Equality of two definitions for transverse momentum dependent parton distribution functions

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Equality of Two Definitions for Transverse Momentum Dependent Parton Distribution Functions

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We compare recent, seemingly different, approaches to TMD-factorization (due to Echevarria, Idilbi, and Scimemi and to Collins), and show that they are the same, apart from an apparent difference in their definition of the $\overline{\text{MS}}$ renormalization scheme.

I. INTRODUCTION

In a number of processes, notably the Drell-Yan process at small transverse momentum, it is important to use transverse-momentum-dependent (TMD) parton distribution functions (pdfs) and an associated TMD factorization theorem of QCD. However, there is a variety of different formulations in the literature, with seemingly very different methodologies and different definitions of the TMD pdfs.

Some of the differences even amount to apparent incompatibilities. For example, in the formalism developed by Collins, Soper, and Sterman (CSS) [1, 2], TMD pdfs depend not only on the essential kinematic parameters x and k_T, and on a renormalization scale μ, but also on an extra auxiliary scale ζ. An essential role is played by the Collins-Soper (CS) evolution equation [1] of the pdfs with respect to ζ. In contrast, in the recent work by Echevarria, Idilbi, and Scimemi (EIS) [3] in the context of soft-collinear effective theory (SCET), there appears to be no corresponding parameter. Indeed, in a phenomenological application of this work by Echevarria, Idilbi, Schäfer, and Scimemi [4], there even appears the statement that they "do not have to solve Collins-Soper equation". Furthermore, in the recent improvement of the CSS method by one of us (JCC) in Ref. [5], the presence of the ζ parameter appears to be intimately associated with the use of non-light-like Wilson lines in JCC's gauge-invariant definition of the TMD pdfs. But all the Wilson lines in EIS's definition are light-like (with the aid of their δ-regulator in a limit that the regulator is removed). Finally, the definitions in the two methods appear very different: In the JCC method the finite TMD pdf is [5, Eq. (13.106)] a renormalized convolution product of a bare TMD pdf with a square root of three bare soft factors, two of them with the characteristic non-light-like Wilson lines. But in the EIS definition [3, Eq. (2.13)], the square root involves only one bare soft factor. JCC and EIS do agree on the presence of the square root.

In this paper, we show that nevertheless the EIS and JCC definitions give exactly the same TMD pdfs (and hence the same hard scattering factors in TMD factorization properties). We show how the EIS definitions do in fact have a parameter ζ that EIS have identified with Q^2. Thus their TMD pdfs obey a CS equation. Although the application of the EIS work in [4] does not explicitly use the CS equation, it actually uses equivalent results, the most notable of which are the results of Korchemsky and Radyushkin [6] on the Q-dependence of the anomalous dimension of a cusp anomalous dimension. We show how the JCC definition of the TMD pdfs can be cast into exactly the form of the EIS definition, in which no non-light-like Wilson lines appear and in which there is only a single bare soft factor instead of three. It should be noted that, in both cases, the bare factors are separately divergent because of the presence of light-like Wilson lines, which must be regulated, and a limit taken with the regulator removed. The EIS and JCC differ in the preferred method to regulate light-like Wilson lines. The EIS form of definition requires coordination of the rates at which are removed the regulators of different light-like Wilson lines.

EIS do say in [4] that "one can argue that the two [methods] are equivalent", but they do not provide any details of the argument.[1]

We expect that the methods developed in this paper can be applied to analyze and/or relate many other methods for formulating TMD factorization. A list of some relevant papers beyond those cited above is [8–14].

The reader needs to be very aware that the JCC method we use here differs in very important details from the earlier CS formalism on which it is based, and also that there was a change in candidate definitions of collinear factors in the course of the derivations in [5]. The definitions that we use are of the form given in [5, Sec. 10.11] for the Sudakov form factor, and in [5, Sec. 13.15.3] for TMD pdfs (with a modification to use past-pointing Wilson lines for the Drell-Yan process, as summarized at the beginning of [5, Sec. 14.5]).

The paper is structured as follows: In Sect. II we introduce notation and enumerate the common features of the EIS and JCC methods. In Sect. III, we prove that

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[1] See also [7], which appeared as we were completing this paper.

the EIS and JCC methods both give the same result for the cross section. We prove in Sect. IV that the JCC and EIS TMD PDFs are equal. In Sect. V, we identify the parameter in the EIS TMD PDFs that corresponds to ζ in the JCC method. We summarize our results in Sect. VII.

II. COMMON OVERALL STRUCTURE

We work with the Drell-Yan process: production of a high-mass lepton pair of 4-momentum q from a single virtual electroweak boson in a collision of hadrons of momenta p_A and p_B. The invariant mass of the lepton pair is $Q = \sqrt{q^\mu q_\mu}$, its rapidity is y, and its transverse momentum is q_T.

When $q_T \ll Q$, TMD factorization as formulated by both EIS and JCC is that the cross section is proportional to a Fourier transform in transverse coordinate space of TMD pdfs:

$$\frac{d\sigma}{d^4q} = \sum_{i,j} \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T \cdot \boldsymbol{q}_T}$$
$$\tilde{f}_{i/A}(x_A, b_T)\, \tilde{f}_{i/B}(x_B, b_T)\, H_{ij}(Q) + \text{p.s.c.} \quad (1)$$

Here, the sum is over the relevant parton flavors, $\tilde{f}_{i/A}$ and $\tilde{f}_{j/B}$ are Fourier transforms into transverse coordinate space of TMD PDFs, H_{ij} is a hard scattering, and "p.s.c." denotes "power-suppressed corrections". The corrections to the formula are either suppressed by a power of q_T/Q or by a power of M/Q, where M denotes a typical scale for light hadronic masses; those corrections suppressed only by a power of q_T/Q can be handled by normal factorization methods with integrated pdfs, as by the Y term in the CSS formalism.

To simplify the notation, we write the above structure as

$$\sigma = ABH, \quad (2)$$

with all the kinematic variables, flavor indices, and corrections left implicit. The intuitive interpretation is that the factors A and B contain dependence on momenta collinear to each of the corresponding beam particles.

As for the kinematics, we have $x_A x_B s = Q^2$ and the rapidity of the Drell-Yan pair in the center-of-mass frame of the beams is $y = \frac{1}{2}\ln\frac{x_A}{x_B}$. In light-front coordinates, the beam momenta are

$$p_A = \left(p_A^+, \frac{m_A^2}{2p_A^+}, 0_T\right), \qquad p_B = \left(\frac{m_B^2}{2p_B^-}, p_B^-, 0_T\right), \quad (3)$$

with p_A^+ and p_B^- large, so that $s \simeq 2p_A^+ p_B^-$. The plus momentum of the parton from A is $k_A^+ = x_A p_A^+$, and the minus momentum of the parton from B is $k_B^- = x_B p_B^-$.

The above statement of TMD factorization is used by both the JCC and EIS methods. However, in (1) we have indicated only the dependence on kinematic parameters

of the factors $\tilde{f}_{i/A}\ \tilde{f}_{j/B}$, and H_{ij}. In addition all three factors depend on auxiliary parameters. It is common to both methods that all the factors depend on the renormalization scale μ. There is one further parameter for the TMD pdfs in both methods.

In the JCC approach, each collinear factor also depends on a ζ parameter. This is constructed from an arbitrarily chosen rapidity parameter y_n, which has the intuitive purpose of separating right and left moving quanta. For A, the ζ parameter is

$$\zeta_A = 2(k_A^+)^2 e^{-2y_n} = 2(x_A p_A^+)^2 e^{-2y_n}, \quad (4)$$

and for B it is

$$\zeta_B = 2(k_B^-)^2 e^{2y_n} = 2(x_B p_B^-)^2 e^{2y_n}. \quad (5)$$

These two quantities obey $\zeta_A \zeta_B = Q^4$. In both the EIS and JCC approaches, neither the hard scattering H nor the cross section has dependence on the ζ variables.

In the case of the EIS method, the authors' statements and notation suggest that the TMD PDFs depend on μ but not on some other parameter like ζ — see the paragraph above [3, Eq. (1.6)], and see [3, Eq. (2.15)]. In fact, the EIS PDFs do depend on Q as a separate parameter distinct from μ. This can be seen from [3, Eq. (2.17)], where there is Q and μ dependence for the coefficient C_n that relates the TMD PDF to the ordinary integrated PDF. It can also be seen from the fact that the anomalous dimension of their TMD PDF depends on both Q and μ — see [3, Eq. (3.21)]. In later sections of this paper, we will show how the Q^2 parameter of the TMD pdfs with the EIS definition should in fact be identified with the ζ parameter of those with the JCC definition.

Common to both schemes is the property that the product of the two TMD functions is related to a combination of unsubtracted TMD functions and a soft function:

$$AB = \frac{A_0 B_0}{S_0} \times \text{UV renormalization}. \quad (6)$$

Note that all the authors agree on applying UV renormalization in the $\overline{\text{MS}}$ scheme to remove divergences where transverse momenta go to infinity [2]. Such divergences are all regulated by dimensional regularization.

Each of the unsubtracted functions is defined with the aid of light-like Wilson lines, taken as a limit [3]. The A_0 and B_0 are the "unsubtracted" TMD pdfs $\tilde{f}^{\text{unsub}}$ in [5,

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[2] But see Sec. VI for a possible difference in the definitions of the $\overline{\text{MS}}$ scheme.

[3] Note that these Wilson lines must be joined by transverse links at infinity to give exactly gauge-invariant quantities [15, 16]. Only with these transverse links can correct results be obtained in all gauges. Notably, the transverse links are essential to get the correct Sivers asymmetry in light-cone gauge [15, 16]. See Refs. [17, 18] for corresponding statements in SCET.

Eq. (13.106)], or the $\hat{f}_n$ functions in [3, Eq. (3.21)]. The S_0 is a light-cone regulated soft function, defined according to [5, Eq. (13.106)] and [3, Eq. (2.7)]. The division by S_0 cancels double counting of soft gluon contributions between the A_0 and B_0 factors and also cancels the rapidity divergences that would otherwise occur because light-like Wilson lines are used in A_0 and B_0. Since each of the factors in (6) separately contain light cone divergences, a meaning can only be given by applying regulators to the Wilson lines, and then taking the limit of removing the regulators in a manner consistent with factorization.

As explained in [5, Sec. 10.11.1], the limit of light-like Wilson lines and the removal of dimensional regularization do *not* commute. We will always define that the limit light-like Wilson lines is to be applied first. Then $\overline{\text{MS}}$ renormalization is applied and dimensional regularization removed. EIS's work appears to use this order as well.

At this point, the only relevant difference between the EIS and JCC treatments is the choice of method for regulating the divergences that come from using light-like Wilson lines. The JCC method uses non-light-like Wilson lines, and takes the limit that they becomes light-like:

$$A_{\text{JCC}}B_{\text{JCC}} = \lim_{\substack{y_1 \to +\infty \\ y_2 \to -\infty}} \frac{A_{\text{JCC},0}(y_2)\, B_{\text{JCC},0}(y_1)}{S_{\text{JCC},0}(y_1,y_2)}$$
$$\times \text{ UV renormalization.} \quad (7)$$

(This result follows directly from [5, Eqs. (13.106) and (14.31)].) The Wilson lines appear in Feynman diagram calculations as eikonal propagators of the form,

$$\frac{i}{-k^+ + e^{2y_2}k^- + i0}, \qquad \frac{i}{-e^{-2y_1}k^+ + k^- + i0}, \quad (8)$$

together with the appropriate vertex factors. Here, y_1 and y_2 are the finite Wilson line rapidities, and the signs are those appropriate for when the gluon momenta are oriented as in the one-loop graph for the Drell-Yan process in Fig. 1. Note that, in the JCC method the regulated factors are all matrix elements of exactly gauge-invariant operators.

The EIS approach [3] uses what they call a δ-regulator instead of (8), where the Wilson line propagators for regulated light-like lines have the following forms:

$$\frac{i}{-k^+ + i\delta^+}, \qquad \frac{i}{k^- + i\delta^-}. \quad (9)$$

Their equivalent of (7) is

$$A_{\text{EIS}}B_{\text{EIS}} = \lim_{\delta^+,\delta^- \to 0} \frac{A_{\text{EIS},0}(\delta^+)\, B_{\text{EIS},0}(\delta^-)}{S_{\text{EIS},0}(\delta^-,\delta^+)}$$
$$\times \text{ UV renormalization.} \quad (10)$$

That is, instead of using Eq. (8), they replace light-like propagators with ones with fixed non-zero auxiliary terms inserted into the denominators.
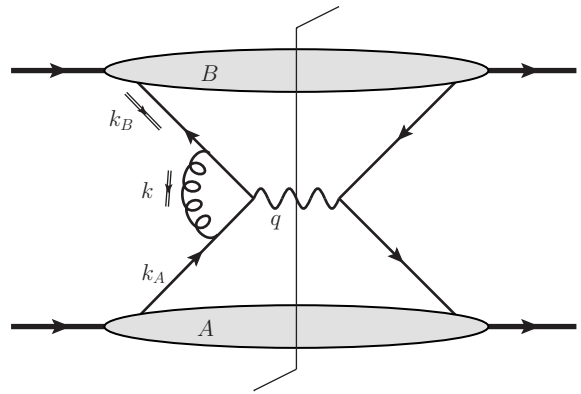

FIG. 1. One-loop graph contributing to the Drell-Yan process.

The right-hand-sides Eqs. (7) and (10) differ only in how the light-like Wilson lines are regulated, and therefore the limit as the regulator is removed must be the same in each case. The same UV renormalization prescription, $\overline{\text{MS}}$, is used. Hence the product of the two TMD pdfs on the left-hand-sides is the same, i.e., AB is the same in the EIS and JCC schemes. The remaining issue is to determine whether the individual TMD pdfs, A and B, are the same in the two schemes.

Our proof that this is indeed the case will use factorization properties for the unsubtracted factors, and one immediate consequence of these factorization properties is that the limits in Eqs. (7) and (10) do exist.

One complication is that EIS also use a corresponding regulator for the quark lines in their calculations, for example, [3, Sec. 3]. This provides a regulator for the collinear divergences associated with quark-gluon interactions in massless QCD. The regulators for the quark lines and the Wilson lines have to correspond: Their regulator parameters of the quark lines are denoted Δ^+ and Δ^-, and they set $\Delta^+ = \delta^+ p_B^-$ and $\Delta^- = \delta^- p_A^+$ (in our notation). This coherence is needed so that the combination $A_0 B_0/S_0$ is a valid approximation to the original graphs for the cross section.

However, for defining the parton densities, and for using (10), one can ignore this issue, provided that we also use dimensional regulation. Since dimensional regularization regulates the collinear and soft divergences, we can take the limits $\Delta^\pm \to 0$ first. Then the limits $\delta^\pm \to 0$ can be taken separately to define the parton densities. These results are compatible with the factorization results obtained below. However, as we will demonstrate, the limits on δ^+ and on δ^- need to be coordinated.

III. FACTORIZATION PRESERVES FINITENESS AND REGULATOR INDEPENDENCE OF $\lim (A_0 B_0/S_0)$

To see explicitly that the limits in (7) and (10) exist and are the same, we apply factorization by the JCC method to express the regulated A_0, B_0 and S_0 in terms

of products of the finite collinear factors in the JCC scheme. Factorization is applied separately for A_0 etc regulated by each of the JCC and EIS methods. (EIS would undoubtedly apply their factorization method instead.)

In all of this paper, we will take for granted that factorization holds for the cross section and for the other situations that we consider. Our aim is to use factorization to relate different formulations.

There are a couple of complications that affect the organization of our proofs:

- In constructing the TMD pdfs there are limiting operations: to remove the regulator(s) on light-like Wilson lines, and to remove dimensional regularization. These limits do not commute — see, e.g., [5, Sec. 10.8]. The definitions are first to remove the regulator of light-like Wilson lines, e.g., in Eqs. (7) and (10), then to apply UV renormalization, and finally to remove dimensional regularization.

- Corresponding to the different steps in taking limits, there are actually three different objects that implement some kind of TMD pdf:

  1. The unsubtracted quantities A_0, etc, as on the right-hand side of Eqs. (7) and (10); to make these finite a regulator must be used on their light-like Wilson lines. Were all divergences absent, these would give the natural definitions of TMD pdfs.

  2. The unsubtracted TMD pdfs are combined with square roots of unsubtracted soft factors, in definitions that we present below, in which the limit is taken that the regulator(s) of the light-like Wilson lines is removed. (The JCC definition continues to have certain other non-light-like Wilson lines.) These we will call "unrenormalized" TMD pdfs, notated A_{unren}, etc, and we will prove these are the same in the JCC and EIS definitions.

  3. Finally UV renormalization is applied and then dimensional regularization removed. The resulting TMD pdfs are those on the left-hand side of Eqs. (7) and (10), they are finite in QCD, and they are the TMD pdfs that are directly used in phenomenological applications.

There are several kinds of divergence that appear at various stages of using factorization.

First are the UV divergences in QCD itself; these are removed by renormalization as part of the definition of QCD.

There are also UV divergences in the definitions of pdfs (and other related quantities). Although these divergences are also removed by essentially conventional renormalization, their status is rather different from those of QCD itself. The definitions of pdfs arise from the application of approximations appropriate to a collinear region

of momentum, and the UV divergences arise from a natural extrapolation of the approximations to all momenta. We will therefore characterize these divergences as "induced divergences". They cancel between the factors in a factorization property (as do the corresponding renormalization factors). Induced UV divergences also arise for the same reasons in the ordinary short-distance operator product expansion (OPE). It is possible to simply cut-off induced divergences. But a cleaner formalism results if one allows the divergences to occur and defines finite operator matrix elements by renormalization.

Then there are the IR and collinear divergences that appear in Feynman-graph calculations with massless quarks and gluons. These would be absent if all the fields have non-zero mass, as can be true in a model theory. They are presumably cutoff by (non-perturbative) confinement in QCD. In perturbatively calculable short-distance quantities, like hard-scattering coefficients, it is useful to neglect masses to simplify calculations. But the IR and collinear divergences must cancel in such quantities, precisely because they are short-distance quantities.

Finally, there are rapidity divergences. These arise when TMD pdfs are defined in terms of operator matrix elements with exactly light-like Wilson lines. In a gauge theory such Wilson lines are a natural consequence of the expansions in small variables for collinear and soft momenta in setting up factorization; for this reason, they occur in both the EIS and JCC methods. Having separately defined operator matrix elements is important for analyzing asymptotic behavior in the form of factorization properties, and for identifying the operator structure of universal non-perturbative objects like pdfs and fragmentation functions. These matrix elements can be studied with non-perturbative methods, and are of intrinsic interest as objects related to hadron structure. Rapidity divergences are another example of induced divergences, i.e., divergences introduced by the approximations used to factorize the cross section. To define finite pdfs, it is possible to cut off the rapidity divergences in some fashion (as in the original CS formalism), but it is also possible to remove them by a kind of generalization of the renormalization that is applied to UV divergences. Both the EIS and JCC formalisms use such a generalized renormalization, implemented by multiplying unsubtracted TMD pdfs by suitable square roots of unsubtracted soft factors (or by some equivalent of this procedure).

Induced divergences, no matter whether they are UV divergences or rapidity divergences, are unphysical in the sense that they do not correspond to regions of momenta that give actual divergences in unapproximated graphs for scattering. Nevertheless they are symptoms of important physical phenomena. They arise when in some region of momentum one expands to low order in powers of a small variable relative to a much large variable. Such expansions lead to the definition of quantities like pdfs. The natural extrapolation of the expansion to all values of the expansion parameter gives the induced di-

vergences. The range of momenta over which the expansion gives a valid approximation to the original graph depends on the energy and other kinematic parameters of the process under consideration.

Thus while induced divergences, including rapidity divergences, are in one sense unphysical, they are also physical in another sense that they are related (by extrapolation) to important, and even dominant, physical effects.

Associated with the generalized renormalization of induced divergences to construct finite pdfs, etc, are typically auxiliary scales like the renormalization scale μ of $\overline{\text{MS}}$ renormalization. Such a scale can be intuitively characterized as the range over which the underlying approximation is used. Since the range over which the approximation is useful is process-dependent, evolution (e.g., by the renormalization group) with respect to the auxiliary parameter(s) quantifies how the actual cross section is related to particular ranges of momenta inside the amplitudes for the process.

The generalized renormalization of rapidity divergences in the JCC scheme leads to the auxiliary parameter ζ that we have already mentioned, with the associated CS evolution equation. As we will see, the EIS method also uses corresponding properties.

A. A_0, B_0 and S_0 in the JCC method

In $A_{\text{JCC},0}$, the hadron and the Wilson line of rapidity y_2 are widely separated in rapidity, so they factor into two collinear pieces:

$$A_{\text{JCC},0}(y_2) = A_{\text{JCC, unren.}}\left(2(k_A^+)^2 e^{-2y_n}\right)$$
$$C_{\text{JCC, unren},W}(y_n - y_2) + \text{error}. \quad (11)$$

The factors are illustrated in Fig. 2. As usual, the leading regions have collinear parts, a soft part, and a hard part. The collinear parts are associated with the hadron and

with the Wilson line of rapidity y_2, and correspond to the factors on the right. No separate factor is needed for the soft part, according to the methods of both JCC and EIS. As usual in the JCC scheme, the collinear factors depend on a rapidity parameter y_n, which we choose to be the same as in (4), although our argument can be generalized to deal with a different value. Notice that we indicate the dependence on the auxiliary parameters and the Wilson line rapidities and not the other kinematic parameters (e.g., x and b_T.)

As for the hard part, it is in fact to be replaced by its lowest order value, which is unity. The reason is that at this stage, we keep dimensional regularization as we take $y_2 \to -\infty$. Now the hard scale is set by the total energy of the process, which we can parameterize by $k_A^+ e^{-y_2}$. With dimensional regularization in $4 - 2\epsilon$ dimensions, higher orders in the hard scattering are suppressed by a power $1/(k_A^+ e^{-y_2})^{\text{constant}\times\epsilon}$, and therefore vanish when $y_2 \to -\infty$ at fixed ϵ. (If we removed dimensional regularization and renormalized everything before taking $y_2 \to -\infty$, we would get differently organized results; this is the non-commutativity of limits mentioned earlier.) Since we have not yet removed dimensional regularization, we do not yet need to apply UV renormalization. Consequently, we are working with the unrenormalized collinear factors, as indicated by the notation on the right-hand-side of Eq. (11).

Apart from the issue of UV renormalization, the collinear factor $A_{\text{JCC, unren.}}$ is the same as in the JCC version of Drell-Yan factorization. That is, it is the TMD pdf associated with hadron A, defined as in [5, Sec. 13.15.3], but with past-pointing Wilson lines, as appropriate for the Drell-Yan process that we work with here. It includes factors involving the square roots of soft factors, and in two of these factors, there is a Wilson line with a finite rapidity, y_n. The quantity $C_{\text{JCC},W}$ is a corresponding collinear factor associated with the Wilson line of rapidity y_2 in the left-hand-side.

We recall the definition ([5, Eq. (13.106)]) of A_{JCC}:

$$A_{\text{JCC, unren}}(\zeta_A) = \lim_{\substack{y_1' \to +\infty \\ y_2' \to -\infty}} A_{\text{JCC},0}(y_2') \sqrt{\frac{S_{\text{JCC},0}(y_1' - y_n)}{S_{\text{JCC},0}(y_1' - y_2') \, S_{\text{JCC},0}(y_n - y_2')}}$$

$$= A_{\text{JCC},0}(-\infty) \sqrt{\frac{S_{\text{JCC},0}(+\infty - y_n)}{S_{\text{JCC},0}(+\infty - (-\infty)) \, S_{\text{JCC},0}(y_n - (-\infty))}}, \quad (12)$$

where the meaning of the second line is given by the limits in the previous line. The corresponding definition of $C_{\text{JCC},W}$ is

$$C_{\text{JCC, unren},W}(y_n - y_2) = \lim_{\substack{y_1' \to +\infty \\ y_2' \to -\infty}} S_{\text{JCC},0}(y_1', y_2) \sqrt{\frac{S_{\text{JCC},0}(y_n - y_2')}{S_{\text{JCC},0}(y_1' - y_2') \, S_{\text{JCC},0}(y_1' - y_n)}}$$

$$= S_{\text{JCC},0}(+\infty - y_2) \sqrt{\frac{S_{\text{JCC},0}(y_n - (-\infty))}{S_{\text{JCC},0}(+\infty - (-\infty)) \, S_{\text{JCC},0}(+\infty - y_n)}}, \quad (13)$$
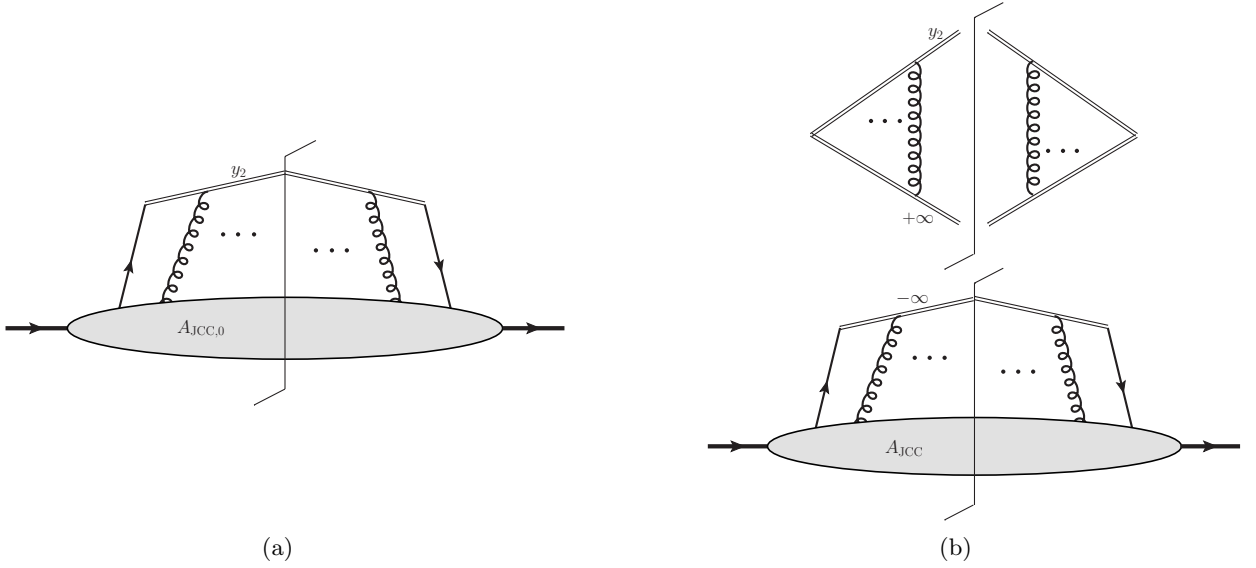
FIG. 2. For the factorization in (11): (a) the left-hand-side, and (b) the *bare* collinear factors for the right-hand-side. Note that the two factors in (b) also have generalized renormalization by square roots of soft factors that are *not* shown here.

where the role of y_1' and y_2' is exchanged compared with the definition of A_{JCC}.

Invariance under boosts in the z direction shows that the dependence of A_{JCC} on k_A^+ and y_n is only via the combination $(k_A^+)^2 e^{-2y_n}$. Similarly, the dependence of $C_{\text{JCC},W}$ on y_n and y_2 is only via $y_n - y_2$. The derivation of (11) is simply a version of the proofs of factorization for DIS and Drell-Yan cross sections that is given in Ref. [5].

We now apply the CS equation to obtain the y_n-dependence of $A_{\text{JCC}}\big(2(k_A^+)^2 e^{-2y_n}\big)$ and of $C_{\text{JCC},W}(y_n - y_2)$ at fixed values of their arguments. The CS equations are

$$\frac{\partial A_{\text{JCC, unren}}\big(2(k_A^+)^2 e^{-2y_n}\big)}{\partial y_n} = -A_{\text{JCC, unren}}\big(2(k_A^+)^2 e^{-2y_n}\big)\,\tilde{K}_{\text{unren}}, \tag{14}$$

$$\frac{\partial C_{\text{JCC, unren},W}(y_n - y_2)}{\partial y_n} = C_{\text{JCC, unren},W}(y_n - y_2)\,\tilde{K}_{\text{unren}}, \tag{15}$$

where $\tilde{K}$ is the CS kernel which depends on b_T and μ, but not the other variables. These equations have the immediate implication that the product of A and C in (11) is independent of y_n. The tilde in $\tilde{K}$ merely indicates that this is the object in transverse position space, not in transverse momentum space.

Let M be some chosen fixed reference mass scale. We use (14) to express $A_{\text{JCC}}\big(2(k_A^+)^2 e^{-2y_n}\big)$ in terms of its value with $2(k_A^+)^2 e^{-2y_n}$ replaced by M^2:

$$A_{\text{JCC, unren}}\big(2(k_A^+)^2 e^{-2y_n}\big) = A_{\text{JCC, unren}}\big(M^2\big)\exp\left[\left(-\ln\frac{\sqrt{2}k_A^+}{M} - y_n\right)\tilde{K}_{\text{unren}}\right]. \tag{16}$$

Similarly,

$$C_{\text{JCC, unren},W}(y_n - y_2) = C_{\text{JCC, unren},W}(0)\exp\left[(y_n - y_2)\,\tilde{K}_{\text{unren}}\right]. \tag{17}$$

Applying these results to Eq. (11) allows us to write $A_{\text{JCC},0}(y_2)$ in terms of the collinear factors at fixed values of their arguments, and thus to determine the y_2 dependence of $A_{\text{JCC},0}(y_2)$ for large negative y_2:

$$A_{\text{JCC},0}(y_2) = A_{\text{JCC, unren}}\big(M^2\big)\exp\left[\left(\ln\frac{\sqrt{2}k_A^+}{M} - y_2\right)\tilde{K}_{\text{unren}}\right]C_{\text{JCC, unren},W}(0) + \text{error}, \tag{18}$$

with the error vanishing when $y_2 \to -\infty$.

Exactly analogous steps show that the other unsubtracted quantities obey

$$B_{\text{JCC},0}(y_1) = B_{\text{JCC, unren}}\big(M^2\big)\exp\left[\left(y_1 - \ln\frac{\sqrt{2}k_B^-}{M}\right)\tilde{K}_{\text{unren}}\right]C_{\text{JCC, unren},W}(0) + \text{error}, \tag{19}$$

and

$$S_{\text{JCC},0}(y_1, y_2) = C_{\text{JCC, unren},W}(0) \, \exp\Big[(y_1 - y_2)\tilde{K}_{\text{unren}}\Big] \, C_{\text{JCC},W}(0) + \text{error}. \tag{20}$$

The collinear factors associated with the Wilson lines are the same for all the Wilson lines. This is shown simply by exchanging the $+$ and $-$ coordinates.

Inserting all of these results into $A_0 B_0/S_0$ shows that the dependence on y_1 and y_2 cancels, as do the factors of $C_{\text{JCC},W}$. Therefore, the limit $y_1 \to +\infty$ and $y_2 \to -\infty$ exists. It is just

$$\lim_{\substack{y_1 \to +\infty \\ y_2 \to -\infty}} \frac{A_{\text{JCC},0}(y_2) \, B_{\text{JCC},0}(y_1)}{S_{\text{JCC},0}(y_1, y_2)} = A_{\text{JCC, unren}}(M^2) \, B_{\text{JCC, unren}}(M^2) \, \exp\Big[\ln \frac{Q^2}{M^2} \tilde{K}_{\text{unren}}\Big]$$

$$= A_{\text{unren}}(\zeta_A) \, B_{\text{unren}}(\zeta_B), \tag{21}$$

given that $2k_A^+ k_B^- = Q^2$ and $\zeta_A \zeta_B = Q^4$. The last result is exactly as in the CSS factorization formula, with the JCC definition of the collinear factors.

B. A_0, B_0 and S_0 in the EIS method

For the EIS method, we again apply the JCC factorization again to A_0, but with a changed regulator for the Wilson line. Instead of the non-light-like Wilson line of rapidity y_2 that is used in (11), EIS use a δ-regulator. The steps for deriving factorization equations like (11) depend on the wide separation of rapidity between the hadron of momentum p_A and the Wilson line in A_0. Thus the structure of factorization is insensitive to the method for regulating the Wilson line (so long as it is a method consistent with factorization), and the proof continues to apply. The factorization formula corresponding to (18) is then

$$A_{\text{EIS},0}(\delta^+) = A_{\text{JCC, unren}}\big(2(k_A^+)^2 e^{-2y_n}\big) \, C_{\text{EIS, unren},W}\big(y_n - \ln \delta^+\big) + \text{error}, \tag{22}$$

where now the error vanishes as $\delta^+ \to 0$. In obtaining this equation, we have again used dimensional regularization. This regulates any soft and collinear divergences associated with massless quark and gluon lines, and therefore we have removed the Δ-regulator that EIS use. The collinear factor A_{JCC} is the same as in (11), because it is associated with the same collinear subgraphs associated with the hadron. But the different regulator for the Wilson line implies that the C_W factor is different.

This factor, $C_{\text{EIS, unren},W}(y_n - \ln \delta^+)$, depends only on the combination $y_n - \ln \delta^+$ but not on y_n and δ^+ separately, because of boost invariance. To see this result explicitly, we examine the relevant Wilson line propagators:

$$\frac{i}{-e^{-2y_n}k^+ + k^- + i0}, \qquad \frac{i}{-k^+ + i\delta^+}. \tag{23}$$

Suppose we simultaneously change y_n to $y_n + \delta y$ and δ^+ to $\delta^+ e^{\delta y}$, so that $y_n - \ln \delta^+$ is unchanged. Then we change variables on all loop momentum integrals in $C_{\text{EIS, unren},W}$ by corresponding boost factors: k^+ to $k^+ e^{\delta y}$, and k^- to $k^- e^{-\delta y}$. This has effectively implemented a boost, and C is boost invariant, so its value is unchanged.

In accordance with the JCC factorization method, $C_{\text{EIS, unren},W}$ is defined by

$$C_{\text{EIS, unren},W}\big(y_n - \ln \delta^+\big) = \lim_{\substack{y_1' \to +\infty \\ y_2' \to -\infty}} C_{\text{EIS},0}\big(y_1' - \ln \delta^+\big) \sqrt{\frac{S_{\text{JCC},0}(y_n, y_2')}{S_{\text{JCC},0}(y_1', y_2') \, S_{\text{JCC},0}(y_1', y_n)}}$$

$$= C_{\text{EIS},0}\big(+\infty - \ln \delta^+\big) \sqrt{\frac{S_{\text{JCC},0}(y_n, -\infty)}{S_{\text{JCC},0}(+\infty, -\infty) \, S_{\text{JCC},0}(+\infty, y_n)}}. \tag{24}$$

The bare factor is defined with one Wilson line regulated by the EIS method and one regulated by the JCC method.

The CS equations are the same as before, since they arise from the same $S_{\text{JCC},0}$ factors under the square root. Hence, instead of (18), we get

$$A_{\text{EIS},0}(\delta^+) = A_{\text{JCC, unren}}(M^2) \, \exp\left[\left(\ln \frac{\sqrt{2}k_A^+}{M} - \ln \frac{\delta^+}{M}\right) \tilde{K}_{\text{unren}}\right] C_{\text{EIS, unren},W}(M) + \text{error}. \tag{25}$$
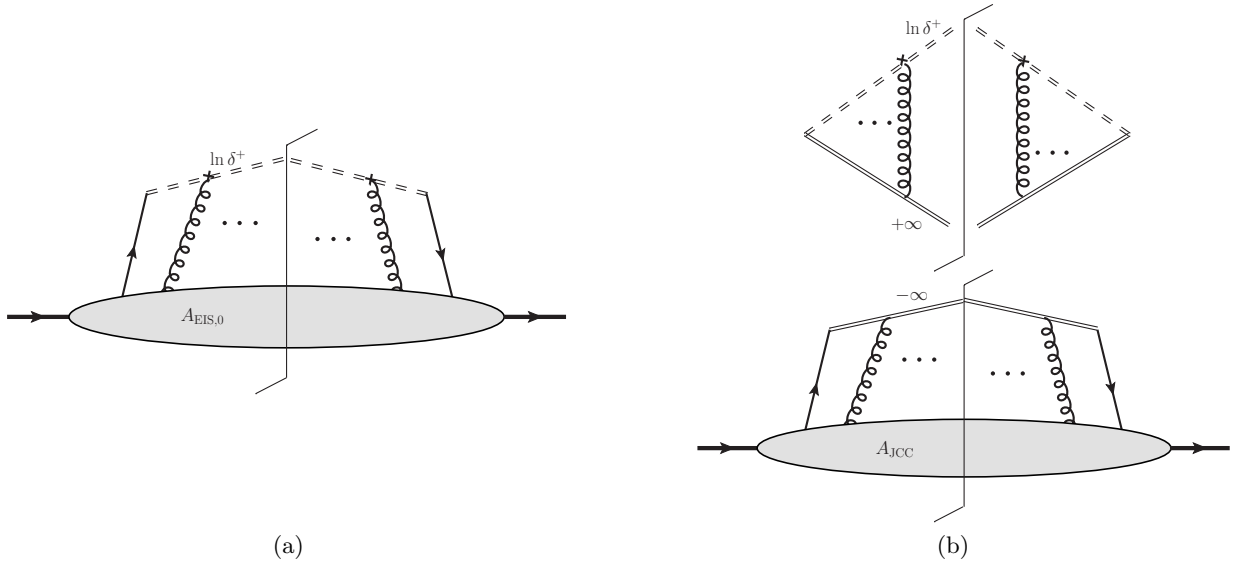
FIG. 3. For the factorization in (22): (a) the left-hand-side, and (b) the *bare* collinear factors for the right-hand-side. The dashed-double lines with cross-vertices represent the δ-regulated Wilson lines of the EIS method.

This and the corresponding factorizations for B_0 and S_0 have errors that vanish as $\delta^{\pm} \to 0$. The collinear factor for the Wilson line changed from that in (11) and (18) because of the changed regulator. Note that the reference scale for the $C_{\text{EIS, unren},W}(M)$ in Eq. (25) is now a mass M, since the regulator is a mass scale δ^+. But the A factor is the same as in the JCC method.

The formulas for B_0 and S_0 are

$$B_{\text{EIS},0}\big(\delta^-\big) = C_{\text{EIS, unren},W}\big(-\ln\delta^- - y_n\big)\ B_{\text{JCC, unren}}\big(2(k_B^-)^2 e^{2y_n}\big) + \text{error}$$

$$= C_{\text{EIS, unren},W}(M)\exp\left[\left(\ln\frac{\sqrt{2}k_B^-}{M} - \ln\frac{\delta^-}{M}\right)\tilde{K}_{\text{unren}}\right]B_{\text{JCC, unren}}\big(M^2\big) + \text{error}, \qquad (26)$$

$$S_{\text{EIS},0}\big(\delta^-, \delta^+\big) = C_{\text{EIS, unren},W}\big(-\ln\delta^- - y_n\big)\ C_{\text{EIS, unren},W}\big(y_n - \ln\delta^+\big) + \text{error}$$

$$= C_{\text{EIS, unren},W}(M)\exp\left[\left(\ln\frac{M^2}{\delta^-\delta^+}\right)\tilde{K}_{\text{unren}}\right]C_{\text{EIS, unren},W}(M) + \text{error}, \qquad (27)$$

Hence, the limit of $A_0 B_0/S_0$ is the same as in the JCC method:

$$\lim_{\delta^+,\delta^- \to 0}\frac{A_{\text{EIS},0}(\delta^+)\,B_{\text{EIS},0}(\delta^-)}{S_{\text{EIS},0}(\delta^-,\delta^+)} = A_{\text{JCC, unren}}\big(M^2\big)\,B_{\text{JCC, unren}}\big(M^2\big)\exp\left[\left(\ln\frac{Q^2}{M^2}\right)\tilde{K}_{\text{unren}}\right]. \qquad (28)$$

As before, we have a cancellation of the collinear factors for the Wilson lines and of the dependence on the regulator parameters. Errors go to zero in the limit that the regulator is removed. Hence, we get the same finite result with the EIS method as with the JCC method.

The above derivation used the JCC organization of factorization to express the combination $A_0 B_0/S_0$ of bare quantities in the EIS method for the Drell-Yan cross section in terms of A and B in the JCC scheme. But this same product is the product of parton densities in the EIS scheme. Hence from Eq. (28), it follows that the product AB is the same in both methods. The only question now is whether the individual factors A and B are the same.

The only caveat in the above derivation arises because the derivation of factorization uses gauge invariance heavily, at least beyond simple one-loop graphs. But when Wilson lines in gauge-invariant operators are replaced by regulated Wilson lines in the EIS method, the resulting operators are not gauge invariant. This could potentially cause some problems. It needs to be investigated whether any problems actually occur. It is likely that any resulting corrections are suppressed by a power of the regulator parameters $\delta^{\pm}$, and will not affect the final results.

IV. THE JCC METHOD AND THE FORMULA $A = A_0/\sqrt{S_0}$

The unrenormalized collinear factors in the JCC method are defined by

$$A_{\text{JCC, unren}}(\zeta_A) = \lim_{\substack{y_1 \to +\infty \\ y_2 \to -\infty}} A_{\text{JCC},0}(y_2) \sqrt{\frac{S_{\text{JCC},0}(y_1, y_n)}{S_{\text{JCC},0}(y_1, y_2) \, S_{\text{JCC},0}(y_n, y_2)}} \tag{29}$$

$$B_{\text{JCC, unren}}(\zeta_B) = \lim_{\substack{y_1 \to +\infty \\ y_2 \to -\infty}} B_{\text{JCC},0}(y_1) \sqrt{\frac{S_{\text{JCC},0}(y_n, y_2)}{S_{\text{JCC},0}(y_1, y_2) \, S_{\text{JCC},0}(y_1, y_n)}}, \tag{30}$$

where ζ_A and ζ_B are defined by Eqs. (4) and (5). Evidently the product of $A_{\text{JCC, unren}}$ and $B_{\text{JCC, unren}}$ is exactly (the appropriate limit of) $A_0 B_0 / S_0$, a result we have already seen in Eqs. (21) and (28).

In these formulas, the limits $y_1 \to +\infty$, $y_2 \to -\infty$ can be taken independently. But the rate at which the rapidities go to infinity can also be coordinated. Let us do this so that the rapidity intervals are the same in both of the S factors that have a y_n argument, i.e., let us set

$$y_1 = y_n + \Delta y, \qquad y_2 = y_n - \Delta y. \tag{31}$$

Then we take the limit $\Delta y \to \infty$. Now,

$$S_{\text{JCC},0}(y_1, y_n) = S_{\text{JCC},0}(y_n + \Delta y, y_n) = S_{\text{JCC},0}(y_n, y_n - \Delta y) = S_{\text{JCC},0}(y_n, y_2), \tag{32}$$

where the middle equality is obtained by exchanging the $+$ and $-$ coordinates, by applying charge conjugation to exchange the color charges of the Wilson lines, and by then applying a boost. After that, we apply UV renormalization.

We therefore obtain

$$A_{\text{JCC}}(\zeta_A) = \lim_{\substack{y_1 \to +\infty \\ y_2 \to -\infty}} \frac{A_{\text{JCC},0}(y_2)}{\sqrt{S_{\text{JCC},0}(y_1, y_2)}} \times \text{UV renormalization factor}, \tag{33}$$

$$B_{\text{JCC}}(\zeta_B) = \lim_{\substack{y_1 \to +\infty \\ y_2 \to -\infty}} \frac{B_{\text{JCC},0}(y_1)}{\sqrt{S_{\text{JCC},0}(y_1, y_2)}} \times \text{UV renormalization factor}, \tag{34}$$

where we require the limits to be taken with $\frac{1}{2}(y_1 + y_2) = y_n$ fixed. We can also write these with the limit of a single variable:

$$A_{\text{JCC}}(\zeta_A) = \lim_{y_2 \to -\infty} \frac{A_{\text{JCC},0}(y_2)}{\sqrt{S_{\text{JCC},0}(2y_n - y_2, y_2)}} \times \text{UV renormalization factor}, \tag{35}$$

$$B_{\text{JCC}}(\zeta_B) = \lim_{y_1 \to +\infty} \frac{B_{\text{JCC},0}(y_1)}{\sqrt{S_{\text{JCC},0}(y_1, 2y_n - y_1)}} \times \text{UV renormalization factor}. \tag{36}$$

These formulas are simpler than the original JCC formulas, because they have two factors instead of four, and they are of the form of the EIS definitions except for the method of regulation of the light-like Wilson lines.

V. EIS TMD PDFS HAVE A ζ PARAMETER AND ARE EQUAL TO THE JCC PDFS

The EIS PDFs are defined by

$$A_{\text{EIS}} = \lim_{\delta^+, \delta^- \to 0} \frac{A_{\text{EIS},0}(\delta^+)}{\sqrt{S_{\text{EIS},0}(\delta^-, \delta^+)}} \times \text{UV renormalization factor}, \tag{37}$$

$$B_{\text{EIS}} = \lim_{\delta^+, \delta^- \to 0} \frac{B_{\text{EIS},0}(\delta^-)}{\sqrt{S_{\text{EIS},0}(\delta^-, \delta^+)}} \times \text{UV renormalization factor}. \tag{38}$$

Compared with EIS's actual definition [3, Eq. (2.13)], we have changed the names of factors, we have made explicit the regulator parameters, and we have omitted the Fourier transform needed to get the TMD PDFs in transverse momentum space.

In their statement of the definitions, EIS did not originally specify the relative rates at which δ^+ and δ^- go to zero. However, in their actual calculations — see [3, Sec. 3.1], they set $\delta^+ = \delta^-$. In fact, this is not a boost invariant

condition, and, as we now show, it suffices to keep the ratio δ^+/δ^- fixed as δ^+ and δ^- go to zero. From the perspective of factorization alone, any choice of fixed δ^+/δ^- gives a distinct but equally valid definition in the limit of $\delta^+, \delta^- \to 0$. The choice of $\delta^+ = \delta^-$, plus the choice to work in a frame where $k_A^- = k_B^- = Q$, is what allows EIS to recover logarithms of Q^2 in their explicit calculation (e.g. [3, Eq. 3.10]), despite the apparent absence of the scale Q^2 inside their TMD PDF definitions.

We apply (25) and (27) to (37), and apply UV renormalization, to get

$$A_{\mathrm{EIS}} = \lim_{\delta^+, \delta^- \to 0} A_{\mathrm{JCC}}(M^2) \exp\left[\left(\ln \frac{\sqrt{2}k_A^+}{M} + \frac{1}{2}\ln\frac{\delta^-}{\delta^+}\right)\tilde{K}\right]. \tag{39}$$

Evidently, the ratio δ^-/δ^+ finite and nonzero for the limit to be well-defined. Equation (39) is *exactly* equal to the JCC PDF, A_{JCC}, with

$$\frac{\delta^+}{\delta^-} = e^{2y_n}. \tag{40}$$

Then, in exact correspondence to the JCC scheme, we define

$$\zeta_A = 2(k_A^+)^2 e^{-2y_n} = 2(k_A^+)^2 \frac{\delta^-}{\delta^+}, \tag{41}$$

and A_{EIS} depends on ζ_A. Thus, the δ^+, δ^- regulation scheme of Ref. [3] is exactly equivalent to the JCC scheme if we identify the rapidity parameter as $y_n = \ln\sqrt{\delta^+/\delta^-}$.

An exactly corresponding equation applies to the other PDF:

$$\begin{aligned}
B_{\mathrm{EIS}}(\zeta_B) &= \lim_{\delta^+, \delta^- \to 0} B_{\mathrm{JCC}}(M^2) \exp\left[\left(\ln \frac{\sqrt{2}k_B^-}{M} + \frac{1}{2}\ln\frac{\delta^+}{\delta^-}\right)\tilde{K}\right] \\
&= B_{\mathrm{JCC}}(M^2) \exp\left[\left(\ln \frac{\sqrt{2}k_B^-}{M} + y_n\right)\tilde{K}\right],
\end{aligned} \tag{42}$$

with just an exchange of the roles of the $+$ and $-$ coordinates. Now we have

$$\zeta_B = 2(k_B^-)^2 e^{2y_n} = 2(k_B^-)^2 \frac{\delta^+}{\delta^-}. \tag{43}$$

The limit taken by EIS with $\delta^+ = \delta^-$ simply corresponds to the case $y_n = 0$, for which $\zeta_A = \zeta_B = Q^2$, provided that one uses a frame where $k_A^+ = k_B^-$. Since the EIS PDFs have a ζ argument, and since they equal the JCC PDFs, they must also obey a CS equation. In applications, the auxiliary parameters are to be evolved to values that give good or optimal accuracy for the perturbative calculations that are relevant for a given process at a given hard scale. For the hard scattering, this entails setting μ and $\sqrt{\zeta}$ to Q (or to within a a finite factor of Q). In this sense the two auxiliary parameters can be treated as tied together. (In fact only the product $\zeta_A \zeta_B$ of the two ζs is relevant.)

Although EIS [3] claim not to use the CS equation — see e.g., their Sec. 10 — they use an equivalent result, obtained in their Sec. 5 for the Q-dependence of their TMD pdfs. They label this result resummation, but the result is exactly equivalent to the use of the CS equation, the RG equation that is already part of their formalism, and the small-b expansion of TMD pdfs. The results are the same as in the CS method (in the JCC version), as can be seen, for example, by comparing their Eq. (5.7) with Sec. 13.13.1 of Ref. [5]. EIS's restrict their results to the region $\Lambda \ll q_T \ll Q$. However, with the exception of the small-b expansion, the results also apply for all small q_T as well, as is proved in Ref. [5].

Furthermore, EIS use here that the logarithm of the TMD PDFs are linear in $\ln(Q^2/\mu^2)$. (See [3, Eq. (5.4)] and the discussion that follows.) In fact, this linearity is a trivial consequence of the CS equation.

With the identifications in Eqs. (39-43), EIS find a one-loop anomalous dimension [3, Eq. (3.21,3.22)] that is the same as the JCC one.

EIS, in a recent preprint [7], advocate setting $y_n = 0$ ($\delta^+ = \delta^-$) from the outset. While this is legitimate, we find it non-optimal for the following reasons. First, fixing a particular value for y_n amounts to fixing a choice of reference frame with respect to boosts along the beam axis. In the case of $y_n = 0$, it is the rest frame of the Drell-Yan lepton pair. This choice treats the hadrons

asymmetrically because, although it corresponds to the parton rest frame, $k_A^+ = k_B^- = Q$, it is not the rest frame of the incoming hadrons, so the fractional momenta x_A and x_B will in general be very different, so it does not correspond to a particularly natural frame with regard to the TMD pdfs. More importantly, setting y_n to zero does not eliminate the CS evolution, but rather shifts it entirely into the dependence on k_A^+ in Eq. (41) and k_B^- in Eq. (43). In fact, CS evolution is boost-independent, which is a particular advantage of the JCC approach, so we view it as preferable to leave the frame unspecified in the implementation of evolution.

VI. DIFFERENT VERSIONS OF $\overline{\text{MS}}$ RENORMALIZATION

However a possible problem is that calculations of the hard part do not agree: Compare [19, Eq. (6)] with [3, Eq. (7.5)], where the π^2 terms are different by a factor of 7/6. We suspect that this is due to a difference in the definitions of the $\overline{\text{MS}}$ scheme for UV renormalization. JCC [5, Sec. 3.3.2] defines one-loop counterterms to have the structure

$$S_\epsilon \times \text{sum of poles in } \epsilon, \qquad (44)$$

where

$$S_\epsilon = \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}. \qquad (45)$$

A more commonly used definition would replace S_ϵ by $(4\pi e^{-\gamma_E})^\epsilon$, which differs by a term of order ϵ^2. Normal UV renormalization only needs at most one factor of $1/\epsilon$ per loop, and the change in the definition of the $\overline{\text{MS}}$ scheme is irrelevant for such quantities. But this is no longer true when there are UV divergences with a double pole $1/\epsilon^2$ per loop, as in TMD pdfs with the EIS and JCC definitions. The modified definition Eq. (45) was made specifically with this situation in mind, since the factor S_ϵ is common to all one-loop transverse-momentum integrals in dimensional regularization. The use of (45) completes the simplifications of the results of one-loop calculations provided by the $\overline{\text{MS}}$ scheme. We have checked that the difference in the coefficient of π^2 is consistent with this change in the definition of the $\overline{\text{MS}}$ scheme. However, not enough details are given in [3] for us to verify definitively that this is the correct explanation of the difference between the reported hard-scattering coefficients.

VII. SUMMARY

In this article, we have shown that two seemingly different methods of formulating TMD-factorization — one originating in ideas from SCET, and one based on subtractive methods of pQCD — are actually equivalent. That the same conclusions are found from two seemingly different approaches provides important support for the general validity of the TMD-factorization formalism.

To clarify this equivalence, and to understand the origin of the superficial disparity between the two approaches, it is necessary to comment on some differences between what we have referred to as the EIS method in Eq. (10) and the original presentation of the EIS formalism in Ref. [3]: We have naturally assumed it to be implicit in Ref. [3] that the limit $\delta^+, \delta^- \to 0$ for the regulators of the Wilson lines is to be taken, although this appears not to be explicitly stated. Now the calculations in Ref. [3] rely on separate regulators Δ^+ and Δ^- for the quark lines, as well as regulators δ^+ and δ^- for the Wilson lines. The two kinds of regulator were tied by the conditions $\Delta^+ = \delta^+ p_B^-$ and $\Delta^- = \delta^- p_A^+$, so that in massless calculations, an appropriate soft approximation is valid. The Δ regulators cut off infrared divergences associated with the quark lines. If some other cutoff is imposed, for example, by nonzero masses, then the Δ regulator can be removed without being tied to the δ regulator. In our general discussion, we have assumed that some other infrared cutoff is present throughout, and we have set $\Delta^+ = \Delta^- = 0$ from the outset. We note that the consistency of δ-regulator formalism is derived in Ref. [3] from an examination of one-loop graphs with on-shell external partons. It is not clear to us how complete the derivations Ref. [3] are beyond that order. For the purposes of this paper, we have taken as our starting point the assumption that overall factorization with the δ-regulators remains consistent in a general derivation at arbitrary order.

We made one other explicit extension to the EIS definition. EIS [3] imposed the condition $\delta^+ = \delta^-$. But this is boost dependent. We have instead allowed the ratio δ^+/δ^- to remain unspecified in Eq. (10) (its variation is responsible for the CS equation), and have observed that the EIS definition continues to work in this case[4].

Given the consistency assumption, we find not only that the methods (EIS and JCC) are equivalent, but that the different definitions give exactly equal TMD pdfs, with the exception of different definitions of the $\overline{\text{MS}}$ scheme, as explained around Eq. (45). A criticism that is sometimes made of the CSS formalism is that it is prohibitively difficult or complicated for practical use, particular in the context of probing non-perturbative structure. However, we believe that the most recent version, presented in Ref. [5], has eliminated the main issues responsible for such difficulties, and now provides a maximally user-friendly TMD formalism. At least in the manual one-loop calculations we have performed, the length of the calculations is little different. The remaining complexities are those of ordinary Feynman graph calculations.

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[4] We thank Markus Diehl for bringing to our attention the issues discussed in this and the previoius paragraph.

We conclude that since approaches to TMD factorization that are apparently widely different nevertheless give the same TMD pdfs, this supports the idea that there is a particularly natural kind of preferred definition of TMD pdfs in QCD. This gives good current and near-future prospects [20–24] for applications of TMD-factorization in a broad variety of phenomenological studies. There remain differences in how the different groups analyze the non-perturbative large-b region in the solution of the evolution equations. We leave the analysis of these differences to future work. The TMD pdfs themselves, once given an operator definition, are unambiguous entities in QCD.

However, for precise phenomenological work there needs to be agreement on the definition of the $\overline{\text{MS}}$ scheme; differences in its definition matter for TMD pdfs even though they do not affect simpler quantities (where the UV divergences give only at most one factor of $1/\epsilon$ per loop). But this issue is not a matter of principle.

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