case it is the position space that becomes discrete[2]. If one properly defines a Hamiltonian on this discrete space, one can represent various dynamics and one can produce in the continuum limit (where the discretization becomes small) various continuum Hamiltonians (for example the free particle)[3]. For wave packets localized in momentum space the topology has only a small effect on the dynamics and one obtains the usual time evolution in the continuum limit.

One can also consider the case where both position space and momentum space are discrete. In fact this may be the more familiar case to most physicists. When the compact manifolds are taken to be (discrete) circles, then the phase operators which replace the usual position and momentum operators obey the Weyl algebra[4]. The study of the algebra is extensive in physics.

These ideas can be extended to quantum field theory. Indeed the idea of quantum field theory defined on a configuration space which is a compact manifold has a rich history. In Kaluza-Klein theories the spacetime symmetries of extra dimensions can be seen as internal symmetries in the four-dimensional theory. Compactified dimensions have played a major role in recent years in string theory and later in particle physics phenomenology where possible extra dimensions can be rendered unobservable at low energies by making them sufficiently small or by making the metric describing them highly warped. Many studies assume that each extra dimension is topologically a circle to make the analysis tractable. Some of the physical effects of these extra dimensions can be accounted for by modifying the propagator to include corrections arising from the possibility of winding around the compact direction [5]. One can also define a quantum field on discrete spacetime coordinates. This approach, known as lattice gauge theory, can be formulated as one with the momentum space, as the dual to the configuration space, being a compact space. Lattice physicists usually describe this feature as integration over a Brillouin zone, but here we would like to emphasize the interpretation in terms of a compactification of momentum space. One then obtains a quantum field theory defined on discrete spacetime coordinates (or lattice), and again some of the physical effects are represented by a modified propagator. In lattice gauge theory the discretization is a technical device invented to obtain approximate results which become increasingly accurate as one approaches the continuum limit.

In this paper we investigate a lattice propagator with one discretized dimension in configuration space. We investigate the renormalization of the theory, calculate some results at the one-loop level, and analyze the analytic properties of scattering amplitudes. For this purpose it is necessary and interesting to consider momentum scales which correspond to distance scales much *smaller* than the lattice spacing. We consider a scalar field theory for simplicity. Since we assume weak coupling the calculations here are most similar to those which are performed in lattice gauge theory to understand the continuum limit[6].

Attempts to formulate theories of quantum gravity often contain the idea of a minimal length either explicitly or implicitly. In some models spacetime is formulated directly in terms of some quantum degrees of freedom whose characteristic size is given by the Planck scale. In string theory the physical extent of the string means that experimentally one cannot in principle probe length

 $^{{}^*}Electronic \ address: \ \tt berger@indiana.edu$

[†]Electronic address: nyamatsu@indiana.edu

scales less than the Planck scale in a scattering experiment. These physical requirements may serve as motivation for studying at theories which have an ultraviolet cutoff (say from some kind of discretization), but nevertheless admit practical calculations.

II. SCALAR QUANTUM FIELD THEORY ON A DISCRETE CONFIGURATION SPACE

We will consider a real scalar field with a momentum space defined as $M^3 \times S^1$ with coordinates (p^0, p^1, p^2, p^3) . This corresponds to defining a Hamiltonian on the configuration space with one dimension having an equal and discrete interval spacing,

$$H = \int \frac{dxdy}{(2\pi)^2} \sum_{n} \left[\Pi(x, y, z_n)^2 + \left(\frac{\partial \Phi(x, y, z_n)}{\partial x} \right)^2 + \left(\frac{\partial \Phi(x, y, z_n)}{\partial y} \right)^2 + \left(\frac{\Phi(x, y, z_{n+1}) - \Phi(x, y, z_n)}{z_{n+1} - z_n} \right)^2 + m^2 \Phi(x, y, z_n)^2 + \lambda \Phi(x, y, z_n)^4 \right],$$
(1)

where $\ell \equiv z_{n+1} - z_n$. The requirement of equal spacing is not required, but makes the calculations that follow tractable and the usual propagator description applicable. Also unrequired is the assumption that the discretized derivative involves only adjacent lattice sites. The position space fields $\Phi(x, y, z_n)$ and $\Pi(x, y, z_n)$ can be written by the momentum space fields $\varphi(p)$ and $\pi(p)$

$$\Phi(x, y, z_n) = \int \frac{d^3 p}{(2\pi)^3} \varphi(p) e^{i(p_1 x + p_2 y + p_3 z_n)}, \quad (2)$$

$$\Pi(x, y, z_n) = \int \frac{d^3p}{(2\pi)^3} \pi(p) e^{i(p_1 x + p_2 y + p_3 z_n)}.$$
 (3)

Substituting these relation to the Hamiltonian, we obtain the Hamiltonian on momentum spaces with one compactified momentum space,

$$H = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \bigg[\pi(p)\pi(-p) + \left\{ p_1^2 + p_2^2 + \frac{4}{\ell^2} \sin^2\left(\frac{\ell p_3}{2}\right) + m^2 \right\} \varphi(p)\varphi(-p) \bigg].$$
(4)

not including the interaction. We emphasize that the p_3 integration is over a circle of circumference $2\pi/\ell$. This is performed in a frame we call the preferred frame which is also the one in which the Hamiltonian is defined. If one boosts in the 3-direction the standard Lorentz contraction will cause the separation between the adjacent points to decrease. The preferred frame therefore corresponds to the one where the points are defined to be separated by the largest distance at equal times. In momentum space the integration corresponds to one where the cylinder is parameterized such that the p_3 direction

is defined in this preferred frame. Clearly the generalization that the momentum space manifold has to be treated with open patches for the integration. Here we avoid this technical problem by staying in the preferred frame at all times during the calculation.

The Hamiltonian leads to the free propagator

$$\tilde{D}(p) = \frac{i}{p_0^2 - p_1^2 - p_2^2 - \frac{4}{\ell^2}\sin^2\left(\frac{\ell p_3}{2}\right) - m^2}.$$
 (5)

We also make sure that this gives the ordinary propagator on four dimensional Minkowski space for $\ell \to 0$

$$\tilde{D}(p) = \frac{i}{p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2} = \frac{i}{p^2 - m^2}.$$
 (6)

The propagator in Eq. (5) is properly defined as periodic for a p_3 coordinate defined on a circle.

The discretization of the spatial coordinate can also be thought of as a means of regularization of the ultraviolet divergences. For the case $M^3 \times S^1$ the effect is to replace divergences in a four-dimensional theory with those of a three-dimensional one. So by the standard power counting arguments, the quadratic divergence of the scalar self-energy is softened to a logarithmic divergence and so on. Just as the regulated divergences cancel in physical observables in a renormalized field theory, low energy cross sections and other measurables are independent of the scale ℓ to leading order. The ℓ -dependent corrections are, however, calculable, and here represent small effects of the physical scale of discretization.

One can of course proceed to discretize the other spatial directions and even the time direction as is usually done in lattice theory. We restrict our attention to the case of one discrete coordinate because it is sufficient to highlight the features we want to demonstrate.

The appearance of the sine function in the propagator is a consequence of assuming only nearest neighbor interactions in the 3-direction. We can consider more generally any periodic function $f(p_3^2)$ instead and relate the propagator to an effective Hamiltonian. This discussion is sufficient to consider only the one discrete spatial dimension, and we can formally write

$$H = \int \frac{dp}{2\pi} \left[\pi^2 + f(p)\varphi^2 \right]. \tag{7}$$

We shall examine the constraints on this function f(p). First since f(p) satisfies a periodic condition $p \to p + 2\pi/\ell$, we can write

$$f(p) = f(e^{iqp\ell}),\tag{8}$$

where q is any integer. Second, f(p) is real in order to satisfy the hermiticity of Hamiltonian.

$$f(p) = f\left(\cos\left(qp\ell\right), \sin\left(qp\ell\right)\right). \tag{9}$$

Third, invariance under the parity transformation requires f(p) = f(-p), so that

$$f(p) = f\left(\cos\left(qp\ell\right)\right). \tag{10}$$

Fourth, f(p) is equal to zero for p = 0

$$f(p) = f\left(\sin^2\left(\frac{qp\ell}{2}\right)\right). \tag{11}$$

Finally, f(p) is equal to p^2 when ℓ goes to zero. Then

$$f(p) = \frac{1}{\sum_{q=1} a_q} \sum_{q=1} \frac{4a_q}{(q\ell)^2} \sin^2\left(\frac{qp\ell}{2}\right), \quad (12)$$

where the a_q 's are real number and $1/\sum_{q=1} a_q$ is a normalization factor needed to normalize the factor of p^2 equal to one when ℓ goes to zero.

We can write a general Hamiltonian satisfying the above constraints

$$H = \int \frac{dp}{2\pi} \left[\pi^2 + \frac{1}{\sum_{q=1} a_q} \sum_{q=1} \frac{4a_q}{(q\ell)^2} \sin^2\left(\frac{qp\ell}{2}\right) \varphi^2 + m^2 \varphi^2 \right]$$
$$= \sum_n \left[\Pi^2(z_n) + \frac{1}{\sum_{q=1} a_q} \sum_{q=1} a_q \left(\frac{\Phi(z_{n+q}) - \Phi(z_n)}{z_{n+q} - z_n}\right)^2 + m^2 \Phi(z_n)^2 \right], \tag{13}$$

Therefore the interpretation is that the nearest neighbor coupling in Eq. (1) becomes a Hamiltonian which includes couplings between lattice sites of any separation (represented here by $q\ell$).

III. LOOP EFFECTS

In this section we investigate, using the theory with discretized space, the full propagator and a scattering cross section using Feynman rules derived from Eq. (1). The lattice propagator in Eq. (5) can be understood as a tree level propagator defined for a periodic momentum space, or alternatively it can be expanded in terms of an infinite series of operator insertions in the usual continuum propagator. These insertions can be understood as those contained in field theory containing Lorentz-violating coefficients[7], but we do not pursue this further here. We define $-iM^2(p)$ as the sum of all 1-particle-irreducible(1PI) insertions into the propagator. The full 2-point function is given by the geometric series,

$$G(p) = \frac{i}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2).$$
 (14)

In this expression the four-momentum squared is defined on $M^3 \times S^1$ as

$$p^{2} = p_{0}^{2} - p_{1}^{2} - p_{2}^{2} - \frac{4}{\ell^{2}} \sin^{2}\left(\frac{\ell p_{3}}{2}\right), \qquad (15)$$

This guarantees that the propagator has the correct properties to describe an asymptotic state. The mass-shell condition is expressed as $p^2 = m^2$. Since Lorentz invariance is broken, an interesting consequence is that the wave function renormalization Z of the propagator must be a function of p_3 , so it is not a constant.

We can define renormalization conditions for $M^3 \times S^1$. A subtlety that emerges is that one must specify a renormalization point for the momentum in the 3-direction since the Lorentz symmetry is broken. For $M^3 \times S^1$, we adopt the renormalization conditions that the pole in this full propagator occur at $\tilde{p}^2 = p_0^2 - p_1^2 - p_2^2 = m^2$ and $p_3 = 0$ and have residue 1.

$$M^{2}(p)\big|_{\tilde{p}^{2}=m^{2},p_{3}=0}=0,$$
(16)

$$\left. \frac{d}{d\tilde{p}^2} M^2(p) \right|_{\tilde{p}^2 = m^2, p_3 = 0} = 0, \tag{17}$$

$$\left. \frac{d^n}{d(p_3^2)^n} M^2(p) \right|_{\tilde{p}^2 = m^2, p_3 = 0} = 0, \tag{18}$$

where $n = 1, 2, 3, \cdots$. These conditions guarantee that the when the 1PI graphs are summed to give the full propagator in Eq. (14), this propagator has the form of a free particle as an asymptotic state. An alternative way of expressing this is that the wave function renormalization is a function of p_3^2 . If one expands out in powers of p_3^2 there will be an infinite number of wave function counterterms with one associated with each term in the expansion.

Latter we will evaluate the $2 \rightarrow 2$ scattering amplitude, so we will specify the appropriate renormalization condition here. The renormalized scattering amplitude is usually defined to take a certain value at some kinematic point, such as $s = 4m^2, t = u = 0$, which then determines the renormalized coupling λ . One has the usual kinematic constraint of the Mandelstam variables $s+t+u = 4m^2$ using conservation of 4-momentum. Again with the loss of Lorentz invariance one must impose a renormalization condition which specifies the renormalized coupling in a scattering amplitude oriented in a certain way with respect to the underlying lattice in space. One must also specify the values of the 3-components of the momentum. There are two values that must be specified. For scattering $\phi(p_i)\phi(p'_i) \to \phi(p_f)\phi(p'_f)$, define $p_3^s = (p_1 + p_2)_3^2, p_3^t = (p_1 - p_3)_3^2$, and $p_3^u = (p_1 - p_4)_3^2$. Specification of these quantities for the renormalization condition then properly and unambiguously defines the renormalized coupling. For the 4-point function on $M^3 \times S^1$, we use the following condition

$$iM_4(p)|_{s=4m^2, t=u=0, (p_i+p'_i)_3=(p_i-p_f)_3=(p_i-p'_f)_3=0} = -i\lambda,$$
(19)

where s, t, u are Mandelstam variables defined in the usual way in terms of squared four-momenta. Using momentum conservation the conditions on the 3-components of the momenta is equivalent to the condition $p_{i3} = p'_{i3} = p_{f3} = p'_{f3} = 0$ (as measured in the preferred frame). Since the Lorentz symmetry is broken, a definition of the renormalized λ requires these additional specifications to uniquely define the renormalization point.

We calculate one-loop correction of two point function derived by Φ^4 interaction on $M^3 \times S^1$. Since the Lorentz symmetry is broken there must appear different wave function renormalization constants for the M^3 and the S^1 directions. These can be derived in the standard way from the Lagrangian corresponding to the Hamiltonian defining our theory.

$$-iM^{2}(p) = \frac{-i\lambda}{2} \int_{-\infty}^{+\infty} \frac{d^{3}k}{(2\pi)^{3}} \int_{-\pi/\ell}^{+\pi/\ell} \frac{dk_{3}}{2\pi} \\ \times \frac{i}{k_{0}^{2} - k_{1}^{2} - k_{2}^{2} - \frac{4}{\ell^{2}} \sin^{2}\left(\frac{\ell k_{3}}{2}\right) - m^{2}} \\ + i\left(\tilde{p}^{2}\delta_{Z} + \sum_{n=1}^{\infty} \frac{1}{n!} (p_{3}^{2})^{n} \delta_{Z_{3}n} - \delta_{m}\right), \quad (20)$$

where δ_m , δ_Z and δ_{Z_3n} are counterterms, and the treelevel propagator is

$$\frac{i}{p_0^2 - p_1^2 - p_2^2 - \frac{4}{\ell^2}\sin^2\left(\frac{\ell p_3}{2}\right) - m^2}.$$
 (21)

After some calculation, we obtain

$$-iM^{2}(p) = \frac{i\lambda m}{4\pi^{2}\ell}E\left(-\frac{4}{\ell^{2}m^{2}}\right) + i\left(\tilde{p}^{2}\delta_{Z} + \sum_{n=1}^{\infty}\frac{1}{n!}(p_{3}^{2})^{n}\delta_{Z_{3}n} - \delta_{m}\right)$$
(22)

where E(x) is the elliptic integral of the second kind, and we applied the dimensional regularization scheme and $\overline{\text{MS}}$ for three-dimensional Minkowski spaces. The one loop diagram induced from the $\lambda \Phi^4$ term has no momentum dependence. For $1/\ell \gg m$, we obtain

$$\delta_Z = 0, \tag{23}$$

$$\delta_{Z_3n} = 0, \tag{24}$$

$$\delta_m = \frac{\ell \lambda m}{4\pi^2 \ell} E\left(-\frac{4}{\ell^2 m^2}\right)$$
$$= \frac{\lambda}{32\pi^2} \left[\frac{16}{\ell^2} + m^2 \log\left(\frac{4}{\ell^2 m^2}\right) + m^2 \log 2\right] + O\left(\ell^2\right),$$
(25)

where we used

$$E(-x) = \sqrt{x} + \frac{1}{\sqrt{x}} \left[-\frac{1}{4} \log\left(\frac{1}{x}\right) + \frac{1}{4} \log 2 \right] + O(1/x^{3/2})$$
(26)

for $x \to \infty$. There is no wave function renormalization for either the usual continuum case M^4 or the discrete case $M^3 \times S^1$ at the one-loop level. An interesting feature of the propagator is that the loop corrections require a set of wavefunction renormalization constants δ_{Z_3n} . These constants are required to produce a renormalized propagator which is periodic in p_3 and satisfies the renormalization conditions. Nevertheless the wavefunction renormalization factors δ_{Z_3n} are not independent of each other since they must conspire to produce a renormalized propagator defined on $M^3 \times S^1$ even away from its pole.

As already remarked, at the one-loop level there is no momentum dependence (and thus no wave function renormalization) for the propagator in the ϕ^4 theory. There is nothing interesting to say about its analytic structure. Therefore let us now proceed to calculate fourpoint function on $M^3 \times S^1$. We should first define the renormalization condition as setting λ equal to the magnitude of the scattering amplitude at zero momentum. We can write the amplitude as

$$iM_4 = -i\lambda + (-i\lambda)^2 \left[iV(s; p_3^s) + iV(t; p_3^t) + iV(u; p_3^u) \right] - i\delta_{\lambda},$$
(27)

where

,

$$(-i\lambda)^{2} \cdot iV(p^{2}; p_{3})$$

$$:= \int \frac{d^{4}k}{(2\pi)^{4}} \frac{(-i\lambda)^{2}}{2} \frac{i}{k_{0}^{2} - k_{1}^{2} - k_{2}^{2} - \frac{4}{\ell^{2}} \sin^{2}\left(\frac{\ell k_{3}}{2}\right) - m^{2}}{i}$$

$$\times \frac{i}{(k_{0} + p_{0})^{2} - (k_{1} + p_{1})^{2} - (k_{2} + p_{2})^{2} - \frac{4}{\ell^{2}} \sin^{2}\left(\frac{\ell (k_{3} + p_{3})}{2}\right) - m^{2}}$$
(28)

We wish to examine the analytic properties of this amplitude. It is well known that in the continuum there is a branch cut which represents the dispersive part of the four-point diagram. It accounts for the imaginary part of the amplitude and appears when the value of p^2 is sufficient to create real (not virtual) particle in the loop.

Our intent here is to take seriously the theory defined on a lattice (in one discrete dimension $M^3 \times S^1$) and investigate the branch cut structure of the propagator. Typically lattice physicists are interested in a situation where one is close to the continuum limit ($p^2 \ll 1/\ell^2$). In the continuum a branch point appears at the real particle thresholds, and this branch point must appear in its usual place in the discretized theory. In the continuum the branch cut extending away from the branch point goes to infinity. We wish to demonstrate in this paper that in the discretized theory the branch cut ends on another branch point. This new branch point arises from the fact that the momentum space is a circle.

After we integrate out three dimensional Minkowski space-time M^3 by using dimensional regularization, we obtain the following integral

$$V(\tilde{p}^{2}, p_{3}) = -\frac{1}{32\pi^{2}} \int_{0}^{1} dx \int_{-\pi/\ell}^{+\pi/\ell} dk_{3} \frac{1}{\left[(1-x)\frac{4}{\ell^{2}}\sin^{2}\left(\frac{\ell k_{3}}{2}\right) + x\frac{4}{\ell^{2}}\sin^{2}\left(\frac{\ell (k_{3}+p_{3})}{2}\right) + m^{2} - x(1-x)\tilde{p}^{2}\right]^{1/2}}$$
$$= -\frac{1}{32\pi^{2}} \int_{0}^{1} dx \int_{-\pi/2}^{+\pi/2} dy \frac{2/\ell}{\left[\frac{4}{\ell^{2}}\left\{(1-x)\sin^{2}y + x\sin^{2}(y+z)\right\} + m^{2} - x(1-x)\tilde{p}^{2}\right]^{1/2}},$$
(29)

where $y = \ell k_3/2$, $z = \ell p_3/2$ and $\tilde{p}^2 = p_0^2 - p_1^2 - p_2^2$. For illustration of the fundamental analytic properties of the propagator we choose $\tilde{p}^2 \gg 1/\ell^2$ and $p_3 = 0$. For this subset of cases the theory effectively has no degrees of freedom in the 3-direction. In this limit one expects the theory to behave as one effectively in 2 + 1 dimensions as the lattice spacing is much larger than the inverse momenta. The integral in Eq. (29) becomes

$$\int_{-\pi/2}^{+\pi/2} dy \frac{1}{\left(\tilde{\Delta} + \sin^2 y\right)^{1/2}} = \frac{2}{\sqrt{1 + \frac{1}{\tilde{\Delta}}}\sqrt{\tilde{\Delta}}} K\left(\frac{1}{1 + \tilde{\Delta}}\right)$$
$$= \frac{2}{\sqrt{\tilde{\Delta}}} K\left(-\frac{1}{\tilde{\Delta}}\right), \qquad (30)$$

where K(x) is the elliptic integral of the first kind and the dimensionless quantity $\tilde{\Delta} = \ell^2 \Delta/4 = \ell^2 (m^2 - x(1 - x)\tilde{p}^2)/4.$

The behavior of the elliptic integral $K(1/\bar{\Delta})$ in the complex plane is shown in Fig. 1. There are two branch points. The one at $\tilde{\Delta} = 0$ is the usual branch point associated with the onset of the dispersive behavior of the propagator which survives taking the continuum limit. The other branch point at $\tilde{\Delta} = -1$ corresponds to the onset of dispersive behavior in the dimensionally reduced 2+1 theory where the 3-direction degrees of freedom are frozen out. It is clear that the branch point here (and in more general cases) emerges when the denominator of the integrand in Eq. (30) passes through zero. The appropriate $i\epsilon$ prescription in the propagator (hidden in the treatment above) places one it on one side of the branch cut as usual.

It is worthwhile investigating the nature of the branch point that arises at $\tilde{\Delta} = -1$. The branch point at $\tilde{\Delta} = 0$ occurs for small loop momentum k_3 for which the integrand in Eq. (29) is

$$\frac{1}{\left[\frac{4}{\ell^2}\sin^2\left(\frac{\ell k_3}{2}\right) + m^2 - x(1-x)\tilde{p}^2\right]^{1/2}} \rightarrow \frac{1}{\left[k_3^2 + m^2 - x(1-x)\tilde{p}^2\right]^{1/2}},$$
(31)

which is of the same nature of the branch point in the continuum case. In fact it becomes this branch point when $\ell \to 0$. If one now examines the integral for a four-point diagram in Eq. (29) (again for the case $p_3 = 0$)



FIG. 1: The branch cut of the elliptic integral.

near the point $\tilde{\Delta} = -1$, one has

$$\frac{1}{\left[\frac{4}{\ell^2}\sin^2\left(\frac{\ell k_3}{2}\right) + m^2 - x(1-x)\tilde{p}^2\right]^{1/2}} \rightarrow \frac{1}{\left[-k_3^2 + \left(m^2 + \frac{4}{\ell^2}\right) - x(1-x)\tilde{p}^2\right]^{1/2}}, \quad (32)$$

which has the interpretation as a branch cut with momentum-squared $-k_3^2$ and an effective mass $M^2 = m^2 + 4/\ell^2$. When the continuum limit $\ell \to 0$ is taken this mass approaches infinity. The effect of the minus sign is to reorient the direction of the branch cut so that it extends to the right, and the $i\epsilon$ prescription places the amplitude on one side of the cut. The conclusion here is that if one takes the field theory description to remain valid for all momentum scales rather than being an effective theory, then the branch cut ends at a new branch point.

The new branch point that arises is associated with the existence of a maximum contribution to the energy from the discretized 3-direction[8]

$$E_k^2 = k_1^2 + k_2^2 + \frac{4}{\ell^2} \sin^2\left(\frac{\ell k_3}{2}\right) + m^2, \qquad (33)$$

At this new point the propagator of the theory recovers a Lorentz-type symmetry of the same size as occurs in the continuum limit. One can understand the nature of the new branch point from a more physical point of view. Consider 2 to 2 scattering at tree-level which is related to the discontinuity across the branch cut in the one-loop amplitude. For the case when the 3-components of the final-state momenta are near their maxima, define

$$\ell p_3^3 = \ell \tilde{p}_3^3 + \pi \ell p_4^3 = \ell \tilde{p}_4^3 + \pi , \qquad (34)$$

so that $\ell \tilde{p}_3^3$ and $\ell \tilde{p}_4^3$ are small. By momentum conservation $\ell p_3^3 + \ell p_4^3 = \ell \tilde{p}_3^3 + \ell \tilde{p}_4^3 + 2\pi$ which is physically equivalent to $\ell \tilde{p}_3^3 + \ell \tilde{p}_4^3$. Setting the 1- and 2-components to zero to examine the threshold, one has

$$s = (p_3^0 + p_4^0)^2 - \frac{4}{\ell^2} \sin^2\left(\frac{\ell \tilde{p}_3^3 + \ell \tilde{p}_4^3}{2}\right) , \qquad (35)$$

and

$$m^{2} = (p_{i}^{0})^{2} - \frac{4}{\ell^{2}} \left(1 - \sin^{2} \frac{\ell \tilde{p}_{i}^{3}}{2} \right) \qquad i = 3, 4.$$
 (36)

This indicates that the new branch point at $s = 4(m^2 + 4/\ell^2)$ is associated with the scattering of "large" momenta $\ell p_3^3 \approx \ell p_4^3 \approx \pi$ which by momentum conservation yields a "small" momentum. This is a novel feature of a theory with a maximum momentum and should be a generic one which persists beyond the illustration considered in this paper. It is enforced by a modified version of two-particle phase space.

One can reintroduce nonzero values of p_3 and perform the integral in Eq. (29). An expansion in ℓp_3 yields coefficients which involve elliptic integrals. The amplitude is presented here as an expansion appropriate for small ℓ^2 , i.e. close to the continuum,

$$iM_4 = -i\lambda - \lambda^2 \left[iV(\tilde{s}, p_3^s) + iV(\tilde{t}, p_3^t) + iV(\tilde{s}, p_3^u) \right] - i\delta_\lambda,$$
(37)

where $\tilde{s} = (\tilde{p}_i + \tilde{p}')^2$, etc. The counter term is

$$\delta_{\lambda} = -\lambda \left[V(4m^2, 0) - 2V(0, 0) \right],$$
 (38)

and

$$iM_{4} = -i\lambda + \frac{i\lambda^{2}}{32\pi^{2}} \int_{0}^{1} dx \left[-\log\left(\frac{\Delta_{s}}{\Delta_{4m^{2}}}\right) + \frac{1}{4} \left\{ \tilde{\Delta}_{s} \log\left(\frac{\tilde{\Delta}_{\tilde{s}}}{16}\right) - \tilde{\Delta}_{4m^{2}} \log\left(\frac{\tilde{\Delta}_{4m^{2}}}{16}\right) \right\} \\ -x(1-x)z^{2} \log\left(\frac{\tilde{\Delta}_{\tilde{s}}}{16}\right) + \frac{1}{2} \left(\tilde{\Delta}_{s} - \tilde{\Delta}_{4m^{2}}\right) \\ -\frac{7}{4}x(1-x)z^{2} + \frac{1}{24\tilde{\Delta}_{\tilde{s}}}x(1-x)(8 - 45x + 45x^{2})z^{4} \\ + \frac{1}{24\tilde{\Delta}_{\tilde{s}}^{2}}x^{2}(1-x)^{2}(-8 + 35x - 35x^{2})z^{6} + O(z^{8};\ell^{2}) + (s\leftrightarrow t) + (s\leftrightarrow u) \right] \\ = -i\lambda + \frac{i\lambda^{2}}{32\pi^{2}} \int_{0}^{1} dx \left[-\log\left(\frac{\Delta_{s}}{\Delta_{4m^{2}}}\right) - \log\left(\frac{\Delta_{t}}{m^{2}}\right) - \log\left(\frac{\Delta_{u}}{m^{2}}\right) \\ + \frac{\ell^{2}}{16} \left\{ \Delta_{s} \log \Delta_{\tilde{s}} + \Delta_{t} \log \Delta_{\tilde{t}} + \Delta_{u} \log \Delta_{\tilde{u}} - \Delta_{4m^{2}} \log \Delta_{4m^{2}} - 2m^{2} \log m^{2} \right\} \\ - \frac{\ell^{2}}{4}x(1-x) \left[p_{3}^{s^{2}} \log\left(\frac{\ell^{2}\Delta_{\tilde{s}}}{64}\right) + p_{3}^{t^{2}} \log\left(\frac{\ell^{2}\Delta_{\tilde{t}}}{64}\right) + p_{3}^{u^{2}} \log\left(\frac{\ell^{2}\Delta_{\tilde{u}}}{64}\right) \right] \\ - \frac{7}{16}\ell^{2}x(1-x)(p_{3}^{s^{2}} + p_{3}^{t^{2}} + p_{3}^{u^{2}}) + \frac{1}{96}\ell^{2}x(1-x)(8 - 45x + 45x^{2}) \left(\frac{p_{3}^{s4}}{\Delta_{\tilde{s}}} + \frac{p_{3}^{t4}}{\Delta_{\tilde{t}}} + \frac{p_{3}^{u4}}{\Delta_{\tilde{u}}}\right) \\ + \frac{1}{96}\ell^{2}x^{2}(1-x)^{2}(-8 + 35x - 35x^{2}) \left(\frac{p_{3}^{s6}}{\Delta_{\tilde{s}}^{2}} + \frac{p_{3}^{t6}}{\Delta_{\tilde{u}}^{2}} + \frac{p_{3}^{u6}}{\Delta_{\tilde{u}}^{2}}\right) + O(p_{3}^{8};\ell^{2}) \right].$$
(39)

One could in principle use this expression to obtain corrections to the usual one-loop cross section coming from the nonzero discretization (ℓ) . Rather than pursue this direction (which, if ℓ is taken to be the Planck length to perhaps model some discretization arising from a theory of quantum gravity, gives experimentally small and therefore uninteresting results), we comment on the form of the expression which follows from its analytical properties in the complex plane. The appearance of terms of the form $\ell^2 \Delta \log(\Delta)$ are indicative of the new branch point that exists at $\Delta = -1$ that terminates the usual branch cut in the amplitude that appears in the continNotice that the behavior of the scattering amplitude in Eq. (29) arises precisely because the momentum integral is performed over a circle. The finite extent of the branch cut would survive the inclusion of higher order quantum corrections because the renormalized propagator involves a periodic function of p_3 . The compactification of momentum space introduces another branch point on which the usual branch cut evident in the continuum can terminate. This structure is also independent of how the renormalization conditions are chosen. It should also clear that the existence of two branch points will also be the case if one discretizes more of the spatial directions as they arise when the denominator in the integral for the scattering amplitude passes through zero.

IV. CONCLUSION

We have considered the quantum corrections in a theory with one spatial direction put on a lattice. This coupling is represented by a periodic function in the compactified momentum. This way of viewing the momentum space is dual to the notion of a compactified con-

The propagator and the four-point diagrams in the interacting theory were computed (to one loop). The analytic structure of the scattering amplitudes involves a branch cut which originates in the usual place when considered in terms of the continuum limit. This branch point is associated with the threshold for real particles as required by the optical theorem. However, rather than extending to infinity in momentum space, it terminates at another branch point which results as a direct consequence of the discretization of the spatial direction. The mathematical nature of the lattice corrections to the continuum are dictated in part by the new branch point. A fundamental theory addressing Planck scale physics should account how the branch cut extends into regions which involve Planck scale momenta. In the toy model we considered here, we can employ field theory at all length scales, and (at least in this case) the branch cut terminates at a another branch point.

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