Area-angular momentum-charge inequality for stable marginally outer trapped surfaces in 4D Einstein-Maxwell-dilaton theory

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Phys. Rev. D **87**, 024016 — Published 7 January 2013

DOI: 10.1103/PhysRevD.87.024016

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gravity, the low energy string theory [16, 17], Kaluza-Klein theory [18], as well as in some theories with gradient spacetime torsion [19].

The field equations of Einstein-Maxwell-dilaton gravity with matter are presented below in eq. (1). A characteristic feature of this theory is the coupling between the scalar field (dilaton) φ and the electromagnetic field F_{ab} and this coupling is governed by a parameter γ (called dilaton coupling parameter). The static and stationary isolated black holes in 4D Einstein-Maxwell-dilaton theory were extensively studied in various aspects during the last two decades. The classification of the isolated stationary, axisymmetric, asymptotically flat black holes with a connected horizon in Einstein-Maxwell-dilaton gravity was given in [20] for dilaton coupling parameter γ satisfying $0 \leq \gamma^2 \leq 3$. The static asymptotically flat Einstein-Maxwelldilaton black holes (without axial symmetry and horizon connectedness assumption) were classified in [21]. The sector of stationary Einstein-Maxwell-dilaton black holes with dilaton coupling parameter beyond the critical value $\gamma^2 = 3$ is extremely difficult to be analyzed analytically. Most probably the black hole uniqueness is violated in this sector as the numerical investigations imply [22].

In the present paper we derive some inequalities between the area, the angular momentum and the charges for dynamical black holes in Einstein-Maxwell-dilaton gravity with a non-negative dilaton potential and with a matter energy-momentum tensor satisfying the dominant energy condition.

2 Basic notions and setting the problem

Let $(\mathcal{M}, g)$ be a 4-dimensional spacetime satisfying the Einstein-Maxwell-dilaton-matter equations

$$\begin{aligned}
&R_{ab} - \frac{1}{2}Rg_{ab} = 2\nabla_a\varphi\nabla_b\varphi - g_{ab}\nabla^c\varphi\nabla_c\varphi + 2e^{-2\gamma\varphi}\left(F_{ac}F_b{}^c - \frac{g_{ab}}{4}F_{cd}F^{cd}\right)\\
&-2V(\varphi)g_{ab} + 8\pi T_{ab},\\
&\nabla_{[a}F_{bc]} = 0,\\
&\nabla_a\left(e^{-2\gamma\varphi}F^{ab}\right) = 4\pi J^b,\\
&\nabla_a\nabla^a\varphi = -\frac{\gamma}{2}e^{-2\gamma\varphi}F_{ab}F^{ab} + \frac{dV(\varphi)}{d\varphi},
\end{aligned} \tag{1}$$

where g_{ab} is the spacetime metric and ∇_a is its Levi-Civita connection, $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ is the Einstein tensor. F_{ab} is the Maxwell tensor and J^a is the current. The dilaton field is denoted by φ, $V(\varphi)$ is its potential and γ is the dilaton coupling parameter governing the coupling strength of the dilaton to the electromagnetic field. The matter energy-momentum tensor is T_{ab}. We assume that T_{ab} satisfies the dominant energy condition. Concerning the dilaton potential, we assume that it is non-negative ($V(\varphi) \geq 0$).

Further we consider a closed orientable 2-dimensional spacelike surface $\mathcal{B}$ smoothly embedded in the spacetime $\mathcal{M}$. The induced metric on $\mathcal{B}$ and its Levi-Civita connection are denoted by q_{ab} and D_a. In order to describe the extrinsic geometry of $\mathcal{B}$ we introduce the normal outgoing and ingoing null vectors l^a and k^a with the normalization condition $g(l,k) = l^a k_a = -1$. The extrinsic geometry then is characterized by the expansion Θ^l, the shear σ^l_{ab} and the normal fundamental form Ω^l_a associated with the outgoing null normal l^a and defined as follows

$$\Theta^l = q^{ab}\nabla_a l_b, \tag{2}$$

$$\sigma^l_{ab} = q^c_a q^d_b \nabla_c l_d - \frac{1}{2}\Theta^l q_{ab}, \tag{3}$$

$$\Omega^l_a = -k^c q^d_a \nabla_d l_c. \tag{4}$$

In what follows we require $\mathcal{B}$ to be a marginally outer trapped surface (i.e. $\Theta^l = 0$) and $\mathcal{B}$ to be stable (or spacetime stably outermost in more formal language) [27]–[25],[4]. The last condition means that there exists an outgoing vector $V^a = \lambda_1 l^a - \lambda_2 k^a$ with functions $\lambda_1 \geq 0$ and $\lambda_2 > 0$ such that $\delta_V \Theta^l \geq 0$, with δ_V being the deformation operator on $\mathcal{B}$ [24]–[26]. In simple words the deformation operator describes the infinitesimal variations of the geometrical objects on $\mathcal{B}$ under an infinitesimal deformation of $\mathcal{B}$ along the flow of the vector V^a.

As an additional technical assumption we require $\mathcal{B}$ to be invariant under the action of $U(1)$ group with a Killing generator η^a. We assume that the Killing vector η^a is normalized to have orbits with a period 2π. Also we require that $\mathcal{B}$ is axisymmetrically stable[1] and $\pounds_\eta l^a = \pounds_\eta k^a = 0$ and $\pounds_\eta \Omega^l_a = \pounds_\eta \tilde{F}_{ab} = \pounds_\eta \varphi = 0$, where $\tilde{F}$ is the projection of the Maxwell 2-form on $\mathcal{B}$.

From the axisymmetric stability condition one can derive the following important inequality valid for every axisymmetric function α on $\mathcal{B}$ [4]

$$\int_{\mathcal{B}} \left[|D\alpha|^2_q + \frac{1}{2} R_{\mathcal{B}} \alpha^2 \right] dS \geq \int_{\mathcal{B}} \left[\alpha^2 |\Omega^\eta|^2_q + \alpha\beta |\sigma^l|^2_q + G_{ab} \alpha l^a (\alpha k^b + \beta l^b) \right] dS, \tag{5}$$

where $|.|_q$ is the norm with respect to the induced metric q_{ab}, dS is the surface element measure on $\mathcal{B}$, $R_{\mathcal{B}}$ is the scalar curvature of $\mathcal{B}$, $\Omega^\eta = \eta^a \Omega^l_a$ and $\beta = \alpha\lambda_1/\lambda_2$.

At this stage we can use the field equations (1) which gives

$$\int_{\mathcal{B}} \left[|D\alpha|^2_q + \frac{1}{2} R_{\mathcal{B}} \alpha^2 \right] dS \geq \int_{\mathcal{B}} \Big\{ \alpha^2 |\Omega^\eta|^2_q + \alpha\beta |\sigma^l|^2_q + \alpha^2 |D\varphi|^2_q + 2\alpha^2 V(\varphi) \tag{6}$$
$$+ 2\alpha\beta (l^a \nabla_a \varphi)^2 + \alpha^2 e^{-2\gamma\varphi} \left[E^2_\perp + B^2_\perp \right] + 2\alpha\beta e^{-2\gamma\varphi} (i_l F)_a (i_l F)^a + 8\pi T_{ab} \alpha l^a (\alpha k^b + \beta l^b) \Big\} dS,$$

where $E_\perp = i_k i_l F$ and $B_\perp = i_k i_l \star F$. All terms on the right hand side of the above inequality are non-negative. Indeed, for the last term we have $8\pi T_{ab} \alpha l^a (\alpha k^b + \beta l^b) \geq 0$ since the energy-momentum tensor of matter satisfies the dominant energy condition. We also have $2\alpha\beta e^{-2\gamma\varphi} (i_l F)_a (i_l F)^a \geq 0$ since the electromagnetic field satisfies the null energy condition and $\alpha\beta \geq 0$.

Considering now the inequality for $\alpha = 1$ and applying the Gauss-Bonnet theorem[2] we find that the Euler characteristic of $\mathcal{B}$ satisfies

$$Euler(\mathcal{B}) > 0, \tag{7}$$

[1] i.e. axisymmetric and stable with axisymmetric functions λ_1 and λ_2.

[2] According to the Gauss-Bonnet theorem, the Euler characteristic is given by $Euler(\mathcal{B}) = \frac{1}{2}\int_{\mathcal{B}} R_{\mathcal{B}} dS = 4\pi(1-g)$ where g is the genus of $\mathcal{B}$.

which shows that the topology of $\mathcal{B}$ is that of a 2-dimensional sphere S^2.

Discarding the following non-negative terms $\alpha\beta|\sigma^l|_q^2$, $2\alpha^2 V(\varphi)$, $2\alpha\beta(l^a\nabla_a\varphi)^2$, $2\alpha\beta e^{-2\gamma\varphi}(i_lF)_a(i_lF)^a$ and $8\pi T_{ab}\alpha l^a(\alpha k^b+\beta l^b)$ we obtain

$$\int_{\mathcal{B}}\left[|D\alpha|_q^2+\frac{1}{2}R_{\mathcal{B}}\alpha^2\right]dS \geq \int_{\mathcal{B}}\alpha^2\left\{|\Omega^\eta|_q^2+|D\varphi|_q^2+e^{-2\gamma\varphi}\left[E_\perp^2+B_\perp^2\right]\right\}dS. \tag{8}$$

Proceeding further we write the induced metric on $\mathcal{B}$ in the form

$$dl^2 = e^{2C-\sigma}d\theta^2+e^\sigma\sin^2\theta d\phi^2, \tag{9}$$

where C is a constant. The absence of conical singularities requires $\sigma|_{\theta=0}=\sigma|_{\theta=\pi}=C$. It is easy to see that the area of $\mathcal{B}$ is given by $\mathcal{A}=4\pi e^C$. Regarding the 1-form Ω^l_a, we may use the Hodge decomposition

$$\Omega^l = *d\omega + d\varsigma, \tag{10}$$

where $*$ is the Hodge dual on $\mathcal{B}$, and ω and ς are regular axisymmetric functions on $\mathcal{B}$. Then we obtain

$$\Omega^\eta = i_\eta * d\omega \tag{11}$$

since ς is axisymmetric and $i_\eta d\varsigma = \pounds_\eta\varsigma = 0$.

We can also introduce electromagnetic potentials Φ and Ψ on $\mathcal{B}$ defined by[3]

$$d\Phi = B_\perp * \eta, \tag{12}$$
$$d\Psi = e^{-2\gamma\varphi}E_\perp * \eta. \tag{13}$$

It turns out useful to introduce another potential χ instead of ω which is defined by

$$d\chi = 2Xd\omega - 2\Phi d\Psi + 2\Psi d\Phi, \tag{14}$$

where $X = q_{ab}\eta^a\eta^b$ is the norm of the Killing field η^a. This step is necessary in order to bring the functional $I_*[X^A]$ defined below, in the same formal form as in the stationary case.

The electric charge Q and the magnetic charge P associated with $\mathcal{B}$ are defined as follows

$$Q = \frac{1}{4\pi}\int_{\mathcal{B}} e^{-2\gamma\varphi}E_\perp dS, \tag{15}$$
$$P = \frac{1}{4\pi}\int_{\mathcal{B}} B_\perp dS. \tag{16}$$

We also define the angular momentum J associated with $\mathcal{B}$

$$J = \frac{1}{8\pi}\int_{\mathcal{B}}\Omega^\eta dS + \frac{1}{8\pi}\int_{\mathcal{B}}\left(\Phi e^{-2\gamma\varphi}E_\perp - \Psi B_\perp\right)dS, \tag{17}$$

[3] We denote the Killing vector field η and its naturally corresponding 1-form with the same letter.

where the first integral is the contribution of the gravitational field, while the second integral is the contribution due to the electromagnetic field [20].

Using the definitions of the potentials Ψ, Φ and χ one can show that the electric charge, the magnetic charge and the angular momentum are given by

$$Q = \frac{\Psi(\pi) - \Psi(0)}{2}, \;\; P = \frac{\Phi(\pi) - \Phi(0)}{2}, \;\; J = \frac{\chi(\pi) - \chi(0)}{8}. \tag{18}$$

Since the potentials Ψ, Φ and χ are defined up to a constant, without loss of generality we put $\Psi(\pi) = -\Psi(0) = Q$, $\Phi(\pi) = -\Phi(0) = P$ and $\chi(\pi) = -\chi(0) = 4J$.

Going back to the inequality (8), choosing $\alpha = e^{C-\sigma/2}$, and after some algebra we obtain

$$\begin{aligned} 2(C+1) \geq \frac{1}{2\pi}\int_{\mathcal{B}} \Big\{ & \sigma + \frac{1}{4}|D\sigma|^2 + \frac{1}{4X^2}|D\chi + 2\Phi D\Psi - 2\Psi D\Phi|^2 \\ & + \frac{1}{X}e^{-2\gamma\varphi}|D\Phi|^2 + \frac{1}{X}e^{2\gamma\varphi}|D\Psi|^2 + |D\varphi|^2 \Big\} dS_0, \end{aligned} \tag{19}$$

where the norm $|\,.\,|$ and the surface element dS_0 are with respect to the standard usual round metric on S^2. Taking into account that $\mathcal{A} = 4\pi e^C$ the above inequality is transformed to the following inequality for the area

$$\mathcal{A} \geq 4\pi e^{(I[X^A]-2)/2}, \tag{20}$$

where the functional $I[X^A]$, with $X^A = (X, \chi, \Phi, \Psi, \varphi)$, is defined by the right hand side of (19), i.e.

$$\begin{aligned} I[X^A] = \frac{1}{2\pi}\int_{\mathcal{B}} \Big\{ & \sigma + \frac{1}{4}|D\sigma|^2 + \frac{1}{4X^2}|D\chi + 2\Phi D\Psi - 2\Psi D\Phi|^2 \\ & + \frac{1}{X}e^{-2\gamma\varphi}|D\Phi|^2 + \frac{1}{X}e^{2\gamma\varphi}|D\Psi|^2 + |D\varphi|^2 \Big\} dS_0. \end{aligned} \tag{21}$$

In order to bring the action into a form more suitable for the further investigation we express $D\sigma$ by the norm of the Killing field η (i.e. $e^{\sigma} = X/\sin^2\theta$) and introduce a new independent variable $\tau = \cos\theta$. In this way we obtain

$$\begin{aligned} I[X^A] = \quad & \int_{-1}^{1} \Bigg\{ \frac{d}{d\tau}(\sigma\tau) + 1 + (1-\tau^2)\Bigg[\frac{1}{4X^2}\left(\frac{dX}{d\tau}\right)^2 + \frac{1}{4X^2}\left(\frac{d\chi}{d\tau} + 2\Phi\frac{d\Psi}{d\tau} - 2\Psi\frac{d\Phi}{d\tau}\right)^2 \\ & + \frac{e^{-2\gamma\varphi}}{X}\left(\frac{d\Phi}{d\tau}\right)^2 + \frac{e^{2\gamma\varphi}}{X}\left(\frac{d\Psi}{d\tau}\right)^2 + \left(\frac{d\varphi}{d\tau}\right)^2\Bigg] - \frac{1}{1-\tau^2}\Bigg\} d\tau. \end{aligned} \tag{22}$$

At this stage we introduce the strictly positive definite metric[4]

$$dL^2 = G_{AB}dX^A dX^B = \frac{dX^2 + (d\chi + 2\Phi d\Psi - 2\Psi d\Phi)^2}{4X^2} + \frac{e^{-2\gamma\varphi}d\Phi^2 + e^{2\gamma\varphi}d\Psi^2}{X} + d\varphi^2 \tag{23}$$

[4]It is worth mentioning that the continuous rotational $O(2)$ symmetry in the case of Einstein-Maxwell gravity degenerates here to the discrete symmetry $\pm\Phi \longleftrightarrow \pm\Psi$ and $\varphi \longleftrightarrow -\varphi$.

on the 5-dimensional Riemannian manifold $\mathcal{N} = \{(X,\chi,\Phi,\Psi,\varphi) \in \mathbb{R}^5; X > 0\}$. In terms of this metric the functional $I[X^A]$ is written in the form

$$I[X^A] = \int_{-1}^{1} \left\{ \frac{d}{d\tau}(\sigma\tau) + 1 + (1-\tau^2) G_{AB} \frac{dX^A}{d\tau}\frac{dX^B}{d\tau} - \frac{1}{1-\tau^2} \right\} d\tau. \tag{24}$$

Let us summarize the results obtained so far in the following

Lemma 1. *Let $\mathcal{B}$ be a smooth, spacetime stably outermost axisymmetric marginally outer trapped surface in a spacetime satisfying the Einstein-Mawell-dilaton-matter equations (1). If the matter energy-momentum tensor satisfies the dominant energy condition and the dilaton potential is non-negative, then the area of $\mathcal{B}$ satisfies the inequality*

$$\mathcal{A} \geq 4\pi e^{(I[X^A]-2)/2}, \tag{25}$$

where the functional $I[X^A]$ is given by (24) with a metric G_{AB} defined by (23).

In order to put a tight lower bound for the area we should solve the variational problem for the minimum of the functional $I[X^A]$ with appropriate boundary conditions if the minimum exists at all. Since the first two terms in $I[X^A]$ are in fact boundary terms, the minimum of $I[X^A]$ is determined by the minimum of the reduced functional

$$I_\star[X^A] = \int_{-1}^{1} \left[(1-\tau^2) G_{AB} \frac{dX^A}{d\tau}\frac{dX^B}{d\tau} - \frac{1}{1-\tau^2} \right] d\tau. \tag{26}$$

In order to perform the minimizing procedure we have to specify in which class of functions $X^A = (X,\chi,\Phi,\Psi,\varphi)$, the functional $I_\star[X^A]$ is varied. From a physical point of view the relevant class of functions is specified by the natural requirements $(\chi,\Phi,\Psi,\varphi) \in C^\infty[-1,1]$, $\sigma = \ln\left(\frac{X}{1-\tau^2}\right) \in C^\infty[-1,1]$ with boundary conditions $\Phi(\tau=-1) = -\Phi(\tau=1) = P$, $\Psi(\tau=-1) = -\Psi(\tau=1) = Q$ and $\chi(\tau=-1) = -\chi(\tau=1) = 4J$.

Lemma 2. *For dilaton coupling parameter satisfying $0 \leq \gamma^2 \leq 3$, there exists a unique smooth minimizer of the functional $I[X^A]$ (respectively $I_\star[X^A]$) with the prescribed boundary conditions.*

Proof. Let us consider the "truncated" functional

$$I_\star[X^A][\tau_2,\tau_1] = \int_{\tau_1}^{\tau_2} \left[(1-\tau^2) G_{AB} \frac{dX^A}{d\tau}\frac{dX^B}{d\tau} - \frac{1}{1-\tau^2} \right] d\tau \tag{27}$$

with boundary conditions $X^A(\tau_1), X^A(\tau_2)$ for $-1 < \tau_1 < \tau_2 < 1$. Introducing the new variable $t = \frac{1}{2}\ln\left(\frac{1+\tau}{1-\tau}\right)$ the truncated action takes the form

$$I_\star[X^A][t_2,t_1] = \int_{t_1}^{t_2} \left[G_{AB} \frac{dX^A}{dt}\frac{dX^B}{dt} - 1 \right] dt, \tag{28}$$

which is just a modified version of the geodesic functional in the Riemannian space $(\mathcal{N}, G_{AB})$. Consequently the critical points of our functional are geodesics in $\mathcal{N}$. However, it was shown in [20] that for $0 \leq \gamma^2 \leq 3$ the Riemannian space $(\mathcal{N}, G_{AB})$ is simply connected, geodesically complete and with negative sectional curvature. Therefore for fixed points $X^A(t_1)$ and $X^A(t_2)$ there exists a unique minimizing geodesic connecting these points. Hence we conclude that the global minimizer of $I_\star[X^A][t_2,t_1]$ exists and is unique for $0 \le \gamma^2 \le 3$. Since $(\mathcal{N}, G_{AB})$ is geodesically complete the global minimizer of $I_\star[X^A][t_2,t_1]$ can be extended to a global minimizer of $I_\star[X^A]$ and $I[X^A]$. In more detail the proof goes as follows. Let us put $\tau_2(\varepsilon) = 1-\varepsilon, \tau_1(\varepsilon) = -1+\varepsilon$ (i.e. $t_2(\varepsilon) = -t_1(\varepsilon) = \frac{1}{2}\ln\left(\frac{2-\varepsilon}{\varepsilon}\right)$) where ε is a small positive number ($\varepsilon > 0$) and consider the truncated functional

$$I_\varepsilon[X^A] = \int_{\tau_1(\varepsilon)}^{\tau_2(\varepsilon)} \left[\frac{d}{d\tau}(\sigma\tau)+1\right] d\tau + I_*[X^A][\tau_2(\varepsilon),\tau_1(\varepsilon)] =$$
$$\sigma[\tau_2(\varepsilon)]\tau_2(\varepsilon) - \sigma[\tau_1(\varepsilon)]\tau_1(\varepsilon) + 2(1-\varepsilon) + I_*[X^A][\tau_2(\varepsilon),\tau_1(\varepsilon)] \tag{29}$$

with boundary conditions $X^A(\tau_1(\varepsilon)) = X_1^A(\varepsilon)$ and $X^A(\tau_2(\varepsilon)) = X_2^A(\varepsilon)$, and with $\lim_{\varepsilon\to 0} X_1^A(\varepsilon) = (0, 4J, P, Q, \varphi_-)$ and $\lim_{\varepsilon\to 0} X_2^A(\varepsilon) = (0, -4J, -P, -Q, \varphi_+)$. Here $\varphi_\pm$ are defined by $\varphi_\pm = \varphi(\tau = \pm 1)$.

Consider now the unique minimizing geodesic Γ_ε in $\mathcal{N}$ between the points $X_1^A(\varepsilon)$ and $X_2^A(\varepsilon)$. Then we have

$$I_\varepsilon[X^A] \ge \sigma[\tau_2(\varepsilon)]|_{\Gamma_\varepsilon}\tau_2(\varepsilon) - \sigma[\tau_1(\varepsilon)]|_{\Gamma_\varepsilon}\tau_1(\varepsilon) + 2(1-\varepsilon) + I_*[X^A][\tau_2(\varepsilon),\tau_1(\varepsilon)]|_{\Gamma_\varepsilon} \tag{30}$$

where the right hand side of the above inequality is evaluated on the geodesic Γ_ε. Since $\lambda_\varepsilon^2 = G_{AB}\frac{dX^A}{dt}\frac{dX^B}{dt}$ is a constant on the geodesic Γ_ε we find

$$I_*[X^A][\tau_2(\varepsilon),\tau_1(\varepsilon)]|_{\Gamma_\varepsilon} = \int_{t_1(\varepsilon)}^{t_2(\varepsilon)} \left[G_{AB}\frac{dX^A}{dt}\frac{dX^B}{dt} - 1\right] dt = \left(\lambda_\varepsilon^2 - 1\right)\left(t_2(\varepsilon) - t_1(\varepsilon)\right). \tag{31}$$

The nest step is to evaluate λ_ε. This can be done by evaluating $G_{AB}\frac{dX^A}{dt}\frac{dX^B}{dt}$ at the boundary points which are in a small neighborhood of the poles $\tau = \pm 1$. First we write $\lambda_\varepsilon^2 = G_{AB}\frac{dX^A}{dt}\frac{dX^B}{dt}$ in the form

$$\begin{aligned}\lambda_\varepsilon^2 = \quad & \frac{(1-\tau^2)^2}{4X^2}\left(\frac{dX}{d\tau}\right)^2 + \frac{(1-\tau^2)^2}{4X^2}\left(\frac{d\chi}{d\tau} + 2\Phi\frac{d\Psi}{d\tau} - 2\Psi\frac{d\Phi}{d\tau}\right)^2 + \\ & \frac{(1-\tau^2)^2}{X}e^{-2\gamma\varphi}\left(\frac{d\Phi}{d\tau}\right)^2 + \frac{(1-\tau^2)^2}{X}e^{2\gamma\varphi}\left(\frac{d\Psi}{d\tau}\right)^2 + (1-\tau^2)^2\left(\frac{d\varphi}{d\tau}\right)^2.\end{aligned} \tag{32}$$

Within the class of function we consider, the terms associated with X and φ have the following behavior in a small neighborhood of the poles $\tau = \pm 1$, namely

$$\frac{(1-\tau^2)^2}{4X^2}\left(\frac{dX}{d\tau}\right)^2 = 1 + O(\varepsilon), \tag{33}$$
$$(1-\tau^2)^2\left(\frac{d\varphi}{d\tau}\right)^2 = O(\varepsilon^2). \tag{34}$$

The terms associated with Φ and Ψ behave as

$$\frac{(1-\tau^2)^2}{X}e^{-2\gamma\varphi}\left(\frac{d\Phi}{d\tau}\right)^2 = O(\varepsilon), \tag{35}$$

$$\frac{(1-\tau^2)^2}{X}e^{2\gamma\varphi}\left(\frac{d\Psi}{d\tau}\right)^2 = O(\varepsilon). \tag{36}$$

In order to find the behavior of the term associated with the potential χ, we should notice that $\partial/\partial\chi$ is a Killing vector for the metric G_{AB} and consequently we have the following conservation law

$$\frac{1}{4X^2}\left(\frac{d\chi}{dt}+2\Phi\frac{d\Phi}{dt}-2\Psi\frac{d\Phi}{dt}\right) = \frac{1-\tau^2}{4X^2}\left(\frac{d\chi}{d\tau}+2\Phi\frac{d\Phi}{d\tau}-2\Psi\frac{d\Phi}{d\tau}\right) = const_{\varepsilon}. \tag{37}$$

Using this we obtain that the term associated with χ is equal to $4\,const_{\varepsilon}^2 X^2$ which shows that it behaves as $O(\varepsilon^2)$. Therefore we can conclude that $\lambda_{\varepsilon}^2 - 1 = O(\varepsilon)$ which gives

$$\lim_{\varepsilon\to 0} I_*[X^A][\tau_2(\varepsilon),\tau_1(\varepsilon)]|_{\Gamma_\varepsilon} = \lim_{\varepsilon\to 0}\left(\lambda_\varepsilon^2-1\right)\left(t_2(\varepsilon)-t_1(\varepsilon)\right) = 0. \tag{38}$$

In this way, from (30) we have

$$\begin{aligned} &I[X^A] = \lim_{\varepsilon\to 0} I_\varepsilon[X^A] \geq \\ &\lim_{\varepsilon\to 0}\left\{\sigma[\tau_2(\varepsilon)]|_{\Gamma_\varepsilon}\tau_2(\varepsilon) - \sigma[\tau_1(\varepsilon)]|_{\Gamma_\varepsilon}\tau_1(\varepsilon) + 2(1-\varepsilon) + I_*[X^A][\tau_2(\varepsilon),\tau_1(\varepsilon)]|_{\Gamma_\varepsilon}\right\} \end{aligned} \tag{39}$$

and therefore

$$I[X^A] \geq 2\sigma_p + 2 \tag{40}$$

where σ_p is the value of $\sigma(\tau)$ on the poles. This completes the proof.

Even in the cases when the global minimizer of $I[X^A]$ exists, there is another serious problem in Einstein-Maxwell-dilaton gravity. In Einstein-Maxwell gravity the lower bound for the area is clear from physical considerations and there is a completely explicit solution realizing it, namely the extremal Kerr-Newman solution. So the approach is to formally prove that the area of the extremal Kerr-Newman solution is indeed the lower bound. The situation in Einstein-Maxwell-gravity is rather different. Contrary to the Einstein-Maxwell case where the Euler-Lagrange equations can be solved explicitly, in Einstein-Maxwell-dilaton gravity the corresponding Euler-Lagrange equations are not integrable for general dilaton coupling parameter γ. So it is very difficult to find explicitly the sharp lower bound for the area in Einstein-Maxwell-dilaton gravity with arbitrary γ. That is why our approach here should be different in comparison with the Einstein-Maxwell gravity.

3 Area-angular momentum-charge inequality for critical dilaton coupling parameter

The main result in this section is the next theorem:

Theorem 1. *Let $\mathcal{B}$ be a smooth, spacetime stably outermost axisymmetric marginally outer trapped surface in a spacetime satisfying the Einstein-Mawell-dilaton-matter equations (1) with a dilaton coupling parameter $\gamma^2 = 3$. If the matter energy-momentum tensor satisfies the dominant energy condition and the dilaton potential is non-negative, then the area of $\mathcal{B}$ satisfies the inequality*

$$\mathcal{A} \geq 8\pi\sqrt{|Q^2P^2 - J^2|}, \tag{41}$$

where Q, P and J are the electric charge, the magnetic charge and the angular momentum associated with $\mathcal{B}$. The equality is saturated only for the extremal stationary near horizon geometry of the $\gamma^2 = 3$ Einstein-Maxwell-dilaton gravity with $V(\varphi) = 0$ and $T_{ab} = 0$.

Proof. For the critical coupling $(\mathcal{N}, G_{AB})$ is a symmetric space with a negative sectional curvature [20]. In fact $\mathcal{N}$ is $SL(3,R)/O(3)$ symmetric space and therefore its metric can be written in the form

$$dL^2 = \frac{1}{8}Tr\left(M^{-1}dMM^{-1}dM\right), \tag{42}$$

where the matrix M is symmetric, positive definite and $M \in SL(3,R)$. After tedious calculations it can be shown that

$$M = e^{\frac{2}{3}\sqrt{3}\varphi}\begin{pmatrix} X + 4\Phi^2 e^{-2\sqrt{3}\varphi} + X^{-1}(\chi + 2\Phi\Psi)^2 & 2e^{-2\sqrt{3}\varphi}\Phi + 2X^{-1}(\chi + 2\Phi\Psi)\Psi & X^{-1}(\chi + 2\Phi\Psi) \\ 2e^{-2\sqrt{3}\varphi}\Phi + 2X^{-1}(\chi + 2\Phi\Psi)\Psi & e^{-2\sqrt{3}\varphi} + 4\Psi^2 X^{-1} & 2\Psi X^{-1} \\ X^{-1}(\chi + 2\Phi\Psi) & 2\Psi X^{-1} & X^{-1} \end{pmatrix}.$$

In terms of the matrix M the functional $I[X^A]$ becomes

$$I[X^A] = \int_{-1}^{1}\left\{\frac{d}{d\tau}(\sigma\tau) + 1 + \frac{1}{8}(1-\tau^2)Tr\left(M^{-1}\frac{dM}{d\tau}\right)^2 - \frac{1}{1-\tau^2}\right\}d\tau. \tag{43}$$

The Euler-Lagrange equations are then equivalent to the following matrix equation

$$\frac{d}{d\tau}\left((1-\tau^2)M^{-1}\frac{dM}{d\tau}\right) = 0. \tag{44}$$

Hence we find

$$(1-\tau^2)M^{-1}\frac{dM}{d\tau} = 2A, \tag{45}$$

where A is a constant matrix. From $\det M = 1$ it follows that $TrA = 0$. Integrating further we obtain

$$M = M_0 \exp\left(\ln\frac{1+\tau}{1-\tau}A\right), \tag{46}$$

where M_0 is a constant matrix with the same properties as M and satisfying $A^T M_0 = M_0 A$. Since M_0 is positive definite it can be written in the form $M_0 = BB^T$ for some matrix B with $|\det B| = 1$ and this presentation is up to an orthogonal matrix O (i.e. $B \to BO$). This freedom can be used to diagonalize the matrix $B^T AB^{T\ -1}$. So we can take $B^T AB^{T\ -1} = diag(a_1, a_2, a_3)$ and we obtain

$$M = B \begin{pmatrix} \left(\frac{1+\tau}{1-\tau}\right)^{a_1} & 0 & 0 \\ 0 & \left(\frac{1+\tau}{1-\tau}\right)^{a_2} & 0 \\ 0 & 0 & \left(\frac{1+\tau}{1-\tau}\right)^{a_3} \end{pmatrix} B^T. \tag{47}$$

The eigenvalues a_i can be found by comparing the singular behavior of the left hand and right hand side of (47) at $\tau \to \pm 1$. Doing so we find, up to renumbering, that $a_1 = 0$, $a_2 = -1$ and $a_3 = 1$. The matrix B can be found by imposing the boundary conditions which gives

$$B = \begin{pmatrix} -\frac{P^2 Q e^{-\frac{1}{\sqrt{3}}(\varphi_+ + \varphi_-)}}{\sqrt{|P^2Q^2 - J^2|}} & (2J + PQ)e^{\frac{1}{\sqrt{3}}\varphi_- - \frac{1}{2}\sigma_p} & (-2J + PQ)e^{\frac{1}{\sqrt{3}}\varphi_+ - \frac{1}{2}\sigma_p} \\ -\frac{J e^{-\frac{1}{\sqrt{3}}(\varphi_+ + \varphi_-)}}{\sqrt{|P^2Q^2 - J^2|}} & Q e^{\frac{1}{\sqrt{3}}\varphi_- - \frac{1}{2}\sigma_p} & -Q e^{\frac{1}{\sqrt{3}}\varphi_+ - \frac{1}{2}\sigma_p} \\ \frac{P e^{-\frac{1}{\sqrt{3}}(\varphi_+ + \varphi_-)}}{2\sqrt{|P^2Q^2 - J^2|}} & \frac{1}{2} e^{\frac{1}{\sqrt{3}}\varphi_- - \frac{1}{2}\sigma_p} & \frac{1}{2} e^{\frac{1}{\sqrt{3}}\varphi_+ - \frac{1}{2}\sigma_p} \end{pmatrix}, \tag{48}$$

where

$$\varphi_\pm = \varphi(\tau = \pm 1), \tag{49}$$

$$\sigma_p = \lim_{\tau \to \pm 1} \ln\left(\frac{X}{1-\tau^2}\right) = \sigma(\tau = \pm 1). \tag{50}$$

Taking into account that $|\det B| = 1$ we find

$$e^{\sigma_p} = 2\sqrt{|P^2Q^2 - J^2|}. \tag{51}$$

Now we are ready to evaluate the minimum of the functional $I[X^A]$. Substituting (45) in (43) we see that the last two terms cancel each other and we find

$$I_{min}[X^A] = 2\sigma_p + 2 = 2\ln\left(2\sqrt{|P^2Q^2 - J^2|}\right) + 2. \tag{52}$$

Substituting further this result in (20) we finally obtain

$$\mathcal{A} \geq 8\pi\sqrt{|P^2Q^2 - J^2|}. \tag{53}$$

The extremal stationary near horizon geometry is in fact defined by equation (44), by the same boundary conditions and by the same class of functions as those in the variational problem. Therefore, it is clear that the equality is saturated only for the extremal stationary near horizon geometry. This completes the proof.

Remark. The case $P^2Q^2 = J^2$ is formally outside of the class of functions we consider. In the language of stationary solutions, it corresponds to an extremal (naked) singularity with zero area.

It is interesting to note that when $PQ = 0$, but $P^2 + Q^2 \neq 0$, the lower bound of the area depends only on the angular momentum but not on the nonzero charge in contrast with the Einstein-Maxwell gravity.

4 Area-angular momentum-charge inequality for dilaton coupling parameter $0 \leq \gamma^2 \leq 3$

As we mentioned above, finding of sharp lower bound for the area $\mathcal{A}$ in the case of arbitrary γ seems to be very difficult. However, an important estimate for the area can be found for dilaton coupling parameter satisfying $0 \leq \gamma^2 \leq 3$. The result is given by the following

Theorem 2. *Let $\mathcal{B}$ be a smooth, spacetime stably outermost axisymmetric marginally outer trapped surface in a spacetime satisfying the Einstein-Mawell-dilaton-matter equations (1) with a dilaton coupling parameter γ, satisfying $0 \leq \gamma^2 \leq 3$. If the matter energy-momentum tensor satisfies the dominant energy condition and the dilaton potential is non-negative, then for every γ in the given range, the area of $\mathcal{B}$ satisfies the inequality*

$$\mathcal{A} \geq 8\pi\sqrt{|Q^2P^2 - J^2|}, \tag{54}$$

where Q, P and J are the electric charge, the magnetic charge and the angular momentum associated with $\mathcal{B}$. The equality is saturated for the extremal stationary near horizon geometry of the $\gamma^2 = 3$ Einstein-Maxwell-dilaton gravity with $V(\varphi) = 0$ and $T_{ab} = 0$.

Proof. Let us first focus on the case $0 < \gamma^2 \leq 3$ and consider the metric

$$\begin{aligned} d\tilde{L}^2 &= \tilde{G}_{AB}dX^AdX^B \\ &= \frac{dX^2 + (d\chi + 2\Phi d\Psi - 2\Psi d\Phi)^2}{4X^2} + \frac{e^{-2\gamma\varphi}d\Phi^2 + e^{2\gamma\varphi}d\Psi^2}{X} + \frac{\gamma^2}{3}d\varphi^2 \end{aligned} \tag{55}$$

and the associated functional

$$\tilde{I}[X^A] = \int_{-1}^{1}\left\{\frac{d}{d\tau}(\sigma\tau) + 1 + (1-\tau^2)\tilde{G}_{AB}\frac{dX^A}{d\tau}\frac{dX^B}{d\tau} - \frac{1}{1-\tau^2}\right\}d\tau. \tag{56}$$

It is easy to see that $I[X^A] \geq \tilde{I}[X^A]$ and therefore

$$\mathcal{A} \geq 4\pi e^{(\tilde{I}[X^A]-2)/2}. \tag{57}$$

Redefining now the scalar field $\tilde{\varphi} = \frac{\gamma}{\sqrt{3}}\varphi$ we find that the functional $\tilde{I}[X^A]$ reduces to the functional $I[X^A]$ for the critical coupling $\gamma^2 = 3$. Hence we conclude that

$$\mathcal{A} \geq 8\pi\sqrt{|Q^2P^2 - J^2|} \tag{58}$$

for every γ with $0 < \gamma^2 \leq 3$.

The case $\gamma = 0$ needs a separate investigation. Fortunately, it can be easily reduced to the pure Einstein-Maxwell case. Indeed, it is not difficult to see that for $\gamma = 0$ we have

$$I[X^A] \geq I^{EM}[X^A], \tag{59}$$

where $I^{EM}[X^A]$ is the functional for the pure Einstein-Maxwell gravity. In Einstein-Maxwell gravity it was proven in [6] that $\mathcal{A} \geq 8\pi\sqrt{J^2 + \frac{1}{4}(Q^2 + P^2)^2}$ which gives $\mathcal{A} \geq 8\pi\sqrt{J^2 + \frac{1}{4}(Q^2 + P^2)^2} \geq 8\pi\sqrt{|Q^2P^2 - J^2|}$.

Finally, it is worth noting that, as a direct consequence of Lemma 2, for every fixed γ we obtain the following inequality

$$\mathcal{A} \geq \mathcal{A}_{NHG} \tag{60}$$

where $\mathcal{A}_{NHG}$ is the area associated with the extremal stationary near horizon geometry of Einstein-Maxwell-dilaton gravity with $V(\varphi) = 0$ and $T_{ab} = 0$, for the corresponding γ.

5 Discussion

In the present paper we derived area-angular momentum-charge inequalities for stable marginally outer trapped surfaces in the four dimensional Einstein-Maxwell-dilaton theory for values of the dilaton coupling parameter less than or equal to the critical value. The coupling of the dilaton to the Maxwell field leads in general to inequalities that can be rather different from that in the Einstein-Maxwell gravity. Some estimates for the sector $\gamma^2 > 3$ could be found if we impose the additional condition on the dilaton potential to be convex. We leave this study for the future.

Given the current interest in the higher dimensional gravity it is interesting to extend the area-angular momentum-charge inequalities to higher dimensions. This is almost straightforward in the case of Einstein equations [27]. However, in the case of Einstein-Maxwell and Einstein-Maxwell-dilaton gravity the extensions of the inequalities is difficult. The central reason behind that is the fact that even the stationary axisymmetric Einstein-Maxwell equations are not integrable in higher dimensions [28]. Nevertheless, some progress can be made and the results will be presented elsewhere [29].

Acknowledgments: This work was partially supported by the Bulgarian National Science Fund under Grants DMU-03/6, and by Sofia University Research Fund under Grant 148/2012.

References

[1] A. Acena, S. Dain and M.E. Gabach Clement, Class. Quant. Grav. **28** 105014 (2011); [arXiv:1012.2413[gr-qc]].

[2] S. Dain and M. Reiris. Phys. Rev. Lett. **107**, 051101 (2011); [arXiv:1102.5215[gr-qc]].

[3] M.E. Gabach Clement, [arXiv:1102.3834[gr-qc]].

[4] J. L. Jaramillo, M. Reiris and S. Dain, Phys. Rev. **D84**, 121503 (2011); [arXiv:1106.3743[gr-qc]].

[5] S. Dain, J. L. Jaramillo and M. Reiris, Class. Quantum Grav. 29, 035013 (2012); [arXiv:1109.5602[gr-qc]].

[6] M.E. Gabach Clement and J.L. Jaramillo, [arXiv:1111.6248[gr-qc]].

[7] W. Simon. Class. Quant. Grav. **29**, 062001 (2012); [arXiv:1109.6140[gr-qc].]

[8] M. E. Gabach Clement, J. L. Jaramillo and M. Reiris, [arXiv:1207.6761[gr-qc]].

[9] M. Ansorg, J. Hennig and C. Cederbaum. Gen. Rel. Grav. **43**, 1205 (2011); [arXiv:1005.3128[gr-qc]].

[10] J. Hennig, M. Ansorg and C. Cederbaum. Class. Quantum Grav. **25** 162002 (2008);[arXiv:0805.4320[gr-qc]].

[11] J. Hennig, C. Cederbaum and M. Ansorg, Commun. Math. Phys. **293**, 449 (2010); [arXiv:0812.2811[gr-qc]].

[12] S. Dain. Class. Quant. Grav. **29**, 073001 (2012), [arXiv:1111.3615[gr-qc]].

[13] P. Chrusciel, M. Eckstein, L. Nguyen and S. Szybka, Class. Quant. Grav. **28**, 245017 (2011).

[14] M. Mars, Class. Quant. Grav. **29**, 145019 (2012).

[15] J. Jaramillo, Class. Quant. Grav. **29**, 177001 (2012).

[16] G. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988).

[17] D. Garfinkle, G. Horowitz and A. Strominger, Phys. Rev. **D43**, 3140 (1991); **D4**5, 3888, 1992 (E).

[18] D. Maison, Gen. Rel. and Grav. **10**, 717 (1979).

[19] S. Hojman, M. Rosenbaum and M. Ryan, Phys. Rev. **D17**, 3141 (1978).

[20] S. Yazadjiev, Phys. Rev. **D82**, 124050 (2010); [arXiv:1009.2442[hep-th]]

[21] A. Masood-ul-Alam, Class. Quant. Grav. **10**, 2649 (1993); M. Mars and W. Simon, Adv. Theor. Math. Phys. **6**, 279 (2003).

[22] B. Kleihaus, J. Kunz and F. Navarro-Lerida, Phys. Rev. **D69**, 081501 (2004); [arXiv:0309082[gr-qc]].

[23] S. Hayward, Phys. Rev. **D49**, 6467 (1994).

[24] L. Andersson, M. Mars and W. Simon. Phys. Rev. Lett. **95**, 111102 (2005);[arXiv:0506013[gr-qc].]

[25] L. Andersson, M. Mars and W. Simon. Adv. Theor. Math. Phys., **12(4)**, 853 (2008); [arXiv:0704.2889[gr-qc]].

[26] I. Booth and S. Fairhurst, Phys. Rev. **D77**, 084005 (2008); [arXiv:0708.2209[gr-qc]].

[27] S. Hollands, Class. Quant. Grav. **29**, 065006 (2012); [arxiv:1110.5814[gr-qc]].

[28] S. Yazadjiev, JHEP **1106**, 083 (2011); [arXiv:1104.0378 [hep-th]].

[29] S. Yazadjiev, In preparation