Wigner-Eckart theorem in cosmology: Bispectra for total-angular-momentum waves

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The Wigner-Eckart Theorem in Cosmology: Bispectra For TAM Waves

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Total-angular-momentum (TAM) waves provide a set of basis functions for scalar, vector, and tensor fields that can be used in place of plane waves and that reflect the rotational symmetry of the spherical sky. Here we discuss three-point correlation functions, or bispectra in harmonic space, for scalar, vector, and tensor fields in terms of TAM waves. The Wigner-Eckart theorem dictates that the expectation value, assuming statistical isotropy, of the product of three TAM waves is the product of a Clebsch-Gordan coefficient (or Wigner-3j symbol) times a function only of the total-angular-momentum quantum numbers. Here we show how this works, and we provide explicit expressions relating the bispectra for TAM waves in terms of the more commonly used Fourier-space bispectra. This formalism will be useful to simplify calculations of projections of three-dimensional bispectra onto the spherical sky.

I. INTRODUCTION

Cosmological measurements have over the past several decades made the notion of a period of inflationary expansion in the early Universe particularly appealing. The principal aim of early-Universe cosmology has thus become the elucidation of the new physics responsible for inflation. While the simplest single-field slow-roll (SFSR) inflation models predict primordial perturbations to be very nearly Gaussian [1], they do require some nonvanishing departures from Gaussianity [2]. Moreover, just about any extension of SFSR models, or embedding of toy SFSR models into more realistic models [3–5], or alternatives or additions to inflation [6] lead to larger departures from Gaussianity. There has thus been ever-growing attention focused on the search for non-Gaussianity.

The vast majority of the literature on departures from Gaussianity focuses on the bispectrum (the three-point correlation function in Fourier space) for the primordial curvature, a scalar quantity [7]. However, inflation also predicts primordial gravitational waves (tensor metric perturbations) [8], and some inflationary models involve the introduction of vector fields [9]. There are also discussions of primordial magnetic fields [10]. Just as there may arise non-Gaussianity in the primordial scalar perturbation, there may also be non-Gaussianity in these vector fields, in magnetic fields [11, 12], and in the gravitational-wave background [13–15]. Additional work on non-Gaussianities with vector and scalar fields can be found in Refs. [16].

Non-Gaussianity in the primordial scalar perturbation is most commonly parametrized in terms of a bispectrum, the expectation value for the product of three Fourier modes of wavevectors $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$, of the fields under consideration. If the fields under consideration include vector or tensor fields, then the bispectrum will also depend on some contractions of the polarization vectors/tensors for these fields.

Many cosmological observations, however, are performed on a spherical sky, and many observables (e.g., CMB temperature/polarization) depend only on an angular position on the sky. While others may depend on three-dimensional positions, astronomical measurements discriminate between the two angular coordinates and a radial coordinate. Comparison of measured quantities with theoretical Fourier-space bispectra therefore necessarily require projection of the three-dimensional Fourier-space bispectra onto the two-dimensional spherical sky. The bispectra projected onto the spherical sky depend on the three multipoles l_1, l_2, and l_3, associated with the three spherical-harmonic coefficients being correlated. Such projections necessarily involve the complications associated with spherical harmonics, Clebsch-Gordan coefficients, and Wigner-3j symbols, and sometimes even Wigner-6j and Wigner-9j symbols.

In a recent paper [17], we developed a total-angular-momentum (TAM) formalism to provide sets of basis functions for scalar, vector, and tensor fields in three spatial dimensions that reflect the rotational symmetry of the Universe about any given point (taken to be our location). The purpose of these TAM waves is to incorporate the rotational symmetry of the sky from the start, rather than start with plane waves which are later projected onto the sky. These TAM waves generalize the more familiar tensor spherical harmonics [18–20], which provide a basis for tensor functions on the two-sphere, to a basis for functions in three-dimensional Euclidean space. The TAM waves of Ref. [17] are also provided for scalar and vector fields, and for all five components of a traceless tensor, not just the two transverse components.

Each TAM wave is labeled by a wavenumber k, total-angular-momentum quantum numbers J, and azimuthal quantum number M. There are three sets of TAM waves for each kJM for vector fields (to reflect the three components of a vector field) and five sets of TAM waves for each kJM for a symmetric traceless rank-2 tensor. We provided in Ref. [17] several different sets of bases, for a given JM, for these three vector TAM waves and five tensor TAM waves. In the orbital-angular-momentum (OAM) basis, the three vector basis functions for a given JM have orbital angular momentum $l = J-1, J, J+1$, and the five tensor basis functions have $l = J-2, J-1, J, J+1, J+2$. In the second

basis, the three vector basis functions for a given JM include a longitudinal (L) mode and two transverse modes (E and B) of opposite parity. The five tensor functions in this basis include a longitudinal (L) mode, two vector modes (VE and VB) of opposite parity, and two transverse-tensor modes (TE and TB) of opposite parity. The third basis represents vector fields of a given JM in terms of states of helicity $\lambda = 0, \pm 1$ and tensor fields in terms of modes of helicity $\lambda = 0, \pm 1, \pm 2$. The most general scalar, vector, or tensor field can then be written in terms of these TAM waves, rather than Fourier modes. Spherical-sky observables are then obtained more naturally from these TAM waves than from Fourier waves.

In this paper we calculate the bispectra for TAM waves in terms of the more commonly used Fourier-space bispectra. As we will see, the Wigner-Eckart theorem guarantees that the expectation value of the product of three TAM-wave coefficients depends on M_1, M_2, and M_3 only through the Clebsch-Gordan coefficient (or equivalently Wigner-3j symbol or Gaunt integral); there may then be some dependence on J_1, J_2, and J_3 in a prefactor in addition to that in the Clebsch-Gordan coefficient. The specific form for the prefactor will depend on whether the bispectrum is for scalar, vector, and/or tensor fields as well as on the tensorial nature of the bispectrum (i.e., how the indices on the polarization vector/tensors are contracted). The principal results of this paper are thus the specific forms for these prefactors for a variety of bispectra. These results will facilitate the calculation of observables, such as angular bispectra, for theories that involve such correlations, particularly those that involve vector and/or tensor fields.

We begin in Section II with a brief review of the total-angular-momentum wave (TAM) formalism, and we use the notation and conventions of Ref. [17] throughout. Section III presents the bispectra for three scalar TAM waves. Section IV follows to calculate bispectra involving two transverse-vector TAM waves, in the E/B basis, and one scalar TAM wave. Section V presents bispectra of two transverse-traceless tensor TAM waves plus one scalar TAM wave. Section VI then deals with the correlation of one symmetric traceless tensor TAM wave with two scalar TAM waves, where the traceless tensor can be a transverse-traceless, vectorial, or longitudinal. The same results apply also to correlation of one traceless tensor TAM wave and two longitudinal vector TAM waves. Section VII provides concluding remarks and provides a list of the equation numbers for the central results that relate the Fourier-space and TAM-wave three-point functions. Appendix A presents helicity-basis overlap integrals that are used earlier in the paper. We discuss a semi-classical picture to understand the structure of the TAM-wave bispectra in Appendix B.

II. REVIEW OF TOTAL ANGULAR MOMENTUM WAVES

Here we very briefly re-introduce total-angular-momentum (TAM) waves for scalar, vector, and tensor fields. We follow throughout the convention and notation of Ref. [17] and refer the reader there for more details.

TAM waves are eigenfunctions of eigenvalue $-k^2$ of the Laplacian that are also eigenstates of quantum numbers J and M of total angular momentum squared and its third component. The scalar TAM waves are denoted by $\Psi^k_{(JM)}(\mathbf{x})$. TAM vector waves of orbital angular momentum l are specified by $\Psi^{\prime k}_{(JM)a}(\mathbf{x})$, where $a = \{x, y, z\}$ is the vector index and $l = J-1, J, J+1$. TAM tensor waves of orbital angular momentum l are specified by $\Psi^{\prime k}_{(JM)ab}(\mathbf{x})$, where a and b are the tensor indices and $l = J-2, J-1, J, J+1, J+2$. The scalar, vector, and tensor TAM waves are distinguished by the number of indices.

These TAM waves satisfy the completeness relations,

$$\sum_{JM} \int \frac{k^2\,dk}{(2\pi)^3} \left[4\pi i^J \Psi^k_{(JM)}(\mathbf{x})\right]^* \left[4\pi i^J \Psi^k_{(JM)}(\mathbf{x}')\right] = \delta_D(\mathbf{x}-\mathbf{x}'), \tag{1}$$

$$\sum_{JMl} \int \frac{k^2\,dk}{(2\pi)^3} \left[4\pi i^l \Psi^{l,k}_{(JM)a}(\mathbf{x})\right]^* \left[4\pi i^l \Psi^{l,k}_{(JM)}{}^a(\mathbf{x}')\right] = \delta_D(\mathbf{x}-\mathbf{x}'), \tag{2}$$

$$\sum_{JMl} \int \frac{k^2\,dk}{(2\pi)^3} \left[4\pi i^l \Psi^{l,k}_{(JM)ab}(\mathbf{x})\right]^* \left[4\pi i^l \Psi^{l,k}_{(JM)}{}^{ab}(\mathbf{x}')\right] = \delta_D(\mathbf{x}-\mathbf{x}'). \tag{3}$$

These relations imply that an arbitrary scalar, vector, or tensor field can be expanded, respectively,

$$\phi(\mathbf{x}) = \sum_{JM} \int \frac{k^2\,dk}{(2\pi)^3} \phi^k_{(JM)}\, 4\pi i^J \Psi^k_{(JM)}(\mathbf{x}), \tag{4}$$

$$V_a(\mathbf{x}) = \sum_{JMl} \int \frac{k^2\,dk}{(2\pi)^3} V^{l,k}_{(JM)}\, 4\pi i^l \Psi^{l,k}_{(JM)a}(\mathbf{x}), \tag{5}$$

$$T_{ab}(\mathbf{x}) = \sum_{JMl} \int \frac{k^2\,dk}{(2\pi)^3} T^{l,k}_{(JM)}\, 4\pi i^l \Psi^{l,k}_{(JM)ab}(\mathbf{x}), \tag{6}$$

in terms of TAM waves and expansion coefficients,

$$\phi^k_{(JM)} = \int d^3x\, \phi(\mathbf{x}) \left[4\pi i^J \Psi^k_{(JM)}(\mathbf{x})\right]^*, \tag{7}$$

$$V^{l,k}_{(JM)} = \int d^3x\, V^a(\mathbf{x}) \left[4\pi i^l \Psi^{l,k}_{(JM)a}(\mathbf{x})\right]^*, \tag{8}$$

$$T^{l,k}_{(JM)} = \int d^3x\, T^{ab}(\mathbf{x}) \left[4\pi i^l \Psi^{l,k}_{(JM)a}(\mathbf{x})\right]^*. \tag{9}$$

$$\tag{10}$$

By using the plane-wave expansions,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{JM} 4\pi i^J \left[Y^*_{(JM)}(\hat{\mathbf{k}})\right] \Psi^k_{(JM)}(\mathbf{x}), \tag{11}$$

$$\hat{\varepsilon}_a(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{JMl} 4\pi i^l \left[\hat{\varepsilon}^b(\hat{\mathbf{k}}) Y^{l\,*}_{(JM)b}(\hat{\mathbf{k}})\right] \Psi^{l,k}_{(JM)a}(\mathbf{x}), \tag{12}$$

$$\hat{\varepsilon}_{ab}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{JMl} 4\pi i^l \left[\hat{\varepsilon}^{cd}(\hat{\mathbf{k}}) Y^{l\,*}_{(JM)cd}(\hat{\mathbf{k}})\right] \Psi^{l,k}_{(JM)ab}(\mathbf{x}), \tag{13}$$

and orthogonality of the TAM waves,

$$\int d^3x \left[4\pi i^{J'} \Psi^{k'}_{(J'M')}(\mathbf{x})\right]^* \left[4\pi i^J \Psi^k_{(JM)}(\mathbf{x})\right] = \delta_{JJ'}\delta_{MM'} \frac{(2\pi)^3}{k^2} \delta_D(k-k'), \tag{14}$$

$$\int d^3x \left[4\pi i^{l'} \Psi^{l',k'\,a}_{(J'M')}(\mathbf{x})\right]^* \left[4\pi i^l \Psi^{l,k}_{(JM)a}(\mathbf{x})\right] = \delta_{ll'}\delta_{JJ'}\delta_{MM'} \frac{(2\pi)^3}{k^2} \delta_D(k-k'), \tag{15}$$

$$\int d^3x \left[4\pi i^{l'} \Psi^{l',k'\,ab}_{(J'M')}(\mathbf{x})\right]^* \left[4\pi i^l \Psi^{l,k}_{(JM)ab}(\mathbf{x})\right] = \delta_{ll'}\delta_{JJ'}\delta_{MM'} \frac{(2\pi)^3}{k^2} \delta_D(k-k'), \tag{16}$$

we can also write the expansion coefficients for scalar, vector and tensor waves as

$$\phi^k_{(JM)} = \int d^2\hat{\mathbf{k}}\, \tilde{\phi}(\mathbf{k}) Y^*_{(JM)}(\hat{\mathbf{k}}), \tag{17}$$

$$V^{l,k}_{(JM)} = \int d^2\hat{\mathbf{k}}\, \tilde{V}^a(\mathbf{k}) Y^{l\,*}_{(JM)a}(\hat{\mathbf{k}}), \tag{18}$$

$$T^{l,k}_{(JM)} = \int d^2\hat{\mathbf{k}}\, \tilde{T}^{ab}(\mathbf{k}) Y^{l\,*}_{(JM)ab}(\hat{\mathbf{k}}). \tag{19}$$

Similarly, if we choose to decompose the vector field into $L/E/B$ waves and the tensor fields into $L/VE/VB/TE/TB$ waves, which we label by α, then we also have

$$V^{\alpha,k}_{(JM)} = \int d^3x\, V^a(\mathbf{x}) \left[4\pi i^J \Psi^{\alpha,k}_{(JM)a}(\mathbf{x})\right]^* = \int d^2\hat{\mathbf{k}}\, \tilde{V}^a(\mathbf{k}) Y^{\alpha\,*}_{(JM)a}(\hat{\mathbf{k}}), \tag{20}$$

$$T^{\alpha,k}_{(JM)} = \int d^3x\, T^{ab}(\mathbf{x}) \left[4\pi i^J \Psi^{\alpha,k}_{(JM)a}(\mathbf{x})\right]^* = \int d^2\hat{\mathbf{k}}\, \tilde{T}^{ab}(\mathbf{k}) Y^{\alpha\,*}_{(JM)ab}(\hat{\mathbf{k}}). \tag{21}$$

Finally, we can write the $L/E/B$ vector TAM waves in terms of the $L/E/B$ vector spherical harmonics as

$$\begin{aligned}
\Psi^{B,k}_{(JM)a}(\mathbf{x}) &= j_J(kr) Y^B_{(JM)a}(\hat{\mathbf{n}}), \\
\Psi^{E,k}_{(JM)a}(\mathbf{x}) &= -i\left[j'_J(kr) + \frac{j_J(kr)}{kr}\right] Y^E_{(JM)a}(\hat{\mathbf{n}}) - i\sqrt{J(J+1)}\frac{j_J(kr)}{kr} Y^L_{(JM)a}(\hat{\mathbf{n}}), \\
\Psi^{L,k}_{(JM)a}(\mathbf{x}) &= -i\sqrt{J(J+1)}\frac{j_J(kr)}{kr} Y^E_{(JM)a}(\hat{\mathbf{n}}) - i j'_J(kr) Y^L_{(JM)a}(\hat{\mathbf{n}}),
\end{aligned} \tag{22}$$

and the $L/VE/VB/TE/TB$ tensor TAM waves in terms of the $L/VE/VB/TE/TB$ tensor spherical harmonics as

$$\begin{aligned}
\Psi^{L,k}_{(JM)ab}(\mathbf{x}) =& -\frac{1}{2}\left(j_J(kr)+3j''_J(kr)\right)Y^L_{(JM)ab}(\hat{\mathbf{n}}) - \sqrt{3J(J+1)}f_J(kr)Y^{VE}_{(JM)ab}(\hat{\mathbf{n}}) \\
& -\frac{1}{2}\sqrt{\frac{3(J+2)!}{(J-2)!}}\frac{j_J(kr)}{(kr)^2}Y^{TE}_{(JM)ab}(\hat{\mathbf{n}}), \\
\Psi^{VE,k}_{(JM)ab}(\mathbf{x}) =& -\sqrt{3J(J+1)}f_J(kr)Y^L_{(JM)ab}(\hat{\mathbf{n}}) - \left(j_J(kr)+2j''_J(kr)+2f_J(kr)\right)Y^{VE}_{(JM)ab}(\hat{\mathbf{n}}) \\
& -\sqrt{(J-1)(J+2)}\left(f_J(kr)+2\frac{j_J(kr)}{(kr)^2}\right)Y^{TE}_{(JM)ab}(\hat{\mathbf{n}}), \\
\Psi^{TE,k}_{(JM)ab}(\mathbf{x}) =& -\frac{1}{2}\sqrt{\frac{3(J+2)!}{(J-2)!}}\frac{j_J(kr)}{(kr)^2}Y^L_{(JM)ab}(\hat{\mathbf{n}}) - \sqrt{(J-1)(J+2)}\left(f_J(kr)+2\frac{j_J(kr)}{(kr)^2}\right)Y^{VE}_{(JM)ab}(\hat{\mathbf{n}}) \\
& -\frac{1}{2}\left(-j_J(kr)+j''_J(kr)+4f_J(kr)+6\frac{j_J(kr)}{(kr)^2}\right)Y^{TE}_{(JM)ab}(\hat{\mathbf{n}}), \\
\Psi^{VB,k}_{(JM)ab}(\mathbf{x}) =& -i\left(j'_J(kr)-\frac{j_J(kr)}{kr}\right)Y^{VB}_{(JM)ab}(\hat{\mathbf{n}}) - i\sqrt{(J-1)(J+2)}\frac{j_J(kr)}{kr}Y^{TB}_{(JM)ab}(\hat{\mathbf{n}}), \\
\Psi^{TB,k}_{(JM)ab}(\mathbf{x}) =& -i\sqrt{(J-1)(J+2)}\frac{j_J(kr)}{kr}Y^{VB}_{(JM)ab}(\hat{\mathbf{n}}) - i\left(j'_J(kr)+2\frac{j_J(kr)}{kr}\right)Y^{TB}_{(JM)ab}(\hat{\mathbf{n}}),
\end{aligned} \tag{23}$$

where $f_J(x) = (j_J(x)/x)'$. For later use, we define additional radial functions for vector TAM waves with $\alpha, \beta = E, L$ by

$$\Psi^{\alpha,k}_{(JM)a}(\mathbf{x}) \equiv -i\sum_\beta j^{(\alpha,\beta)}_{J,v}(kr)Y^\beta_{(JM)a}(\hat{\mathbf{n}}), \tag{24}$$

and for tensor TAM waves with $\alpha, \beta = L, VE, TE$ by

$$\Psi^{\alpha,k}_{(JM)ab}(\mathbf{x}) \equiv -\sum_\beta j^{(\alpha,\beta)}_{J,t}(kr)Y^\beta_{(JM)ab}(\hat{\mathbf{n}}), \tag{25}$$

and for tensor TAM waves with $\alpha, \beta = TB, VB$ by

$$\Psi^{\alpha,k}_{(JM)ab}(\mathbf{x}) \equiv -i\sum_\beta j^{(\alpha,\beta)}_{J,t}(kr)Y^\beta_{(JM)ab}(\hat{\mathbf{n}}). \tag{26}$$

The precise forms for $j^{(\alpha,\beta)}_{J,v}$ can be read off Eq. (22) and for $j^{(\alpha,\beta)}_{J,t}$ from Eq. (23). Some of these radial profiles have appeared in previous full-sky studies in cosmology. For example in the line-of-sight approach to CMB polarization, the full-sky EE and BB power spectra, Eq.(29) of Ref. [20]), contain radial kernels identical to the functions $j^{(TE,TE)}_{J,t}(x)$ and $j^{(TB,TB)}_{J,t}(x)$ defined here.

A. From TAM-bispectrum to angular bispectrum

Throughout this paper, we will calculate TAM bispectra of the form,

$$\left\langle X^{\alpha_1,\,k_1}_{J_1M_1}Y^{\alpha_2,\,k_2}_{J_2M_2}Z^{\alpha_3,\,k_3}_{J_3M_3}\right\rangle, \tag{27}$$

where X, Y, and Z represent generic scalar, vector and tensor fields, and α_i represents the particular mode of interest ($L/E/B$ for vector fields and $L/VE/VB/TE/TB$ for tensor fields). The TAM-wave bispectra above are related to the angular bispectra for various observables as follows. Suppose we observe some projection of a three-dimensional field $X(\mathbf{x})$. The spherical-harmonic coefficients for the projection can be written in terms of the Fourier components $\tilde{X}(\mathbf{k})$ of the field as

$$a^{\alpha\,X}_{lm} = \int\frac{d^3k}{(2\pi)^3}g_X(k)\tilde{X}(\mathbf{k})\mathcal{Y}^{\alpha\,*}_{lm}(\hat{\mathbf{k}}) = \int\frac{k^2dk}{(2\pi)^3}g_X(k)X^{\alpha,\,k}_{(lm)}, \tag{28}$$

in terms of a transfer function $g_X(k)$ which, assuming statistical isotropy, is a function only of the wavevector magnitude k. Here we suppress tensor indices, as the end result (as will be spelled out more clearly in the rest of the paper) is the same for scalar, vector and tensor fields, and $\mathcal{Y}^\alpha_{lm}$ generically represents the corresponding α modes of scalar, vector, and tensor spherical harmonics. The observable angular bispectrum is then related to the TAM bispectrum as

$$\left\langle a^{\alpha_1,X}_{J_1M_1} a^{\alpha_2,Y}_{J_2M_2} a^{\alpha_3,Z}_{J_3M_3} \right\rangle = \int \frac{k_1^2 dk_1}{(2\pi)^3} g_X(k_1) \int \frac{k_2^2 dk_2}{(2\pi)^3} g_Y(k_2) \int \frac{k_3^2 dk_3}{(2\pi)^3} g_Z(k_3) \left\langle X^{\alpha_1,k_1}_{J_1M_1} Y^{\alpha_2,k_2}_{J_2M_2} Z^{\alpha_3,k_3}_{J_3M_3} \right\rangle. \tag{29}$$

For example, if we take $X(\mathbf{x})$ to be Bardeen's curvature perturbation $\Phi(\mathbf{x})$, and the a^X_{lm} to be CMB-temperature spherical-harmonic coefficients, then the appropriate transfer function is

$$g_T(k) = 4\pi i^l g_{Tl}(k), \tag{30}$$

in terms of the radiation transfer function $g_{Tl}(k)$ [21]. If the a^X_{lm} are taken to be the density of galaxies on the sky, then the transfer function is

$$g_g(k) = 4\pi i^l b_g \left(\frac{2}{3}\frac{k^2 T(k)}{H_0^2 \Omega_m}\right) \int dz W_g(z) D(z) j_l[k d_A(z)], \tag{31}$$

with galaxy bias b_g, matter transfer function $T(k)$,, linear growth factor $D(z)$, comoving angular=diameter distance $d_A(z)$, and redshift distribution $W_g(z)$ of galaxies normalized to $\int dz W_g(z) = 1$. Here, H_0 and Ω_m are, respectively, the Hubble and matter-density parameters at present.

III. SCALAR BISPECTRUM

Before turning to vector and tensor fields, for which the TAM formalism provides the most substantial advantage, we begin by way of introduction with scalar bispectra in the TAM formalism.

The bispectrum $B_{sss}(k_1, k_2, k_3)$ for the scalar field is usually defined in terms of Fourier modes as

$$\left\langle \tilde\phi(\mathbf{k}_1)\tilde\phi(\mathbf{k}_2)\tilde\phi(\mathbf{k}_3) \right\rangle = (2\pi)^3 \delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) B_{sss}(k_1,k_2,k_3), \tag{32}$$

where the Dirac delta function arises as a consequence of statistical homogeneity, and the dependence of the bispectrum only on the magnitudes k_1, k_2, and k_3 as a consequence of statistical isotropy. The subscript 'sss' is used to denote the bispectrum for three scalar fields, to distinguish it from the bispectra to be discussed below that involve vector and/or tensor fields.

If the scalar bispectrum is defined as in Eq. (32), then the expectation value for the product of three TAM-wave coefficients are given by

$$\left\langle \phi^{k_1}_{(J_1M_1)} \phi^{k_2}_{(J_2M_2)} \phi^{k_3}_{(J_3M_3)} \right\rangle = \int d^2\hat{\mathbf{k}}_1 \int d^2\hat{\mathbf{k}}_2 \int d^2\hat{\mathbf{k}}_3 \left\langle \tilde\phi(\mathbf{k}_1)\tilde\phi(\mathbf{k}_2)\tilde\phi(\mathbf{k}_3) \right\rangle Y^*_{(J_1M_1)}(\hat{\mathbf{k}}_1) Y^*_{(J_2M_2)}(\hat{\mathbf{k}}_2) Y^*_{(J_3M_3)}(\hat{\mathbf{k}}_3). \tag{33}$$

We then substitute Eq. (32) and expand the Dirac delta function, using the plane-wave expansion, Eq. (11), as

$$\begin{aligned}(2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) &= \int d^3x\, e^{-i\mathbf{k}_1\cdot\mathbf{x}} e^{-i\mathbf{k}_2\cdot\mathbf{x}} e^{-i\mathbf{k}_3\cdot\mathbf{x}} \\ &= \int d^3x \sum_{l_i m_i} (4\pi)^3 (-i)^{l_1+l_2+l_3} j_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) \\ &\quad \times Y_{(l_1m_1)}(\hat{\mathbf{k}}_1) Y^*_{(l_1m_1)}(\hat{\mathbf{n}}) Y_{(l_2m_2)}(\hat{\mathbf{k}}_2) Y^*_{(l_2m_2)}(\hat{\mathbf{n}}) Y_{(l_3m_3)}(\hat{\mathbf{k}}_3) Y^*_{(l_3m_3)}(\hat{\mathbf{n}}),\end{aligned} \tag{34}$$

to eliminate the $\hat{\mathbf{k}}_i$ integrals. We can then write the TAM-wave bispectrum as

$$\left\langle \phi^{k_1}_{(J_1M_1)} \phi^{k_2}_{(J_2M_2)} \phi^{k_3}_{(J_3M_3)} \right\rangle = (4\pi)^3 (-i)^{l_1+l_2+l_3} B_{sss}(k_1,k_2,k_3) \int d^3x\, \Psi^{k_1\,*}_{(J_1M_1)}(\mathbf{x}) \Psi^{k_2\,*}_{(J_2M_2)}(\mathbf{x}) \Psi^{k_3\,*}_{(J_3M_3)}(\mathbf{x}), \tag{35}$$

in terms of an overlap of three TAM waves. Below we will see that the TAM-wave three-point functions will always depend, even with vector and tensor fields, on overlaps of three TAM waves. Since the scalar TAM waves are $\Psi^k_{JM}(\mathbf{x}) = j_J(kr) Y_{JM}(\hat{\mathbf{x}})$, the overlap is simply,

$$\int d^3x\, \Psi^{k_1}_{J_1M_1}(\mathbf{x}) \Psi^{k_2}_{J_2M_2}(\mathbf{x}) \Psi^{k_3}_{J_3M_3}(\mathbf{x}) = \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} J_{J_1J_2J_3}(k_1,k_2,k_3), \tag{36}$$

where

$$\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} = \int d^2\hat{\mathbf{n}} Y_{(l_1 m_1)}(\hat{\mathbf{n}}) Y_{l_2 m_2}(\hat{\mathbf{n}}) Y_{l_3 m_3}(\hat{\mathbf{n}}) = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad (37)$$

is the Gaunt integral, and

$$J_{l_1 l_2 l_3}(k_1, k_2, k_3) = \int_0^\infty r^2\, dr\, j_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r). \quad (38)$$

Therefore,

$$\left\langle \phi^{k_1}_{(J_1 M_1)} \phi^{k_2}_{(J_2 M_2)} \phi^{k_3}_{(J_3 M_3)} \right\rangle = (4\pi)^3 (-i)^{l_1+l_2+l_3} \mathcal{G}^{J_1 J_2 J_3}_{M_1 M_2 M_3} J_{J_1 J_2 J_3}(k_1, k_2, k_3) B_{sss}(k_1, k_2, k_3). \quad (39)$$

We have thus evaluated the TAM-wave three-point correlation function in terms of the overlap of three TAM waves. We then arrive at our final result, Eq. (39), which provides a simple expression for the TAM-wave three-point function in terms of the Fourier-space bispectrum, by evaluating that overlap. The factors connecting the Fourier-space and TAM-wave three-point functions are the easily evaluated and familiar Gaunt integral, and the radial integral, Eq. (38). The Gaunt integral is proportional to a Wigner-3j symbol and the Clebsch-Gordan coefficient. Its presence here is a result of the Wigner-Eckart theorem, and we will see in the Sections below that the expectation value for the product of any three TAM waves—not just scalar TAM waves—will be proportional to a Clebsch-Gordan coefficient times a function only of J_1, J_2, and J_3. We choose to write the results in terms of the Gaunt factor, rather than Clebsch-Gordan coefficients or Wigner-3j symbols simply because it provides for more compact expressions.

The radial integral in Eq. (39) can be evaluated numerically using standard techniques [22–24], but often an end result, which requires an additional integral over the k_i, can be obtained more quickly by changing the orders of those k_i integrals and the radial integral in Eq. (38).

Below we will generalize Eq. (39) to three-point functions involving vector and tensor fields as well.

IV. TWO TRANSVERSE VECTORS AND A SCALAR

We now proceed with the analogous calculation for the three-point correlation function that involves two transverse-vector fields and a scalar. The transverse vector field $V^a(\mathbf{x})$ satisfies $\nabla_a V^a = 0$, or in Fourier space, $k_a \tilde{V}^a(\mathbf{k}) = 0$. Such a vector-vector-scalar correlation arises, for example, in models in which a magnetic field (the transverse-vector field) is produced during inflation and thus correlated with the inflaton field (the scalar field) [11].

Our goal is to calculate the TAM-wave bispectra,

$$\left\langle V^{\alpha_1, k_1}_{(J_1 M_1)} V^{\alpha_2, k_2}_{(J_2 M_2)} \phi^{k_3}_{(J_3 M_3)} \right\rangle, \quad (40)$$

for $\{\alpha_1, \alpha_2\} = \{E, B\}$ that correspond to a given Fourier-space bispectrum.

A. Fourier-space bispectra

We begin by discussing the Fourier-space bispectrum. Each Fourier component of the transverse-vector field can be written

$$\tilde{V}^a(\mathbf{k}) = \sum_{\lambda=\pm} \hat{\varepsilon}^a_\lambda(\mathbf{k}) \tilde{V}_\lambda(\mathbf{k}), \quad (41)$$

in terms of two helicity-basis polarization vectors $\hat{\varepsilon}_\lambda(\mathbf{k})$ and helicity Fourier amplitudes $\tilde{V}_\lambda(\mathbf{k})$. The helicity basis vectors can be written as $\hat{\varepsilon}_\pm(\mathbf{k}) = \left[\hat{\varepsilon}_1(\mathbf{k}) \pm i\hat{\mathbf{k}} \times \hat{\varepsilon}_1(\mathbf{k})\right]/\sqrt{2}$, where $\hat{\varepsilon}_1$ is any unit vector in the plane perpendicular to $\mathbf{k}$. The most general Fourier-space vector-vector-scalar three-point function can therefore be written,

$$\left\langle \tilde{V}^a(\mathbf{k}_1) \tilde{V}^b(\mathbf{k}_2) \tilde{\phi}(\mathbf{k}_3) \right\rangle = \sum_{\lambda_1 \lambda_2} \hat{\varepsilon}^a_{\lambda_1}(\mathbf{k}_1) \hat{\varepsilon}^b_{\lambda_2}(\mathbf{k}_2) \left\langle \tilde{V}_{\lambda_1}(\mathbf{k}_1) \tilde{V}_{\lambda_2}(\mathbf{k}_2) \tilde{\phi}(\mathbf{k}_3) \right\rangle. \quad (42)$$

Given the orthonormality of the polarization vectors, this relation can then be inverted to provide the three-point correlation,

$$\left\langle \tilde{V}_{\lambda_1}(\mathbf{k}_1)\tilde{V}_{\lambda_2}(\mathbf{k}_2)\tilde{\phi}(\mathbf{k}_3)\right\rangle = \hat{\varepsilon}^a_{\lambda_1}(\mathbf{k}_1)\hat{\varepsilon}^b_{\lambda_2}(\mathbf{k}_2)\left\langle V_a(\mathbf{k}_1)V_b(\mathbf{k}_2)\phi(\mathbf{k}_3)\right\rangle, \tag{43}$$

for the helicity amplitudes.

Now consider the tensor structure of the three-point function on the right side of Eq. (43). By momentum conservation, it most generally depends on k_1^a and k_2^a. Among all structures one can possibly construct, g^{ab}, $k_1^a k_1^b$, $k_1^a k_2^b$, $k_2^a k_1^b$ and $k_2^a k_2^b$, terms proportional to k_a^1 or k_2^b have no contribution to Eq. (43) under helicity basis, and hence to the bispectrum through Eq. (42).[1] The most general parity-conserving three-point function can then be written,

$$\left\langle \tilde{V}^a(\mathbf{k}_1)\tilde{V}^b(\mathbf{k}_2)\tilde{\phi}(\mathbf{k}_3)\right\rangle = (2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\left[g^{ab}B^{(1)}_{vvs}(k_1,k_2,k_3) + k_2^a k_1^b B^{(2)}_{vvs}(k_1,k_2,k_3)\right], \tag{44}$$

in terms of two vector-vector-scalar bispectra $B^{(1)}_{vvs}(k_1,k_2,k_3)$ and $B^{(2)}_{vvs}(k_1,k_2,k_3)$. Two different bispectra arise because there are two components for each polarization vector.

To demonstrate our formalism, we take the example of non-zero $B^{(1)}_{vvs}(k_1,k_2,k_3)$, which may arise from a local three-point interaction of the form $\phi(\mathbf{x})|\mathbf{V}(\mathbf{x})|^2$. The other bispectrum $B^{(2)}_{vvs}(k_1,k_2,k_3)$, which can arise as the vector-vector-scalar correlation between the vector eletromagnetic potential and the inflaton from a $\phi F_{\mu\nu}F^{\mu\nu}$ coupling [11], is left to future work.

Therefore, the Fourier-space bispectrum we will consider here will be of the form,

$$\left\langle V^a(\mathbf{k}_1)V^b(\mathbf{k}_2)\phi(\mathbf{k}_3)\right\rangle = (2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)g^{ab}B_{vvs}(k_1,k_2,k_3). \tag{45}$$

B. TAM-wave bispectra

We now calculate the three-point correlation function,

$$\left\langle V^{\alpha_1,k_1}_{(J_1M_1)}V^{\alpha_2,k_2}_{(J_2M_2)}\phi^{k_3}_{(J_3M_3)}\right\rangle = \int d^2\hat{\mathbf{k}}_1 d^2\hat{\mathbf{k}}_2 d^2\hat{\mathbf{k}}_3\, Y^{\alpha_1\,*}_{(J_1M_1)a}(\hat{\mathbf{k}}_1)Y^{\alpha_2\,*}_{(J_2M_2)b}(\hat{\mathbf{k}}_2)Y^{*}_{(J_3M_3)}(\hat{\mathbf{k}}_3)\left\langle \tilde{V}^a(\mathbf{k}_1)\tilde{V}^b(\mathbf{k}_2)\phi(\mathbf{k}_3)\right\rangle, \tag{46}$$

for the E/B TAM-wave coefficients, where $\alpha_1,\alpha_2=\{E,B\}$, and we have used Eq. (20). The first step is to expand the Dirac delta function in Eq. (45), and using the identity,

$$\sum_{J'M'}(-i)^{J'}j_{J'}(kr)Y^*_{(J'M')}(\hat{\mathbf{n}})\int d^2\hat{\mathbf{k}}\,Y^{\alpha\,*}_{(JM)a}(\hat{\mathbf{k}})Y_{(J'M')}(\hat{\mathbf{k}}) = (-i)^J\left[\Psi^{\alpha,k}_{(JM)a}(\mathbf{x})\right]^*, \tag{47}$$

we find that

$$\left\langle V^{\alpha_1,k_1}_{(J_1M_1)}V^{\alpha_2,k_2}_{(J_2M_2)}\phi^{k_3}_{(J_3M_3)}\right\rangle = (4\pi)^3(-i)^{J_1+J_2+J_3}B_{vvs}(k_1,k_2,k_3)\int d^3x\,\Psi^{\alpha_1,k_1\,*a}_{(J_1M_1)}(\mathbf{x})\Psi^{\alpha_2,k_2\,*}_{(J_2M_2)a}(\mathbf{x})\Psi^{k_3\,*}_{(J_3M_3)}(\mathbf{x}). \tag{48}$$

That is, the bispectrum of three TAM coefficients is proportional to the overlap of three TAM waves.

These overlap integrals can be evaluated with the projection of vector TAM waves in Eq. (22). The angular parts of the overlap integral can be evaluated by writing the $L/E/B$ vector spherical harmonics in terms of helicity spherical harmonics as

$$Y^{\pm1}_{(JM)a} = \left(Y^E_{(JM)a} \pm iY^B_{(JM)a}\right)/\sqrt{2}, \qquad Y^L_{(JM)a} = Y^0_{(JM)a}. \tag{49}$$

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[1] If parity violation is allowed, there may be additional terms constructed with the antisymmetric tensor ϵ_{abc}, but we restrict our attention to parity-preserving terms.

We can then use the helicity-harmonic integrals in Appendix A 1 to obtain the spherical-harmonic overlap integrals,

$$\begin{aligned}
\int d^2\hat{\mathbf{n}} Y^{L,a}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{L}_{(J_2M_2)a}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_2}, \\
\int d^2\hat{\mathbf{n}} Y^{B,a}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{B}_{(J_2M_2)a}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= \int d^2\hat{\mathbf{n}} Y^{E,a}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{E}_{(J_2M_2)a}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) \\
&= -\frac{1+(-1)^{J_1+J_2+J_3}}{2}\mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}\frac{\langle J_1 1 J_2,-1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}, \\
\int d^2\hat{\mathbf{n}} Y^{B,a}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{E}_{(J_2M_2)a}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= i\left[\frac{1-(-1)^{J_1+J_2+J_3}}{2}\right]\mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}\frac{\langle J_1 1 J_2,-1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle},
\end{aligned} \tag{50}$$

and we also have that

$$Y^{L}_{(J_1M_1)a}(\hat{\mathbf{n}}) Y^{E/B,a}_{(J_2M_2)}(\hat{\mathbf{n}}) = 0. \tag{51}$$

From these we obtain that the nonzero TAM-wave overlap integrals for $J_1+J_2+J_3=$ even are

$$\int d^3x \Psi^{B,k_1}_{(J_1M_1)a}(\mathbf{x})\Psi^{B,k_2,a}_{(J_2M_2)}(\mathbf{x})\Psi^{k_3}_{(J_3M_3)}(\mathbf{x}) = -\mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}\frac{\langle J_1 1 J_2,-1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle} J_{J_1J_2J_3}(k_1,k_2,k_3), \tag{52}$$

$$\begin{aligned}
\int d^3x \Psi^{E,k_1}_{(J_1M_1)a}(\mathbf{x})\Psi^{E,k_2,a}_{(J_2M_2)}(\mathbf{x})\Psi^{k_3}_{(J_3M_3)}(\mathbf{x}) &= \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}\left[\frac{\langle J_1 1 J_2,-1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}\int r^2\, dr\, j^{(E,E)}_{J_1,v}(k_1 r)\, j^{(E,E)}_{J_2,v}(k_2 r)\, j_{J_3}(k_3 r)\right. \\
&\quad \left. - \int r^2\, dr\, j^{(E,L)}_{J_1,v}(k_1 r)\, j^{(E,L)}_{J_2,v}(k_2 r)\, j_{J_3}(k_3 r)\right],
\end{aligned} \tag{53}$$

and when $J_1+J_2+J_3=$ odd,

$$\int d^3x \Psi^{B,k_1}_{(J_1M_1)a}(\mathbf{x})\Psi^{E,k_2,a}_{(J_2M_2)}(\mathbf{x})\Psi^{k_3}_{(J_3M_3)}(\mathbf{x}) = \frac{\langle J_1 1 J_2-1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}\mathcal{G}^{J_1J_2J_3}_{M_1M_2M_2}\int r^2\, dr\, j_{J_1}(k_1 r)\, j^{(E,E)}_{J_2,v}(k_2 r)\, j_{J_3,v}(k_3 r). \tag{54}$$

Here, we have used the radial functions defined in Eq. (24). Note that strictly speaking, the Gaunt integral vanishes if $J_1+J_2+J_3=$ odd, as the Wigner-3j symbol $\begin{pmatrix} J_1 & J_2 & J_3 \\ 0 & 0 & 0 \end{pmatrix}$ vanishes. In every case where a Gaunt integral appears with $J_1+J_2+J_3=$ odd, though, it appears divided by a Clebsch-Gordan coefficient $\langle J_1 0 J_2 0|J_3 0\rangle$ which also contains the same Wigner-3j symbol, and thus cancels that in the numerator. We choose to write things in this slightly unconventional way to enable compact, but still unambiguous, expressions.

Eq. (48), together with the overlap integrals in Eqs. (52), (53), and (54), provide the B-B-scalar, E-E-scalar, and B-E-scalar TAM-wave bispectra in terms of $B_{vvs}(k_1,k_2,k_3)$, the Fourier space vector-vector-scalar bispectrum.

V. TWO TRANSVERSE-TRACELESS TENSORS AND ONE SCALAR

We now consider a three-point correlation involving two transverse-traceless tensors and one scalar. Such a correlation arises, for example, in inflation if the tensor modes are gravitational waves and the scalar field is the curvature perturbation [13].

Our goal here will be to calculate the TAM-wave bispectra,

$$\left\langle T^{\alpha_1,k_1}_{(J_1M_1)} T^{\alpha_2,k_2}_{(J_2M_2)} \phi^{k_3}_{(J_3M_3)} \right\rangle, \tag{55}$$

for $\{\alpha_1,\alpha_2\}=\{TE,TB\}$ that correspond to a given Fourier-space bispectrum. Here we will take the Fourier-space tensor-tensor-scalar three-point function to be of the form

$$\left\langle \tilde{T}^{ab}(\mathbf{k}_1)\tilde{T}^{cd}(\mathbf{k}_2)\tilde{\phi}(\mathbf{k}_3)\right\rangle = (2\pi)^3\delta^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) g^{ac}g^{bd} B_{tts}(k_1,k_2,k_3), \tag{56}$$

that arises in single-field slow-roll inflation [13], in terms of a tensor-tensor-scalar bispectrum $B_{tts}(k_1,k_2,k_3)$. The tensor fields T^{ab} are transverse, $\nabla^a\tilde{T}_{ab}(\mathbf{x})=0$, and traceless, $g^{ab}\tilde{T}_{ab}(\mathbf{x})=0$.

We calculate the three-point correlation function of TE/TB TAM coefficients directly from Eq. (21):

$$\left\langle T^{\alpha_1,k_1}_{(J_1M_1)} T^{\alpha_2,k_2}_{(J_2M_2)} \phi^{k_3}_{(J_3M_3)} \right\rangle = \int d^2\hat{\mathbf{k}}_1 d^2\hat{\mathbf{k}}_2 d^2\hat{\mathbf{k}}_3 \, Y^{\alpha_1\,*}_{(J_1M_1)ab}(\hat{\mathbf{k}}_1) Y^{\alpha_2\,*}_{(J_2M_2)cd}(\hat{\mathbf{k}}_2) Y^{*}_{(J_3M_3)}(\hat{\mathbf{k}}_3) \left\langle \tilde{T}^{ab}(\mathbf{k}_1)\tilde{T}^{cd}(\mathbf{k}_2)\phi(\mathbf{k}_3)\right\rangle, \quad (57)$$

where $\alpha_1, \alpha_2 = \{L, E, B\}$. Expanding the Dirac delta function in Eq. (56) with Eq. (34) and using the identity,

$$\sum_{J'M'} (-i)^{J'} j_{J'}(kr) Y^{*}_{(J'M')}(\hat{\mathbf{n}}) \int d^2\hat{\mathbf{k}}\, Y^{\alpha\,*}_{(JM)ab}(\hat{\mathbf{k}}) Y_{(J'M')}(\hat{\mathbf{k}}) = (-i)^J \left[\Psi^{\alpha,k}_{(JM)ab}(\mathbf{x})\right]^*, \quad (58)$$

we find that

$$\left\langle T^{\alpha_1,k_1}_{(J_1M_1)} T^{\alpha_2,k_2}_{(J_2M_2)} \phi^{k_3}_{(J_3M_3)} \right\rangle = (4\pi)^3 (-i)^{J_1+J_2+J_3} B_{tts}(k_1,k_2,k_3) \int d^3x\, \Psi^{\alpha_1,k_1\,*ab}_{(J_1M_1)}(\mathbf{x}) \Psi^{\alpha_2,k_2\,*}_{(J_2M_2)ab}(\mathbf{x}) \Psi^{k_3\,*}_{(J_3M_3)}(\mathbf{x}). \quad (59)$$

That is, the bispectrum of three TAM-wave coefficients is again proportional to the overlap of three TAM waves.

Our task now is to evaluate the overlap integrals,

$$\int d^3x\, \Psi^{\alpha_1,k_1}_{(J_1M_1)ab}(\mathbf{x}) \Psi^{\alpha_2,k_2,ab}_{(J_2M_2)}(\mathbf{x}) \Psi^{k_3}_{(J_3M_3)}(\mathbf{x}), \quad (60)$$

for $\alpha_1, \alpha_2 = TE, TB$. With the the decomposition in Eq. (94) of Ref. [17], we only have to calculate the overlap of three tensor spherical harmonics

$$\int d^2\hat{n} I^{ab,cd} Y^{\alpha_1}_{(J_1M_1)ab}(\hat{\mathbf{n}}) Y^{\alpha_2}_{(J_2M_2)cd}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}), \quad (61)$$

for $\alpha_1, \alpha_2 = L, VE/VB, TE/TB$. Here the constant tensor $I^{ab,cd}$ is the identity tensor acting on symmetric traceless tensors. In Appendix A 2, we calculate these overlaps in the helicity basis. We then use the relations,

$$Y^{\pm 2}_{(JM)ab} = \frac{1}{\sqrt{2}}\left(Y^{TE}_{(JM)ab} \pm i Y^{TB}_{(JM)ab}\right), \quad Y^{\pm 1}_{(JM)ab} = \frac{1}{\sqrt{2}}\left(Y^{VE}_{(JM)ab} \pm i Y^{VB}_{(JM)ab}\right), \quad Y^{0}_{(JM)ab} = Y^{L}_{(JM)ab}, \quad (62)$$

to find that

$$\begin{aligned}
\int d^2\hat{\mathbf{n}}\, Y^{L,ab}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{L}_{(J_2M_2)ab}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}, & (63)\\
\int d^2\hat{\mathbf{n}}\, Y^{VB,ab}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{VB}_{(J_2M_2)ab}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= \int d^2\hat{\mathbf{n}} Y^{VE,ab}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{VE}_{(J_2M_2)ab}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}})\\
&= -\frac{1+(-1)^{J_1+J_2+J_3}}{2} \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \frac{\langle J_1 1 J_2, -1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}, & (64)\\
\int d^2\hat{\mathbf{n}}\, Y^{VB,ab}_{(J_1M_1)} Y^{VE}_{(J_2M_2)ab}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= i\frac{1-(-1)^{J_1+J_2+J_3}}{2} \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \frac{\langle J_1 1 J_2, -1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}, & (65)\\
\int d^2\hat{n} Y^{TB,ab}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{TB}_{(J_2M_2)ab}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= \int d^2\hat{\mathbf{n}} Y^{TE,ab}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{TE}_{(J_2M_2)ab}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}})\\
&= \frac{1+(-1)^{J_1+J_2+J_3}}{2} \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \frac{\langle J_1 2 J_2, -2|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}, & (66)\\
\int d^2\hat{\mathbf{n}}\, Y^{TB,ab}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{TE}_{(J_2M_2)ab}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= -i\frac{1-(-1)^{J_1+J_2+J_3}}{2} \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \frac{\langle J_1 2 J_2, -2|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}. & (67)
\end{aligned}$$

We now use Eq. (94) of Ref. [17], along with the orthonormality conditions,

$$Y^{L}_{(J_1M_1)ab}(\hat{\mathbf{n}}) Y^{VE/VB,ab}_{(J_2M_2)}(\hat{\mathbf{n}}) = 0, \quad Y^{L}_{(J_1M_1)ab}(\hat{\mathbf{n}}) Y^{TE/TB,ab}_{(J_2M_2)}(\hat{\mathbf{n}}) = 0, \quad Y^{VE/VB}_{(J_1M_1)ab}(\hat{\mathbf{n}}) Y^{TE/TB,ab}_{(J_2M_2)}(\hat{\mathbf{n}}) = 0, \quad (68)$$

to find that for $J_1+J_2+J_3=$ even,

$$\int d^3x\,\Psi^{TB,k_1}_{(J_1M_1)ab}(\mathbf{x})\Psi^{TB,k_2,ab}_{(J_2M_2)}(\mathbf{x})\Psi^{k_3}_{(J_3M_3)}(\mathbf{x})=\mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}\int r^2dr j_{J_3}(k_3r)\left[\frac{\langle J_1 1J_2,-1|J_30\rangle}{\langle J_10J_20|J_30\rangle}j^{(TB,VB)}_{J_1,t}(k_1r)j^{(TB,VB)}_{J_2,t}(k_2r)\right.$$
$$\left.-\frac{\langle J_1 2J_2,-2|J_30\rangle}{\langle J_10J_20|J_30\rangle}j^{(TB,TB)}_{J_1,t}(k_1r)j^{(TB,TB)}_{J_2,t}(k_2r)\right], \tag{69}$$

$$\int d^3x\,\Psi^{TE,k_1}_{(J_1M_1)ab}(\mathbf{x})\Psi^{TE,k_2,ab}_{(J_2M_2)}(\mathbf{x})\Psi^{k_3}_{(J_3M_3)}(\mathbf{x})=\mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}\int r^2dr j_{J_3}(k_3r)\left[j^{(TE,L)}_{J_1,t}(k_1r)j^{(TE,L)}_{J_2,t}(k_2r)\right.$$
$$\left.-\frac{\langle J_1 1J_2,-1|J_30\rangle}{\langle J_10J_20|J_30\rangle}j^{(TE,VE)}_{J_1,t}(k_1r)j^{(TE,VE)}_{J_2,t}(k_2r)+\frac{\langle J_1 2J_2,-2|J_30\rangle}{\langle J_10J_20|J_30\rangle}j^{(TE,TE)}_{J_1,t}(k_1r)j^{(TE,TE)}_{J_2,t}(k_2r)\right], \tag{70}$$

and for $J_1+J_2+J_3=$ odd,

$$\int d^3x\,\Psi^{TB,k_1}_{(J_1M_1)ab}(\mathbf{x})\Psi^{TE,k_2,ab}_{(J_2M_2)}(\mathbf{x})\Psi^{k_3}_{(J_3M_3)}(\mathbf{x}) = \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}\int r^2dr j_{J_3}(k_3r)\left[-\frac{\langle J_1 1J_2,-1|J_30\rangle}{\langle J_10J_20|J_30\rangle}j^{(TB,VB)}_{J_1,t}(k_1r)\,j^{(TE,VE)}_{J_2,t}(k_2r)\right.$$
$$\left.+\frac{\langle J_1 2J_2,-2|J_30\rangle}{\langle J_10J_20|J_30\rangle}j^{(TB,TB)}_{J_1,t}(k_1r)\,j^{(TE,TE)}_{J_2,t}(k_2r)\right]. \tag{71}$$

Here, we have used the radial functions defined in Eq. (25) and Eq. (26). Eq. (59), together with the overlap integrals in Eqs. (69), (70), and (71), provides the TB-TB-scalar, TE-TE-scalar, and TB-TE-scalar TAM-wave bispectra in terms of $B_{tts}(k_1,k_2,k_3)$, the Fourier-space tensor-tensor-scalar bispectrum.

VI. ONE SYMMETRIC TRACELESS TENSOR AND TWO SCALARS (OR TWO LONGITUDINAL VECTORS)

We now calculate the three-point correlation that involves one symmetric traceless tensor and two scalars. The traceless tensor may be a longitudinal (L) mode, one of the two vector modes (VE or VB), or one of the two transverse-tensor (TE or TB) modes. The three-dimensional galaxy-survey observables of such a three-point tensor-scalar-scalar correlation were studied in Ref. [15]. Moreover, a tensor-scalar-scalar correlation between inflationary gravitational waves (the tensor mode) and the primordial curvature perturbation (the scalar mode) arises generically during inflation [13]. The TAM-wave bispectra we present here may be useful in calculating full-sky observables associated with these couplings.

To be precise, we consider here a tensor-scalar-scalar bispectrum $B_{sst}(k_1,k_2,k_3)$ defined by

$$\left\langle\widetilde{\nabla^a\phi}(\mathbf{k}_1)\widetilde{\nabla^b\phi}(\mathbf{k}_2)\tilde{T}^{cd}(\mathbf{k}_3)\right\rangle=(2\pi)^3\delta^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)g^{ac}g^{bd}k_1k_2B_{sst}(k_1,k_2,k_3). \tag{72}$$

Given that $V^a=\nabla^a\phi$ is a longitudinal-vector field, we can also write this three-point function as a vector-vector-tensor bispectrum $B_{vvt}(k_1,k_2,k_3)$, where here "vector" is a longitudinal vector, of the form,

$$\left\langle\tilde{V}^a(\hat{\mathbf{k}}_1)\tilde{V}^b(\hat{\mathbf{k}}_2)\tilde{T}^{cd}(\hat{\mathbf{k}}_3)\right\rangle=(2\pi)^3\delta^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)g^{ac}g^{bd}B_{vvt}(k_1,k_2,k_3). \tag{73}$$

The three-point function of one TAM coefficient and two scalar TAM coefficients can be calculated from

$$\left\langle V^{L,k_1}_{(J_1M_1)}V^{L,k_2}_{(J_2M_2)}T^{\alpha,k_3}_{(J_3M_3)}\right\rangle=(4\pi)^3(-i)^{J_1+J_2+J_3}B_{vvt}(k_1,k_2,k_3)\int d^3x\Psi^{L,k_1\ a}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2\ b}_{(J_2M_2)}(\mathbf{x})\Psi^{\alpha,k_3}_{(J_3M_3)ab}(\mathbf{x}), \tag{74}$$

and using Eq. (72) and $\phi^k_{JM}=iV^{L,k}_{(JM)}/k$,

$$\left\langle\phi^{k_1}_{(J_1M_1)}\phi^{k_2}_{(J_2M_2)}T^{\alpha,k_3}_{(J_3M_3)}\right\rangle=-(4\pi)^3(-i)^{J_1+J_2+J_3}B_{sst}(k_1,k_2,k_3)\int d^3x\Psi^{L,k_1\ a}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2\ b}_{(J_2M_2)}(\mathbf{x})\Psi^{\alpha,k_3}_{(J_3M_3)ab}(\mathbf{x}). \tag{75}$$

That is, we only need to calculate the overlap,

$$\int d^3x\Psi^{L,k_1\ a}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2\ b}_{(J_2M_2)}(\mathbf{x})\Psi^{\alpha,k_3}_{(J_3M_3)ab}(\mathbf{x})=\int d^3x\left[\frac{i}{k_1}\nabla^a\Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\right]\left[\frac{i}{k_2}\nabla^b\Psi^{k_2}_{(J_2M_2)}(\mathbf{x})\right]\Psi^{\alpha,k_3}_{(J_3M_3)ab}(\mathbf{x}), \tag{76}$$

for $\alpha=L,VE,VB,TE,TB$. Below we discuss the three cases where the traceless tensor is longitudinal (L), vectorial (VE/VB), or transverse (TE/TB).

A. The transverse-traceless components

We start with transverse-traceless tensors, $\alpha = TE, TB$. Since the transverse-traceless fields are divergence-free, we can integrate by parts in the integral in Eq. (76) to get,

$$\frac{1}{k_1 k_2}\int d^3x \Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\left[\nabla_a\nabla_b \Psi^{L,k_2}_{(J_2M_2)}(\mathbf{x})\right]\Psi^{\alpha,k_1,ab}_{(J_3M_3)}(\mathbf{x}). \tag{77}$$

We next use the definition [17],

$$\Psi^{L,k}_{(JM)ab}(\mathbf{x}) = \sqrt{\frac{3}{2}}\left(\frac{1}{k^2}\nabla_a\nabla_b + \frac{1}{3}g_{ab}\right)\Psi^{k}_{(JM)}(\mathbf{x}), \tag{78}$$

of longitudinal-tensor TAM waves, and the fact that the tensor mode is traceless, to rewrite the three-wavefunction overlap as,

$$\int d^3x \Psi^{L,k_1\ a}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2\ b}_{(J_2M_2)}(\mathbf{x})\Psi^{\alpha,k_3}_{(J_3M_3)ab}(\mathbf{x}) = \sqrt{\frac{1}{6}}\left[\frac{k_2}{k_1}\int d^3x \Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2}_{(J_2M_2)ab}(\mathbf{x})\Psi^{\alpha,k_3,\,ab}_{(J_3M_3)}(\mathbf{x}) + (1\leftrightarrow 2)\right], \tag{79}$$

for $\alpha = TE, TB$. What we now need to calculate is the overlap of a transverse tensor, a longitudinal tensor, and a scalar. Using Eq. (23) and Eqs. (63)–(67), we find that when $J_1 + J_2 + J_3 =$ even,

$$\begin{aligned}\int d^3x \Psi^{L,k_1\ a}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2\ b}_{(J_2M_2)}(\mathbf{x})\Psi^{TE,k_3}_{(J_3M_3)ab}(\mathbf{x}) &= \frac{1}{\sqrt{6}}\mathcal{G}^{J_3J_2J_1}_{M_3M_2M_1}\left\{\frac{k_2}{k_1}\int r^2 dr j_{J_1}(k_1r)\left[j^{(L,L)}_{J_2,t}(k_2r)\, j^{(TE,L)}_{J_3,t}(k_3r)\right.\right.\\ &\left.\left. -\frac{\langle J_3 1 J_2,-1|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle} j^{(L,VE)}_{J_2,t}(k_2r)\, j^{(TE,VE)}_{J_3,t}(k_3r) + \frac{\langle J_3 2 J_2,-2|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle} j^{(L,TE)}_{J_2,t}(k_2r)\, j^{(TE,TE)}_{J_3,t}(k_3r)\right] + (1\leftrightarrow 2)\right\},\end{aligned} \tag{80}$$

and when $J_1 + J_2 + J_3 =$ odd,

$$\begin{aligned}\int d^3x \Psi^{L,k_1\ a}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2\ b}_{(J_2M_2)}(\mathbf{x})\Psi^{TB,k_3}_{(J_3M_3)ab}(\mathbf{x}) &= \frac{1}{\sqrt{6}}\mathcal{G}^{J_3J_2J_1}_{M_3M_2M_1}\left\{\frac{k_2}{k_1}\int r^2 dr j_{J_1}(k_1r)\right.\\ &\left.\times\left[-\frac{\langle J_3 1 J_2,-1|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle} j^{(L,VE)}_{J_2,t}(k_2r)\, j^{(TB,VB)}_{J_3,t}(k_3r) + \frac{\langle J_3 2 J_2,-2|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle} j^{(L,TE)}_{J_2,t}(k_2r)\, j^{(TB,TB)}_{J_3,t}(k_3r)\right] + (1\leftrightarrow 2)\right\}.\end{aligned} \tag{81}$$

Here, again, we use the radial functions defined in Eq. (25) and Eq. (26).

B. The vector components

Next we consider the two vector modes, $\alpha = VE, VB$. The TAM waves for the vector components of the symmetric trace-free tensor can be related to the TAM waves for transverse-vector fields [17] through

$$\frac{i}{k}\nabla_a \Psi^{VB,ab}_{(JM)}(\mathbf{x}) = -\frac{1}{\sqrt{2}}\Psi^{B,b}_{(JM)}(\mathbf{x}), \quad \frac{i}{k}\nabla_a \Psi^{VE,ab}_{(JM)}(\mathbf{x}) = -\frac{1}{\sqrt{2}}\Psi^{E,b}_{(JM)}(\mathbf{x}). \tag{82}$$

Since the $\Psi^{TE/TB,k}_{(JM)ab}$ are traceless, we can do the overlap integral in Eq. (76) by integrating by parts to obtain

$$\begin{aligned}\int d^3x \Psi^{L,k_1}_{(J_1M_1)a}(\mathbf{x})\Psi^{L,k_2}_{(J_2M_2)b}(\mathbf{x})\Psi^{\alpha,k_3,ab}_{(J_3M_3)}(\mathbf{x}) &= \sqrt{\frac{1}{6}}\left[\frac{k_2}{k_1}\int d^3x \Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2}_{(J_2M_2)ab}(\mathbf{x})\Psi^{\alpha,k_3,\,ab}_{(J_3M_3)}(\mathbf{x}) + (1\leftrightarrow 2)\right]\\ &+\frac{1}{\sqrt{8}}\left[\frac{k_3}{k_1}\int d^3x \Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2}_{(J_2M_2)b}(\mathbf{x})\Psi^{\alpha,k_3,b}_{(J_3M_3)}(\mathbf{x}) + (1\leftrightarrow 2)\right],\end{aligned} \tag{83}$$

where $\alpha = TE, TB$ for tensor-valued TAM waves, and $\alpha = E, B$ for vector-valued TAM waves. Following the same techniques as in previous Sections, we find for $J_1 + J_2 + J_3 =$even,

$$\int d^3x \Psi^{L,k_1}_{(J_1M_1)a}(\mathbf{x})\Psi^{L,k_2}_{(J_2M_2)b}(\mathbf{x})\Psi^{VE,k_3,ab}_{(J_3M_3)}(\mathbf{x}) = \sqrt{\frac{1}{6}}\mathcal{G}^{J_3J_2J_1}_{M_3M_2M_1}\left[\frac{k_2}{k_1}\int r^2 dr j_{J_1}(k_1r)\left\{j^{(L,L)}_{J_2,t}(k_2r)j^{(VE,L)}_{J_3,t}(k_3r)\right.\right.$$
$$\left.\left. -\frac{\langle J_3 1 J_2, -1|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle}j^{(L,VE)}_{J_2,t}(k_2r)j^{(VE,VE)}_{J_3,t}(k_3r) + \frac{\langle J_3 2 J_2, -2|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle}j^{(L,TE)}_{J_2,t}j^{(VE,TE)}_{J_3,t}\right\} + (1 \leftrightarrow 2)\right]$$
$$-\frac{1}{\sqrt{8}}\mathcal{G}^{J_3J_2J_1}_{M_3M_2M_1}\left[\frac{k_3}{k_1}\int r^2 dr j_{J_1}(k_1r)\left\{j^{(L,L)}_{J_2,v}(k_2r)j^{(E,L)}_{J_3,v}(k_3r) - \frac{\langle J_3 1 J_2, -1|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle}j^{(L,E)}_{J_2,v}(k_2r)j^{(E,E)}_{J_3,v}(k_3r)\right\} + (1 \leftrightarrow 2)\right], \tag{84}$$

and for $J_1 + J_2 + J_3 =$odd,

$$\int d^3x \Psi^{L,k_1}_{(J_1M_1)a}(\mathbf{x})\Psi^{L,k_2}_{(J_2M_2)b}(\mathbf{x})\Psi^{VB,k_3,ab}_{(J_3M_3)}(\mathbf{x}) = \sqrt{\frac{1}{6}}\mathcal{G}^{J_3J_2J_1}_{M_3M_2M_1}\left[\frac{k_2}{k_1}\int r^2 dr j_{J_1}(k_1r)\right.$$
$$\left.\times\left\{-\frac{\langle J_3 1 J_2, -1|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle}j^{(L,VE)}_{J_2,t}(k_2r)j^{(VB,VB)}_{J_3,t}(k_3r) + \frac{\langle J_3 2 J_2, -2|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle}j^{(L,TE)}_{J_2,t}j^{(VB,TB)}_{J_3,t}\right\} + (1 \leftrightarrow 2)\right]$$
$$+\frac{1}{\sqrt{8}}\mathcal{G}^{J_3J_2J_1}_{M_3M_2M_1}\left[\frac{k_3}{k_1}\frac{\langle J_3 1 J_2, -1|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle}\int r^2 dr j_{J_1}(k_1r)j^{(L,E)}_{J_2,v}(k_2r)j_{J_3}(k_3r) + (1 \leftrightarrow 2)\right]. \tag{85}$$

Again, we have used the radial functions defined in Eq. (25) and Eq. (26).

C. Longitudinal part

Finally, we consider the longitudinal mode of the traceless tensor field. Once again, we can write the longitudinal tensor-valued TAM wave function $\Psi^{L,k}_{(JM)ab}(\mathbf{x})$ in terms of differential operators acting on scalar-valued TAM waves $\Psi^{k}_{(JM)}(\mathbf{x})$, as shown in Eq. (78). We can also put the two longitudinal vector-valued TAM wave functions into this operator form. Integrating by parts allows us to move around those gradient operators, and we re-write the overlap of three TAM waves as

$$\int d^3x \Psi^{L,k_1,a}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2,b}_{(J_2M_2)}(\mathbf{x})\Psi^{L,k_3}_{(J_3M_3)ab}(\mathbf{x}) = \frac{1}{\sqrt{6}}\int d^3x \left\{\frac{k_2}{k_1}\Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2,ab}_{(J_2M_2)}(\mathbf{x})\Psi^{L,k_3}_{(J_3M_3)ab}(\mathbf{x})\right.$$
$$\left. +\frac{k_3}{k_1}\Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2,a}_{(J_2M_2)}(\mathbf{x})\Psi^{L,k_3}_{(J_3M_3)a}(\mathbf{x}) + (1 \leftrightarrow 2)\right\}. \tag{86}$$

The first term is the overlap of two longitudinal tensors and one scalar, and the second term is the overlap of two longitudinal vectors and one scalar. The integrals are non-zero only if $J_1 + J_2 + J_3 =$ even, and the end result is

$$\int d^3x \Psi^{L,k_1,a}_{(J_1M_1)}(\mathbf{x})\Psi^{L,k_2,b}_{(J_2M_2)}(\mathbf{x})\Psi^{L,k_3}_{(J_3M_3)ab}(\mathbf{x}) = \frac{1}{\sqrt{6}}\mathcal{G}^{J_3J_2J_1}_{M_3M_2M_1}\left\{\int r^2 dr j_{J_1}(k_1r)\left[\frac{k_2}{k_1}\left(j^{(L,L)}_{J_2,t}(k_2r)\, j^{(L,L)}_{J_3,t}(k_3r)\right.\right.\right.$$
$$\left. -\frac{\langle J_3 1 J_2, -1|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle}j^{(L,VE)}_{J_2,t}(k_2r)\, j^{(L,VE)}_{J_3,t}(k_3r) + \frac{\langle J_3 2 J_2, -2|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle}j^{(L,TE)}_{J_2,t}(k_2r)\, j^{(L,TE)}_{J_3,t}(k_3r)\right)$$
$$\left.\left. -\frac{k_3}{k_1}\left(j^{(L,L)}_{J_2,v}(k_2r)\, j^{(L,L)}_{J_3,v}(k_3r) - \frac{\langle J_3 1 J_2, -1|J_1 0\rangle}{\langle J_3 0 J_2 0|J_1 0\rangle}j^{(L,E)}_{J_2,v}(k_2r)\, j^{(L,E)}_{J_3,v}(k_3r)\right)\right] + (1 \leftrightarrow 2)\right\}, \tag{87}$$

where the various radial functions have been defined in Eq. (25) in Eq. (26).

We have thus calculated the overlaps of a traceless-tensor TAM wave and two scalar (or longitudinal-vector) TAM waves for the longitudinal, vector, and transverse-tensor components. These results, when inserted into Eq. (74) or Eq. (74), give the TAM bispectra for a traceless tensor field with either two longitudinal fields or two scalar fields.

VII. CONCLUSIONS

In this paper, we have calculated several bispectra for scalar, vector, and tensor fields in the total-angular-momentum formalism. We began with the scalar-scalar-scalar bispectrum. We then considered an example of a vector-vector-scalar (where here vector is a transverse vector) bispectrum and a tensor-tensor-scalar (where here tensor is a transverse-traceless tensor). The vector-vector-scalar bispectrum is of the form that may arise from the correlation of a magnetic field with a scalar field [11], while the tensor-tensor-scalar correlation is precisely the same form that arises in inflation [13]. We obtained bispectra for TAM waves in the E/B basis (for vector fields) and the TE/TB basis (for the tensor fields) through intermediate steps that involved the TAM-wave helicity basis. We then moved on to calculate the TAM-wave three-point function for a tensor-scalar-scalar correlation that comes from a tensor-scalar-scalar bispectrum of precisely the same form that arises in inflation [13]. For completeness (and to follow through in Ref. [25] on CMB signatures of correlations of the form considered in Ref. [15]), we have also considered the bispectra for two scalars and either the longitudinal of vector components of the traceless tensor field.

type of correlation	Fourier-space bispectrum	TAM-wave result
scalar-scalar-scalar	(32)	(39)
vector-vector-scalar (transverse vectors)	(45)	(46) and (52)–(54)
tensor(T)-tensor(T)-scalar	(56)	(59) and (69)–(71)
tensor(T)-scalar-scalar	(72)	(75), (80), and (81)
tensor(T)-vector-vector (longitudinal vectors)	(73)	(74), (80), and (81)
tensor(V)-scalar-scalar	(72)	(75), (80), and (81)
tensor(V)-vector-vector (longitudinal vectors)	(73)	(74), (84), and (85)
tensor(L)-scalar-scalar	(72)	(75) and (87)
tensor(L)-vector-vector (longitudinal vectors)	(73)	(74) and (87)

TABLE I: List of the types of three-point functions we consider, the equation where the Fourier-space bispectrum we consider is defined, and the equations that contain the central results for the TAM-wave three-point functions. We distinguish between longitudinal and transverse vectors. The labels T, V and L refer to transverse tensorial, transverse vectorial and longitudinal part of a traceless tensor field respectively.

In the plane-wave formalism, the three-point correlation in Fourier space is parametrized by a bispectrum $B(k_1, k_2, k_3)$ that depends on the three wavenumber magnitudes k_1, k_2, k_3 only, multiplied by a "momentum-conserving" Dirac delta function, a consequence of statistical homogeneity. In the TAM formalism, generically the three-point correlation is parametrized by exactly the same bispectrum, but multiplied by an "angular-momentum-conserving" Clebsch-Gordan coefficient (which we write in shorthand as a Gaunt factor), a consequence of statistical isotropy (the Wigner-Eckart theorem), and an integral over radial profiles which does not depend on the azimuthal quantum numbers. Statistical homogeneity along the radial direction is encoded in the integral over the radial profiles.

Since the E/B and TE/TB TAM waves have definite parity, parity conservation is manifest in the bispectrum (determined by whether $J_1 + J_2 + J_3$ is even or odd). This is to be contrasted with the plane-wave formalism, where the two linear polarizations of a transverse-vector or a transverse-traceless-tensor are identified with each other by a simple rotation about the direction of wave propagation. In the TAM formalism, the E/B modes have different geometrical factors and different radial integrals in the bispectrum, despite the fact that their power spectra must coincide by statistical isotropy [17]. This means that E/B modes correlate to other fields differently when cubic interactions are taken into consideration. They thus have different bispectra.

Our survey of bispectra was not at all exhaustive. We showed in Section IV A, for example, that vector-vector-scalar correlations can be parametrized, assuming parity invariance, in terms of two bispectra, and we then considered one of those. There may likewise be additional types of bispectra involving tensor fields and beyond those that we considered here. There are also additional forms for bispectra that may arise if we allow for parity violation, and in this case, the helicity-basis TAM waves we developed in Ref. [17] should be particularly appropriate.

The value of the TAM-wave bispectra we have discussed here becomes apparent when we realize that many cosmological measurements are performed on the full sky, or over wide-angle surveys. Angular correlations are then decomposed into angular power spectra C_J (usually written as C_l) parametrized by multipoles J. The advantage of TAM waves is that once the proper scalar, vector, or tensor TAM waves are identified, rotational symmetry guarantees that the observable C_J will receive contributions only from TAM waves with that same J. In this sense, TAM waves provide a more natural choice of basis functions than the conventionally used plane waves. The advantage for three-point functions is that once the proper TAM waves are identified, the results we have derived in this paper provide the TAM-wave angular bispectra in terms of the more commonly seen Fourier-space bispectra. Since observable angular bispectra will be obtained from projections of the TAM-wave bispectra, the $J_1, M_1, J_2, M_2, J_3, M_3$ dependences of the

angular bispectra can be read off directly from the results we have presented here. For calculating CMB observables from bispectra involving vector or tensor fields, the TAM approach is particularly powerful in that equivalent results can be obtained without any Wigner-6j or Wigner-9j symbol, nor does any additional summation over the orbital angular momentum arise, as opposed to using plane waves.

One example of the utility of this formalism will be provided in forthcoming work [25] where we calculate the bipolar power spectrum of the cosmic microwave background that arises from scalar, vector, and tensor distortions to the local density two-point autocorrelation function of the form discussed in Ref. [15]. In other work [26] we will show how these TAM-wave bispectra can be used to construct estimators from wide-angle galaxy surveys (including redshift-space distortions) for the types of distortions considered in Ref. [15].

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Appendix A: TAM-wave overlap integrals in the helicity basis

In this Section, we calculate the overlap of three TAM waves in the helicity basis. We first calculate the overlap of two transverse-vector TAM waves and one scalar TAM wave in Section A 1 and then the overlap of two transverse-tensor TAM waves and one scalar TAM wave in Section A 2.

Our starting point will be the overlap [27],

$$\int d^2\hat{\mathbf{n}}\, {}_{s_1}Y_{(l_1m_1)}(\hat{\mathbf{n}})_{s_2}Y_{(l_2m_2)}(\hat{\mathbf{n}})_{s_3}Y_{(l_3m_3)}(\hat{\mathbf{n}}) = (-1)^{l_1+l_2+l_3+s_3}\mathcal{G}^{l_1l_2l_3}_{m_1m_2m_3}\frac{\langle l_1s_1l_2s_2|l_3-s_3\rangle}{\langle l_10l_20|l_30\rangle}, \tag{A1}$$

of three spin-s spherical harmonics, which holds for $s_1+s_2+s_3=0$.

1. Two transverse vector TAM-waves and one scalar TAM wave

Let us first calculate

$$\int d^2\hat{\mathbf{n}} g^{ab} Y^{\lambda_1}_{(J_1M_1)a}(\hat{\mathbf{n}})\, Y^{\lambda_2}_{(J_2M_2)b}(\hat{\mathbf{n}})\, Y_{(J_3M_3)}(\hat{\mathbf{n}}), \tag{A2}$$

for $\lambda_1,\lambda_2 = 0,\pm1$. We use the completeness relation for the spherical basis,

$$g^{ab} = \sum_\lambda \hat{\varepsilon}^a_\lambda(\hat{\mathbf{n}})\hat{\varepsilon}^{b*}_\lambda(\hat{\mathbf{n}}) = \sum_\lambda (-1)^\lambda\, \hat{\varepsilon}^a_\lambda(\hat{\mathbf{n}})\hat{\varepsilon}^b_{-\lambda}(\hat{\mathbf{n}}). \tag{A3}$$

The g^{ab} on the far left does not really depend on $\hat{\mathbf{n}}$, but the decomposition can be done at each $\hat{n}$. Then, we find

$$\begin{aligned}\int d^2\hat{\mathbf{n}} g^{ab} Y^{\lambda_1}_{(J_1M_1)a}(\hat{\mathbf{n}})Y^{\lambda_2}_{(J_2M_2)b}(\hat{\mathbf{n}})Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= \sum_\lambda (-1)^\lambda \int d^2\hat{\mathbf{n}}\hat{\varepsilon}^a_\lambda(\hat{\mathbf{n}})\hat{\varepsilon}^b_{-\lambda}(\hat{\mathbf{n}})Y^{\lambda_1}_{(J_1M_1)a}(\hat{\mathbf{n}})Y^{\lambda_2}_{(J_2M_2)b}(\hat{\mathbf{n}})Y_{(J_3M_3)}(\hat{\mathbf{n}}) \\ &= (-1)^{\lambda_1}\,\delta_{\lambda_1,-\lambda_2}\int d^2\hat{\mathbf{n}}_{-\lambda_1}Y_{(J_1M_1)}(\hat{\mathbf{n}})_{\lambda_1}Y_{(J_2M_2)}(\hat{\mathbf{n}})Y_{(J_3M_3)}(\hat{\mathbf{n}}).\end{aligned} \tag{A4}$$

Here, we use Eq. (52) of Ref. [17] to relate the vector spherical harmonics in the helicity basis to the spin-weighted spherical harmonics [18]

$$\hat{\varepsilon}^a_{\lambda'}(\hat{\mathbf{n}})Y^\lambda_{(JM)a}(\hat{\mathbf{n}}) = {}_{-\lambda}Y_{(JM)}(\hat{\mathbf{n}})\delta_{\lambda\lambda'}. \tag{A5}$$

Since the sum of the three spins in the three spin-weighted spherical harmonics in Eq. (A4) is zero, we use Eq. (A1) to calculate the angular integral of three helicity-basis TAM waves to be,

$$\int d^2\hat{\mathbf{n}} g^{ab} Y^{\lambda_1}_{(J_1M_1)a}(\hat{\mathbf{n}})Y^{\lambda_2}_{(J_2M_2)b}(\hat{\mathbf{n}})Y_{(J_3M_3)}(\hat{\mathbf{n}}) = (-1)^{\lambda_1}\,\delta_{\lambda_1,-\lambda_2}\mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}\frac{\langle J_1\lambda_1J_2,-\lambda_1|J_30\rangle}{\langle J_10J_20|J_30\rangle}. \tag{A6}$$

Due to the Kronecker delta, the only non-zero combinations are

$$\begin{aligned}
\int d^2\hat{\mathbf{n}} g^{ab} Y^0_{(J_1M_1)a}(\hat{\mathbf{n}}) Y^0_{(J_2M_2)b}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3}, \\
\int d^2\hat{\mathbf{n}} g^{ab} Y^{+1}_{(J_1M_1)a}(\hat{\mathbf{n}}) Y^{-1}_{(J_2M_2)b}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= -\mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \frac{\langle J_1 1 J_2, -1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}, \\
\int d^2\hat{\mathbf{n}} g^{ab} Y^{-1}_{(J_1M_1)a}(\hat{\mathbf{n}}) Y^{+1}_{(J_2M_2)b}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= -\mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \frac{\langle J_1, -1, J_2 1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}.
\end{aligned} \tag{A7}$$

2. Two transverse tensor TAM-waves and one scalar TAM wave

We now calculate the helicity-basis overlap integral,

$$\int d^2\hat{\mathbf{n}} I^{ab,cd} Y^{\lambda_1}_{(J_1M_1)ab}(\hat{\mathbf{n}}) Y^{\lambda_2}_{(J_2M_2)cd}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}), \tag{A8}$$

for $\lambda_1, \lambda_2 = 0, \pm 1, \pm 2$. With the completeness relation,

$$I^{ab,cd} = \sum_\lambda \hat{\varepsilon}^{ab}_\lambda(\hat{\mathbf{n}}) \hat{\varepsilon}^{cd*}_\lambda(\hat{\mathbf{n}}) = \sum_\lambda (-1)^\lambda \hat{\varepsilon}^{ab}_\lambda(\hat{\mathbf{n}}) \hat{\varepsilon}^{cd}_{-\lambda}(\hat{\mathbf{n}}), \tag{A9}$$

and Eq. (100) of Ref. [17] (the relation between tensor spherical harmonics and the spin-weight $s = 2$ spherical harmonics),

$$\hat{\varepsilon}^{ab}_{\lambda'}(\hat{\mathbf{n}}) Y^\lambda_{(JM)ab}(\hat{\mathbf{n}}) = {}_{-\lambda}Y_{(JM)}(\hat{\mathbf{n}}) \delta_{\lambda\lambda'}, \tag{A10}$$

the integration becomes

$$\begin{aligned}
\int d^2\hat{\mathbf{n}}\, I^{ab,cd} Y^{\lambda_1}_{(J_1M_1)ab}(\hat{\mathbf{n}}) Y^{\lambda_2}_{(J_2M_2)cd}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) &= \sum_\lambda (-1)^\lambda \int d^2\hat{n} \hat{\varepsilon}^{ab}_\lambda(\hat{\mathbf{n}}) \hat{\varepsilon}^{cd}_{-\lambda}(\hat{\mathbf{n}}) Y^{\lambda_1}_{(J_1M_1)ab}(\hat{\mathbf{n}}) Y^{\lambda_2}_{(J_2M_2)cd}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) \\
&= (-1)^{\lambda_1} \delta_{\lambda_1,-\lambda_2} \int d^2\hat{\mathbf{n}}_{-\lambda_1} Y_{(J_1M_1)}(\hat{\mathbf{n}})_{\lambda_1} Y_{(J_2M_2)}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}).
\end{aligned} \tag{A11}$$

Then we find

$$\int d^2\hat{\mathbf{n}}\, Y^{\lambda_1,ab}_{(J_1M_1)}(\hat{\mathbf{n}}) Y^{\lambda_2}_{(J_2M_2)ab}(\hat{\mathbf{n}}) Y_{(J_3M_3)}(\hat{\mathbf{n}}) = (-1)^{\lambda_1} \delta_{\lambda_1,-\lambda_2} \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \frac{\langle J_1 \lambda_1 J_2, -\lambda_1|J_3 0\rangle}{\langle J_1 0 J_2 0|J_3 0\rangle}. \tag{A12}$$

Appendix B: A semi-classical interpretation

In this Appendix, we briefly discuss a semi-classical picture which allows us to interpret our results for the three-point function involving two transverse vectors and one scalar in the limit of large angular momentum.

In Sec. IV, we presented three-point correlations involving two transverse vectors and one scalar, namely Eq. (46), together with the overlap integrals in Eqs. (52), (53), and (54). Some re-arrangment of those results enables us to rewrite them as

$$\begin{aligned}
\left\langle V^{E,k_1}_{(J_1M_1)} V^{E,k_2}_{(J_2M_2)} \phi^{k_3}_{(J_3M_3)} \right\rangle &= -(4\pi)^3 (-i)^{J_1+J_2+J_3} B_{vvs}(k_1,k_2,k_3) \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \\
&\times \int r^2 dr \left[\frac{[1]+[2]-[3]}{2\sqrt{[1][2]}} j^{(E,E)}_{J_1,v}(k_1 r) j^{(E,E)}_{J_2,v}(k_2 r) + \sqrt{[1][2]} \frac{j_{J_1}(k_1 r)}{k_1 r} \frac{j_{J_2}(k_2 r)}{k_2 r} \right] j_{J_3}(k_3 r),
\end{aligned} \tag{B1}$$

$$\begin{aligned}
\langle V^{B,k_1}_{(J_1M_1)} V^{B,k_2}_{(J_2M_2)} \phi^{k_3}_{(J_3M_3)} \rangle &= (4\pi)^3 (-i)^{J_1+J_2+J_3} B_{vvs}(k_1,k_2,k_3) \frac{[1]+[2]-[3]}{2\sqrt{[1][2]}} \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \\
&\times \int r^2 dr j_{J_1}(k_1 r) j_{J_2}(k_2 r) j_{J_3}(k_3 r),
\end{aligned} \tag{B2}$$

$$\langle V^{B,k_1}_{(J_1M_1)} V^{E,k_2}_{(J_2M_2)} \phi^{k_3}_{(J_3M_3)} \rangle = -(4\pi)^3 (-i)^{J_1+J_2+J_3} B_{vvs}(k_1,k_2,k_3) \mathcal{G}^{J_1J_2J_3}_{M_1M_2M_3} \frac{\langle J_1 0 J_2 0 | J_3 - 1, 0\rangle}{\langle J_1 0 J_2 0 | J_3 0\rangle} \sqrt{\frac{2J_3+1}{2J_3-1}}$$
$$\times \frac{\sqrt{(-J_1+J_2+J_3)(J_1-J_2+J_3)(J_1+J_2-J_3+1)(J_1+J_2+J_3+1)}}{2\sqrt{[1][2]}}$$
$$\times \int r^2 dr j_{J_1}(k_1 r) j^{(E,E)}_{J_2,v}(k_2 r) j_{J_3}(k_3 r). \tag{B3}$$

Here we have introduced the shorthand notation $[i] \equiv J_i(J_i+1)$ for $i = 1, 2, 3$. These three-point functions have now been put into forms comparable to the three-point function, Eq. (39), for three scalars. We have a bispectrum function in terms of wave vectors with some suitable multiplying prefactor, a Gaunt integral in accord with the Wigner-Eckart theorem, and an integral of three radial profiles. For a vector-vector-scalar correlation, we have an extra geometric factor made of angular momenta J_1, J_2, and J_3. In the limit of large angular momenta, i.e. $J_1, J_2, J_3 \gg 1$, these factors are reduced to cosine and sine

$$\frac{[1]+[2]-[3]}{2\sqrt{[1][2]}} \approx \cos\theta_{12}, \qquad \frac{\sqrt{(-J_1+J_2+J_3)(J_1-J_2+J_3)(J_1+J_2-J_3+1)(J_1+J_2+J_3+1)}}{2\sqrt{[1][2]}} \approx \sin\theta_{12}. \tag{B4}$$

where θ_{12} is the angle between J_1 and J_2, in a triangle whose three sides have length J_1,J_2 and J_3, repectively.

These geometric factors, interpreted in a semi-classical way, can be attributed to the vector structure of vector TAM waves. First consider the case of two B-mode transverse vectors. The three-point function is proportional to the overlap,

$$\int d^3x \Psi^{k_1,B,a}_{(J_1M_1)}(\mathbf{x}) \Psi^{k_2,B}_{(J_2M_2)a}(\mathbf{x}) \Psi^{k_3}_{(J_3M_3)}(\mathbf{x}) = \int d^3x \left(\frac{-i\hat{\mathbf{L}}}{\sqrt{[1]}} \Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\right) \cdot \left(\frac{-i\hat{\mathbf{L}}}{\sqrt{[2]}} \Psi^{k_2}_{(J_2M_2)}(\mathbf{x})\right) \Psi_{(J_3M_3)}(\mathbf{x})$$
$$\approx -\frac{1}{\sqrt{[1][2]}} \int d^3x \mathbf{J}_1 \Psi_{(J_1M_1)} \cdot \mathbf{J}_2 \Psi_{(J_2M_2)} \Psi_{(J_3M_3)}$$
$$= -\frac{1}{\sqrt{[1][2]}} \sqrt{[1]}\sqrt{[2]} \cos\theta_{12} \int d^3x \Psi_{(J_1M_1)} \Psi_{(J_2M_2)} \Psi_{(J_3M_3)}$$
$$= -\cos\theta_{12} \int d^3x \Psi_{(J_1M_1)} \Psi_{(J_2M_2)} \Psi_{(J_3M_3)}, \tag{B5}$$

where according to the construction in Ref. [17] we have used $\Psi^{k,B}_{(JM)a}(\mathbf{x}) = K_a \Psi^k_{(JM)}(\mathbf{x})/\sqrt{J(J+1)}$, with $K_a = -i\hat{L}_a$ and $\hat{L}_a$ being the orbital angular-momentum operator. In the second line, we associate two classical angular-momentum vectors $\mathbf{J}_1, \mathbf{J}_2$, of magnitude $J_1(J_1+1)$ and $J_2(J_2+1)$ respectively, with the first two TAM waves. Note that $\mathbf{J}_1$ and $\mathbf{J}_2$ are not fixed vectors, since they precess about the z axis. But by angular-momentum conservation they always differ by a third angular-momentum vector $\mathbf{J}_3$ of magnitude $J_3(J_3+1)$ that is associated with the third TAM wave, which also precesses about the z axis, as shown in Fig. 1. With this picture, the cosine factor then arises naturally.

The case of one B mode and one E mode can be analyzed similarly,

$$\int d^3x \Psi^{k_1,B,a}_{(J_1M_1)}(\mathbf{x}) \Psi^{k_2,E}_{(J_2M_2)a}(\mathbf{x}) \Psi^{k_3}_{(J_3M_3)}(\mathbf{x}) = \int d^3x \left(\frac{-i\hat{\mathbf{L}}}{\sqrt{[1]}} \Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\right) \cdot \left(\frac{\nabla \times \hat{\mathbf{L}}}{k_2\sqrt{[2]}} \Psi^{k_2}_{(J_2M_2)}(\mathbf{x})\right) \Psi_{(J_3M_3)}(\mathbf{x})$$
$$\approx \frac{1}{\sqrt{[1][2]}} \int d^3x \mathbf{J}_1 \Psi_{(J_1M_1)} \cdot \left(\hat{\mathbf{k}}_2 \times \mathbf{J}_2 \Psi_{(J_2M_2)}\right) \Psi_{(J_3M_3)}$$
$$= \left(\hat{\mathbf{J}}_1 \cdot \left(\hat{\mathbf{k}}_2 \times \hat{\mathbf{J}}_2\right)\right) \int d^3x \Psi_{(J_1M_1)} \Psi_{(J_2M_2)} \Psi_{(J_3M_3)}$$
$$= -\left(\left(\hat{\mathbf{J}}_1 \times \hat{\mathbf{J}}_2\right) \cdot \hat{\mathbf{k}}_2\right) \int d^3x \Psi_{(J_1M_1)} \Psi_{(J_2M_2)} \Psi_{(J_3M_3)}, \tag{B6}$$

where this time we have also used $\Psi^{k,B}_{(JM)a}(\mathbf{x}) = M_a \Psi^k_{(JM)}(\mathbf{x})/\sqrt{J(J+1)}$, with $M_a = i\varepsilon_{abc}\nabla^b K^c/k$. We clearly see a factor $|\hat{\mathbf{J}}_1 \times \hat{\mathbf{J}}_1| = \sin\theta_{12}$, which is the sine we find in the large-J limit.

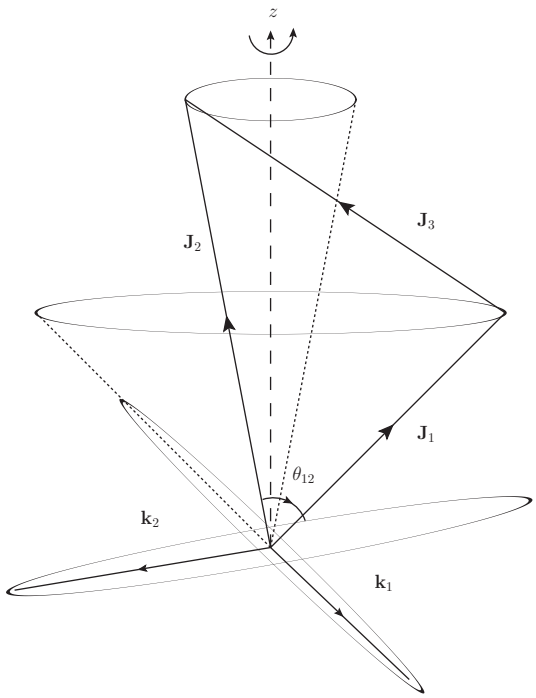

FIG. 1: Geometry of the semi-classical picture: three classical angular-momentum vectors $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}_3$ form a triangle and precess about the chosen z-axis, associated with the three TAM waves of wave numbers k_1, k_2, and k_3 respectively. The angle θ_{12} between $\mathbf{J}_1$ and $\mathbf{J}_2$, however, remains the same. The wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, which are not definitive for TAM waves, are perpendicular to angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$, respectively.

To close, we consider the case of two E-mode vectors

$$\begin{aligned}\int d^3x \Psi^{k_1,E,a}_{(J_1M_1)}(\mathbf{x})\Psi^{k_2,E}_{(J_2M_2)a}(\mathbf{x})\Psi^{k_3}_{(J_3M_3)}(\mathbf{x}) &= \int d^3x \left(\frac{\nabla\times\hat{\mathbf{L}}}{k_1\sqrt{[1]}}\Psi^{k_1}_{(J_1M_1)}(\mathbf{x})\right)\cdot\left(\frac{\nabla\times\hat{\mathbf{L}}}{k_2\sqrt{[2]}}\Psi^{k_2}_{(J_2M_2)}(\mathbf{x})\right)\Psi_{(J_3M_3)}(\mathbf{x}) \\ &\approx -\frac{1}{\sqrt{[1][2]}}\int d^3x \left(\hat{\mathbf{k}}_1\times\mathbf{J}_1\right)\Psi_{(J_1M_1)}\cdot\left(\hat{\mathbf{k}_2}\times\mathbf{J}_2\Psi_{(J_2M_2)}\right)\Psi_{(J_3M_3)} \\ &\approx \left(\hat{\mathbf{k}}_1\times\hat{\mathbf{J}}_1\right)\cdot\left(\hat{\mathbf{k}}_2\times\hat{\mathbf{J}}_2\right)\int d^3x\Psi_{(J_1M_1)}\Psi_{(J_2M_2)}\Psi_{(J_3M_3)} \\ &= \left[\left(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}_2}\right)\left(\hat{\mathbf{J}}_1\cdot\hat{\mathbf{J}}_2\right)-\left(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{J}}_2\right)\left(\hat{\mathbf{J}}_1\cdot\hat{\mathbf{k}}_2\right)\right]\int d^3x\Psi_{(J_1M_1)}\Psi_{(J_2M_2)}\Psi_{(J_3M_3)}.\end{aligned} \tag{B7}$$

We then recognize a term proportional to $\hat{\mathbf{J}}_1\cdot\hat{\mathbf{J}}_2$, i.e. $\cos\theta_{12}$, and a second term which has no such factor. It closely resembles the structure in the exact result Eq. (B1).

So far, we have restricted our discussion to the three-point function involving two transverse vectors and one scalar. Besides, the semi-classical picture we have proposed is still insufficient to quantitatively pin down the correct forms of the radial integral and the J_1, J_2, J_3 dependence. However, we may gain insight from this picture for other three-point functions, such as the ones involving transverse-tensor field.

[1] A. H. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); A. A. Starobinsky, Phys. Lett. B **117**, 175 (1982); J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[2] T. Falk, R. Rangarajan and M. Srednicki, Astrophys. J. **403**, L1 (1993) [arXiv:astro-ph/9208001]; A. Gangui *et al.*, Astrophys. J. **430**, 447 (1994) [arXiv:astro-ph/9312033]; A. Gangui, Phys. Rev. D **50**, 3684 (1994) [arXiv:astro-ph/9406014]; V. Acquaviva, N. Bartolo, S. Matarrese and A. Riotto, Nucl. Phys. B **667**, 119 (2003) [arXiv:astro-ph/0209156].

[3] T. J. Allen, B. Grinstein and M. B. Wise, Phys. Lett. B **197**, 66 (1987); L. A. Kofman and D. Y. Pogosian, Phys. Lett. B **214**, 508 (1988); D. S. Salopek, J. R. Bond and J. M. Bardeen, Phys. Rev. D **40**, 1753 (1989); A. D. Linde and V. F. Mukhanov, Phys. Rev. D **56**, 535 (1997) [arXiv:astro-ph/9610219]; P. J. E. Peebles, Astrophys. J. **510**, 523 (1999) [arXiv:astro-ph/9805194]; P. J. E. Peebles, Astrophys. J. **510**, 531 (1999) [arXiv:astro-ph/9805212].

[4] S. Mollerach, Phys. Rev. D **42**, 313 (1990); A. D. Linde and V. F. Mukhanov, Phys. Rev. D **56**, 535 (1997) [arXiv:astro-ph/9610219]; D. H. Lyth and D. Wands, Phys. Lett. B **524**, 5 (2002) [arXiv:hep-ph/0110002]; T. Moroi and T. Takahashi, Phys. Lett. B **522**, 215 (2001) [Erratum-ibid. B **539**, 303 (2002)] [arXiv:hep-ph/0110096]; D. H. Lyth, C. Ungarelli and D. Wands, Phys. Rev. D **67**, 023503 (2003) [arXiv:astro-ph/0208055]; K. Ichikawa *et al.*, arXiv:0802.4138 [astro-ph]; K. Enqvist, S. Nurmi, O. Taanila and T. Takahashi, arXiv:0912.4657 [astro-ph.CO]; K. Enqvist and T. Takahashi, JCAP **0912**, 001 (2009) [arXiv:0909.5362 [astro-ph.CO]]; K. Enqvist and T. Takahashi, JCAP **0809**, 012 (2008) [arXiv:0807.3069 [astro-ph]]; K. Enqvist and S. Nurmi, JCAP **0510**, 013 (2005) [arXiv:astro-ph/0508573]; A. L. Erickcek, M. Kamionkowski and S. M. Carroll, arXiv:0806.0377 [astro-ph]; A. L. Erickcek, C. M. Hirata and M. Kamionkowski, Phys. Rev. D **80**, 083507 (2009) [arXiv:0907.0705 [astro-ph.CO]]; S. Hannestad, T. Haugboelle, P. R. Jarnhus and M. S. Sloth, arXiv:0912.3527 [hep-ph].

[5] G. R. Dvali and S. H. H. Tye, Phys. Lett. B **450**, 72 (1999) [arXiv:hep-ph/9812483]; P. Creminelli, JCAP **0310**, 003 (2003) [arXiv:astro-ph/0306122]; M. Alishahiha, E. Silverstein and D. Tong, Phys. Rev. D **70**, 123505 (2004) [arXiv:hep-th/0404084].

[6] A. Vilenkin and E. P. S. Shellard "Cosmic Strings and other Topological Defects," (Cambridge University Press, Cambridge, 1994); R. Durrer, New Astron. Rev. **43**, 111 (1999); R. Durrer, M. Kunz and A. Melchiorri, Phys. Rept. **364**, 1 (2002) [arXiv:astro-ph/0110348]; A. H. Jaffe, Phys. Rev. D **49**, 3893 (1994) [arXiv:astro-ph/9311023]; A. Silvestri and M. Trodden, Phys. Rev. Lett. **103**, 251301 (2009) [arXiv:0811.2176 [astro-ph]]; M. Hindmarsh, C. Ringeval and T. Suyama, Phys. Rev. D **80**, 083501 (2009) [arXiv:0908.0432 [astro-ph.CO]]; M. Hindmarsh, C. Ringeval and T. Suyama, arXiv:0911.1241 [astro-ph.CO]; D. M. Regan and E. P. S. Shellard, arXiv:0911.2491 [astro-ph.CO]; N. Turok and D. N. Spergel, Phys. Rev. Lett. **66**, 3093 (1991); D. G. Figueroa, R. R. Caldwell and M. Kamionkowski, arXiv:1003.0672 [astro-ph.CO].

[7] N. Bartolo *et al.*, Phys. Rept. **402**, 103 (2004) [arXiv:astro-ph/0406398].

[8] V. A. Rubakov, M. V. Sazhin and A. V. Veryaskin, Phys. Lett. B **115**, 189 (1982); R. Fabbri and M. d. Pollock, Phys. Lett. B **125**, 445 (1983); L. F. Abbott and M. B. Wise, Nucl. Phys. B **244**, 541 (1984); A. A. Starobinsky, Sov. Astron. Lett. **11**, 133 (1985).

[9] A. Lewis, Phys. Rev. **D70**, 043518 (2004). [astro-ph/0403583]; K. Dimopoulos, M. Karciauskas, D. H. Lyth and Y. Rodriguez, JCAP **0905**, 013 (2009) [arXiv:0809.1055 [astro-ph]]; A. Golovnev and V. Vanchurin, Phys. Rev. D **79**, 103524 (2009) [arXiv:0903.2977 [astro-ph.CO]]; N. Bartolo, E. Dimastrogiovanni, S. Matarrese and A. Riotto, JCAP **0911**, 028 (2009) [arXiv:0909.5621 [astro-ph.CO]]; B. Himmetoglu, C. R. Contaldi and M. Peloso, Phys. Rev. Lett. **102**, 111301 (2009) [arXiv:0809.2779 [astro-ph]]; B. Himmetoglu, C. R. Contaldi and M. Peloso, Phys. Rev. D **80**, 123530 (2009) [arXiv:0909.3524 [astro-ph.CO]].

[10] M. S. Turner, L. M. Widrow, Phys. Rev. **D37**, 2743 (1988); B. Ratra, Astrophys. J. **391**, L1 (1992); A. Kandus, K. E. Kunze and C. G. Tsagas, Phys. Rept. **505**, 1 (2011) [arXiv:1007.3891 [astro-ph.CO]].

[11] R. R. Caldwell, L. Motta and M. Kamionkowski, Phys. Rev. D **84**, 123525 (2011) [arXiv:1109.4415 [astro-ph.CO]]; L. Motta and R. R. Caldwell, Phys. Rev. D **85**, 103532 (2012) [arXiv:1203.1033 [astro-ph.CO]].

[12] N. Barnaby, R. Namba and M. Peloso, Phys. Rev. D **85**, 123523 (2012) [arXiv:1202.1469 [astro-ph.CO]]; N. Bartolo, S. Matarrese, M. Peloso and A. Ricciardone, arXiv:1210.3257 [astro-ph.CO].

[13] J. M. Maldacena, JHEP **0305**, 013 (2003) [astro-ph/0210603].

[14] D. Seery, M. S. Sloth and F. Vernizzi, JCAP **0903**, 018 (2009) [arXiv:0811.3934 [astro-ph]].

[15] D. Jeong and M. Kamionkowski, arXiv:1203.0302 [astro-ph.CO].

[16] M. Shiraishi, S. Yokoyama, D. Nitta, K. Ichiki and K. Takahashi, Phys. Rev. D **82**, 103505 (2010) [arXiv:1003.2096 [astro-ph.CO]]; M. Shiraishi, D. Nitta, S. Yokoyama, K. Ichiki and K. Takahashi, Prog. Theor. Phys. **125**, 795 (2011) [arXiv:1012.1079 [astro-ph.CO]]; M. Shiraishi, D. Nitta, S. Yokoyama, K. Ichiki and K. Takahashi, Phys. Rev. D **82**, 121302 (2010) [Erratum-ibid. D **83**, 029901 (2011)] [arXiv:1009.3632 [astro-ph.CO]]; M. Shiraishi, D. Nitta, S. Yokoyama, K. Ichiki and K. Takahashi, Phys. Rev. D **83**, 123003 (2011) [arXiv:1103.4103 [astro-ph.CO]]; M. Shiraishi and S. Yokoyama, Prog. Theor. Phys. **126**, 923 (2011) [arXiv:1107.0682 [astro-ph.CO]]; M. Shiraishi, D. Nitta and S. Yokoyama, Prog. Theor. Phys. **126**, 937 (2011) [arXiv:1108.0175 [astro-ph.CO]]; M. Shiraishi, D. Nitta, S. Yokoyama and K. Ichiki, JCAP **1203**, 041 (2012) [arXiv:1201.0376 [astro-ph.CO]]; M. Shiraishi, JCAP **1206**, 015 (2012) [arXiv:1202.2847 [astro-ph.CO]]; X. Gao, T. Kobayashi, M. Shiraishi, M. Yamaguchi, J. 'i. Yokoyama and S. Yokoyama, arXiv:1207.0588 [astro-ph.CO]; M. Shiraishi, S. Saga and S. Yokoyama, arXiv:1209.3384 [astro-ph.CO]; I. Brown and R. Crittenden, Phys. Rev. D **72**, 063002 (2005) [astro-ph/0506570]; S. Choudhury and S. Pal, arXiv:1210.4478 [hep-th]; P. McFadden and K. Skenderis, JCAP **1106**, 030 (2011) [arXiv:1104.3894 [hep-th]]; A. E. Gumrukcuoglu, B. Himmetoglu and M. Peloso, Phys. Rev. D **81**, 063528 (2010) [arXiv:1001.4088 [astro-ph.CO]]; P. Adshead and E. A. Lim, Phys. Rev. D **82**, 024023 (2010) [arXiv:0912.1615 [astro-ph.CO]]; J. Soda, H. Kodama and M. Nozawa, JHEP **1108**, 067 (2011) [arXiv:1106.3228 [hep-th]]; X. Gao, T. Kobayashi, M. Yamaguchi and J. Yokoyama, Phys. Rev. Lett. **107**, 211301 (2011) [arXiv:1108.3513 [astro-ph.CO]]; J. M. Maldacena and G. L. Pimentel, JHEP **1109**, 045 (2011) [arXiv:1104.2846 [hep-th]].

[17] L. Dai, M. Kamionkowski, and D. Jeong, arXiv:1209.0761 [astro-ph]. To appear in Phys. Rev. D.

[18] E. Newman and R. Penrose, J. Math. Phys. **7**, 863 (1966); J. N. Goldberg *et al.*, J. Math. Phys. **8**, 2155 (1967).

[19] M. Kamionkowski, A. Kosowsky and A. Stebbins, Phys. Rev. D **55**, 7368 (1997) [arXiv:astro-ph/9611125]; M. Kamionkowski, A. Kosowsky and A. Stebbins, Phys. Rev. Lett. **78**, 2058 (1997) [arXiv:astro-ph/9609132]; U. Seljak and M. Zaldarriaga, Phys. Rev. Lett. **78**, 2054 (1997) [arXiv:astro-ph/9609169]; A. Stebbins, astro-ph/9609149;

M. Kamionkowski, A. Babul, C. M. Cress and A. Refregier, Mon. Not. Roy. Astron. Soc. **301**, 1064 (1998) [astro-ph/9712030].
[20] M. Zaldarriaga and U. Seljak, Phys. Rev. D **55**.1830 (1997) [astro-ph/9609170].
[21] E. Komatsu and D. N. Spergel, Phys. Rev. D **63** (2001) 063002 [astro-ph/0005036].
[22] R. Mehrem, J. Londergan and M. Macfarlane, J. Phys. A: Math. Gen. **24**, 1435-1453 (1991)
[23] L. M. Wang and M. Kamionkowski, Phys. Rev. D **61**, 063504 (2000) [arXiv:astro-ph/9907431];
[24] M. Toyoda and T. Ozaki, Comp. Phys. Comm. **181**, 277-282 (2010)
[25] L. Dai, M. Kamionkowski, and D. Jeong, in preparation.
[26] P. Trivedi, L. Dai, D. Jeong, and M. Kamionkowski, in preparation
[27] W. Hu, Phys. Rev. D **62**, 043007 (2000) [astro-ph/0001303].