Simple models with non-Abelian moduli on topological defects

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Simple Models with Non-Abelian Moduli on Topological Defects

M. Shifman

William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA

Abstract

I explain how conventional topological defects – the Abrikosov-Nielsen-Olesen (ANO) string and domain walls – can acquire non-Abelian moduli localized on their world sheets. The set-up is conceptually similar and generalizes that used by Witten for cosmic strings [1].

1 Introduction

The discovery of non-Abelian strings [2] (i.e. those with non-Abelian moduli fields on the string world sheet) paved the way to many applications, both in supersymmetric [3] and non supersymmetric theories [4]. The models supporting strings with non-Abelian moduli which are discussed in the literature are rather advanced, especially in the supersymmetric case. At the same time the physical essence of the phenomenon is quite simple. Here I explain the occurrence of the non-Abelian moduli in a simple set-up. All we need is: (i) the bulk theory having an unbroken *global* non-Abelian symmetry G and supporting topological defects (e.g. strings or domain walls); and (ii) the breaking of above global symmetry G down to a *global* subgroup H on the given defect. These two requirements can be easily implemented. Then, at the classical level we will have $\nu = \dim G - \dim H$ zero modes of an "orientational" type, localized on the given topological defect, in addition to familiar translational modes (which as a rule are decoupled from orientational [9]). At low energies the corresponding moduli fields are described by a sigma model with the G/H coset space as the target space. Quantization of the low-energy G/H sigma model on the world sheet of the topological defect under consideration may or may not lift the zero modes.

I will assume weak coupling justifying quasiclassical treatment. A few sample models to be considered below (and similar) may play a role of the Ginzburg-Landau models in future applications. The theoretical set up to be presented below is conceptually similar and generalizes that used by Witten for cosmic strings [1]. Note that some generalizations of Witten's construction were considered in the 1980s and early 90s [10]. They do not overlap with the non-Abelian moduli construction I will discuss.

2 Non-Abelian string from Abrikosov-Nielsen-Olesen vortex

First, I will be briefly outline the construction of the ANO string which has no moduli on its world sheet other than translational (see e.g. [5]). The ANO string is a soliton in the U(1) gauge theory with a single charged scalar field whose vacuum expectation value breaks U(1) spontaneously. The model is described by the Lagrangian

$$\mathcal{L}_{\rm v} = -\frac{1}{4e^2}F_{\mu\nu}^2 + |\mathcal{D}^\mu\phi|^2 - V(\phi) \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \qquad \mathcal{D}_\mu\phi = (\partial_\mu - iA_\mu)\phi\,. \tag{2}$$

The potential energy $V(\phi)$ must be chosen in such a way as to ensure Higgsing of U(1) in the bulk,

$$V = \lambda\left(|\phi|^2 - v^2\right)^2\,, \tag{3}$$

where v is assumed to be real and positive (no loss of generality). In the vacuum in the unitary gauge

$$A_\mu = 0, \qquad \phi = v\,. \tag{4}$$

The U(1) photon is Higgsed and acquires the mass

$$m_\gamma = \sqrt{2}ev\,, \tag{5}$$

$\mathrm{Im}\,\phi$ is eaten by the Higgs mechanism, while $\mathrm{Re}\,\phi(x) = v + \eta(x)/\sqrt{2}$, where the real scalar field $\eta(x)$ is not eaten up by the photon. Its mass is

$$m_\eta = 2\sqrt{\lambda}\,v\,. \tag{6}$$

We will assume that $m_\eta > m_\gamma$, but not much larger, i.e. $m_\eta \sim m_\gamma$. This is not crucial, though.

Now, as well-known, this model supports topologically stable Indeed, let us first consider all non-singular field configurations that are static (time-independent) in the gauge $A_0 = 0$. Then the energy functional takes the form

$$\begin{aligned} \mathcal{E}[\vec{A}(\vec{x}),\phi(\vec{x})] &= \int dz \int d^2x \left[\frac{1}{4e^2}F_{ij}F_{ij} + |\mathcal{D}_i\phi|^2 + V(\phi)\right] \\ &= L \times \int d^2x \left[\frac{1}{4e^2}F_{ij}F_{ij} + |\mathcal{D}_i\phi|^2 + V(\phi)\right], \end{aligned} \tag{7}$$

where $L\to\infty$ is the string length (it is assumed to be oriented along the z axis, while the integral in the second line presents the string tension T. Requiring T to be finite we observe that $V(\phi)\to 0$ at $|\vec{x}_\perp|\to\infty$, i.e.

$$|\phi| \to v \quad \text{at} \quad |\vec{x}_\perp| \to \infty\,. \tag{8}$$

Let us choose a circle of large radius R (eventually $R \to \infty$) centered at $x = y = 0$. The absolute value of ϕ on this circle must be v, however, the phase of the field ϕ is not fixed by the condition $\int d^2x\, V(\vec{x}_\perp) < \infty$. The minimal ANO string is obtained if

$$\phi = v\, e^{i\alpha} \tag{9}$$

on the large circle. Here α is the polar angle in the x, y plane.

The *ansatz* (9) combined with the requirement of finiteness of the kinetic term of the ϕ field generates a gauge potential in the perpendicular plane, $A_\perp \to A_i$ ($i = 1, 2$). At large distances from the string axis A_i is pure gauge,

$$A_i = \partial_i \alpha = -\varepsilon_{ij} \frac{x_j}{r^2}\,, \quad i, j = 1, 2\,, \tag{10}$$

where ε_{ij} is the two-dimensional Levi-Civita tensor. It is clear that then both $\mathcal{D}_i\phi$ and F_{ij} fall off at infinity faster than $1/r^2$ (in fact, they fall off exponentially fast), and the integral for T converges,

$$T \sim v^2\,. \tag{11}$$

It is easy to see that this vortex carries a (quantized) magnetic flux in the z direction. It has two zero modes (translational modes in the x and y directions) which become sterile moduli fields on the string world sheet.

The full elementary elementary vortex solution is parametrized by two profile functions $\varphi(r)$ and $f(r)$ as follows:

$$\phi(x) = v\, \varphi(r) e^{i\,\alpha}\,, \qquad A_i(x) = -\frac{1}{n_e} \varepsilon_{ij}\, \frac{x_j}{r^2}\, \left[1 - f(r)\right]\,, \tag{12}$$

where $r = |x_\perp| = \sqrt{x^2 + y^2}$ is the distance from the string axis and α is the polar angle, as above.

The boundary conditions for the profile functions are rather obvious from the form of the *ansatz* (12). At large distances

$$\varphi(\infty) = 1\,, \quad f(\infty) = 0\,. \tag{13}$$

The absence of singularities at the origin requires

$$\varphi(0) = 0\,, \quad f(0) = 1\,. \tag{14}$$

Thus, in the vortex core the ϕ expectation value vanishes and the original U(1) gauge symmetry is restored. The transverse size of the ANO flux tube $\sim 1/m_{\eta,\gamma}$.

So far, this an absolutely standard construction. Now I will extend it. Introduce a triplet field χ^i (here $i = 1, 2, 3$) endowed with a (globally) SO(3) invariant interaction,

$$\mathcal{L}_\chi = \partial_\mu \chi^i \, \partial^\mu \chi^i - U(\chi, \phi) \,, \tag{15}$$

$$U = \gamma \left[\left(-\mu^2 + |\phi|^2 \right) \chi^i \chi^i + \beta \left(\chi^i \chi^i \right)^2 \right] , \tag{16}$$

where γ is a (positive) coupling constant. For simplicity I will assume $\beta > 1$ and the field χ to be real. The parameter μ is real and positive, with the condition

$$\mu < v \,, \tag{17}$$

but not much smaller. For the validity of our consideration we must require

$$\gamma \mu^2 \gg \lambda v^2 \,, \tag{18}$$

so that the length scale of variation of the η, γ fields is larger than that of the χ fields.

In the bulk the expectation value of ϕ does not vanish, $|\phi| = v$. Equation (16) implies then that χ is stable, no vacuum condensate of χ develops, and the global O(3) symmetry remains unbroken. At the same time, in the vortex core ϕ vanishes destabilizing the χ field which develops an expectation value,

$$\chi^2 = \frac{\mu^2}{2\beta} \,, \tag{19}$$

implying, in turn, that in the core the O(3) symmetry is spontaneously broken. While the absolute value of χ^i is fixed at $|\chi^i_*| = \mu/\sqrt{2\beta}$ by the energy minimization the angular orientation of the vector χ^i is arbitrary. Independently of χ^i_*, the pattern of the symmetry breaking in the string core is

$$O(3) \to O(2) \,. \tag{20}$$

Correspondingly, the vortex solution (more exactly, its χ component) will depend on two moduli whose dynamics is determined by the $O(3)/O(2) = CP(1)$ coset. Differentiating the solution with respect to these two collective coordinates we get the explicit form of the zero modes. The low-energy theory on the string world sheet is the $CP(1)$ model for the orientational moduli fields (in addition to two decoupled translational moduli fields), namely,

$$\mathcal{L}_{\rm sws} = \frac{1}{\beta}\left(\partial^a n^i\right)\left(\partial_a n^i\right), \qquad n^i n^i = 1\,, \qquad a = 0, 3\,. \tag{21}$$

The subscript sws means string world sheet.

Classically and in perturbation theory the above moduli fields are massless. However, from the Coleman theorem we know that massless non-sterile boson fields cannot exist in two dimensions [6]. And indeed, the exact solution of the $CP(1)$ model (which is asymptotically free in the UV, but strongly coupled in the IR [7]) exhibits a mass gap generation and complete restoration of O(3). The would-be Nambu-Goldstone (NG) bosons on the string world sheet become quasi-NG bosons, provided the mass scale Λ which is nonperturbatively generated in CP(1) is small,

$$\Lambda \ll v\,. \tag{22}$$

If $\beta \gg 1$ the above condition is met.

3 Non-Abelian moduli fields on domain walls

The general idea is the same as in Sect. 2: an unbroken global symmetry in the vacuum combined with a domain wall which breaks a part of the above global symmetry in its core. As a pedagogical example we will consider the same set-up (15) and (16) in conjunction with the simplest model of the complex field ϕ supporting an appropriate domain wall.

Such a model is given by the Lagrangian (see e.g. [5])

$$\mathcal{L}_{\rm w} = (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - V(\phi, \phi^\dagger)\,, \tag{23}$$

where

$$V(\phi, \bar\phi) = \left|\frac{\partial\, W(\phi)}{\partial \phi}\right|^2, \qquad W(\phi) = \frac{m^2}{\lambda}\,\phi - \frac{\lambda}{3}\,\phi^3\,, \tag{24}$$

and the constants m and λ are assumed to be real and positive.

The potential (24) implies two degenerate classical vacua,

$$\phi_* = \pm v\,, \qquad v \equiv \frac{m}{\lambda}\,. \tag{25}$$

Both vacua are physically equivalent. This equivalence could be explained by the spontaneous breaking of the Z_2 symmetry, $\phi \to -\phi$, present in the action. The static field configuration interpolating between the two degenerate vacua is the domain wall (which is topologically stable). Assume for definiteness that the wall lies in the xy plane. The wall tension $T_{\rm w}$ (the energy per unit area $T_{\rm w} = E_{\rm w}/A$) is

$$T = \frac{8m^3}{3\lambda^2}\,. \tag{26}$$

while the wall solution takes the form

$$\phi_{\rm w} = \frac{m}{\lambda}\tanh(mz)\,, \tag{27}$$

implying that the wall thickness is $\sim m^{-1}$.

Now we combine $\mathcal{L}_{\rm w}$ from Eq. (23) with $\mathcal{L}_\chi$ from Eqs. (15) and (16). In much the same way as in Section 2, in the bulk the ϕ condensate, $|\phi_*|^2 = v^2$, guarantees stability of the χ field. The global O(3) symmetry is unbroken.

At the same time, in the wall core where ϕ is close to zero (see Eq. (27) at z close to zero) the χ field becomes unstable and develops an expectation value, see Eq. (19). Thus, in the wall core the global symmetry breaking (20) occurs, giving rise to massless $O(3)/O(2)$ moduli described by the Lagrangian (21) on the wall world sheet.

There are two distinction, however. First the index a now runs over $a = 0, 1, 2$, i.e. the world sheet theory is three-dimensional. This distinction implies, in turn, that the world sheet theory is admittedly non-renormalizable low-energy effective theory and, as a result, less infrared-dependent. There is no reason to expect that the classical masslessness of the n^i fields will be lifted at the quantum level. In other words, the O(3) symmetry of the bulk will not be restored on the wall implying that the moduli fields will remain exactly massless NG bosons localized on the wall. Needless to say, in addition to the n^i fields, one (sterile) translational NG field is localized on the wall too. The corresponding sigma model will have a factorized structure with no coupling between the transnational and orientational moduli fields. Such factorization is common [9].

4 Other patterns

The same strategy as above can be easily used to obtain a variety of non-Abelian moduli localized on topological defects, such as domain walls or strings. Here I will give an extra example.

Instead of the O(3) triplet χ^i field as in (15) and (16) let us consider an SU(2) doublet η^p of complex scalar fields ($p = 1, 2$). We will couple this doublet to $\mathcal{L}_{\rm w}$ analogously to (15) and (16),

$$\mathcal{L}_\eta = \partial_\mu \bar\eta_i \, \partial^\mu \eta^i - U_\eta(\bar\eta, \eta, \phi) \,, \tag{28}$$

$$U_\eta = \gamma \left[\left(-\mu^2 + |\phi|^2 \right) \bar\eta_p \eta^p + \beta \left(\bar\eta_p \eta^p \right)^2 \right] , \tag{29}$$

It is obvious that this theory has a global $SU(2)$ symmetry, by construction. Actually, it has an $SU(2) \times SU(2)$ symmetry [8] due to the fact that $SU(2)$ is a quasi-real group. To see that this is indeed the case let us introduce a two-by-two matrix X,

$$X = \left(\begin{array}{cc} \eta^1 & -(\eta^2)^* \\ \eta^2 & (\eta^1)^* \end{array} \right) . \tag{30}$$

In terms of X the Lagrangian (28), (29) takes the form [8]

$$\mathcal{L} = \frac{1}{2} {\rm Tr} \left(\partial^\mu X \right)^\dagger \left(\partial_\mu X \right) - \gamma \left[\left(-\mu^2 + |\phi|^2 \right) \frac{1}{2} \, {\rm Tr}\, X^\dagger X + \frac{\beta}{4} \left({\rm Tr}\, X^\dagger X \right)^2 \right] . \tag{31}$$

This Lagrangian is obviously invariant under the transformation

$$X(x) \to U X(x) M^{-1} \,, \tag{32}$$

where U and M are arbitrary x-independent matrices from SU(2). The $SU(2) \times SU(2)$ symmetry is apparent. In the vacuum where $|\phi_*| > \mu$ the expectation value of X vanishes. The full $SU(2) \times SU(2)$ symmetry is unbroken. If, however, $|\phi_*| \sim 0$ the value of X approaches

$$X_* = \frac{\mu}{\sqrt{2\beta}} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \tag{33}$$

(corresponding to $\eta^1 = 1, \ \eta^2 = 0$). This vacuum expectation value breaks $SU(2)_L$ and $SU(2)_R$, but the diagonal global SU(2) corresponding to $U = M$ in (32) remains unbroken.

Now, we combine this Lagrangian with (23). Repeating the argumentation of the previous sections we expect that in the core of the wall the global symmetry is broken down to SU(2), with three NG modes in the coset

$$\frac{SU(2) \times SU(2)}{SU(2)} \,. \tag{34}$$

In other words, if a solution with (33) in the core exists, there should exist a family of solutions with the unit matrix replaced by U, an arbitrary two-by-two matrix from SU(2). These matrices are parametrized by three parameters (moduli). When they are endowed by the world sheet coordinate dependence, they become moduli fields localized on the topological defect (i.e. the moduli fields in 1+2 dimensions, if localized on the wall, and in 1+1 dimension, if localized on the vortex).

This is the standard chiral model for pions, whose Lagrangian can be written as

$$\mathcal{L} = \frac{1}{4g^2}\,\mathrm{Tr}\,\partial^\mu U \partial_\mu U^\dagger \,. \qquad (35)$$

In 1+1 dimensions this model is renormalizable and, moreover, asymptotically free [11]. A mass gap is expected to be generated at strong coupling, due to infrared interactions. In 1+2 dimensions this model is non-renormalizable; no mass gap generation is expected.

5 Conclusions

A few simple models discussed above demonstrate the fact that the occurrence of non-Abelian moduli localized on topological defects is a common and simple phenomenon, rather than an exotic and rare possibility.

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References

[1] E. Witten, Nucl. Phys. B **249**, 557 (1985).

[2] A. Hanany and D. Tong, JHEP **0307**, 037 (2003) [hep-th/0306150]; R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B **673**, 187 (2003) [hep-th/0307287]; M. Shifman, A. Yung, Phys. Rev. **D70**, 045004 (2004). [hep-th/0403149]; A. Hanany and D. Tong, JHEP **0404**, 066 (2004) [hep-th/0403158].

[3] M. Shifman and A. Yung, *Supersymmetric Solitons*, (Cambridge University Press, 2009).

[4] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **71**, 045010 (2005) [hep-th/0412082]; A. Gorsky and V. Mikhailov, Phys. Rev. D **76**, 105008 (2007) [arXiv:0707.2304 [hep-th]]; E. Nakano, M. Nitta and T. Matsuura, Prog. Theor. Phys. Suppl. **174**, 254 (2008) [arXiv:0805.4539 [hep-ph]]; M. Eto, M. Nitta and N. Yamamoto, Phys. Rev. Lett. **104**, 161601 (2010) [arXiv:0912.1352 [hep-ph]]; A Yung, Unpublished, 2009; A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **83**, 085027 (2011) [arXiv:1101.1120 [hep-ph]].

[5] M. Shifman, *Advanced Topics in Quantum Field Theory*, (Cambridge University Press, 2012), Section 57.

[6] S. R. Coleman, Commun. Math. Phys. **31**, 259 (1973).

[7] A. M. Polyakov and A. A. Belavin, JETP Lett. **22**, 245 (1975).

[8] M. Shifman and A. Vainshtein, Nucl. Phys. B **362**, 21 (1991).

[9] A. Losev, M. A. Shifman and A. I. Vainshtein, Phys. Lett. B **522**, 327 (2001) [hep-th/0108153]; New J. Phys. **4**, 21 (2002) [hep-th/0011027].

[10] M. Hindmarsh, Phys. Lett. B **225**, 127 (1989);
M. G. Alford, K. Benson, S. R. Coleman, J. March-Russell and F. Wilczek, Nucl. Phys. B **349**, 414 (1991).

[11] H. Leutwyler and M. A. Shifman, Int. J. Mod. Phys. A **7**, 795 (1992).