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Bulk Properties of a Fermi Gas in a Magnetic Field

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We calculate the number density, energy density, transverse pressure, longitudinal pressure, and magnetization of an ensemble of spin one-half particles in the presence of a homogenous background magnetic field. The magnetic field direction breaks spherical symmetry causing the pressure transverse to the magnetic field direction to be different than the pressure parallel to it. We present explicit formulae appropriate at zero and finite temperature for both charged and uncharged particles including the effect of the anomalous magnetic moment. We demonstrate that the resulting expressions satisfy the canonical relations, $\Omega = -P_\parallel$ and $P_\perp = P_\parallel - MB$, with $M = -\partial\Omega/\partial B$ being the magnetization of the system. We numerically calculate the resulting pressure anisotropy for a gas of protons and a gas of neutrons and demonstrate that the inclusion of the anomalous magnetic increases the level of pressure anisotropy in both cases.

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I. INTRODUCTION

The determination of the bulk properties of a Fermi gas in the presence of a magnetic field is important for understanding neutron stars and the early-time dynamics of the quark gluon plasma created in relativistic heavy ion collisions. In the presence of a uniform magnetic field, both the matter and the field contributions to the space-like components of the energy-momentum tensor become anisotropic. The degree of pressure anisotropy increases as the magnitude of the magnetic field increases. In this paper we revisit the calculation of the bulk properties of a Fermi gas of spin one-half particles in a uniform magnetic field with the goal of unambiguously determining the pressure anisotropy from first principles including the effect of the anomalous magnetic moment.

As mentioned above, there is currently considerable interest in the behavior of matter in the presence of high magnetic fields. Neutron stars, for example, are known to possess high magnetic fields. More specifically, magnetars [1–7] are believed to have surface magnetic fields as strong as $10^{14} - 10^{15}$ Gauss. Based on such surface magnetic fields, one could expect magnetic fields in the interior of magnetars to be on the order of $10^{16} - 10^{19}$ Gauss. There have been many previous studies of the effect of magnetic fields on neutron stars and magnetars focusing on the effect of magnetic fields on the equation of state of the matter composing the star including hadronic matter, quark matter, and hybrid stars composed of hadronic matter with a quark matter core [8–48].

Among these references some authors have simply assumed that the system continues to be describable in terms of an energy density and an isotropic pressure derivable from standard thermodynamic relations, while other authors have included the fact that the background magnetic field breaks the spherical symmetry of the system. The breaking of the spherical symmetry has two distinct contributions: (i) the matter contribution to the energy-momentum tensor and (ii) the field contribution to the energy-momentum tensor. For charged particles the presence of a magnetic field causes the pressure transverse to and longitudinal to the local magnetic field direction to be different, with the level of pressure anisotropy increasing monotonically with the magnitude of the magnetic field. The same occurs for uncharged particles that have a non-vanishing anomalous magnetic moment as we will demonstrate.

There have been dynamical models of neutron stars which have attempted to include the effect of high magnetic fields on the three-dimensional structure of neutron stars [49–54]. Some of these studies have self-consistently included modifications of the general relativistic metric necessary to describe the breaking of spherical symmetry by the neutron star's magnetic field. However, to the best of our knowledge there has not been a study which has simultaneously included the general relativity aspects, effects of magnetic fields on the equation of state, and effects of pressure anisotropy on the static and dynamical properties of a high-magnetic-field neutron star. In order to complete this program it is necessary to first understand all sources of pressure anisotropy due to magnetic fields.

Another area in which there has been a significant amount of attention focused on the behavior of matter subject to high magnetic fields is the consideration of the first fm/c after the collision of two high-Z ions in a relativistic heavy ion collision. Because of the large number of protons in the colliding nuclei, magnetic fields on the order of $10^{18} - 10^{19}$ Gauss are expected to be generated at early times after the initial nuclear impact [55–59]. The existence of such high magnetic fields prompted many research groups to study how the finite temperature deconfinement and chiral phase transitions are affected by the presence of a background magnetic field. These studies have included direct numerical investigations using lattice quantum chromodynamics (QCD) [60–63] and theoretical investigations using a variety of methods including, for example, perturbative QCD studies, model studies, and string-theory inspired anti-de Sitter/conformal field theory (AdS/CFT) correspondence studies [64–87].

In order to have more a comprehensive understanding of the behavior of matter in a background magnetic field, we begin with the basics and study Fermi gases consisting of charged and uncharged spin one-half particles including the effect of the anomalous magnetic moment. Many of the results obtained here are already available in the literature; however, the results for the transverse pressure including the effect of the anomalous magnetic moment have not appeared previously. For sake of completeness, we present the results for all of the components of the matter contribution to the energy-momentum tensor with and without anomalous magnetic moment as a point of reference for future applications. In this paper we consider systems at both zero and finite temperature. For zero temperature systems, we demonstrate by explicit calculation that the grand potential $\Omega = \epsilon - \mu n = -P_\parallel$ where ϵ is the energy density, n is the number density, $P_\parallel$ is the pressure along the direction of the background magnetic field, and μ is the chemical potential. For finite temperature systems one also finds that $\Omega = -P_\parallel$.

We then show that, both with and without anomalous magnetic moment, the resulting expressions satisfy the canonical relation $P_\perp = P_\parallel - MB$, where $P_\perp$ is the pressure transverse to the magnetic field direction and $M = -\partial\Omega/\partial B$ is the magnetization of the system. Evaluating the resulting expressions numerically, we demonstrate that the magnitude of the pressure anisotropy is larger when one takes into account the anomalous magnetic moment, however, as the temperature of the system increases the pressure anisotropy decreases.

The structure of the paper is as follows. In Sec. II we introduce the basic formulae necessary to calculate the bulk properties of an ensemble of particles using quantum field theory. In Sec. III we present the resulting formulae for charged particles with and without anomalous magnetic moment. In Sec. IV we present the corresponding formulae for uncharged particles. In Sec. V we compare the numerical evaluation of the transverse and longitudinal pressures. In Sec. VI we present our conclusions and an outlook for the future. Finally, in Apps. A and B we present a quantum field theory derivation of the necessary components of the energy-momentum tensor for charged and uncharged particles.

II. GENERALITIES

In the presence of fields, the energy-momentum tensor can be decomposed into matter and field contributions

$$T^{\mu\nu} = T^{\mu\nu}_{\text{matter}} + T^{\mu\nu}_{\text{fields}} \,. \tag{1}$$

If there is only a background magnetic field B pointing along the z-direction, then the field contribution to the energy-momentum tensor takes the form $T^{\mu\nu}_{\text{fields}} = \text{diag}(B^2/2, B^2/2, B^2/2, -B^2/2)$.[1] Since this contribution is well-understood, we do not spend more time discussing it in this paper. Instead, we focus on $T^{\mu\nu}_{\text{matter}}$ for a system composed of spin one-half fermions. In what follows, the bulk properties of the system (energy density, transverse pressure, etc.) are understood to specify the components of $T^{\mu\nu}_{\text{matter}}$ in the local rest frame of the system.

The matter contribution to the bulk properties of a system can be expressed in terms of the one-particle distribution function f. We m consider a single particle type with mass m and charge q and sum over the spin polarizations. The results obtained can be straightforwardly extended to a system consisting of multiple particle types. We present a derivation of the necessary components of the energy-momentum tensor in Apps. A and B. Summarizing the results, one finds that the local rest frame number density, energy density, longitudinal pressure, and transverse pressure can be expressed in terms of the following integrals of the one-particle distribution function

$$n = \sum_s \int_k f \,, \tag{2}$$

$$\epsilon = T^{00} = \sum_s \int_k E f \,, \tag{3}$$

$$P_\parallel = T^{zz} = \sum_s \int_k \frac{k_z^2}{E} f \,, \tag{4}$$

$$\begin{aligned} P_\perp &= \frac{1}{2}\left(T^{xx} + T^{yy}\right) \\ &= \sum_s \int_k \frac{1}{E}\left[\frac{1}{2}\frac{k_\perp^2 \bar{m}(\nu)}{\sqrt{m^2+k_\perp^2}} - s\kappa B \bar{m}(\nu)\right] f \,, \end{aligned} \tag{5}$$

where we have singled out the z (parallel) direction for future application, $\bar{m}^2(\nu) \equiv (\sqrt{m^2+k_\perp^2} - s\kappa B)^2$, $k_\perp^2$ is the (discretized) transverse momentum, $\sum_s$ represents a sum over spin polarizations, κ represents the anomalous magnetic moment, and $\int_k$ is a properly normalized (sum-)integration over momenta which we will define separately for charged and uncharged particles. For charged particles with vanishing anomalous magnetic moment, the expressions above were first derived in Ref. [8]. For charged particles with finite anomalous magnetic moment the expressions for the number density and energy density above were first derived in Ref. [11]. Here we extend the treatment to include uncharged particles and independently compute the transverse and longitudinal pressures in the case of finite anomalous magnetic moment.

[1] This is the form in Heaviside-Lorentz natural units. In Gaussian natural units, when converting the magnetic field to GeV^2, the magnetic field is increased by a factor of $\sqrt{4\pi}$ and the components of the energy momentum-tensor are divided by a factor of 4π to compensate, e.g. $\epsilon_B = B^2/8\pi$.

We note that in order to include interactions, one should use the interaction-corrected expression for the particle's dispersion relation. In the mean-field approximation, this amounts to including corrections to the bare mass of the particle being considered, e.g. $m \to m^*$. The resulting effective mass can depend on the chemical potential and temperature. In what follows we indicate the effective mass of the particle as m assuming that interaction corrections could be absorbed into the mass.[2]

III. CHARGED PARTICLES

In the presence of a uniform external magnetic field pointing in the z-direction, the transverse momenta of particles with an electric charge q are restricted to discrete Landau levels with $k_\perp^2 = 2\nu|q|B$ where $\nu \geq 0$ is an integer [88] and one has

$$\int_k \to \frac{|q|B}{(2\pi)^2} \sum_n \int_{-\infty}^{\infty} dk_z \,, \tag{6}$$

where the sum over n represents a sum over the discretized orbital angular momentum of the particle in the transverse plane. For spin one-half particles the orbital angular momentum n is related to ν via [88]

$$\nu = n + \frac{1}{2} - \frac{s}{2}\frac{q}{|q|} \,, \tag{7}$$

where $s = \pm 1$ is the spin projection of the particle along the direction of the magnetic field and q is the charge.[3]

An additional consequence of the quantization is that the total energy of a charged particle becomes quantized [90]

$$\begin{aligned} E &= \sqrt{k_z^2 + ((m^2 + 2\nu|q|B)^{1/2} - s\kappa B)^2} \,, \\ &= \sqrt{k_z^2 + \bar{m}^2(\nu)} \,, \end{aligned} \tag{8}$$

where $\kappa = \kappa_i \mu_N$ with κ_i being the coupling strength for the anomalous magnetic moment times the magneton, and $\bar{m}^2(\nu) \equiv (\sqrt{m^2 + 2\nu|q|B} - s\kappa B)^2$.

A. Zero temperature

At zero temperature the one-particle distribution function is given by a Heaviside theta function

$$f(E) = \Theta(\mu - E) \,, \tag{9}$$

where μ is the chemical potential (Fermi energy).

1. Zero anomalous magnetic moment

We begin by considering the case with no anomalous magnetic moment, i.e. $\kappa = 0$. In terms of the chemical potential, μ, the maximum k_z is defined via (8)

$$k_{z,F}(\nu) = \sqrt{\mu^2 - 2\nu|q|B - m^2} \,. \tag{10}$$

In addition, in the sum over the Landau levels one must guarantee that the quantity under the square root in (10) is positive. This requires $\bar{m}^2 \leq \mu^2$ which results in

$$\nu \leq \nu_{\max} = \left\lfloor \frac{\mu^2 - m^2}{2|q|B} \right\rfloor , \tag{11}$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$ is the largest integer less than or equal to x.

Using the above, we can write down an expression for the number density using (2) and (6) to obtain [19, 25]

$$\begin{aligned} n &= \frac{|q|B}{(2\pi)^2} \sum_{s=\pm1} \sum_{n=0}^{\nu \leq \nu_{\max}} \int_{-\infty}^{\infty} dk_z \, \Theta(\mu - E) \,, \\ &= \frac{|q|B}{2\pi^2} \sum_{s=\pm1} \sum_{n=0}^{\nu \leq \nu_{\max}} \int_0^{k_{z,F}} dk_z \,, \\ &= \frac{|q|B}{2\pi^2} \sum_{s=\pm1} \sum_{n=0}^{\nu \leq \nu_{\max}} k_{z,F}(\nu) \,. \end{aligned} \tag{12}$$

Note that the upper limit on the n sum is set in terms of the maximum Landau level and that ν depends on n and s via Eq. (7). Note that the $\kappa = 0$ degeneracy factor for a given Landau level is automatically taken into account by the dual sum over spin and angular momentum.

Similarly, one can evaluate the energy density to obtain [19, 25]

$$\begin{aligned} \epsilon &= \frac{|q|B}{2\pi^2} \sum_{s=\pm1} \sum_{n=0}^{\nu \leq \nu_{\max}} \int_0^{k_{z,F}} dk_z \sqrt{k_z^2 + \bar{m}^2(\nu)} \,, \\ &= \frac{|q|B}{4\pi^2} \sum_{s=\pm1} \sum_{n=0}^{\nu \leq \nu_{\max}} \Bigg[\mu \, k_{z,F}(\nu) \\ &\qquad + \bar{m}^2(\nu) \log\left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)} \right) \Bigg] . \end{aligned} \tag{13}$$

Next, we consider the parallel pressure $P_\parallel$ and obtain [19]

$$\begin{aligned} P_\parallel &= \frac{|q|B}{2\pi^2} \sum_{s=\pm1} \sum_{n=0}^{\nu \leq \nu_{\max}} \int_0^{k_{z,F}} dk_z \frac{k_z^2}{\sqrt{k_z^2 + \bar{m}^2(\nu)}} \,, \\ &= \frac{|q|B}{4\pi^2} \sum_{s=\pm1} \sum_{n=0}^{\nu \leq \nu_{\max}} \Bigg[\mu \, k_{z,F}(\nu) \\ &\qquad - \bar{m}^2(\nu) \log\left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)} \right) \Bigg] . \end{aligned} \tag{14}$$

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[2] In the following, spherical symmetry is broken by a uniform magnetic field. Due to this, the effective mass could, in principle, also depend on the angle of particle momentum relative to the magnetic field direction. We do not take this possibility into account in this work.

[3] The present calculation is valid only for spin one-half particles. Spin zero, one and three-half particles, described respectively by the Klein-Gordon, Proca and Rarita-Schwinger equations are affected differently by the magnetic field [86, 89].

Note that using (12), (13), and (14) it is straightforward to see that $\epsilon + P_\parallel = \mu n$ and hence $\Omega = \epsilon - \mu n = -P_\parallel$.

Finally, we consider the transverse pressure $P_\perp$ and obtain

$$P_\perp = \frac{|q|B}{4\pi^2} \sum_{s=\pm 1} \sum_{n=0}^{\nu \le \nu_{\max}} 2\nu |q| B \int_0^{k_{z,F}} dk_z \, \frac{1}{\sqrt{k_z^2 + \bar{m}^2(\nu)}} \,, \\ = \frac{|q|^2 B^2}{2\pi^2} \sum_{s=\pm 1} \sum_{n=0}^{\nu \le \nu_{\max}} \nu \, \log\left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)}\right). \tag{15}$$

Numerically the results for $P_\parallel$ and $P_\perp$ are different for any value of B; however, they only become significantly different for very large B. Using Eq. (11), for example, we see that when $B > (\mu^2 - m^2)/2|q|$, only the lowest Landau level contributes to the sums and one obtains

$$\lim_{B\to\infty} P_\parallel = \frac{|q|B}{4\pi^2}\left[\mu \, k_F - m^2 \log\left(\frac{\mu + k_F}{m}\right)\right], \tag{16}$$

where $k_F \equiv \sqrt{\mu^2 - m^2}$. The transverse pressure on the other hand vanishes in this limit

$$\lim_{B\to\infty} P_\perp = 0 \,. \tag{17}$$

A relationship between $P_\parallel$ and $P_\perp$ can be established by evaluating the magnetization of the system $M \equiv -\partial\Omega/\partial B = \partial P_\parallel/\partial B$ [91]. Performing the necessary derivatives[4] of the parallel pressure one finds $M = (P_\parallel - P_\perp)/B$. Rearranging gives $P_\perp = P_\parallel - MB$ which is the canonical relationship one finds in the literature between the transverse and longitudinal pressures.

2. Nonzero anomalous magnetic moment

We now turn to the case of nonzero anomalous magnetic moment. In this case the expressions for $k_{z,F}$ and $\nu_{\max}$ must be adjusted to

$$k_{z,F} = \sqrt{\mu^2 - ((m^2 + 2\nu|q|B)^{1/2} - s\kappa B)^2} \,, \tag{18}$$

$$\nu_{\max} = \left\lfloor \frac{(\mu + s\kappa B)^2 - m^2}{2|q|B} \right\rfloor . \tag{19}$$

With these two modifications Eqs. (12), (13), and (14) are unchanged, but one should note that $\nu_{\max}$ now depends on the spin alignment s.

[4] Formally one should use left or right derivatives in the vicinity of magnetic field magnitudes where $\nu_{\max}$ changes under infinitesimal variation.

The transverse pressure, however, is modified when there is a non-vanishing anomalous magnetic moment

$$P_\perp = \frac{|q|B^2}{2\pi} \sum_{s=\pm 1} \sum_{n=0}^{\nu \le \nu_{\max}} \left[\frac{|q|\nu \bar{m}(\nu)}{\sqrt{m^2 + 2\nu|q|B}} - s\kappa \bar{m}(\nu)\right] \\ \times \int_0^{k_{z,F}} dk_z \, \frac{1}{\sqrt{k_z^2 + \bar{m}^2(\nu)}} \,. \\ = \frac{|q|B^2}{2\pi} \sum_{s=\pm 1} \sum_{n=0}^{\nu \le \nu_{\max}} \left[\frac{|q|\nu \bar{m}(\nu)}{\sqrt{m^2 + 2\nu|q|B}} - s\kappa \bar{m}(\nu)\right] \\ \times \log\left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)}\right) . \tag{20}$$

Evaluating the magnetization one obtains in this case [25][5]

$$M = \frac{\partial P_\parallel}{\partial B} = \frac{P_\parallel}{B} + \frac{|q|B}{2\pi^2} \sum_{s=\pm 1} \sum_{n=0}^{\nu \le \nu_{\max}} \\ \times \left[s\kappa \bar{m}(\nu) - \frac{|q|\nu \bar{m}(\nu)}{\sqrt{m^2 + 2\nu|q|B}}\right] \log\left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)}\right). \\ = \frac{P_\parallel}{B} - \frac{P_\perp}{B} \,. \tag{21}$$

So one finds once again $P_\perp = P_\parallel - MB$.

B. Finite temperature

We now turn our attention to the case of a finite temperature ensemble of charged particles. In this case the distribution function is

$$f_\pm(E, T, \mu) = \frac{1}{e^{\beta(E \mp \mu)} + 1} \,, \tag{22}$$

where f_+ describes particles, f_- describes anti-particles, and μ is the chemical potential.

1. Zero anomalous magnetic moment

We begin with the number density

$$n_\pm = \frac{|q|B}{(2\pi)^2} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk_z \, f_\pm(E, T, \mu) \,, \tag{23}$$

recalling that $E = \sqrt{k_z^2 + \bar{m}^2(\nu)}$ with $\bar{m}^2(\nu) = m^2 + 2\nu|q|B$. Introducing the variable $x = E \mp \mu$ we can rewrite

[5] We note that there appear to be some typos in the expression contained in Ref. [25].

$k_z = \sqrt{(x\pm\mu)^2 - \bar{m}^2(\nu)}$ and using $dk_z = (x\pm\mu)((x\pm\mu)^2 - \bar{m}^2(\nu))^{-1/2}dx$ one obtains

$$n_\pm = \frac{|q|B}{2\pi^2}\sum_{s=\pm 1}\sum_{n=0}^{\infty}\int_{\bar{m}(\nu)\mp\mu}^{\infty} dx\, \frac{(x\pm\mu)f_\pm(x,T,0)}{\sqrt{(x\pm\mu)^2-\bar{m}^2(\nu)}}\,. \tag{24}$$

Next, we consider the energy density. Using the same change of variables as before, one obtains

$$\epsilon_\pm = \frac{|q|B}{2\pi^2}\sum_{s=\pm 1}\sum_{n=0}^{\infty}\int_{\bar{m}(\nu)\mp\mu}^{\infty} dx\, \frac{(x\pm\mu)^2 f_\pm(x,T,0)}{\sqrt{(x\pm\mu)^2-\bar{m}^2(\nu)}}\,. \tag{25}$$

Similarly, one obtains for the longitudinal pressure

$$P_{\parallel,\pm} = \frac{|q|B}{2\pi^2}\sum_{s=\pm 1}\sum_{n=0}^{\infty} \times\int_{\bar{m}(\nu)\mp\mu}^{\infty} dx\, \sqrt{(x\pm\mu)^2-\bar{m}^2(\nu)}\, f_\pm(x,T,0)\,. \tag{26}$$

Finally, one obtains for the transverse pressure

$$P_{\perp,\pm} = \frac{|q|^2B^2}{2\pi^2}\sum_{s=\pm 1}\sum_{n=0}^{\infty}\nu\int_{\bar{m}(\nu)\mp\mu}^{\infty} dx\, \frac{f_\pm(x,T,0)}{\sqrt{(x\pm\mu)^2-\bar{m}^2(\nu)}}\,. \tag{27}$$

Next we consider the magnetization obtained from $M = \partial P_\parallel/\partial B$. In order to do this we apply the fundamental theorem of calculus

$$\frac{d}{dy}\int_{a(y)}^{b} dx\, g(x,y,\cdots) = -a'(y)\, g(a(y),y,\cdots) + \int_{a(y)}^{b} dx\, \frac{dg(x,y,\cdots)}{dy}\,. \tag{28}$$

Using this we can evaluate the derivative of the integral appearing on the second line of (26)

$$\frac{\partial}{\partial B}\left(\int_{m(\nu)\mp\mu}^{\infty} dx\, \sqrt{(x\pm\mu)^2-\bar{m}^2(\nu)}\, f_\pm(x,T,0)\right) = -\bar{m}(\nu)\frac{\partial\bar{m}(\nu)}{\partial B}\int_{\bar{m}(\nu)\mp\mu}^{\infty} dx\, \frac{f_\pm(x,T,0)}{\sqrt{(x\pm\mu)^2-\bar{m}^2(\nu)}}\,, \tag{29}$$

where we have used the fact that in the case at hand the first term on the right-hand side of (28) is zero. Using $\bar{m}\,\partial\bar{m}/\partial B = \frac{1}{2}\partial\bar{m}^2/\partial B = |q|\nu$ we can obtain finally

$$M_\pm = \frac{\partial P_{\parallel,\pm}}{\partial B} = \frac{P_{\parallel,\pm}}{B} - \frac{P_{\perp,\pm}}{B}\,, \tag{30}$$

which is the canonical relation between the transverse pressure, the longitudinal pressure, and the magnetization. Rearranging we obtain $P_{\perp,\pm} = P_{\parallel,\pm} - M_\pm B$ between the perpendicular and parallel pressures at finite temperature in the case that there is no anomalous magnetic moment.

2. Nonzero anomalous magnetic moment

As was the case at zero temperature, when including the anomalous magnetic moment, the primary thing that changes is the mass $\bar{m}^2(\nu) = (\sqrt{m^2+2\nu|q|B} - s\kappa B)^2$. With this change, the expressions for $n_\pm$, $\epsilon_\pm$, and $P_{\parallel,\pm}$ given in Eqs. (24), (25), and (26), respectively, are unchanged. For the transverse pressure, however, one must include additional terms

$$P_{\perp,\pm} = \frac{|q|B^2}{2\pi^2}\sum_{s=\pm 1}\sum_{n=0}^{\infty}\bar{m}(\nu)\left[\frac{|q|\nu}{\sqrt{m^2+2\nu|q|B}} - s\kappa\right] \times\int_{\bar{m}(\nu)\mp\mu}^{\infty} dx\, \frac{f_\pm(x,T,0)}{\sqrt{(x\pm\mu)^2-\bar{m}^2(\nu)}}\,. \tag{31}$$

In addition, when including the anomalous magnetic moment, the magnetization has a different form since

$$\bar{m}(\nu)\frac{\partial\bar{m}(\nu)}{\partial B} = -\bar{m}(\nu)\left[s\kappa - \frac{|q|\nu}{\sqrt{m^2+2\nu|q|B}}\right], \tag{32}$$

which results in

$$M_\pm = \frac{P_{\parallel,\pm}}{B} + \frac{|q|B}{2\pi^2}\sum_{s=\pm 1}\sum_{n=0}^{\infty}\bar{m}(\nu)\left[s\kappa - \frac{|q|\nu}{\sqrt{m^2+2\nu|q|B}}\right] \times\int_{\bar{m}(\nu)\mp\mu}^{\infty} dx\, \frac{f_\pm(x,T,0)}{\sqrt{(x\pm\mu)^2-\bar{m}^2(\nu)}}\,. \tag{33}$$

Once again we see that $P_{\perp,\pm} = P_{\parallel,\pm} - M_\pm B$.

IV. UNCHARGED PARTICLES

In the case that the particle being considered is uncharged, then one does not obtain discrete Landau levels and, as a result,

$$\int_k \to \int \frac{d^3k}{(2\pi)^3}\,, \tag{34}$$

in Eqs. (2)-(5). Prior to proceeding with the calculations, we note that for uncharged particles one has

$$\bar{m}^2 = \left(\sqrt{m^2+k_\perp^2} - s\kappa B\right)^2\,. \tag{35}$$

A. Finite temperature

We first consider the general case of uncharged particles at finite temperature including the effect of the anomalous magnetic moment. The derivation necessary is performed in App. B. Here we summarize the results and list the contributions from particles and anti-particles. The resulting expression for the number density is

$$n_\pm = \frac{1}{2\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, E\, f_\pm(E,T,\mu) \times\left[\hat{k} + s\kappa B\left(\arctan\left(\frac{s\kappa B - m}{\hat{k}}\right)+\frac{\pi}{2}\right)\right], \tag{36}$$

where $\hat{k} \equiv \sqrt{E^2-(m-s\kappa B)^2}$. The energy density is

$$\epsilon_\pm = \frac{1}{2\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, E^2\, f_\pm(E,T,\mu) \times\left[\hat{k} + s\kappa B\left(\arctan\left(\frac{s\kappa B - m}{\hat{k}}\right)+\frac{\pi}{2}\right)\right]. \tag{37}$$

The longitudinal pressure is

$$P_{\parallel,\pm} == \frac{1}{24\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, f_\pm(E,T,\mu) \times\Bigg\{2\hat{k}(s\kappa B - m)(2m+s\kappa B) +E^2\left[4\hat{k}+6s\kappa B\left(\arctan\left(\frac{s\kappa B-m}{\hat{k}}\right)+\frac{\pi}{2}\right)\right]\Bigg\}. \tag{38}$$

The transverse pressure is

$$P_{\perp,\pm} = \frac{1}{6\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, f_\pm(E,T,\mu)(\hat{k}^3 - 3s\kappa B m\hat{k}). \tag{39}$$

Finally, we obtain the magnetization

$$M_\pm = \frac{\kappa}{4\pi^2}\sum_{s=\pm1} s\int_{m-s\kappa B}^{\infty} dE\, f_\pm(E,T,\mu) \times\left[\hat{k}(s\kappa B+m) + E^2\left(\arctan\left(\frac{s\kappa B-m}{\hat{k}}\right)+\frac{\pi}{2}\right)\right]. \tag{40}$$

We see that the magnetization vanishes when $\kappa \to 0$. In addition, with these expressions one finds $P_{\perp,\pm} = P_{\parallel,\pm} - M_\pm B$.

B. Zero temperature

In the zero temperature limit there is only a particle contribution since $\lim_{T\to0} f_-(E,T,\mu) = 0$ for $E \geq 0$ and $\lim_{T\to0} f_+(E,T,\mu) = \Theta(\mu - E)$. Using the results listed in the previous subsection one finds for the number density [25]

$$n = \frac{1}{4\pi^2}\sum_{s=\pm1}\left[\frac{k_F}{3}\left(2k_F^2 - 3s\kappa B\hat{m}\right) -s\kappa B\mu^2\left(\arctan\left(\frac{\hat{m}}{k_F}\right)-\frac{\pi}{2}\right)\right], \tag{41}$$

where $\hat{m} = m - s\kappa B$, $k_F = \sqrt{\mu^2 - \hat{m}^2}$. Similarly the energy density can be obtained in this limit [25]

$$\epsilon = \frac{1}{48\pi^2}\sum_{s=\pm1}\Bigg[k_F\mu(6\mu^2 - 3\hat{m}^2 - 4s\kappa B\hat{m}) -8s\kappa B\mu^3\left(\arctan\left(\frac{\hat{m}}{k_F}\right)-\frac{\pi}{2}\right) -\hat{m}^3(3\hat{m}+4s\kappa B)\log\left(\frac{k_F+\mu}{\hat{m}}\right)\Bigg]. \tag{42}$$

And the longitudinal pressure can also be easily obtained

$$P_\parallel = \frac{1}{48\pi^2}\sum_{s=\pm1}\Bigg[k_F\mu(2\mu^2 - 5\hat{m}^2 - 8s\kappa B\hat{m}) -4s\kappa B\mu^3\left(\arctan\left(\frac{\hat{m}}{k_F}\right)-\frac{\pi}{2}\right) +\hat{m}^3(3\hat{m}+4s\kappa B)\log\left(\frac{k_F+\mu}{\hat{m}}\right)\Bigg]. \tag{43}$$

Using the derived expressions for n, ϵ, and $P_\parallel$ one can show that $\epsilon + P_\parallel = \mu n$ is satisfied explicitly. The transverse pressure is

$$P_\perp = \frac{1}{48\pi^2}\sum_{s=\pm1}\Bigg[k_F\mu\left(2\mu^2 - 5\hat{m}^2 - 12s\kappa B\hat{m} - 12(s\kappa B)^2\right) +3\hat{m}^2(\hat{m}+2s\kappa B)^2\log\left(\frac{k_F+\mu}{\hat{m}}\right)\Bigg]. \tag{44}$$

Finally, evaluating $\partial P_\parallel/\partial B$ one obtains the magnetization in this case [25][6]

$$M = \frac{\kappa}{12\pi^2}\sum_{s=\pm1} s\Bigg[\mu k_F(3s\kappa B+\hat{m}) -\mu^3\left(\arctan\left(\frac{\hat{m}}{k_F}\right)-\frac{\pi}{2}\right) -\hat{m}^2(3s\kappa B+2\hat{m})\log\left(\frac{k_F+\mu}{\hat{m}}\right)\Bigg]. \tag{45}$$

From this result we can once again verify that $P_\perp = P_\parallel - MB$.

V. NUMERICAL RESULTS

In this section we present numerical evaluation of the transverse and longitudinal pressures derived in the previous section. For the numerics that follow we will assume (i) a gas of protons with a mass $m = m_p = 0.939$ GeV, electric charge $q = +e$, and an anomalous magnetic moment of $\kappa = \kappa_p\mu_N = 1.79 \cdot e/(2m_p) =$

[6] We note that there appear to be some typos in the expression contained in Ref. [25].

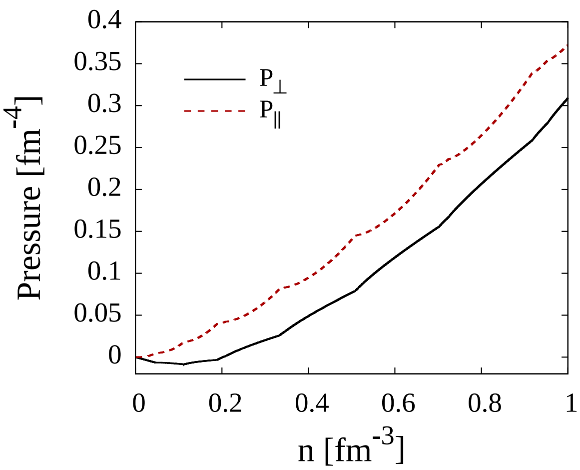

FIG. 1. (Color online) Transverse and longitudinal pressures of a zero temperature gas of protons as a function of the number density. Results include the effect of the proton anomalous magnetic momentum.

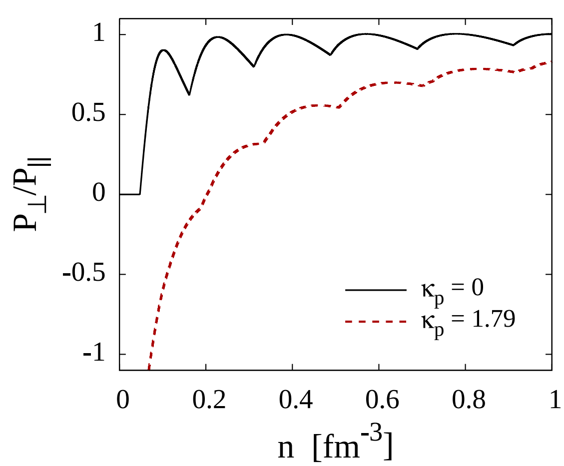

FIG. 2. (Color online) Ratio of transverse and longitudinal pressures of a zero temperature gas of protons as a function of the number density. Results are shown with and without the effect of the proton anomalous magnetic moment.

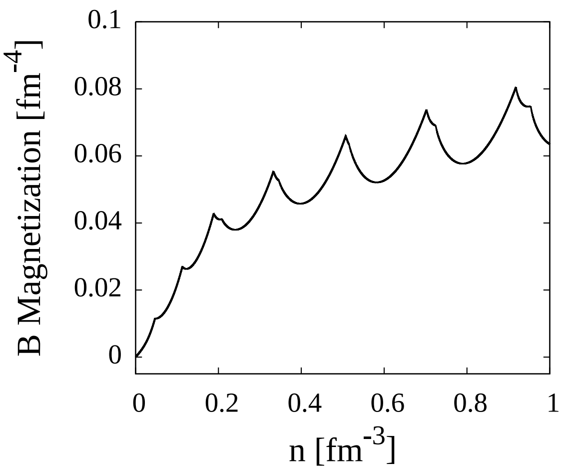

FIG. 3. (Color online) Magnetization of a zero temperature gas of protons times the background magnetic field. Result includes the effect of the proton anomalous magnetic moment.

0.288633 GeV^{-1} in Heaviside-Lorentz natural units [7] and (ii) a gas of neutrons with a mass $m = m_n = 0.939$ GeV, electric charge $q = 0$, and an anomalous magnetic momentum of $\kappa = \kappa_n \mu_N = -1.91 \cdot e/(2m_n) = -0.307983$ GeV^{-1} [38]. In all cases shown we consider a magnetic field magnitude of 5×10^{18} Gauss.

In Fig. 1 we plot the transverse and longitudinal pressures of a zero temperature gas of protons including the effect of the anomalous magnetic moment. The cusps in the curves correspond to threshold crossings for the maximum Landau level. As can be seen from this figure, the transverse and longitudinal pressures are not equal. In addition, one can see from the figure that at low densities the transverse pressure is negative at low densities when there is a non-vanishing anomalous magnetic moment, while the longitudinal pressure remains positive at all densities.

In Fig. 2 we show the ratio of the transverse to longitudinal pressures for a zero temperature gas of protons with and without the effect of the anomalous magnetic moment. In both cases we once again see cusps indicative of Landau level crossings and a vanishing transverse pressure at low densities. From this figure we also see that including the anomalous magnetic moment enhances the pressure anisotropy.

In Fig. 3 we plot the background magnetic field times the magnetization of a zero temperature gas of protons obtained via Eq. (21). We note that there are two distinct sets of cusps visible in Fig. 3. This is due to the fact that, when the effect of the anomalous magnetic moment is included, there are two different Landau level thresholds for particles with spins aligned or anti-aligned with the background magnetic field.

In Fig. 4 we plot the ratio of the transverse pressure to the longitudinal pressure of a gas of protons as a function of the net proton density (particle minus anti-particle) for $T = \{0, 10, 30, 500\}$ MeV. As can be seen from this figure, as the temperature is increased, the cusps associated with Landau level crossings are diminished and the level of the pressure anisotropy also decreases. The highest temperature shown $T = 500$ MeV is on the order of those initially

[7] In Gaussian natural units one has $\mu_N = 0.0454871$ GeV^{-1} which is the Heaviside-Lorentz value divided by $\sqrt{4\pi}$. Note that if one uses Gaussian natural units, the magnetic field in GeV^2 is scaled by a factor of $\sqrt{4\pi}$ compared to the corresponding Heaviside-Lorentz magnetic field. As a result the product of $\mu_N B$ is independent of the convention chosen.

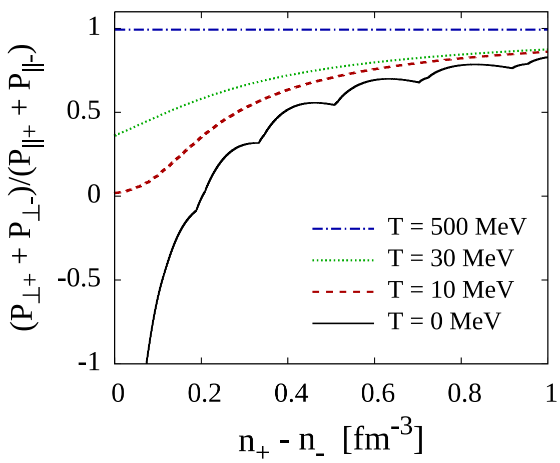

FIG. 4. (Color online) Ratio of transverse to longitudinal pressure of a gas of protons as a function of the net proton density for four different temperatures $T = \{0, 10, 30, 500\}$ MeV. Results include the effect of the proton anomalous magnetic moment.

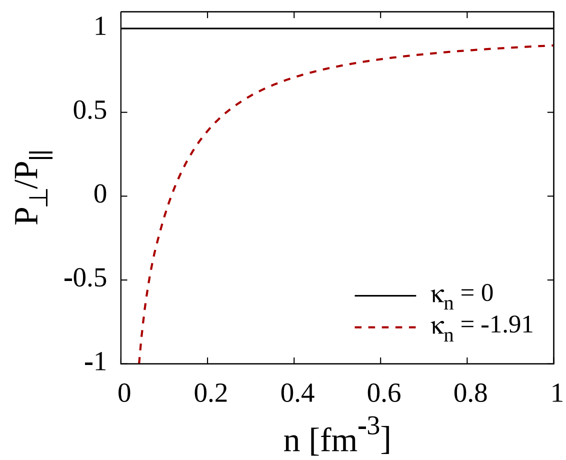

FIG. 5. (Color online) Ratio of transverse and longitudinal pressures of zero temperature gas of neutrons as a function of number density. Results are shown with and without the effect of the neutron anomalous magnetic moment.

generated in relativistic heavy ion collisions at CERN's Large Hadron Collider. As we see, at these high temperatures the pressure anisotropy for charged particles is quite small, $\lesssim 1\%$. However, it should be noted that as the system cools, the pressure anisotropy increases.

We consider next the case of neutral particles, focusing on a specific example of a gas of neutrons. In Fig. 5 we plot the ratio of the transverse to longitudinal pressures of a gas of neutrons as a function of the neutron density with and without the effect of the neutron anomalous magnetic moment. This figure shows that without the anomalous magnetic moment the pressures are completely isotropic; however, when there is a non-vanishing anomalous magnetic moment the pressure anisotropy can be quite sizable. In Fig. 6 we show the ratio of the total particle plus anti-particle transverse to longitudinal pressures. This figure shows that as the temperature of the system increases, the amount of pressure anisotropy, again, decreases.

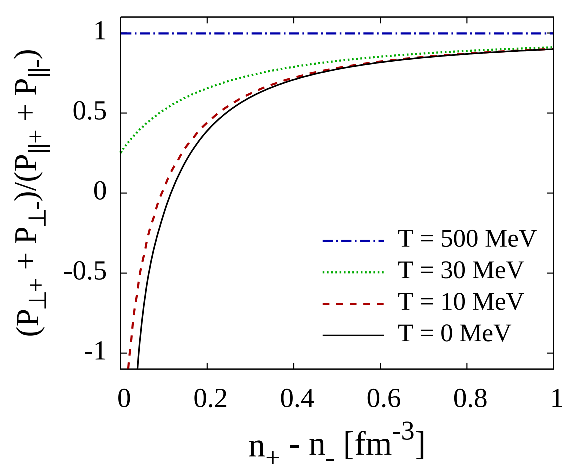

FIG. 6. (Color online) Ratio of transverse and longitudinal pressures of a gas of neutrons as a function of the net neutron density for four different temperature $T = \{0, 10, 30, 500\}$ MeV. Results include the effect of the neutron anomalous magnetic moment.

VI. CONCLUSIONS AND OUTLOOK

In this paper we have revisited the calculation of the matter contribution to the energy-momentum tensor of a Fermi gas of spin one-half particles subject to an external magnetic field. We considered both charged and uncharged particles with and without the effect of the anomalous magnetic moment. For zero temperature systems we demonstrated through explicit calculation that the resulting energy density, number density, and longitudinal pressure satisfy $\epsilon + P_\| = \mu n$. Using the standard definition of the grand potential $\Omega = \epsilon - \mu n$ allowed us to see that, in all cases investigated, the grand potential is related to the longitudinal pressure via $\Omega = -P_\|$ in agreement with previous findings in the literature.

We point out that some of the results contained herein are known in the literature. The results obtained for the transverse pressure of charged and uncharged particles with non-zero anomalous magnetic moment are new. In addition, we have presented in two appendices an explicit derivation of the necessary statistical averages of the energy-momentum tensor, taking into account the anomalous magnetic moment. Using the results obtained, we demonstrated that the standard relationship, $P_\perp = P_\| - MB$, between the transverse pressure, longitudinal pressure, and magnetization of the system holds in all cases considered.

The resulting formulae for the bulk properties can be applied to both zero temperature and finite temperature systems and hence could be useful in understanding the impact of high magnetic fields on the evolution of proto-neutron stars, proto-quark stars, and the matter generated in relativistic heavy ion collisions. Applying the derived formulae to a system of protons we found that there can exist a sizable pressure anisotropy in the matter contribution to the energy-momentum tensor which could have a phenomenological impact. Additionally we found that as the temperature of the system increases, the pressure anisotropy decreases. This is primarily due to the fact that increasing temperature allows higher Landau levels to be partially occupied and hence reduces the discrete effects one sees at zero temperature. For uncharged particles Landau quantization does not play a role and, instead, any pressure anisotropy exhibited comes from a non-vanishing anomalous magnetic moment. Once again as the temperature increases, the pressure anisotropy is reduced. This effect is due to the fact that as the temperature increases high momentum modes become highly occupied which causes momentum terms in the energy to dominate over those associated with the anomalous magnetic moment.

We note that although we presented results applicable to the case of a single particle type, the resulting formulae can be easily applied to the case of a system composed of multiple particle types. Since the contributing particles may have different pressure anisotropies depending on the sign and the magnitude of the anomalous magnetic moment, one must take care to sum over all particle types subject to the necessary conservation laws prior to making quantitative statements about the phenomenological impact of magnetic-field induced pressure anisotropies on dense matter [92]. Finally, we emphasize that although the numerical results shown in the results section assumed a particular magnetic field amplitude, the analytic results derived herein are completely general and as such can be applied to assess the impact of magnetic fields on the bulk properties of matter in a wide variety of situations.

ACKNOWLEDGMENTS

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Appendix A: Energy-momentum tensor

In this appendix we derive the energy-momentum tensor including the effect of the anomalous magnetic moment. For this purpose we will use the method of metric perturbations which allows one to most efficiently compute a symmetric and gauge-invariant energy-momentum tensor. The starting point is the following relation between the variation of the action and the energy momentum tensor

$$\delta\mathcal{S} = \frac{1}{2}\int d^4x\,\sqrt{-g}\,\mathcal{T}^{\mu\nu}\,\delta g_{\mu\nu}\,, \tag{A1}$$

where $g \equiv \det(g_{\mu\nu})$. We proceed in the standard way by writing the action in terms of the Lagrangian density, varying the metric, identifying the energy-momentum tensor by comparison with (A1), and finally taking $g^{\mu\nu} \to \eta^{\mu\nu}$ where $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski-space metric.

We begin with the curved-space Lagrangian density for a spin one-half fermion with charge q in the presence of an external magnetic field including the effect of the anomalous magnetic moment

$$\mathcal{L} = \bar{\psi}(i\not{D} - m + \frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu})\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \tag{A2}$$

where κ is the anomalous magnetic moment and, as usual, $\not{a} \equiv \gamma^\mu a_\mu$, $D_\mu \equiv \frac{1}{2}(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) + \Gamma_\mu + iqA_\mu$ with Γ_μ being the spin connection which is zero in flat space, and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. This allows us to write the covariantized action as $\mathcal{S} = \int d^4x\,\sqrt{-g}\,\mathcal{L} = \mathcal{S}_m + \mathcal{S}_f$ with

$$\mathcal{S}_m = \int d^4x\,\sqrt{-g}\,\bar{\psi}\bigg[\frac{i}{2}\gamma^\alpha D^\beta(g_{\alpha\beta} + g_{\beta\alpha}) - m + \frac{1}{8}\kappa\sigma^{\alpha\beta}F^{\gamma\delta}(g_{\alpha\gamma} + g_{\gamma\alpha})(g_{\beta\delta} + g_{\delta\beta})\bigg]\psi\,, \tag{A3}$$

$$\mathcal{S}_f = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,F^{\alpha\beta}F^{\gamma\delta}g_{\alpha\gamma}g_{\beta\delta}\,, \tag{A4}$$

where we have split the action into matter and field contributions and used the the fact that the metric tensor is symmetric to explicitly symmetrize the matter contribution. First, we evaluate $\delta\mathcal{S}$ making use of the identity $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}$. Note, importantly, that the gamma matrices themselves depend on the metric and therefore one needs to take into account their variation under metric variation. The variation can be computed with [93] or without [94] the use of vierbeins. Computing the variation, identifying $\mathcal{T}^{\mu\nu}$, and taking the limit $g^{\mu\nu} \to \eta^{\mu\nu}$ one finds the following expressions for the matter and field contributions to the energy-momentum tensor in flat space

$$\mathcal{T}_m^{\mu\nu} = \bar{\psi}\bigg[\frac{i}{2}\left(\gamma^\mu D^\nu + \gamma^\nu D^\mu\right) + \frac{1}{2}\kappa\left(\sigma^{\mu\alpha}F^\nu{}_\alpha + \sigma^{\nu\alpha}F^\mu{}_\alpha\right)\bigg]\psi - \eta^{\mu\nu}\mathcal{L}_m\,, \tag{A5}$$

$$\mathcal{T}_f^{\mu\nu} = -F^{\mu\alpha}F^\nu{}_\alpha - \eta^{\mu\nu}\mathcal{L}_f\,, \tag{A6}$$

where $\mathcal{L}_m$ and $\mathcal{L}_f$ are the matter and field contributions to the Lagrangian density corresponding to the first and second terms in Eq. (A2), respectively.

Appendix B: Matter contribution to $T^{\mu\nu}$

In this appendix we derive expressions for the energy-momentum tensor in a uniform background magnetic field. We focus on the matter contribution since the field contribution (A6) is standard. In the rest of this appendix we can therefore ignore the pure gauge field term in the Lagrangian. In flat space the Lagrangian density for a spin one-half fermion with charge q in the presence of an external magnetic field including the effect of the anomalous magnetic moment is

$$\mathcal{L} = \bar{\psi}(i\not{D} - m + \frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu})\psi\,, \tag{B1}$$

where κ is the anomalous magnetic moment and, as usual, $\not{a} \equiv \gamma^\mu a_\mu$, $D_\mu \equiv \frac{1}{2}(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) + iqA_\mu$, and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. The equations of motion for ψ and $\bar{\psi}$ can be determined using

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = 0\,, \\ &\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = 0\,, \end{aligned} \tag{B2}$$

which result in

$$(i\not{\partial} - q\not{A} - m + \frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu})\psi = 0\,, \tag{B3}$$

$$i\partial_\mu\bar{\psi}\gamma^\mu + \bar{\psi}(q\not{A} + m - \frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu}) = 0\,. \tag{B4}$$

We note for application to the calculation of $T^{\mu\nu}$ that if we multiply the first equation from the left by $\bar{\psi}$ we obtain $\mathcal{L} = 0$. This demonstrates that the matter Lagrangian density vanishes when evaluated with solutions which obey the equations of motion. This allows us to simplify Eq. (A5) to

$$\mathcal{T}^{\mu\nu} = \bar{\psi}\Bigg[\frac{i}{2}\left(\gamma^\mu D^\nu + \gamma^\nu D^\mu\right) + \frac{1}{2}\kappa\left(\sigma^{\mu\alpha}F^\nu{}_\alpha + \sigma^{\nu\alpha}F^\mu{}_\alpha\right)\Bigg]\psi\,, \tag{B5}$$

To evaluate the necessary statistical average of $\mathcal{T}^{\mu\nu}$ we first need to solve the equations of motion in order to determine the energy eigenvalues and spinors. The spinors and energy eigenvalues are available in the literature [90, 95–97]; however, we review the derivation for sake of completeness and then use the resulting spinors to evaluate the statistical averages of $\mathcal{T}^{\mu\nu}$. We note that the spinor solutions have been expressed in various different forms in the literature. We present a specific compact form for the spinors, however, we have explicitly verified that using the forms of the spinors presented in Refs. [90, 95–97] yields the same final results.

As in the main body of the text, we choose the magnetic field to point along the z-direction. Choosing the vector potential to be $A^\mu = (0, -By, 0, 0)$ we have $F^{\mu\nu} = B(\delta^{\mu x}\delta^{\nu y} - \delta^{\nu x}\delta^{\mu y})$ and as a result

$$\frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu} = i\kappa B\gamma^x\gamma^y = \kappa B\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} \equiv \kappa B\mathcal{S}_3\,. \tag{B6}$$

Next we write Eq. (B3) in Hamiltonian form by searching for static solutions of the form $\psi = e^{-iEt}\Psi(\mathbf{x})$ which results in the Dirac-Pauli equation [98]

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \gamma^0 m - \kappa B\gamma^0\mathcal{S}_3)\Psi = E\Psi\,, \tag{B7}$$

where $\boldsymbol{\alpha} \equiv \gamma^0\boldsymbol{\gamma}$ and $\boldsymbol{\pi} \equiv -i\boldsymbol{\nabla} - q\mathbf{A}$.

Here we are interested in the diagonal components of $\mathcal{T}^{\mu\nu}$ which for a constant magnetic field are given by

$$\mathcal{T}^{00} = \bar{\psi}\left(i\gamma^0 D^0\right)\psi\,, \tag{B8}$$

$$\mathcal{T}^{xx} = \bar{\psi}\left(i\gamma^x D^x - \kappa B\sigma^{xy}\right)\psi\,, \tag{B9}$$

$$\mathcal{T}^{yy} = \bar{\psi}\left(i\gamma^y D^y - \kappa B\sigma^{xy}\right)\psi\,, \tag{B10}$$

$$\mathcal{T}^{zz} = \bar{\psi}\left(i\gamma^z D^z\right)\psi\,. \tag{B11}$$

1. Charged particles

We now search for the solution of the Dirac-Pauli equation for charged particles. Based on the structure of the equation, we begin by making an ansatz for the bi-spinor Ψ of the form $\Psi(\mathbf{x}) = e^{ik_x x}e^{ik_z z}u_n^{(s)}(y)$ with [90]

$$u_n^{(s)}(y) = \begin{pmatrix} c_1\phi_\nu(y)\\ c_2\phi_{\nu-1}(y)\\ c_3\phi_\nu(y)\\ c_4\phi_{\nu-1}(y)\end{pmatrix}, \tag{B12}$$

where

$$\nu = n + \frac{1}{2} - \frac{s}{2}\frac{q}{|q|}\,, \tag{B13}$$

with $n = 0, 1, 2, \cdots$.[8] The constants c_i above implicitly depend on the spin alignment $s = \pm 1$. The functions ϕ_n are given by

$$\phi_n(\xi) = N_n e^{-\xi^2/2}H_n(\xi)\,, \tag{B14}$$

where the variable ξ is

$$\xi = \sqrt{|q|B}\left(y + \frac{k_x}{qB}\right), \tag{B15}$$

$n \geq 0$ is an integer, H_n is a Hermite polynomial, and $N_n = (qB)^{1/4}(\sqrt{\pi}2^n n!)^{-1/2}$ is a normalization constant

[8] When $\nu = 0$ there could be an issue with the Hermite functions with index $\nu - 1$ not being well defined; however, as we will show below, in this case one finds that the coefficients vanish identically.

which ensures $\int_{-\infty}^{\infty} dy\, \phi_n^2(y) = 1$. Inserting this ansatz and simplifying the Dirac-Pauli equation, one obtains

$$\begin{pmatrix} m-\kappa B & 0 & k_z & k_\nu \\ 0 & m+\kappa B & k_\nu & -k_z \\ k_z & k_\nu & -m+\kappa B & 0 \\ k_\nu & -k_z & 0 & -m-\kappa B \end{pmatrix} \chi = E\chi\,, \tag{B16}$$

where $\chi = (c_1\, c_2\, c_3\, c_4)^{\mathrm{T}}$ and $k_\nu = \sqrt{2|q|B\nu}$. Evaluating the determinant of the matrix on the left we obtain the energy eigenvalues [90]

$$E_s = \pm\sqrt{k_z^2 + (\lambda - s\kappa B)^2}\,, \tag{B17}$$

where $\lambda \equiv \sqrt{m^2 + k_\nu^2}$. The choice of an overall positive sign for the energy eigenvalue above corresponds to particle states and the negative sign to anti-particle states. Without loss of generality we can focus on the positive energy states and, in the end, extend the result to include the necessary contribution from the negative energy states.

The resulting positive energy eigenvectors are

$$\chi^{(s)} = \frac{1}{\sqrt{2\lambda\alpha_s\beta_s}} \begin{pmatrix} s\alpha_s\beta_s \\ -k_z k_\nu \\ s\beta_s k_z \\ \alpha_s k_\nu \end{pmatrix}, \tag{B18}$$

where $\alpha_s \equiv E_s - \kappa B + s\lambda$ and $\beta_s \equiv \lambda + sm$. The overall normalization of the state is fixed by requiring that $\int_{-\infty}^{\infty} dy\, u_n^{(r)\dagger}(\mathbf{x})\, u_m^{(s)}(\mathbf{x}) = 2E_s\delta^{rs}\delta_{nm}$. The general quantum state for positive energy states can now be constructed

$$\psi(x) = \sum_{s=\pm1} \sum\!\!\!\!\!\!\int_k b_s(\mathbf{k}) u^{(s)}(\mathbf{k}) e^{i\tilde{k}_\mu x^\mu}\,, \tag{B19}$$

where $b_s(\mathbf{k})$ is a particle creation operator which obeys

$$\{b_r(\mathbf{p}), b_s^\dagger(\mathbf{k})\} = (2\pi)\delta_{rs}\delta_{nm}\delta(p_z - k_z)\,, \tag{B20}$$

$\mathbf{k} = (n, k_z)$ with $n = 0, 1, 2, \cdots$, $\tilde{k} = (E_k, k_x, 0, k_z)$, and

$$\sum\!\!\!\!\!\!\int_k \equiv \frac{|q|B}{2\pi} \sum_n \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{1}{\sqrt{2E_k}}\,. \tag{B21}$$

Note that the factor of $\sqrt{2E_k}$ in the denominator above is fixed by the spinor normalization used above.

a. Energy Density

To determine the energy density, we begin by evaluating the 00 component of the energy-momentum density which is equivalent to the Hamiltonian density $\mathcal{T}^{00} = \mathcal{H} = i\psi^\dagger\partial_t\psi$. Integrating over space gives the Hamiltonian

$$\begin{aligned} H &= i\int_x \psi^\dagger \partial_t \psi \\ &= i\sum_{r,s}\int_x \sum\!\!\!\!\!\!\int_p \sum\!\!\!\!\!\!\int_k \left[b_r^\dagger(\mathbf{p}) u^{(r)\dagger}(\mathbf{p}) e^{i\tilde{p}_\mu x^\mu} \right] \\ &\qquad \times \left[b_s(\mathbf{k}) u^{(s)}(\mathbf{k})(-iE_k) e^{-i\tilde{k}_\mu x^\mu} \right], \end{aligned} \tag{B22}$$

where $\int_x \equiv \int d^3x$. Using the orthonormality relations listed above one finds

$$H = \frac{|q|B}{2\pi} \sum_{s=\pm1} \sum_n \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} E_k\, b_s^\dagger(\mathbf{k}) b_s(\mathbf{k})\,. \tag{B23}$$

We can now compute the thermal average of the energy using the density matrix ρ

$$\rho = e^{-\beta H + \alpha N}\,, \tag{B24}$$

where is $\beta = 1/T$ is the inverse temperature, $\alpha = \beta\mu$ with μ being the chemical potential, H is the Hamiltonian operator, and

$$N = \frac{|q|B}{2\pi} \sum_{s=\pm1} \sum_n \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} b_s^\dagger(\mathbf{k}) b_s(\mathbf{k})\,, \tag{B25}$$

is the number operator. The statistical average of the Hamiltonian operator gives the energy density

$$\epsilon \equiv \langle H \rangle = \frac{\mathrm{Tr}[\rho \mathrm{H}]}{\mathrm{Tr}[\rho]}\,. \tag{B26}$$

Using the Baker-Campbell-Hausdorff formula one obtains

$$\langle b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) \rangle = \langle b_s(\mathbf{k}) b_s^\dagger(\mathbf{k}) \rangle e^{-\beta(E-\mu)}\,, \tag{B27}$$

which, upon application of the anti-commutation relations for the creation operators, gives the Fermi-Dirac distribution for particles

$$\langle b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) \rangle = \frac{1}{e^{\beta(E_k-\mu)} + 1} = f_+(E_k, T, \mu)\,. \tag{B28}$$

With this we obtain our final expression for the particle contribution to the energy density

$$\epsilon = \langle H \rangle = \frac{|q|B}{2\pi} \sum_{s=\pm1} \sum_n \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} E_k f_+(E_k, T, \mu)\,. \tag{B29}$$

Note that if one includes the anti-particle states, one must normal order the Hamiltonian operator prior to performing the statistical average.

b. Number Density

Based on the above discussion, the number density can easily be seen to be given by

$$n = \langle N \rangle = \frac{|q|B}{2\pi} \sum_{s=\pm1} \sum_n \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} f_+(E_k, T, \mu)\,. \tag{B30}$$

c. Longitudinal Pressure

We now consider the longitudinal pressure which is given by $P_\parallel \equiv \langle \int_x \mathcal{T}^{zz} \rangle$ with

$$\mathcal{T}^{zz} = \frac{i}{2} \left[\bar{\psi}\gamma^z \partial^z \psi - (\partial^z \bar{\psi})\gamma^z \psi \right]. \tag{B31}$$

Plugging in the explicit forms for the spinors we have

$$\begin{aligned}\int_x \mathcal{T}^{zz} = \frac{i}{2}\sum_{r,s}\int_x \sum\!\!\!\!\!\!\int_p \sum\!\!\!\!\!\!\int_k \Big\{ &\left[b_r^\dagger(\mathbf{p}) u^{(r)\dagger}(\mathbf{p}) e^{i\tilde{p}_\mu x^\mu}\right] \\ &\times \gamma^0\gamma^z \left[b_s(\mathbf{k}) u^{(s)}(\mathbf{k})(ik^z) e^{-i\tilde{k}_\mu x^\mu}\right] \\ &- \left[b_r^\dagger(\mathbf{p}) u^{(r)\dagger}(\mathbf{p})(-ip^z) e^{i\tilde{p}_\mu x^\mu}\right] \\ &\times \gamma^0\gamma^z \left[b_s(\mathbf{k}) u^{(s)}(\mathbf{k}) e^{-i\tilde{k}_\mu x^\mu}\right] \Big\}. \end{aligned} \tag{B32}$$

Evaluating the x and p (sum-)integrals, making use of the orthonormality relations and then taking the statistical average gives

$$\begin{aligned} P_\parallel = -\frac{1}{2}\frac{|q|B}{2\pi}\sum_{s=\pm1}\sum_n \int_{-\infty}^{\infty} \frac{dk_z}{2\pi}\frac{k^z}{E_k}\langle b_s^\dagger(\mathbf{k}) b_s(\mathbf{k})\rangle \\ \times \int_{-\infty}^{\infty} dy \left[u^{(s)\dagger}(\mathbf{k})\gamma^0\gamma^z u^{(s)}(\mathbf{k})\right], \end{aligned} \tag{B33}$$

where we have used the fact that $\langle b_r^\dagger(\mathbf{k}) b_s(\mathbf{k})\rangle$ vanishes unless $r = s$. Next we need to evaluate the spinor contraction

$$\int_{-\infty}^{\infty} dy\, u^{(s)\dagger}(\mathbf{k})\gamma^0\gamma^z u^{(s)}(\mathbf{k}) = -2k^z, \tag{B34}$$

which follows from the explicit form of the spinors obtained previously. Using this and rewriting the statistical average of the number operator as a Fermi-Dirac distribution, one obtains

$$P_\parallel = \frac{|q|B}{2\pi}\sum_{s=\pm1}\sum_n \int_{-\infty}^{\infty} \frac{dk_z}{2\pi}\frac{k^z k^z}{E_k} f_+(E_k, T, \mu). \tag{B35}$$

d. Transverse Pressure

We finally turn our attention to the transverse pressure. By rotational symmetry, $P_\perp \equiv \langle \int_x \mathcal{T}^{yy} \rangle = \langle \int_x \mathcal{T}^{xx} \rangle$. Choosing the former, which is somewhat easier to evaluate, we should integrate and statistically average

$$\mathcal{T}^{yy} = \frac{i}{2} \left[\bar{\psi}\gamma^y \partial^y \psi - (\partial^y \bar{\psi})\gamma^y \psi \right] - \kappa B \bar{\psi}\sigma^{xy}\psi. \tag{B36}$$

Plugging in the explicit forms for the spinors we have

$$\begin{aligned}\int_x \mathcal{T}^{yy} = \frac{i}{2}\sum_{r,s}\int_x \sum\!\!\!\!\!\!\int_p \sum\!\!\!\!\!\!\int_k \Big\{ &\left[b_r^\dagger(\mathbf{p}) u^{(r)\dagger}(\mathbf{p}) e^{i\tilde{p}_\mu x^\mu}\right] \\ &\times \gamma^0\gamma^y \left[b_s(\mathbf{k}) \partial^y u^{(s)}(\mathbf{k}) e^{-i\tilde{k}_\mu x^\mu}\right] \\ &- \left[b_r^\dagger(\mathbf{p}) \partial^y u^{(r)\dagger}(\mathbf{p}) e^{i\tilde{p}_\mu x^\mu}\right] \\ &\times \gamma^0\gamma^y \left[b_s(\mathbf{k}) u^{(s)}(\mathbf{k}) e^{-i\tilde{k}_\mu x^\mu}\right] \Big\} \\ -\kappa B \sum_{r,s}\int_x \sum\!\!\!\!\!\!\int_p \sum\!\!\!\!\!\!\int_k &\left[b_r^\dagger(\mathbf{p}) u^{(r)\dagger}(\mathbf{p}) e^{i\tilde{p}_\mu x^\mu}\right] \\ &\times \gamma^0\sigma^{xy} \left[b_s(\mathbf{k}) u^{(s)}(\mathbf{k}) e^{-i\tilde{k}_\mu x^\mu}\right]. \end{aligned} \tag{B37}$$

Evaluating the x and p (sum-)integrals making use of the orthonormality relations and then taking the statistical average gives

$$\begin{aligned} P_\perp = \frac{|q|B}{2\pi}\sum_{s=\pm1}\sum_n \int_{-\infty}^{\infty} \frac{dk_z}{2\pi}\frac{1}{E_k}\langle b_s^\dagger(\mathbf{k}) b_s(\mathbf{k})\rangle \\ \times \Big\{ \frac{i}{2}\int_{-\infty}^{\infty} dy \Big[u^{(s)\dagger}(\mathbf{k})\gamma^0\gamma^y\partial^y u^{(s)}(\mathbf{k}) \\ - \partial^y u^{(s)\dagger}(\mathbf{k})\gamma^0\gamma^y u^{(s)}(\mathbf{k}) \Big] \\ -\kappa B \int_{-\infty}^{\infty} dy\, u^{(s)\dagger}(\mathbf{k})\gamma^0\sigma^{xy} u^{(s)}(\mathbf{k}) \Big\}. \end{aligned} \tag{B38}$$

Integrating by parts one finds that the second term contributes the same as the first. Using the explicit representation of the spinors obtained above one finds

$$\begin{aligned} &\int_{-\infty}^{\infty} dy\, u^{(s)\dagger}(\mathbf{k})\gamma^0\gamma^y\partial^y u^{(s)}(\mathbf{k}) \\ &\quad = -i(c_2c_3 - c_1c_4) \\ &\qquad \times \int_{-\infty}^{\infty} d\xi \left(\phi_{\nu-1}\partial_\xi\phi_\nu - \phi_\nu\partial_\xi\phi_{\nu-1}\right) \\ &\quad = -i\sqrt{2|q|B\nu}(c_2c_3 - c_1c_4) \\ &\quad = -i2|q|B\nu\left(1 - \frac{s\kappa B}{\lambda}\right), \end{aligned} \tag{B39}$$

and

$$\int_{-\infty}^{\infty} dy\, u^{(s)\dagger}(\mathbf{k})\gamma^0\sigma^{xy} u^{(s)}(\mathbf{k}) = 2s(\lambda - s\kappa B). \tag{B40}$$

With this we can write down our final expression for the transverse pressure for charged particles

$$\begin{aligned} P_\perp = \frac{|q|B^2}{2\pi^2}\sum_{s=\pm1}\sum_n \int_{-\infty}^{\infty} dk_z \frac{1}{E_k} f_+(E_k, T, \mu) \\ \times \left[\frac{|q|\nu\bar{m}(\nu)}{\sqrt{m^2 + 2\nu|q|B}} - s\kappa\bar{m}(\nu)\right]. \end{aligned} \tag{B41}$$

where $\bar{m}(\nu) \equiv \sqrt{m^2 + 2\nu|q|B} - s\kappa B$.

2. Uncharged Particles

We now consider the case of uncharged particles. This case is different since the transverse momenta of the particles are not quantized. Starting from the Dirac-Pauli equation (B7) we make an ansatz for the bi-spinor Ψ of the form $\Psi(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} u$ with $u = (c_1\ c_2\ c_3\ c_4)^{\rm T}$. This results in the following matrix equation for u

$$\begin{pmatrix} m-\kappa B & 0 & k_z & k_- \\ 0 & m+\kappa B & k_+ & -k_z \\ k_z & k_- & -m+\kappa B & 0 \\ k_+ & -k_z & 0 & -m-\kappa B \end{pmatrix} u = E u\,, \tag{B42}$$

where $k_\pm \equiv k_x \pm i k_y$. Evaluating the determinant of the matrix on the left we obtain the energy eigenvalues

$$E_s = \pm\sqrt{k_z^2 + (\lambda - s\kappa B)^2}\,, \tag{B43}$$

where now we have $\lambda \equiv \sqrt{m^2 + k_\perp^2}$ with $k_\perp^2 = k_x^2 + k_y^2$. Once again the choice of an overall positive sign corresponds to particle states and negative sign to anti-particle states. We focus on particle states since the result is straightforward to extend to anti-particles.

The resulting positive energy solutions are

$$u^{(s)} = \frac{1}{\sqrt{2\lambda\alpha_s\beta_s}} \begin{pmatrix} s\alpha_s\beta_s \\ -k_z k_+ \\ s\beta_s k_z \\ \alpha_s k_+ \end{pmatrix}, \tag{B44}$$

where as before $\alpha_s \equiv E_s - \kappa B + s\lambda$ and $\beta_s \equiv \lambda + s m$. The overall normalization of the state is fixed in this case by requiring that $u^{(r)\dagger}\, u^{(s)} = 2E_s \delta^{rs}$. The general quantum state for positive energy states can now be constructed

$$\psi(x) = \sum_{s=\pm 1} \int_k \frac{1}{\sqrt{2E_k}}\, b_s(\mathbf{k}) u^{(s)}(\mathbf{k}) e^{i k_\mu x^\mu}\,, \tag{B45}$$

where $b_s(\mathbf{k})$ is a particle creation operator and $\int_k = (2\pi)^{-3}\int d^3k$. Once again the factor of $\sqrt{2E_k}$ in the denominator above is fixed by the spinor normalization used above.

Following the same general procedures used in the charged particle derivation one obtains the following result for the energy density

$$\epsilon = \langle H \rangle = \sum_{s=\pm 1} \int_k E_k f_+(E_k, T, \mu)\,. \tag{B46}$$

The result for the number density is

$$n = \langle N \rangle = \sum_{s=\pm 1} \int_k f_+(E_k, T, \mu)\,. \tag{B47}$$

The result for the parallel pressure is

$$P_\parallel = \langle T^{zz} \rangle = \sum_{s=\pm 1} \int_k \frac{k_z^2}{E_k} f_+(E_k, T, \mu)\,. \tag{B48}$$

And, finally, the result for the transverse pressure $P_\perp = \langle T^{xx} \rangle = \langle T^{yy} \rangle$ is

$$P_\perp = \sum_{s=\pm 1} \int_k \frac{1}{E_k} \left[\frac{1}{2} \frac{k_\perp^2 \bar{m}}{\sqrt{m^2 + k_\perp^2}} - s\kappa B \bar{m} \right] f_+(E_k, T, \mu)\,. \tag{B49}$$

where $\bar{m} = \sqrt{m^2 + k_\perp^2} - s\kappa B$.

In all of the expressions above, we can perform two of the three integrations by making the following change of variables

$$\begin{aligned} k_x &= \sqrt{\lambda^2 - m^2}\cos\phi\,, \\ k_y &= \sqrt{\lambda^2 - m^2}\sin\phi\,, \\ k_z &= \sqrt{E^2 - (\lambda - s\kappa B)^2}\,. \end{aligned} \tag{B50}$$

Evaluating the Jacobian one finds

$$d^3k = \frac{E\lambda}{\sqrt{E^2 - (\lambda - s\kappa B)^2}}\, dE\, d\lambda\, d\phi\,. \tag{B51}$$

With this change of variables we obtain the number density

$$\begin{aligned} n &= \frac{1}{2\pi^2} \sum_{s=\pm 1} \int_{m - s\kappa B}^{\infty} dE\, E\, f_+(E, T, \mu) \\ &\quad \times \int_m^{E + s\kappa B} d\lambda \frac{\lambda}{\sqrt{E^2 - (\lambda - s\kappa B)^2}} \\ &= \frac{1}{2\pi^2} \sum_{s=\pm 1} \int_{m - s\kappa B}^{\infty} dE\, E\, f_+(E, T, \mu) \\ &\quad \times \left[\hat{k} + s\kappa B \left(\arctan\left(\frac{s\kappa B - m}{\hat{k}} \right) + \frac{\pi}{2} \right) \right], \end{aligned} \tag{B52}$$

where $\hat{k} \equiv \sqrt{E^2 - (m - s\kappa B)^2}$. The energy density is given by

$$\begin{aligned} \epsilon &= \frac{1}{2\pi^2} \sum_{s=\pm 1} \int_{m - s\kappa B}^{\infty} dE\, E^2\, f_+(E, T, \mu) \\ &\quad \times \int_m^{E + s\kappa B} d\lambda \frac{\lambda}{\sqrt{E^2 - (\lambda - s\kappa B)^2}} \\ &= \frac{1}{2\pi^2} \sum_{s=\pm 1} \int_{m - s\kappa B}^{\infty} dE\, E^2\, f_+(E, T, \mu) \\ &\quad \times \left[\hat{k} + s\kappa B \left(\arctan\left(\frac{s\kappa B - m}{\hat{k}} \right) + \frac{\pi}{2} \right) \right]. \end{aligned} \tag{B53}$$

For the parallel pressure we obtain

$$
\begin{aligned}
P_{\parallel} &= \frac{1}{2\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, f_{+}(E,T,\mu) \\
&\quad \times \int_{m}^{E+s\kappa B} d\lambda\, \lambda\sqrt{E^2-(\lambda^2-s\kappa B)^2}\,, \\
&= \frac{1}{24\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, f_{+}(E,T,\mu) \\
&\quad \times \Bigg\{ 2\hat{k}(s\kappa B-m)(2m+s\kappa B) \\
&\quad + E^2\left[4\hat{k}+6s\kappa B\left(\arctan\left(\frac{s\kappa B-m}{\hat{k}}\right)+\frac{\pi}{2}\right)\right]\Bigg\}\,,
\end{aligned}
\tag{B54}
$$

and for the perpendicular pressure we obtain

$$
\begin{aligned}
P_{\perp} &= \frac{1}{2\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, f_{+}(E,T,\mu)\int_{m}^{E+s\kappa B} d\lambda \\
&\quad \times \frac{\lambda-s\kappa B}{\sqrt{E^2-(\lambda^2-s\kappa B)^2}}\left[\frac{1}{2}(\lambda^2-m^2)-s\kappa B\lambda\right], \\
&= \frac{1}{6\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, f_{+}(E,T,\mu)(\hat{k}^3-3s\kappa Bm\hat{k})\,.
\end{aligned}
\tag{B55}
$$

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