Continuity and Resurgence: Towards a continuum definition of the CP(N-1) model

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Continuity and Resurgence: towards a continuum definition of the CP(N-1) model

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We introduce a non-perturbative continuum framework to study the dynamics of quantum field theory (QFT), applied here to the CP(N-1) model. We show that the ambiguities in perturbation theory due to infrared renormalons are exactly canceled by corresponding ambiguities in the non-perturbative sector coming from amplitudes of certain non-perturbative objects: neutral bions and bion-anti-bions. This provides an explicit weak-coupling interpretation of the IR-renormalons. We use Écalle's theory of resurgent trans-series and the physical principle of continuity to continuously connect QFT to quantum mechanics, while preventing all intervening rapid cross-overs or phase transitions. The quantum mechanics contains the germ of all non-perturbative data, e.g., mass gap, of the QFT, all of which are calculable. For CP(N-1), the results obtained at arbitrary N are consistent with lattice and large-N results. The trans-series expansion, in which perturbative and non-perturbative effects are intertwined, encapsulates the multi-length-scale nature of the theory, and eliminates all perturbative and non-perturbative ambiguities under consistent analytic continuation of the coupling. A theorem by Pham et al implies that the mass gap is a resurgent function, for which resummation of the semi-classical expansion yields finite exact results in the weakly coupled domain.

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I. INTRODUCTION

The 1970s-80s witnessed an intensive research program on two-dimensional (2D) asymptotically free non-linear sigma models, motivated by their relevance to antiferromagnetic spin systems and four dimensional (4D) QCD [1–6]. Many results were found in the large-N limit, particularly concerning dynamical mass generation and chiral symmetry breaking. However, concrete results in finite-N theories remain scarce [7]; and even at large-N the microscopic mechanism by which a mass gap is generated remains open to date. The inheritance from this époque is a list of deep problems/puzzles about $\mathbb{CP}^{N-1}$. A partial list includes:

1. Invalidity of the dilute instanton gas approximation on $\mathbb{R}^2$. In a theory in which the instanton has size moduli, the dilute instanton gas picture is ill-defined, since it assumes that the typical inter-instanton separation is much larger then instanton size. This is a variant of Coleman's "infrared embarrassment" problem.

2. Perturbation theory leads to a non-Borel-summable divergent series even after regularization and renormalization. Attempts to Borel resum perturbation theory yield a class of ambiguities associated with singularities in the complex Borel plane [8]. While some such ambiguities are cancelled by non-perturbative ambiguities (associated with 2D instanton-anti-instanton events), via a QFT version of the Bogomolny-Zinn-Justin mechanism [9, 10], there are other (more relevant) ambiguities associated with infrared (IR) renormalons [5, 6, 8, 12], and there are no known (semi-classical or otherwise) 2D configurations with which these ambiguities may cancel. Therefore, even Borel resummed perturbation theory by itself is ill-defined, due to the IR renormalons.

3. There is not complete understanding of the microscopic mechanism underlying the large-N mass gap for $\mathbb{CP}^{N-1}$: $m_g = \Lambda = \mu e^{-S_I/N} = \mu e^{-\frac{4\pi}{g^2 N}}$, where μ and Λ are the renormalization and strong scale.

4. The precise connection between large-N results and the instanton gas approximation is only partially understood [2–4, 13].

In this paper we propose a new approach to asymptotically free quantum field theories that directly addresses the first three of these questions. Our approach is based on a mathematical formalism known as resurgent asymptotic analysis [14], in which physical quantities are expanded in terms of trans-series rather than perturbative series. A trans-series is an expansion that unifies both perturbative and non-perturbative expansions, in a manner that is self-consistent under analytic continuation of the various parameters and couplings. Resurgent trans-series have been studied for quantum mechanics [10] and for some simple quantum field theories and matrix models [15], but here we propose an application to the more difficult case of asymptotically free quantum field theories, such as $\mathbb{CP}^{N-1}$ models. More details can be found in [16], and applications to 4D gauge theories in [17, 18].

Resurgent trans-series are physically important because they make perturbation theory consistent for problems with degenerate classical vacua. For example, in quantum mechanical models with degenerate classical vacua, perturbation theory leads to a divergent and non-alternating series [10]. This leads to two immediate problems: (i) analysis of these divergent series (for example, by Borel summation) leads to imaginary contributions to observables (such as the energy) that should be real; (ii) this Borel summation procedure is actually ambiguous, with the ambiguity manifest in the sign of the imaginary non-perturbative contributions. Resurgent trans-series analysis produces an expression for the observable (such as the energy) that unifies the perturbative series with a sum over all non-perturbative contributions. The various terms in this trans-series are inter-related by an infinite ladder of intricate relations, in such a way that the full expression is unique, unambiguous and real. For example, the ambiguous imaginary term arising from a Borel analysis of the perturbative series is exactly cancelled by an identical term coming from the instanton/anti-instanton interaction, which lives in the non-perturbative part of the trans-series. "Resurgence" states that these cancellations occur to all orders, producing a consistent, real and unambiguous result.

This Bogomolny-Zinn-Justin (BZJ) mechanism of cancellation of ambiguities between the non-Borel-summable perturbative expansion and the non-perturbative multi-instanton sector has been explored in some detail for quantum mechanics (QM) problems with degenerate vacua [9–11], but in fact the resurgent structure is a general property of perturbation theory that should also be relevant for quantum field theory (QFT), particularly when there are degenerate vacua. For example, in asymptotically free quantum field theories such as 4D $SU(N)$ gauge theory or 2D $\mathbb{CP}^{N-1}$ theories, there are infrared renormalons that lead to non-Borel summability of perturbation theory. This is a serious problem, because it means that perturbation theory on its own is ill-defined and incomplete, just as is the case for the QM problems with degenerate vacua. But the naive analog of the quantum mechanical BZJ cancellation mechanism does not resolve this problem in the QFT case, for the following reason. The infrared (IR) renormalons lead to non-perturbative exponential factors $\exp\left[-2S_{\rm inst}/\beta_0\right]$, where the exponent involves twice the instanton action (as in QM), but divided by the one-loop beta function, which scales like N. But there are no such semi-classical non-perturbative objects in these theories defined on $\mathbb{R}^4$ or $\mathbb{R}^2$, respectively. Instanton-like configurations have non-perturbative factors of the form $\exp\left[-S_{\rm inst}\right]$, without the $1/N$ factor in the exponent, and so are sub-leading. Therefore, there are no non-perturbative semi-classical amplitudes that can cancel the ambiguous non-perturbative terms arising due to these IR renormalons. This is the puzzle.

Our proposed resolution of this puzzle is motivated by the fact that the instanton gas analysis of the non-perturbative sector is also ill-defined on $\mathbb{R}^4$ or $\mathbb{R}^2$, respectively, because the classical scale invariance implies that instantons of any size have the same action, leading to an infrared divergence in the instanton gas picture. A regularization of the QFT by spatial compactification leads to twisted boundary conditions that produce fractionalized instantons, which are associated with non-perturbative factors of the form $\exp\left[-S_{\rm inst}/N\right]$, appropriate for canceling the ambiguities from the IR renormalon poles. For 4D gauge theory, the dependence is parametrically correct [17, 18], while here we show that for $\mathbb{CP}^{N-1}$ the N dependence matches perfectly the expected N dependence coming from the IR renormalons; for more details see [16]. In Section II we introduce a new order parameter for the spatially compactified $\mathbb{CP}^{N-1}$ model, and show that this spatial compactification is stable and provides a consistent weak-coupling semi-classical window into the confined regime, in contrast to what happens for thermal compactification. In Section III we discuss the fractionalized semi-classical instanton and bion (instanton/anti-instanton molecule) configurations, and show that neutral bion amplitudes have non-perturbative ambiguities. In Section IV we show explicitly (see equation (10)) that the ambiguities in neutral bion amplitudes cancel precisely against the ambiguities coming from IR renormalons, and we compute the non-perturbative mass gap from a microscopic Hamiltonian. In the Conclusions we summarize and comment on outstanding questions.

II. NEW ORDER PARAMETER OF THE $\mathbb{CP}^{N-1}$ MODEL ON $\mathbb{R}\times\mathbb{S}^1_L$

The $\mathbb{CP}^{N-1}$ model is described by a quantum field $n(x)$ in the coset space $\frac{U(N)}{U(N-1)\times U(1)}$. The action is

$$S=\int d^2x\left[\frac{2}{g^2}\left(D_\mu n\right)^\dagger D_\mu n-\frac{i\Theta}{2\pi}\,\epsilon_{\mu\nu}(D_\mu n)^\dagger D_\nu n\right] \tag{1}$$

where $D_\mu=\partial_\mu+iA_\mu$, A_μ is an auxiliary field, and Θ is the topological angle. We consider the $\mathbb{CP}^{N-1}$ model on a spatially compactified cylinder $\mathbb{R}\times\mathbb{S}^1_L$. The $\mathbb{CP}^{N-1}$ model has a global $U(N)$ symmetry, and a local $U(1)$ gauge redundancy, $n\to e^{i\alpha(x)}n$. We parametrize $\mathbb{CP}^{N-1}$ by locally splitting the n-field into a phase and modulus, $n_i=e^{i\varphi_i}r_i$, involving $(N-1)$ phase fields $\{\varphi_1,\dots,\varphi_N\}$, $\sum_{i=1}^N\varphi_i=0\ \mathrm{mod}(2\pi)$ by $U(1)$ gauge redundancy, and $(N-1)$ modulus fields $\{r_1, \ldots, r_N\}$, $\sum_{i=1}^{N} r_i^2 = 1$. We also consider adding N_f species of Dirac fermions, although here we mostly deal with the bosonic $N_f = 0$ theory.

Each $-\partial_\mu \varphi_i \equiv \mathcal{A}_{\mu,i}$ transforms as a "gauge" connection, $\mathcal{A}_{\mu,i} \longrightarrow \mathcal{A}_{\mu,i} - \partial_\mu \alpha$, and we refer to it as the "σ-connection". With spatial compactification on $\mathbb{R}\times\mathbb{S}^1_L$, we define a new order parameter, the "σ-connection holonomy", making a circuit around the compact direction:

$$
\begin{aligned}
({}^L\Omega)_j(x_1) &:= e^{i\int_0^L dx_2\, \mathcal{A}_{2,j}} = e^{i(\varphi_j(x_1,0)-\varphi_j(x_1,L))} \\
{}^L\Omega(x_1) &:= \mathrm{Diag}\Big[({}^L\Omega)_1(x_1), \ldots, ({}^L\Omega)_N(x_1)\Big]
\end{aligned}
\tag{2}
$$

Under an aperiodic global gauge rotation, the line operator ${}^L\Omega(x_1) \to \ e^{i\frac{2\pi k}{N}}\, {}^L\Omega(x_1), \quad k = 1, \ldots, N$, reflecting the theory's global $\mathbb{Z}_N$ center-symmetry. (The fact that it is not $U(1)$ follows from the constraint, $\sum_{i=1}^{N} \varphi_i = 0 \bmod (2\pi)$, i.e., $\det^L \Omega(x_1) = 1$.) The matrix-valued gauge invariant σ-holonomy, ${}^L\Omega(x_1)$, plays an analogous role to the Wilson line in non-abelian $SU(N)$ gauge theory. Crucially, this operator contains more refined data than the familiar $U(1)$ Wilson line in the $\mathbb{CP}^{N-1}$ model: $W = e^{i\int_0^L dx_2 A_2}$.

A typical classical background of the σ-connection holonomy is ${}^L\Omega_{\rm b} = \mathrm{Diag}\Big[e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, \ldots, e^{2\pi i\mu_N}\Big]$. Classically, this background is equivalent to imposing twisted boundary conditions for the $\mathbb{CP}^{N-1}$ fields, of the form $n(x_1, x_2 + L) = \ {}^L\Omega_{\rm b} n(x_1, x_2)$ [19, 20]. Undoing the twist by a field redefinition is equivalent to the substitution, $\partial_\mu \to \partial_\mu + i\,\delta_{\mu 2}\, \frac{2\pi}{L}\,\mathrm{Diag}\Big[\mu_1, \mu_2, \ldots, \mu_N\Big]$, analogous to turning on a Wilson line in compactified $SU(N)$ gauge theory (and further reason to call $-\partial_\mu\varphi_i = \mathcal{A}_{\mu,i}$ σ-connection). In the small-L weak coupling regime, the quantum mechanical stability of a given background can be determined via a one-loop analysis of the potential for the holonomy (2), similar to gauge theory. Integrating out weakly coupled Kaluza-Klein modes, we find

$$
V_\pm[{}^L\Omega] = \frac{2}{\pi L^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(-1 + N_f(\pm 1)^n\right)\left(|\mathrm{tr}\,{}^L\Omega^n| - 1\right)
$$

where $-$ $(+)$ refers to thermal (spatial) compactification, where fermions have anti-periodic (periodic) boundary conditions. For $N_f \geq 0$, the N*-fold degenerate* minima in the thermal case are ${}^L\Omega_0^{\rm thermal} = e^{i\frac{2\pi k}{N}}\mathbf{1}_N, k = 1, \ldots N$, a clumped configuration of holonomy eigenvalues. In sharp contradistinction, for $N_f \geq 1$ and in the spatial case, the minimum is unique, ${}^L\Omega_0^{\rm spatial} = \mathrm{Diag}\left(1, e^{i\frac{2\pi}{N}}, \ldots, e^{i\frac{2\pi(N-1)}{N}}\right)$, a non-degenerate, $\mathbb{Z}_N$-symmetric holonomy, similar to QCD(adj) [21, 22]. The $N_f = 1$ case follows from non-perturbative effects [16].

Since there are no phase transitions (for finite-N) on $\mathbb{R}\times\mathbb{S}^1_L$, one may wonder what qualitative differences these two different backgrounds entail. In the thermal case, the potential at the minimum is the free energy density of the hot $\mathbb{CP}^{N-1}$ model, $\mathcal{F} = V_-[{}^L\Omega_0^{\rm thermal}] = -(2N-2)\frac{\pi}{6}\left(1 + \frac{N_f}{2}\right)T^2 \sim O(N^1)$, the Stefan-Boltzmann result; whereas in the cold regime, $\mathcal{F} \sim O(N^0)$, because the spectral density of physical states is $O(N^0)$. There is a rapid cross-over from a hot deconfined regime to the cold confined regime at the strong scale at finite-N, which becomes a sharp phase transition at $N = \infty$. With spatial compactification, the "free energy" density at small-L is $\mathcal{F} \sim O(N^0)$, just like the cold regime of the thermal theory. Therefore, there is no intervening rapid crossover (finite-N) or phase transition ($N = \infty$) as one dials the radius from large to small. This is the reason that the spatially compactified theory provides, in the small L regime, a weak-coupling semi-classical window into the confined regime [21, 22], whereas the thermally compactified theory provides, at small β, a weak coupling semi-classical window of the deconfined regime [4].

In the pure bosonic theory ($N_f = 0$) in which there is no distinction between the thermal and spatial compactification, a $\mathbb{Z}_N$-symmetric background is unstable. However, one can define a deformed bosonic $\mathbb{CP}^{N-1}$ (analogous to deformed YM) by introducing heavy fermions $m \gg \Lambda$, so that the theory at distances larger than m^{-1} emulates the pure bosonic theory. Then, render the KK-modes sufficiently high such that the heavy fermions appear light with respect to the KK-modes, i.e., $m \lesssim \frac{2\pi}{LN}$. Thus, from the point of view of the one-loop potential, we can use the result of the massless theory, and at the same time, at distances larger than m^{-1}, the theory is the bosonic $\mathbb{CP}^{N-1}$ model on a stable $\mathbb{Z}_N$-symmetric background.

III. KINK-INSTANTONS AND BIONS IN $\mathbb{CP}^{N-1}$ ON $\mathbb{R}\times\mathbb{S}^1_L$

In $\mathbb{CP}^1$, in the small-L regime, the Lagrangian associated with the zero mode of the KK-tower is ($\xi \equiv \frac{2\pi}{N\,L}$):

$$
S^{\rm zero} = \frac{L}{2g^2}\int_{\mathbb{R}} (\partial_t\theta)^2 + \sin^2\theta(\partial_t\phi)^2 + \xi^2\sin^2\theta,
\tag{3}
$$

where the kinetic term describes a particle on $S^2 \sim \mathbb{CP}^1$, and the potential follows from the $\mathbb{Z}_2$ stable background. This action has a semi-classical kink-instanton solution, which we call $\mathcal{K}_1$, interpolating from $\theta = 0$ to $\theta = \pi$, and with action $S_1 = \frac{L}{g^2}(2\xi) = \frac{4\pi}{g^2} \times (\mu_2 - \mu_1) = \frac{S_I}{2}$. Its topological charge is $Q = \frac{1}{2}$. There is also an independent kink-instanton, $\mathcal{K}_2$, with action $S_2 = \frac{S_I}{2}$, interpolating from $\theta = \pi$ to $\theta = 0$, also with topological charge $Q = \frac{1}{2}$. It is important to note that $\mathcal{K}_2$ is not the anti-kink $\overline{\mathcal{K}}_1$, which has $Q = -\frac{1}{2}$. The kink-instanton $\mathcal{K}_2$ is associated with the affine root of the $SU(2)$ algebra [19, 20].

This construction generalizes to $\mathbb{CP}^{N-1}$: there are N types of kinks, associated with the extended root system of the $SU(N)$ algebra, each of which carries a topological charge $Q = \frac{1}{N}$. The amplitude of $\mathcal{K}_i$ has a non-perturbative factor

$$\mathcal{K}_i : e^{-S_0} = e^{-\frac{4\pi}{g^2 N}} = e^{-\frac{4\pi}{\lambda}} \sim e^{-\frac{S_I}{N}}, \quad i = 1, \ldots, N \tag{4}$$

It is crucial to note the appearance of the 't Hooft coupling, $g^2 N \equiv \lambda$, in the amplitude. Thus, the kink-instantons are exponentially more relevant than the 2D instanton: $e^{-\frac{4\pi}{g^2 N}} \gg e^{-\frac{4\pi}{g^2}}$.

At second order in the semiclassical expansion, there are self-dual and non-self-dual configurations. The non-self-dual topological molecules are in one-to-one correspondence with the non-vanishing elements of the extended Cartan matrix $\widehat{A}_{ij}$ of $SU(N)$. For each entry $\widehat{A}_{ij} < 0$ of the extended Cartan matrix, there exists a *charged bion*, $\mathcal{B}_{ij} \sim [\mathcal{K}_i \overline{\mathcal{K}}_j]$, which plays a crucial role in the mass gap of the theory with $N_f \geq 1$, similar to the magnetic bion in QCD(adj) on $\mathbb{R}^3 \times S^1$ [22]. For each diagonal entry, $\widehat{A}_{ii} > 0$, there exists a *neutral bion*, $\mathcal{B}_{ii} \sim [\mathcal{K}_i \overline{\mathcal{K}}_i]$, with zero topological charge,

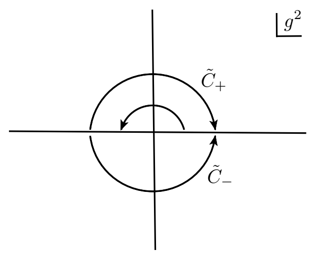

FIG. 1: Defining left (right) Borel sum $\mathbb{B}_{\theta=0^\pm}$, and left (right) neutral bion amplitude $[\mathcal{B}_{ii}]_{\theta=0^\pm}$. The $g^2 > 0$ line is a Stokes ray, the mathematical reason underlying the divergence of perturbation theory.

and indistinguishable from the perturbative vacuum [17, 18]. The $\mathcal{B}_{ii}$ generate a repulsion among the eigenvalues of the holonomy (2). For $g^2 > 0$, the constituents of the neutral bion interact attractively at short distances and the quasi-zero mode integral yields an amplitude which is naively meaningless. However, this is actually a mirror reflection of the "Stokes phenomenon" as will be discussed. We can evaluate, in the $N_f = 0$ theory, the neutral bion amplitude as shown in Fig. 1. First, take $g^2 \to -g^2$, where the neutral bion amplitude is well-defined. Then, analytic continuation along $\tilde{C}_\pm$ to $\theta = 0^\pm$ yields *left* and *right* amplitudes

$$\begin{aligned}[\mathcal{K}_i \overline{\mathcal{K}}_i]_{\theta=0^\pm} =& \mathrm{Re}\, [\mathcal{K}_i \overline{\mathcal{K}}_i] + i\, \mathrm{Im}\, [\mathcal{K}_i \overline{\mathcal{K}}_i]_{\theta=0^\pm} \\ =& \left(\log\left(\frac{g^2 N}{8\pi} \right) - \gamma \right) \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \pm i \frac{16\pi}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \end{aligned} \tag{5}$$

The *absence* of a well-defined $\theta \to 0$ limit means that the semi-classical expansion by itself is also ill-defined. Écalle's resurgent approach is to simultaneously apply this analytic continuation to the Borel summation of the perturbative sector *and* to the non-perturbative sector, in such a way that all ambiguities cancel, yielding an unambiguous and exact "trans-series" result. This mathematical technique has only been partially explored in QFT [15, 18], as most semi-classical studies only capture the first order. In quantum mechanics, this effect is studied in [9, 10]. Surprisingly, in QFT there appears to be universal behavior for the jump in the amplitude of neutral topological defects, arising from analytic continuation of the quasi-zero-mode integrals [16–18, 23].

IV. CANCELLATION OF AMBIGUITIES BETWEEN IR RENORMALONS AND THE NON-PERTURBATIVE SECTOR

The low energy Hamiltonian for (3), dropping ϕ-angle states in the Born-Oppenheimer approximation, is

$$H^{\text{zero}} = -\frac{1}{2} \frac{d^2}{d\theta^2} + \frac{\xi^2}{4g^2} [1 - \cos(2g\theta)] \tag{6}$$

The asymptotic form of the ground state energy, $\mathcal{E}_0(g^2)$, at large-orders in perturbation theory is evaluated in [24], using methods developed by Bender and Wu [25]:

$$\mathcal{E}_0(g^2) \equiv E_0\xi^{-1} = \sum_{q=0}^{\infty} a_{0,q}(g^2)^q, \quad a_{0,q} \sim -\frac{2}{\pi}\left(\frac{1}{4\xi}\right)^q q! \tag{7}$$

The series is "Gevrey-1" [14], non-alternating, and hence non-Borel summable; a manifestation of the fact that we are expanding the ground state energy along a Stokes ray in the complex-g^2 plane. The Borel transform is given by $B\mathcal{E}(t) = -\frac{2}{\pi}\sum_{q=0}^{\infty}\left(\frac{t}{4\xi}\right)^q = -\frac{2}{\pi}\frac{1}{1-\frac{t}{4\xi}}$, and has a pole singularity on the positive real axis $\mathbb{R}^+$ (i.e., non-Borel summability or ambiguity of the sum). However, the series is *right*- and *left*- Borel resummable, given by $\mathcal{S}_{0^\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\ B\mathcal{E}(t)\ e^{-t/g^2}$, where the contours $C_\pm$ pass above (below) the singularity. Equivalently, taking $g^2 \to -g^2$, the series (7) becomes Borel summable. Analytically continuing the sum back along $\tilde{C}_\pm$ yields the two "lateral" Borel sums:

$$\mathcal{S}_{0^\pm}\mathcal{E}(g^2) = \mathrm{Re}\,\mathbb{B}_0 \mp i\frac{16\pi}{g^2N}e^{-\frac{8\pi}{g^2N}} \tag{8}$$

where the imaginary part is the leading non-perturbative ambiguity of resummed perturbation theory, which is $O(e^{-S_{[\mathcal{K}_i\overline{\mathcal{K}}_i]}}) \sim e^{-2S_0} \sim e^{-2S_I/N}$, and $\mathrm{Re}\,\mathbb{B}_0 \sim O(1)$ is the unambiguous real part.

We now come to the crux of the matter: the resummed vacuum energy has an imaginary part, but it is *not* associated with the Dyson *instability*, or *decay of the vacuum*. Rather, this ambiguous imaginary part is a direct reflection of the fact that (resummed) perturbation theory by itself is ill-defined. Furthermore, the ambiguity that we find for $\mathbb{CP}^{N-1}$ on $\mathbb{R}\times\mathbb{S}^1_L$ is parametrically the same as that of the elusive IR-renormalons. Therefore, what was viewed as a problem, in fact becomes a blessing in disguise: consider the Θ-independent part of the vacuum energy density in a trans-series expansion (combining perturbative and non-perturbative terms), and collect unambiguous terms and ambiguous terms together:

$$\begin{aligned}
\mathcal{E}_{0,\mathrm{transseries}}(g^2) &= \sum_{q=0}^{\infty} a_{0,q}(g^2)^q + [\mathcal{B}_{ii}]\sum_{q=0}^{\infty} a_{2,q}(g^2)^q + \ldots \text{(formal)} \\
&\longrightarrow \mathbb{B}_{0,\theta=0^\pm} + [\mathcal{B}_{ii}]_{\theta=0^\pm}\mathbb{B}_{2,\theta=0^\pm} + \ldots (\mathrm{BE-resummation}) \\
&= \mathrm{Re}\,\mathbb{B}_0 + \mathrm{Re}\,[\mathcal{B}_{ii}]\mathrm{Re}\,\mathbb{B}_2 + i[\mathrm{Im}\,\mathbb{B}_{0,\theta=0^\pm} + \mathrm{Re}\,\mathbb{B}_2\mathrm{Im}\,[\mathcal{B}_{ii}]_{\theta=0^\pm}] \\
&= \mathrm{Re}\,\mathbb{B}_0 + \mathrm{Re}\,[\mathcal{B}_{ii}]\mathrm{Re}\,\mathbb{B}_2, \quad \text{up to } e^{-4S_0}
\end{aligned} \tag{9}$$

In our explicit computation, we have taken $\mathbb{B}_2 = a_{2,0} + O(g^2)$, and kept only $a_{2,0}$ for consistency because we are also only accounting for the leading large orders asymptotics in (7). The sum of the left (right) Borel resummation of perturbation theory and non-perturbative left (right) neutral bion amplitude is *unambiguous* at order $e^{-2S_0} = e^{-2S_I/N}$, as encoded in our perturbative-non-perturbative "confluence equation" in (9):

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0^\pm} + \mathrm{Re}\,\mathbb{B}_2\mathrm{Im}\,[\mathcal{B}_{ii}]_{\theta=0^\pm} = 0, \text{ up to } e^{-4S_0} \tag{10}$$

The passage from $\theta = 0^-$ to $\theta = 0^+$ is accompanied by a "Stokes jump" for the Borel resummation (8), which is mirrored by a jump in the neutral bion amplitude in the opposite direction (5) such that the sum of the two gives a unique result, with a smooth limit up to ambiguities at order e^{-4S_0}. Eq.(10) is conjectured to hold in (deformed) Yang-Mills in [17, 18], and here we verify it by explicit computation for $\mathbb{CP}^{N-1}$. Confluence equations are crucial for giving a non-perturbative continuum definition of QFT. We refer to the procedure in (9) as *Borel-Écalle resummation*, after Écalle's seminal work [14], which formalized asymptotic expansions with exponentially small terms (trans-series) and generalized Borel resummation to account for the Stokes phenomenon.

As an application, we calculate a physically interesting non-perturbative quantity. The mass gap is the energy required to excite the system from the ground state to the first excited state. For $\mathbb{CP}^1$, in the standard notation for Mathieu functions, the pair of states $\mathrm{ce}_n(\theta,q)$ and $\mathrm{se}_{n+1}(\theta,q)$ $(q = \frac{\xi^2}{4g^4})$, $n = 0, 1, 2, \ldots$, become degenerate to all orders in perturbation theory: their asymptotic expansions are identical. As $g^2 \to 0$, the splitting $\mathcal{E}(b_{n+1}) - \mathcal{E}(a_n)$ is purely non-perturbative. The mass gap is defined as $m_g = \mathcal{E}(b_1) - \mathcal{E}(a_0)$ and is given by

$$m_g = \frac{8\pi}{g}\left(1 - \frac{7g^2}{16\pi} + O(g^4)\right)e^{-\frac{2\pi}{g^2}} \sim e^{-S_I/2} \tag{11}$$

This also justifies the Born-Oppenheimer approximation, because low-lying states are non-perturbatively split, whereas their separation from the higher states is an order one gap: $m_g \ll \mathcal{E}(a_1) - \mathcal{E}(b_1) \sim \Delta\mathcal{E}_\phi$, where $\Delta\mathcal{E}_\phi$ is the gap in the ϕ-sector in (3). For $\mathbb{CP}^{N-1}$, generalizing the above discussion, we find $m_g \sim \frac{1}{\sqrt{g^2 N}} e^{-\frac{4\pi}{g^2 N}} \sim e^{-S_I/N}$, which is a kink-instanton effect (4). We are not aware of any previous microscopic derivation in $\mathbb{CP}^{N-1}$ of the all-important non-perturbative mass gap $\sim e^{-S_I/N}$. The gap at small-L may be considered as *the germ of the mass gap* for the theory on $\mathbb{R}^2$. At large-N, this agrees with the mass gap obtained by the master field method [1].

V. CONCLUSIONS

We have shown that for the $\mathbb{CP}^{N-1}$ model defined on $\mathbb{R}\times\mathbb{S}^1_L$, the Borel singularities due to infrared renormalons occur at *precisely* the same location as the singularities corresponding to certain non-perturbative objects: neutral bions and bion-anti-bions. We have demonstrated the cancellation of the two ambiguities, from the perturbative and non-perturbative sector, leaving an unambiguous answer. This is the first explicit demonstration of the BZJ mechanism in a non-trivial asymptotically free quantum field theory. We have argued that the spatial compactification leads to a continuity by which this result obtained in the small L regime can be continued smoothly, without any phase transition or rapid crossover, to the large L regime. If this latter claim could be rigorously proven, then we would have a simple and elegant resolution of the first two problems listed in the introduction: the failure of the instanton gas picture for $\mathbb{CP}^{N-1}$, and the ambiguities arising from the non-Borel-summability of perturbation theory due to infrared renormalons. This QFT result is in both qualitative and quantitative agreement with lattice and large-N results.

We have also given an explicit microscopic computation of the mass gap. Notice that in our mass gap result (11) there is a perturbative series multiplying the kink-instanton amplitude, which is itself a divergent asymptotic (non-Borel-summable) series. One can ask what this means. This question has been answered in the mathematics literature, and is the essence of resurgence. An important result by Pham et al, and Delabaere [26], using Écalle's theory of resurgence [14], proves that the semi-classical expansions for the energy levels of the QM double-well and periodic potentials are indeed resurgent functions, resummable to finite, unambiguous, exact results. In the small L regime this is precisely the result we need to argue that these cancellations occur to all orders. Our primary contribution here is that we have found the conditions under which a non-trivial QFT, such as the asymptotically free $\mathbb{CP}^{N-1}$ model, is connected to QM without any rapid cross-over or phase transition; i.e., by guaranteeing continuity. This permits us to derive the germ of all non-perturbative observables in the QFT in the small-$\mathbb{S}^1$ domain using these rigorous QM results of Pham et al. Furthermore, introducing the Θ dependence leads to a 'grading' of the resurgent trans-series structure [16]. We hope that this remarkable connection between QFT and QM may be used to explore other non-perturbative properties of general QFTs, and eventually lead to a fully consistent non-perturbative definition of non-trivial QFTs in the continuum.

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